Chapter 9

Prospect Theory and Mean-Variance Analysis

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The experimental results of prospect theory (PT) reveal suggest that investors make decisions based on change of wealth rather than total wealth, that preferences are S-shaped with a risk-seeking segment, and that probabilities are subjectively distorted. This article shows that while PT's findings are in sharp contradiction to the foundations of mean-variance (MV) analysis, counterintuitively, when diversification between assets is allowed, the MV and PT-efficient sets almost coincide. Thus one can employ the MV optimization algorithm to construct PT-efficient portfolios.

The Markowitz (1952a)-Tobin (1958) mean-variance (MV) rule is probably the most popular investment decision rule under uncertainty in economics and in finance, and it is widely employed by both academics and practitioners. The strength of the MV analysis is that in the case of normal return distributions the choice of *any* expected utility maximizing risk-averse individual will be according to the MV rule.[1] The MV framework is the foundation of the Sharpe (1964)-Lintner (1965) capital asset pricing model (CAPM), which is a cornerstone of modern finance. Moreover, the MV framework provides a very important practical procedure for the construction of efficient portfolios.

While standard economic theory and, in particular, MV analysis assume expected utility maximization and risk aversion, in a breakthrough article Kahneman and Tversky (1979) show that the actual behavior of individuals systematically and consistently violates these

We are grateful to Campbell Harvey and to the anonymous referee of this journal for their many helpful comments and suggestions. Financial support for this study was provided by the Krueger Fund and by the Zagagi Fund. Address correspondence to Haim Levy, Jerusalem School of Business Administration, The Hebrew University, Jerusalem 91905, Israel, or e-mail: mshlevy@mscc.huji.ac.il.

[1] Formally, when the return distributions are normal, the MV rule coincides with second-degree stochastic dominance (SSD) [see Hanoch and Levy (1969) and Levy (1998)]. The MV rule is also valid if one replaces the assumption of normality with the assumption of quadratic utility [Tobin (1958)]. However, this latter assumption is problematic as it implies increasing absolute risk aversion and utility which is decreasing beyond a certain wealth level [see Arrow (1971)]. In contrast, the assumption of normal return distributions is approximately valid for investment horizons longer than one month but shorter than several years. [For very short horizons, the return distribution is "fat tailed," as shown by Fama (1963), Mandelbrot (1963), Mantegna and Stanley (1995), and others].

The Review of Financial Studies Vol. 17, No. 4
doi:10.1093/rfs/hhg062 Advance Access publication October 15, 2003

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assumptions. The alternative framework they present is well known as prospect theory (PT). The three main findings of PT are

(a) Individuals make decisions based on *change* of wealth rather than the *total* wealth, which is in direct contradiction to expected utility theory.
(b) Risk aversion does not globally prevail — individuals are risk-seeking regarding losses.
(c) Individuals distort objective probabilities and subjectively transform them in a systematic way. Tversky and Kahneman (1992) extend PT and suggest cumulative prospect theory, in which the transformation is of the cumulative probability rather than the probability itself. The transformation of cumulative probability allows cumulative prospect theory to avoid violations of first-degree stochastic dominance (FSD). In this study we refer to the cumulative version of PT.

Many studies support the above findings of PT.[2] Only fairly recently, however, researchers have begun to explore the implications of PT for economics and finance.[3] This article investigates the implications of PT to Markowitz's portfolio theory. It is clear that each of PT's main elements is in sharp contradiction to the assumptions of MV analysis: the MV framework assumes expected utility maximization and risk aversion while the above-mentioned elements (a) and (b) are at odds with both of these assumptions. Furthermore, if individuals subjectively distort probabilities, the MV analysis seems inappropriate because, even if the objective probability distributions are normal, the subjective distributions are not. Thus it would seem that the MV framework is completely incompatible with PT. Does PT imply that the MV framework, which is so central in finance, should be abandoned? Can any of the effective MV machinery (e.g., the efficient diversification algorithm it provides) be employed to benefit investors with PT preferences? This article addresses these issues.

We employ stochastic dominance rules to prove our main claim, which is that the PT and MV efficient sets almost coincide. To be more specific, assume ε^1 and ε^2 are two random variables. Then the theory of stochastic

[2] For an extensive review of studies supporting PT, see Edwards (1996). A number of studies reject PT or some of its elements; see, for example, Battalio, Kagel, and Jiranyakul (1990), Casey (1994), Harless and Camerer (1994), and Luce (2000). In two recent experimental studies, Levy and Levy (2002a,b) reject the PT S-shape value function and find support for the Markowitz (1952b) value function, which has a reverse S-shape. It is interesting to note the main results presented here hold also for the set of reverse S-shape value functions. The proof of this statement is available from the authors upon request.

[3] For example, Shefrin and Statman (1985) and Ferris, Haugen, and Makhija (1988) show that PT can explain the empirically observed "disposition effect" — the disposition to sell winning stocks too early and to ride losing stocks too long. Thaler (1985) investigates the implications of PT to marketing. Benartzi and Thaler (1995) show that PT may explain the equity premium puzzle. Gomes (2000) investigates the implication of PT to trading volume, and Barberis, Huang, and Santos (2001) and Levy, DeGiorgi, and Hens (2003) investigate the consequences of PT for asset pricing.

dominance is concerned with the conditions on the CDFs of ε^1 and ε^2 under which

$$\mathrm{E}[u(\varepsilon^1)] < \mathrm{E}[u(\varepsilon^2)]$$

for some set $u \in \mathbf{U}$. The standard concepts of stochastic dominance refer to three situations: (a) $\mathbf{U}$ is the set of all increasing functions (first-degree stochastic dominance, FSD); (b) $\mathbf{U}$ is the set of increasing and concave functions (second-degree stochastic dominance, SSD); (c) $\mathbf{U}$ is the set of increasing and concave functions with a positive third derivative, $u''' > 0$, which reflects a preference for positive skewness (third-degree stochastic dominance, TSD). Recently the theory of stochastic dominance has been extended to the set of functions that are S-shaped, and that include the family of prospect theory value functions (prospect stochastic dominance, PSD), which have been suggested by Kahneman and Tversky (1979) as a better description of individuals' decisions under uncertainty. In this article we link the standard and familiar concepts of stochastic dominance to the new concept of prospect stochastic dominance. In particular, what are the implications of PSD for the traditional mean-variance analysis of Markowitz?

It is well known that SSD is equivalent to mean-variance efficiency when the CDFs are normal. We show that in this case PT efficiency is "almost" identical to mean-variance efficiency. We focus on the canonical case of normal distributions, and offer two results: First, a characterization of PSD efficiency versus MV efficiency when the initial CDFs are not subjectively transformed and the focus on the first two moments is warranted. Second, we characterize PSD efficiency versus MV efficiency when we allow for subjective transformations of the CDFs that preserve FSD. Thus, while in most empirical studies in finance and economics probability distortion is ignored, in this article we analyze MV and PSD efficiency once with no probability distortion and once with probability distortion exactly as suggested by CPT. Note that in this second case the transformed CDFs are not necessarily normal, and may have nonzero skewness and higher odd moments. We show that in both cases the PSD-efficient set is a subset of the (objective) MV-efficient frontier, and it typically almost completely coincides with the MV-efficient set.

The implication of this result is that one can employ the well-known MV optimization techniques to derive the efficient set for PT investors. Both frameworks, MV analysis and PT, are strengthened by this result: the MV framework is shown to be valid for a broader class of preferences, and PT is provided with an algorithm for the construction of PT-efficient portfolios. Thus, despite the sharp contradiction between the assumptions in the PT and MV frameworks, MV analysis turns out to be very central in portfolio optimization even under PT preferences.

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The focus of this article is on the canonical case of normal return distributions. However, as long-horizon empirical return distributions tend to be positively skewed, we also analyze the case of lognormal return distributions. The results for the lognormal case, which are derived in Appendix B, are very similar to those of the normal case. Thus it seems that the results of this article are quite robust and do not hinge on specific assumptions regarding the functional form of the return distribution.

The structure of this article is as follows: In Section 1 we provide a review of the various decision rules employed in the article, and in particular the prospect stochastic dominance rule. Section 2 contains the theoretical results. Section 3 provides an empirical analysis. Section 4 concludes.

1. Investment Decision Rules

This section briefly reviews the main investment decision rules discussed in the article, with a special emphasis on the recently developed and less familiar PSD rule, which is a criterion for dominance by all S-shaped value functions.

1.1 The MV rule

Let F and G be two investments with stochastic outcomes. Investment F dominates G by MV iff

$$\mu_F \geq \mu_G \quad \text{and} \quad \sigma_F \leq \sigma_G, \tag{1}$$

with at least one strict inequality, where μ_F and μ_G denote the expected values of investments F and G, and σ_F and σ_G denote the respective standard deviations. Under normal distributions and risk aversion the MV rule coincides with EU maximization [see Markowitz (1952a) and Tobin (1958)].

1.2 First-degree stochastic dominance

Let F and G be two stochastic investments with cumulative distributions $F(x)$ and $G(x)$, and let $\mathbf{U}_1$ denote the set of all nondecreasing utility functions. Then, F dominates G for all utility functions $U \in \mathbf{U}_1$ (i.e., $EU_F \geq EU_G$) iff

$$F(x) \leq G(x) \quad \text{for all } x, \tag{2}$$

with at least one strict inequality. When the two distributions are normal and $\sigma_1 = \sigma_2$, a dominance by the MV rule coincides with FSD dominance, namely MV $\Leftrightarrow$ FSD. Note that expected utility can be expanded to a Taylor series such that choices depend on all of the distribution moments, for example, mean, variance, skewness, etc. However, if $F(x) \leq G(x)$ for all x, then F is preferred over G by all nondecreasing utility functions, regardless of preference for variance, skewness, and higher moments.

1.3 Second-degree stochastic dominance

Let F and G be two stochastic investments with cumulative distributions $F(x)$ and $G(x)$, and let $\mathbf{U}_2$ denote the set of all nondecreasing risk-averse utility functions ($U' \geq 0$, $U'' < 0$). Then, F dominates G for all $U \in \mathbf{U}_2$, iff

$$I_{\mathrm{SSD}}(x) = \int_{-\infty}^{x} [G(z) - F(z)]dz \geq 0 \quad \text{for all values } x, \tag{3}$$

with at least one strict inequality. When the distributions are normal, Markowitz's MV rule coincides with SSD, namely SSD $\Leftrightarrow$ MV. In other words, in the EU framework with normal distributions and risk aversion, MV is an optimal rule.

For proofs and discussion of FSD and SSD, see Fishburn (1964), Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970). For a survey of SD rules, see Levy (1998). While the above rules are widely used, the recently developed PSD rule [Levy and Wiener (1998)] is still well known. We therefore discuss this rule in some detail below.

1.4 Prospect stochastic dominance

Let F and G be two stochastic investments with cumulative distributions $F(x)$ and $G(x)$, and let $\mathbf{U}_S$ denote the set of all S-shaped utility (or value) functions ($U' \geq 0$ for all $x \neq 0$, $U'' > 0$ for $x < 0$, and $U'' < 0$ for $x > 0$). F dominates G for all $U \in U_s$ iff

$$I_{\mathrm{PSD}} = \int_{\underline{x}}^{\bar{x}} [G(z) - F(z)]dz \geq 0 \quad \text{for all } \underline{x} \leq 0 \leq \bar{x}, \tag{4}$$

with at least one strict inequality.

Graphically F dominates G by PSD if and only if the area enclosed between the two cumulative distributions G and F is positive for any range $[\underline{x}, \bar{x}]$ with $\underline{x} \leq 0$ and $\bar{x} \geq 0$. For a discussion and proof of the PSD rule, see Levy (1998) and Levy and Wiener (1998).

To illustrate the PSD rule, consider the two prospects given in Table 1. Which of these two prospects will various individuals with S-shaped value functions prefer? Tversky and Kahneman (1992) suggest the following

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Table 1
Two alternative investments

F		G	
Probability	Outcome	Probability	Outcome
1/4	−12	1/3	−10
1/4	3.5	1/3	5
1/4	10	1/3	15
1/4	14		

specific form for the value function:

$$V(x) = \begin{cases} x^{\alpha} & \text{if } x \geq 0 \\ \lambda(\ \ x)^{\beta} & \text{if } x < 0 \end{cases}, \tag{5}$$

and they experimentally estimate the parameters of the "typical" individual as $\alpha = 0.88$, $\beta = 0.88$, and $\lambda = 2.25$. It is straightforward to calculate the expected value of the two prospects in Table 1 for such an individual:[4]

$$\mathrm{EV_F} = \tfrac{1}{4}(-2.25)(-(-12))^{0.88} + \tfrac{1}{4}3.5^{0.88} + \tfrac{1}{4}10^{0.88} + \tfrac{1}{4}14^{0.88} = 0.190$$
$$\mathrm{EV_G} = \tfrac{1}{3}(-2.25)(-(-10))^{0.88} + \tfrac{1}{3}5^{0.88} + \tfrac{1}{3}15^{0.88} = -0.703.$$

Thus, as $\mathrm{EV_F} > \mathrm{EV_G}$, this implies that an individual with the S-shaped preferences of Equation (5) with the above parameters would prefer prospect F over G (see footnote 4). What about individuals with different parameters, or with an altogether different functional form of the S-shaped value function? Would they all prefer F, or would some individuals with S-shaped value functions prefer prospect G? The PSD criterion provides an answer.

Figure 1 depicts the cumulative distributions of the two prospects in Table 1. The numbers in the figure denote the areas enclosed between the two cumulative distributions. It is easy to verify from this figure that Equation (4) holds, and therefore F dominates G by PSD. This implies that for these two specific prospects, individuals with any S-shaped function, and not only the specific function given by Equation (5), would prefer F over G. As in the other stochastic dominance rules, the strength of the PSD rule is that it provides a criterion for the preference of all individuals with S-shaped preferences, with no need to know the specific parameters or the exact functional form of the value function. This

[4] For simplicity, in this example we employ the objective probabilities. The possible distortion of probabilities is discussed in the next section, where our general results are derived.

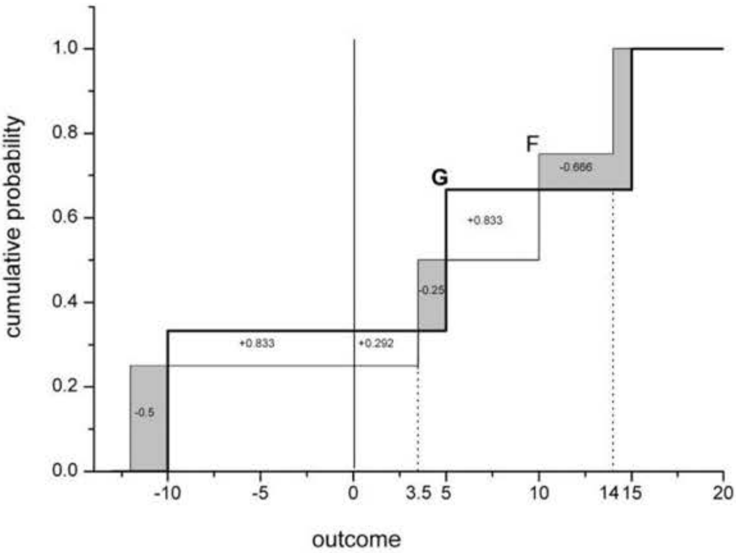

Figure 1
F dominates G by PSD
The figure depicts the cumulative distributions of the two prospects in Table 1. The "+" areas are those where G > F, and the "−" areas are those where the opposite holds. The PSD condition [Equation (4)] states that F dominates G by PSD iff the area enclosed between G and F from $\underline{x}$ to $\bar{x}$ is positive for any $\underline{x} \leq 0 \leq \bar{x}$. It is easy to verify that this condition holds for the two distributions in the figure.

criterion is the basis for deriving the PSD-efficient set in the portfolio context, as discussed below.

Before we turn to the main theoretical results of this article, we would like to discuss two important properties related to the above decision rules, properties which will be employed throughout this article:

(i) All of the above dominance criteria are independent of the initial wealth level. This is natural for PSD, because in PT the value function is defined over the *change* of wealth, and the initial wealth is therefore irrelevant. MV, FSD, and SSD are also independent of the initial wealth level. This is because the addition of the initial wealth W shifts the cumulative distributions by W, without affecting the relationship between the two cumulative distributions under consideration.[5] Thus the efficient set does

[5] The fact that SD and MV rules are not affected by the initial wealth, W, seems to contradict the EU paradigm. However, this is not true. We only claim that the partition into the efficient and inefficient sets is independent of W. However, the choice of the optimal prospect from the efficient set may be affected by W, which is consistent with the EU paradigm.

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not depend on whether one is considering utility functions defined over total wealth or value functions defined over change of wealth.

(ii) From the definitions of the above rules it follows that:

$$\text{FSD} \Rightarrow \text{SSD}$$
$$\text{FSD} \nRightarrow \text{PSD}$$

Thus, by definition, the SSD- and PSD-efficient sets are subsets of the FSD-efficient set. However, there is no obvious mathematical relationship between the PSD- and SSD-efficient sets. The relationship between these two sets, which is the focus of this article, is derived in the next section.

2. The PSD- and MV-Efficient Sets

When considering two alternative investments, generally there is no relationship between MV dominance and PSD. In other words, one can have dominance by MV but not by PSD, or vice versa. Figure 2 exemplifies both of these situations for normal CDFs, with parameters as given in the figure. In Figure 2a, F dominates G by MV, because it has a higher mean and a lower variance. However, F does not dominate G by PSD, because Equation (4) does not hold. For example, take $\underline{x}=0$ and $\bar{x}=1$. It is obvious from Figure 2a that $\int_{\underline{x}=0}^{\bar{x}=1}[G(z)-F(z)]dz<0$, and therefore F does not dominate G by PSD, despite of its MV dominance. Figure 2b exemplifies the opposite case where F dominates G by PSD (the "+ area" is larger than the "− area" and therefore Equation (4) holds), but not by MV ($\sigma_F > \sigma_G$).[6]

In a portfolio context, however, when one can diversify across assets, surprisingly, it turns out that there is a very close relationship between the MV- and PSD-efficient sets.[7] The two theorems below state this relationship. Both theorems are based on the following three standard assumptions:

Assumption 1. Returns are normally distributed.
Assumption 2. Portfolios can be formed without restrictions.
Assumption 3. No two assets are perfectly correlated, $|R_{ij}| < 1$ for all i, j.

[6] Conflicting dominance, that is, F dominates G by MV and G dominates F by PSD, can only be obtained in the special case of $\mu_F = \mu_G$. This is because $\mu_F > \mu_G$ is a necessary condition for the dominance of F over G by both rules. $\mu_F \geq \mu_G$ is obviously a necessary condition for MV dominance. It is also necessary for PSD dominance, because $\mu_F - \mu_G = \int_{-\infty}^{\infty}[G(z)-F(z)]dz$ [see Hanoch and Levy (1969)]. Taking $\underline{x} = -\infty$ and $\bar{x} = \infty$ in Equation (4), we see that $\mu_F \geq \mu_G$ is a necessary condition for the PSD dominance of F over G.

[7] Strictly speaking, PT as postulated by Kahneman and Tversky refers only to "one-shot" lotteries and is silent about individuals' behavior when confronted with a simultaneous series of lotteries, as in the portfolio context. Here we assume that the investor is faced with a universe of portfolios (e.g., those offered by a multitude of mutual funds) and individuals assets alike, and treats each of these as a lottery according to PT.

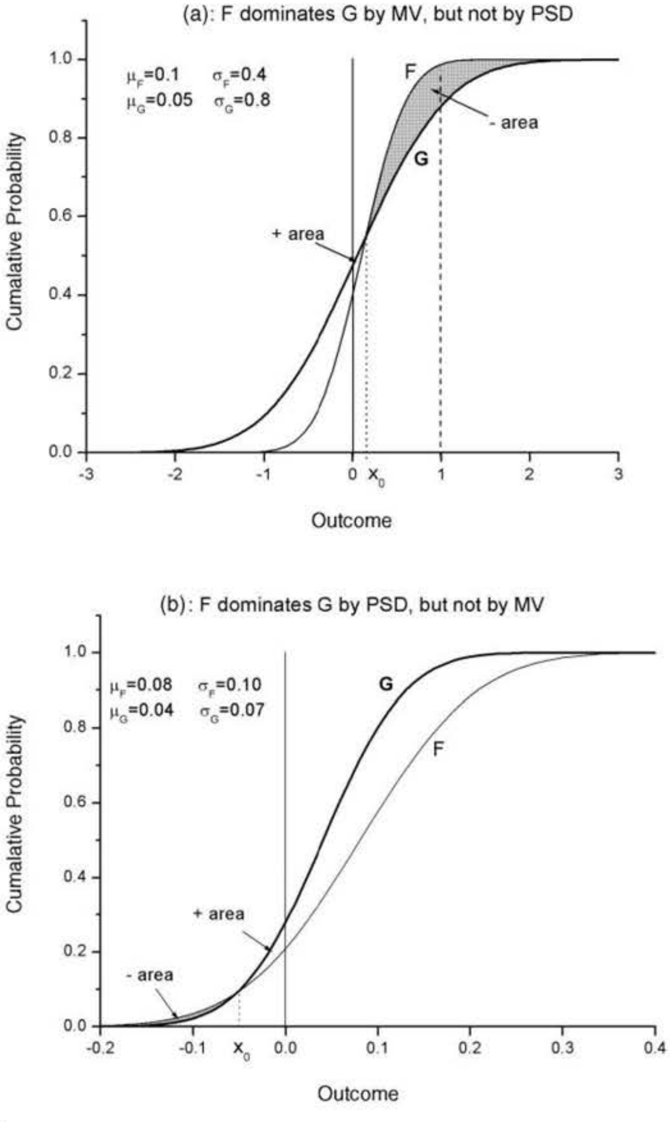

Figure 2
MV dominance and PSD dominance
The figure depicts normal CDFs with parameters as given in the figure. (a) Demonstrates the case where F dominates G by MV, but it does not dominate by PSD, because Equation (4) does not hold (e.g., if one takes $\underline{x} = 0$ and $\bar{x} = 1$). (b) Shows the opposite situation, where F dominates G by PSD [Equation (4) holds] but not by MV (F has a larger variance). In both figures x_0 denotes the point where the two cumulative distributions cross [i.e., $F(x_0) = G(x_0)$].

Theorem 1 derives the PSD-efficient set when the objective probability distributions are employed. Theorem 2 is a generalization to the case where investors may subjectively distort probabilities.

Theorem 1. *Suppose that the objective probability distributions are employed. Then, (i) the PSD-efficient set is a subset of the MV-efficient set, and (ii) the segment of the MV-efficient set which is excluded from the PSD-efficient set is at most the segment between the minimum variance portfolio and the point of tangency from the origin to the frontier (segment Oa in Figure 3).*

Proof. See Appendix A.

While the formal proof of Theorem 1 is given in Appendix A, we provide here the intuition for the theorem's results. The intuition for result (i) is fairly straightforward. This result states that any portfolio that is MV inefficient is also PSD inefficient. To see this, consider a portfolio that is MV inefficient, such as portfolio F′ in Figure 3. For any such inefficient portfolio there exists a portfolio directly above it on the MV frontier with the same standard deviation, but with higher expected return (see portfolio F in Figure 3). While it is obvious that F dominates F′ by the MV rule, F dominates F′ also by PSD [Equation (4) holds], because the CDF of F is the same as that of F′ but shifted to the right (see Figure 4a. Therefore F′ is PSD inefficient (actually F dominates F′ also by FSD; see proof in Appendix A).

The intuition for result (ii) is as follows. Consider the minimum variance portfolio, portfolio O in Figure 3, and consider portfolio O′ slightly above it on the MV frontier. As one moves from O to O′, the expected return increases, but the increase in the standard deviation is only of second order. This slight increase in the standard deviation is enough to rule out the dominance of portfolio O′ over portfolio O by MV, but not by PSD. Indeed, Figure 4b depicts a situation where portfolio O′ PSD dominates portfolio O, and therefore O is PSD inefficient. When one moves upward on the efficient frontier, portfolio O″ may dominate O′ by PSD, etc. However, the further up on the efficient frontier, the smaller the slope of the frontier, that is, a fixed increase in the expected return is accompanied by a growing increase in the standard deviation. Eventually portfolios on the efficient frontier are no longer dominated by portfolios on the frontier to their immediate right.[8] Theorem 1 formally shows that the segment of PSD-inefficient portfolios can be at most the segment between the minimum

[8] We are grateful to the referee for providing this intuition.

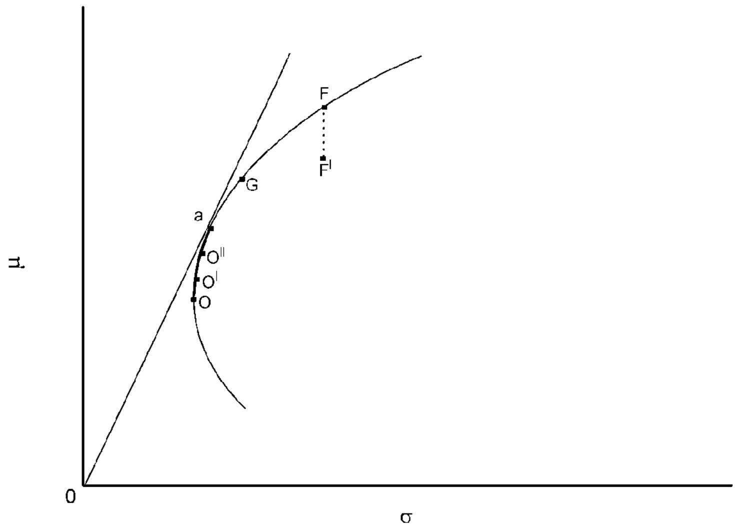

Figure 3
The MV- and PSD-efficient frontiers
When the objective probabilities are employed the PSD-efficient set is a subset of the MV-efficient frontier. The part of the MV-efficient set that is excluded from the PSD-efficient frontier is *at most* segment Oa, where O is the minimum variance portfolio and a is the point of tangency from the origin ($\mu - 0$, $\sigma - 0$) to the MV-efficient frontier.

variance portfolio and the point of tangency from the origin to the frontier (segment Oa in Figure 3).

In Theorem 1 the objective probabilities are employed in the derivation of the PSD-efficient set. However, several researchers have suggested that individuals may subjectively distort probabilities in their decision making. Early studies have modeled this by assuming a direct transformation of the probabilities. This approach has been criticized because it may lead to a violation of FSD, which is considered unrealistic or "too irrational." This problem has been solved in subsequent works by considering the transformation of the cumulative probability distribution. The advantage of this approach is that it does not contradict FSD. For example, any transformation T(F) of the cumulative probability, which is monotonic $T'(F) > 0$ and obeys $T(0) = 0$ and $T(1) = 1$, maintains FSD [i.e., $F >_{FSD} G \Rightarrow T(F) >_{FSD} T(G)$; see Quiggin (1982) and Levy and Wiener (1998)]. Indeed, the extension of PT called cumulative prospect theory (CPT) employs a transformation of the cumulative distributions which maintains FSD [see Tversky and Kahneman (1992)]. Theorem 2 states the relationship between the MV- and PSD-efficient sets when probability distortion takes place.

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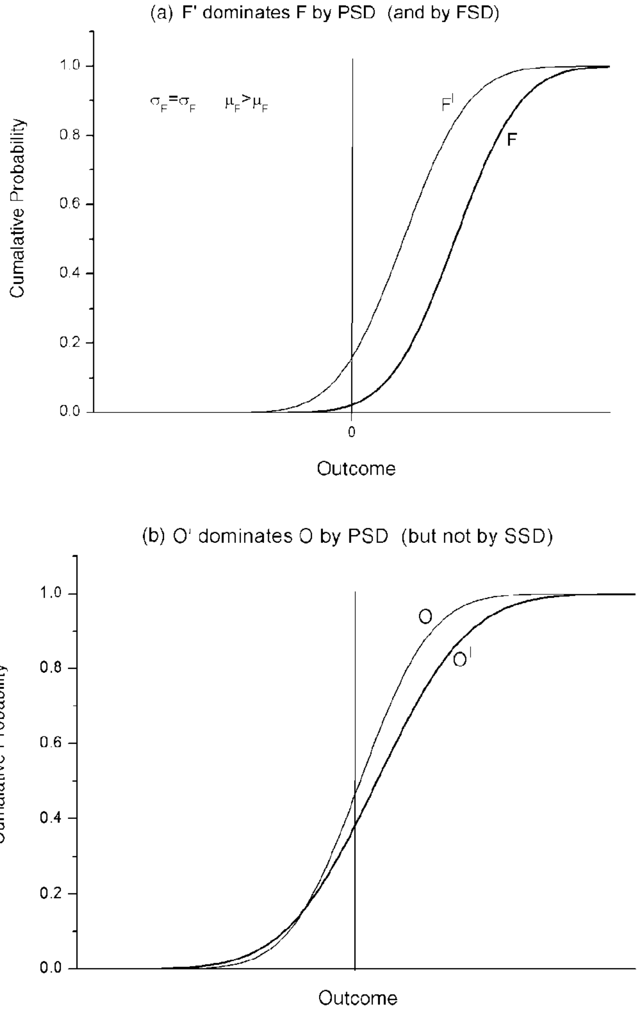

Figure 4
Graphical explanation of Theorem 1
(a) Depicts the CDFs of F′ and F, where F has the same standard deviation as F′, but a higher mean. In this case F dominates F′ by PSD (and actually also by FSD). (b) Shows the CDFs of portfolios O and O′, where O is the minimum variance portfolio and O′ is a portfolio slightly above it on the MV-efficient frontier. O′ has a higher mean, but only a slightly higher standard deviation compared with O, and therefore O′ dominates O by PSD (but not by SSD).

Theorem 2. *Suppose that the objective probabilities are subjectively distorted by any transformation that does not violate FSD, for example, the cumulative prospect theory transformation. Then, the PSD-efficient set is a subset of the MV-efficient set.*

Proof. The proof that the PSD-efficient set is a subset of the MV-efficient frontier is as in Theorem 1. No portfolio interior to the MV-efficient frontier, such as portfolio F′ in Figure 3, can be PSD efficient, because it is FSD dominated by portfolio F on the frontier (see the proof of Theorem 1 in the appendix). As we are considering FSD-maintaining probability transformations, the FSD dominance of F over F′ with the objective probabilities implies that F dominates F′ for every individual with an increasing utility/value function, even if he subjectively distorts the probability distributions. Namely, if F dominates F′ by FSD, then also T(F) dominates T(F′) by FSD. As FSD $\Rightarrow$ PSD, portfolio F′ is dominated by portfolio F, also by PSD. Thus the PSD-efficient set is a subset of the MV-efficient set. ■

Theorem 2 proves that the PSD-efficient set cannot be larger than the MV-efficient set, but like in Theorem 1, it could be strictly smaller. However, unlike the case of Theorem 1, where the objective probabilities are employed, when probabilities are subjectively distorted we cannot generally restrict the location of the PSD-inefficient portfolios to a specific segment of the MV-efficient frontier, that is, the set of MV-efficient portfolios that are not PSD efficient cannot be characterized neatly as a segment between the minimum variance portfolio and the tangency point from the origin. The reason for this is that the probability transformation may change the perceived portfolio mean and standard deviation. Therefore the intuition behind result (ii) of Theorem 1 does not hold, because the *perceived* efficient frontier is not necessarily smooth and with a diminishing slope. As a result, some portfolios that are located on the MV-efficient frontier and which are PSD efficient with the objective probabilities may become PSD inefficient when the probabilities are transformed. Similarly some portfolios that are located on the MV-efficient frontier and are PSD inefficient with the objective probabilities may become PSD efficient with the transformed probabilities. As this claim is not obvious, let us demonstrate it with an example.

Consider two portfolios, F and G, located on the MV-efficient frontier, with normal CDFs and the following parameters: $\mu_F = 0.25$, $\sigma_F = 0.50$; $\mu_G = 0.23$, $\sigma_G = 0.05$. First, note that when the objective probabilities are employed, F is PSD efficient. This follows from the fact that F is necessarily on the segment of the MV-efficient frontier, which is to the right of the point of tangency from the origin to the frontier (point a in Figure 3), because F is on the segment of the frontier where $\frac{\mu}{\sigma}$ is diminishing, that is, $\mu_F > \mu_G$ and $\frac{\mu_F}{\sigma_F} < \frac{\mu_G}{\sigma_G}$. (If F and G were located below point a than we would have had $\frac{\mu_F}{\sigma_F} > \frac{\mu_G}{\sigma_G}$, which does not hold for the parameters in this example.) Thus F is PSD efficient with the *objective probabilities*. (G may be PSD efficient or inefficient, depending on its location on the frontier.) However, when a *subjective probability* transformation is

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employed, F may become PSD inefficient. For example, consider the two CDFs, F and G, with objective probabilities as given in Figure 5a, and the monotonic (and FSD maintaining) probability transformation $T(F) = F^{0.2}$. With this transformation F becomes PSD inefficient G dominates it. This dominance is revealed in Figure 5b, which depicts the transformed CDFs. The figure shows that with the transformed cumulative distributions F is PSD dominated by G: T(F) is above T(G) for $x < 0$, and for $x > 0$ the "+" area is larger than the "−" area [see Figure 5b and Equation (4) with T(F) and T(G) replacing F and G]. Note that as we have a PSD dominance of G over F with the transformed probabilities, it must be that G's perceived mean is larger than the perceived mean of F, $(\mu_{T(F)} < \mu_{T(G)})$ (see footnote 6). Thus when subjective probability transformation is employed, portfolios that are PSD efficient with the objective probabilities may become inefficient, and the segment of the MV-efficient frontier that is PSD inefficient is generally no longer necessarily restricted to be between the minimum variance portfolio and the point of tangency from the origin.

2.1 The role of skewness

We focus in this article on the MV frontier. Some researchers emphasize the importance of skewness and advocate the mean-variance skewness (MVS) criterion [see, e.g., Arditti (1967), Rubinstein (1973), Levy and Sarnat (1984), Harvey and Siddique (2000), and Harvey et al. (2002)]. This section discusses the role of skewness in our analysis.

The article discusses three cases: normal distributions (Theorem 1), transformed normal distributions (Theorem 2), and lognormal distributions (both transformed or not; Theorem 3 in Appendix B). Let us deal with each case separately.

2.1.1 Normal distributions. Here there is no skewness. Theorem 1 proves that all PT investors will choose a portfolio from the MV-efficient frontier. Moreover, in the case of normal distributions, the FSD-, SSD-, and TSD-efficient sets exactly coincide with the MV-efficient frontier: Theorem 1 first shows that the FSD-efficient set is a subset of the MV frontier, and as FSD ⇒ SSD ⇒ TSD, the SSD- and TSD-efficient sets are also subsets of the MV frontier. Second, the fact that, for any two normal distributions with $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$, there is no FSD, SSD, or TSD dominance implies that all portfolios on the MV efficient frontier are FSD, SSD, and TSD efficient. Thus the FSD-, SSD-, TSD-, and MV-efficient sets are exactly identical.

2.1.2 Transformed normal distributions. This case is dealt with in Theorem 2. The theorem proves that all portfolios located below the MV-efficient frontier are inefficient by FSD even if the distributions are

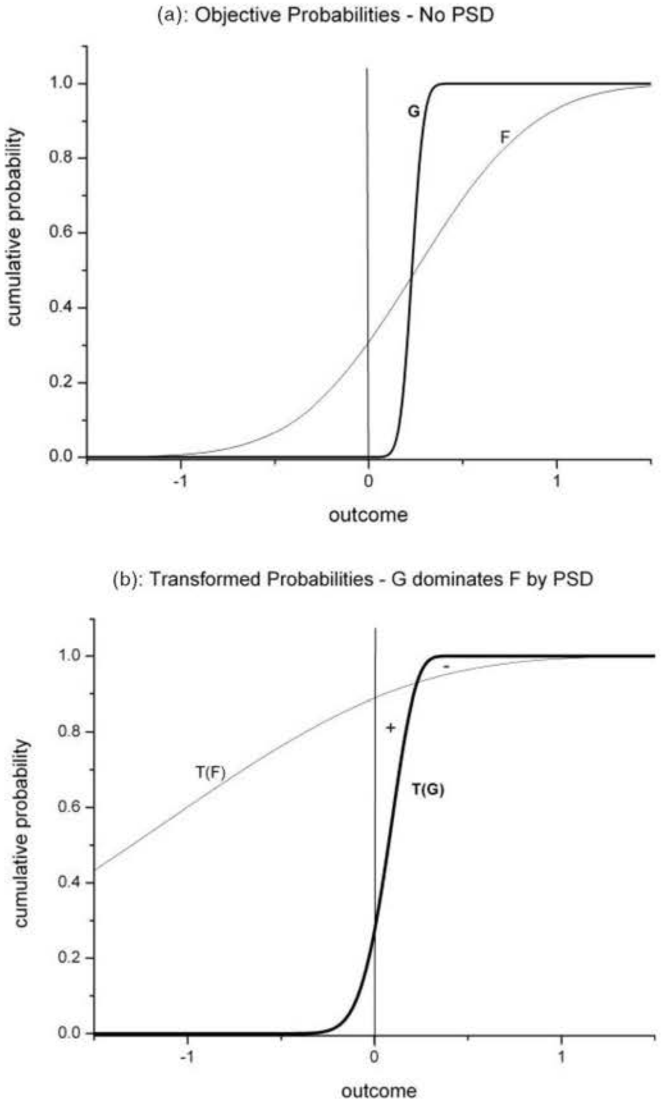

Figure 5
Cumulative probability transformation and PSD
A portfolio that is PSD efficient when the objective probabilities are employed may become inefficient when the transformed probabilities are employed. The figure depicts the CDFs of two hypothetical portfolios on the MV-efficient frontier, F and G, where $\mu_F = 0.25$, $\sigma_F = 0.50$; $\mu_G = 0.23$, $\sigma_G = 0.05$. (a) Depicts the objective probabilities, where no PSD exists. (b) Shows the transformed distributions with the monotonic transformation $T(F) = F^{0.2}$, $T(G) = G^{0.2}$. While F is PSD efficient with the objective probabilities (as shown in the text), when the transformed probabilities are employed, G dominates F by PSD: Equation (4) holds with the reversed roles of F and G (i.e., the "+" area in the figure is larger than the "−" area). Thus, with the transformed probabilities, F is PSD inefficient.

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transformed. As FSD $\Rightarrow$ SSD $\Rightarrow$ TSD, TSD investors will also not choose a portfolio that is not on the MV-efficient frontier. However, in this case the transformed distributions may be skewed and MVS and MV investors may choose different portfolios on the frontier. We cannot tell which portfolios will be chosen unless we know the exact preferences as well as the precise parameters of the transformed distributions (i.e., skewness, which is determined by the specific transformation employed). Moreover, suppose that all investors "switch" from quadratic preferences to cubic preferences. This will effect portfolio demands and the equilibrium prices, and thus the shape of the frontier, but still all investors will choose their portfolios from the MV frontier. In this case it is possible that the SSD-, and in particular the TSD-efficient sets, will be different from the MV-efficient frontier, but Theorem 2 shows that these efficient sets are always subsets of the objective MV-efficient frontier.

2.1.3 Lognormal distributions. Here skewness prevails with the objective distributions, and of course may also prevail with the transformed distributions. If investors have a preference for skewness, skewness may be very important and will be priced. However, Theorem 3 shows that when distributions are lognormal the FSD-efficient set is a subset of the MV-efficient frontier (this result holds both with and without probability transformation; see Appendix B). Hence no investor with increasing utility will choose an MV-inefficient portfolio. In particular, all MVS investors will choose their portfolios from the MV-efficient frontier.

To sum up, in all of the above three cases, MVS and PT investors will generally choose different portfolios, but all of them will choose their portfolios from the objective MV-efficient frontier.

Theorem 1 states that *at most* the lower part of the MV-efficient frontier (segment Oa in Figure 3) is PSD inefficient. Theorem 2 states that the PSD-efficient set is a subset of the MV-efficient frontier, but it does not state which parts of the frontier are PSD inefficient. The important result of these theorems is that in the PT framework, with or without probability transformation, one will always choose his optimal portfolio from Markowitz's MV-efficient set. In the next section we numerically investigate which part of the MV-efficient set is PT efficient by employing the empirical MV frontier and the specific probability transformation suggested by Tversky and Kahneman (1992).

3. The Empirical PT-Efficient Set

Theorems 1 and 2 state that the PT-efficient set is a subset of the MV-efficient frontier, but they provide only partial information (in the case that the objective probabilities are employed) or no information (in the case of probability distortion) regarding the exact part(s) of the MV-efficient frontier that is PSD inefficient. In this section we numerically find the empirical PSD-efficient set with and without probability distortion. We select a random sample of 50 stocks from the Center for Research in Security Prices (CRSP) data set, and employ their monthly rates of return over the period January 1980 to January 2000 to estimate the means and the variance-covariance matrix. These parameters, in turn, are employed to derive the MV-efficient set, as shown in Figure 6. To find out which portfolios on the frontier are PSD inefficient, we go over each portfolio (with discrete increments of 0.0001 in μ) and check its efficiency by comparing it with all portfolios located on the efficient frontier, and numerically testing whether it is PSD dominated.[9] We conduct this analysis twice: once assuming that the return distributions are normal, and once by transforming the normal return distributions with the subjective probability transformation suggested by Tversky and Kahneman (1992):

$$T(F) = \frac{F^{\gamma}}{(F^{\gamma} + (1 - F)^{\gamma})^{1/\gamma}} \tag{6}$$

with γ experimentally estimated as 0.6.[10]

The results are shown in Figure 6. In the case where the objective probabilities are employed, the PSD-efficient set is the MV-efficient frontier except segment Oc. In the case where the PT probability transformation of Equation (6) is employed, the PSD-efficient set is the MV-efficient frontier except segment Ob. In both cases, only a small lower part of the MV-efficient frontier is relegated to the PSD-inefficient set. It is perhaps surprising that when the probabilities are distorted, the part of the MV frontier that is PSD inefficient is even smaller than when the objective probabilities are employed, that is, empirically the probability transformation actually makes the MV- and PSD-efficient sets even more similar.[11]

It is important to point out that while the main focus of this article is on the standard case of normal return distributions, our theoretical and empirical results can be extended to the case of lognormal distributions

[9] The numerical testing of PSD is greatly simplified in the case of normal CDFs (see the lemma in the Appendix). As two normal CDFs with different standard deviations cross exactly once, and as Equation (6) is monotonic, the subjectively transformed cumulative probability distributions also cross exactly once (and at the same x_0). Thus we can employ Equations (A.1) and (A.2) of Appendix A to check for PSD between the transformed distributions without worrying about multiple crossings of the distributions.

[10] Tversky and Kahneman estimate γ separately for gains and losses. For gains they estimate $\gamma = 0.61$, while for losses they estimate $\gamma = 0.69$. Camerer and Ho (1994) estimate the parameter γ as about 0.6.

[11] A possible explanation for this may be as follows: PSD relegates the low-mean, low-variance part of the MV-efficient frontier to the inefficient set. The PT probability transformation of Equation (6) has the effect of increasing the perceived likelihood of low-probability extreme events [see Tversky and Kahneman (1992)]. Thus this transformation shifts probability weight to the extremes and therefore tends to increase the perceived variance. As a result, the range of perceived low-mean, low-variance portfolios becomes smaller under this probability transformation.

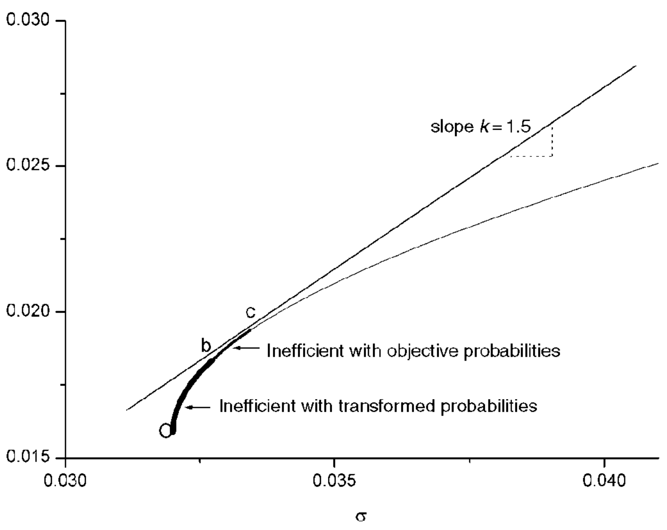

Figure 6
The empirical MV- and PSD-efficient sets
The MV-efficient set is derived for a random sample of 50 stocks from the CRSP monthly return dataset. The PSD-efficient set is found by employing Equations (A.1) and (A.2). O is the minimum variance portfolio. When the objective probabilities are employed, the PSD-efficient set is the MV-efficient frontier excluding segment Oc. This is the same as Baumol's efficient set with $k = 1.5$. When the transformation of cumulative probability, as suggested by cumulative PT, is employed [Equation (6)], the PSD-efficient set is the MV-efficient frontier excluding segment Ob.

as well (see Appendix B). The results for the lognormal case are very similar to those obtained in the normal case, which suggests that the results are robust to the exact mathematical form of the return distributions.

Finally, it is interesting to note that as early as 1963, Baumol suggested an intuitive investment criterion that does not rely on expected utility maximization. Baumol's criterion asserts that F dominates G iff

$$\mu_F \geq \mu_G$$

$$\mu_F - k\sigma_F \geq \mu_G - k\sigma_G,$$

where k is a subjective positive value related to Chebechev's inequality. It turns out that Baumol's efficient set is a subset of the MV-efficient set, relegating to the inefficient set the lower part of the MV-efficient set (like segments Oc or Ob in Figure 6). The size of the efficient set relegated to the inefficient set by Baumol's criterion depends on the value k one selects: the larger k, the smaller Baumol's inefficient set; for $k \to \infty$, the Baumol and the MV-efficient sets coincide. With the actual empirical data corresponding to Figure 6, we investigate what value k is needed in Baumol's criterion such that Baumol's efficient set and the PSD-efficient set with no probability distortion coincide. As we can see from Figure 6, the two efficient sets coincide for $k \approx 1.5$, namely Baumol's lower bound is 1.5 standard deviations from the mean.[12] Of course, this particular value of k is for illustration purposes only, as k is in general a function of the particular dataset under consideration. Thus we see some similarity between Baumol's and the PSD-efficient sets, where both relegate to the inefficient set portfolios with low mean and low variance. However, recall that there is one important difference; while Baumol's rule has no theoretical justification, the PSD-efficient set is based on the foundations of PT.

4. Conclusion

The MV rule is probably the most common investment rule under uncertainty among both academics and practitioners. Moreover, the MV rule is very important as it is the foundation of the cornerstone Sharpe-Lintner capital asset pricing model (CAPM) and other theoretical models.

Kahneman and Tversky (1979), and Tversky and Kahneman (1992), based on experimental results, suggest PT as an alternative to the EU paradigm. According to PT, the value function is S-shaped with a risk-seeking segment, it is a function of *change* of wealth rather than *total* wealth, and in addition, individuals subjectively distort probabilities by a transformation of the cumulative probability. These experimental results seem to be fatal to EU in general and to the MV framework in particular. First, employing change of wealth rather than total wealth contradicts the EU paradigm, and thus casts doubt on the validity of the MV analysis, which is derived in the EU framework. Second, the risk-seeking segment of the PT value function contradicts the risk-aversion assumption, which is the foundation of the MV rule. Finally, when individuals distort probabilities, the perceived distributions are generally not normal, even if the objective distributions are, which again seems to be fatal to the MV analysis.

We show in this study that, in general, when considering two alternative investments F and G, there is no relationship between the PT- and MV-efficient sets: F may dominate G by PSD (which corresponds to PT) and not by MV, and vice versa. This is true even when normality is assumed, let alone in the general case when no restrictions are imposed on the distributions. However, in a portfolio context when diversification between assets is allowed, as is common in the security market, and as suggested by Markowitz (1952a), and when individuals face portfolios

[12] For any given value of k, Baumol's efficient set is the segment of the MV-efficient frontier to the right of the point where a straight line of slope k is tangent to the hyperbola. For the PSD-efficient set obtained with the transformed probability distributions we obtain a slightly higher value of k ($k = 1.7$).

(mutual funds) as well as individual assets, we find that the PT-efficient set is a subset of the MV-efficient frontier. This result holds both for the case where objective probabilities are employed, as well as for the case where probabilities are subjectively transformed, as suggested by cumulative PT. When the objective probabilities are employed, we theoretically show that *at most* a small segment of low-mean low-variance portfolios is relegated to the PSD-inefficient set. With probability transformation the segment of the MV-efficient set relegated to the PT-inefficient set cannot be determined in general. However, empirical analysis reveals that the PSD rule (with and without probability distortion) relegates to the inefficient set a very small segment of the MV-efficient set corresponding to very risk-averse investors. Thus, surprisingly, even with the S-shaped value function, which contains a risk-seeking segment and which is a function of change of wealth rather than total wealth, and even with probability distortion, the PT- and MV-efficient sets almost coincide.

It is interesting to note that PT relegates to the inefficient set some segment of the MV-efficient set in a similar way to the relegation of portfolios to the inefficient set by Baumol's (1963) intuitive criterion. In Baumol's criterion risk is measured by $E\text{-}k\sigma$, where k is a positive constant reflecting the safety requirement of the individual investor. We find in the empirical analysis that the PSD rule relegates to the efficient set the same segment relegated by Baumol's criterion corresponding to the value of approximately $k = 1.5$.

Note that the similarity of the MV- and PT-efficient sets does not necessarily imply that the CAPM equilibrium holds when investors have PT preferences. In particular, for some PT preferences the risk-seeking segment of the utility function may imply infinite borrowing, which contradicts the notion of market equilibrium.

The main results of this article are derived for the canonical case of normal return distributions. However, it is well known that for long investment horizons the return distributions are typically positively skewed. To what extent do the results of this article hold when the return distributions are positively skewed? In order to answer this question we analyze the PT-efficient set in the case of lognormal return distributions (see Appendix B). We find that even in this case the PT- and MV-efficient sets almost coincide. Thus while skewness may be priced (and may perhaps be even more important in the case of PT investors with high sensitivity to losses), the set of PSD-efficient portfolios is very similar to the MV-efficient frontier.

In conclusion, the cumulative PT- and the well-known MV-efficient sets almost coincide. What seems a priori to be a severe contradiction between these two cornerstone paradigms, turns out to be a minimal one. Thus one can employ the existing MV diversification algorithm to construct the PT-efficient set. This is an important step toward bridging between subjects' observed behavior, as manifested by PT, and the standard MV portfolio theory.

Appendix A: Proof of Theorem 1

Before we turn to the proof of Theorem 1, consider the following lemma, which is employed in the proof.

Lemma. *Assume that both* F *and* G *are normal CDFs, such that* $\mu_F > \mu_G$, $\sigma_F > \sigma_G$. *Then* F *dominates* G *by PSD iff the following two conditions hold:*

$$x_0 < 0 \tag{A.1}$$

$$\int_{\infty}^{x_0} [F(z) - G(z)]dz \le \int_{x_0}^{0} [G(z) - F(z)]dz, \tag{A.2}$$

where x_0 *is the crossing point of* F *and* G *(i.e.,* $F(x_0) = G(x_0)$*, recall that two normal CDFs with different variances cross exactly once). In other words, for normal distributions, Equation* (4) *is equivalent to Equations* (A.1) *and* (A.2).

Proof.

Necessity
Equation (A.1) is required for the PSD dominance of F over G. To see this recall that for normal distributions with $\sigma_F > \sigma_G$, F crosses G from above (as in Figure 2b). If $x_0 \ge 0$ (unlike what is shown in Figure 2b), then $I_{PSD} \equiv \int_{\infty}^{0} [G(z) - F(z)]dz < 0$, in violation of the PSD criterion of Equation (4). Equation (A.2) also follows directly from Equation (4) by taking $\underline{x} = -\infty$ and $\bar{x} = 0$, because when $x_0 < 0$, $\int_{-\infty}^{0} [G(z) - F(z)]dz \ge 0$ implies $\int_{-\infty}^{x_0} [F(z) - G(z)]dz \le \int_{x_0}^{0} [G(z) - F(z)]dz$.

Sufficiency
The CDFs of F and G cross exactly once, with F crossing from above ($\sigma_F > \sigma_G$). $x_0 < 0$ ensures that (i) $\int_0^{\bar{x}} [G(z) - F(z)]dz > 0$ for all $\bar{x} > 0$ (because $G > F$ in the positive range; see Figure 2b). Equation (A.2) ensures that (ii) $\int_{\underline{x}}^{0} [G(z) - F(z)]dz > 0$ for all $\underline{x} < 0$, because this expression is minimal for $\underline{x} = -\infty$, and Equation (A.2) states that it is positive even in this case. Taken together, (i) and (ii) imply Equation (4). ■

Note that even though the PSD condition of Equation (4) requires verification for all $\underline{x} \le 0 \le \bar{x}$, in the case of normal CDFs with $\mu_F > \mu_G$ and $\sigma_F > \sigma_G$, this boils down to the verification of only the two conditions [Equations (A.1) and (A.2)]. This result can be extended to the case of any two CDFs that cross only once.

Proof of Theorem 1. Let us first prove that the PSD-efficient set is a subset of the MV-efficient set [claim (a)], and then go on to derive the segment of the MV-efficient frontier which *may* be PSD inefficient [claim (b)].

(a) PSD-efficient set ⊆ MV-efficient set
The MV-efficient set is well known to have the shape of a hyperbola, which is called the efficient frontier [see Roy (1952), Merton (1972), and Roll (1977)]. We claim that any portfolio that is interior to the MV-efficient frontier (i.e., which is MV inefficient) is also PSD inefficient. To see this, consider any portfolio F′ that is MV inefficient (see Figure 3). This portfolio is FSD dominated by portfolio F, which has the same standard deviation, but higher mean (see the FSD condition, [Equation (2)] for the case of normal distributions). As FSD implies dominance by all individuals with increasing utility/value functions, it also implies preference by all PT investors (FSD ⇒ PSD). Hence all PT investors prefer F over

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F′, and therefore F′ is PSD inefficient. As the same argument holds for all portfolios located below the MV-efficient frontier, we conclude that all the portfolios below the MV-efficient frontier are PSD inefficient.

(b) The segment of the MV frontier that may be PSD inefficient

While the MV-efficient frontier coincides with the FSD-efficient set for normal distributions, some portfolios located on the MV-efficient frontier may be relegated to the PSD-inefficient set, because the PSD-efficient set is by definition a subset of the FSD-efficient set. We show below that the PSD-efficient set is the entire MV-efficient frontier excluding *at most* the segment between the minimum variance portfolio and the point of tangency from the origin to the frontier (segment Oa in Figure 3). To see this, consider any two portfolios F and G on the efficient frontier, and assume without loss of generality that $\mu_F > \mu_G$, $\sigma_F > \sigma_G$. Figure A.1 depicts such a situation. As mentioned above, any two cumulative normal distributions with $\sigma_F \neq \sigma_G$ cross exactly once. Denote the crossing point of F and G by x_0 (see Figure 2). G cannot dominate F by PSD because $\mu_F > \mu_G$ (see footnote 6). By the above lemma, F dominates G by PSD iff Equations (A.1) and (A.2) hold. The condition $x_0 < 0$ has a direct geometric interpretation in the MV plane. For two normal distributions the intersection point x_0 of the cumulative distributions is given by the solution of the following equation:

$$\frac{x_0 - \mu_F}{\sigma_F} = \frac{x_0 - \mu_G}{\sigma_G} \tag{A.3}$$

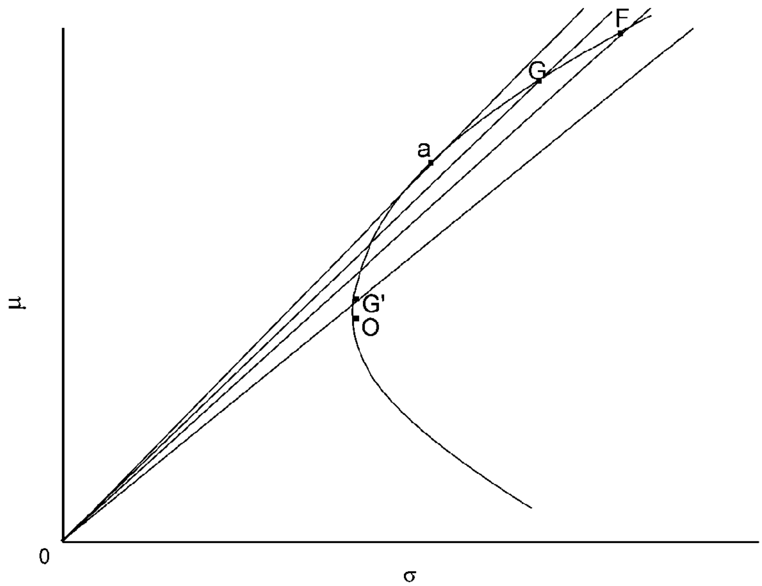

Figure A.1
PSD inefficiency and the MV frontier
Necessary conditions for the PSD dominance of F over G are that F has a higher mean than G, and that the slope from the origin (0,0) to F is higher than the slope to G. Thus all portfolios on the MV frontier to the right of point a are PSD efficient: for any such portfolio there is no other portfolio with both higher mean and higher slope. For example, in the figure neither F nor G dominate the other by PSD.

[see, e.g., Hanoch and Levy (1969)]. Thus x_0 is given by

$$x_0 = \frac{\mu_G \sigma_F - \mu_F \sigma_G}{\sigma_F - \sigma_G}. \tag{A.4}$$

The condition $x_0 < 0$ therefore implies $\mu_G \sigma_F - \mu_F \sigma_G < 0$ (recall that $\sigma_F > \sigma_G$ by construction), or

$$\frac{\mu_F}{\sigma_F} > \frac{\mu_G}{\sigma_G}. \tag{A.5}$$

Thus, Equation (A.5) can replace Equation (A.1) as a condition for the PSD dominance of F over G. Geometrically $\frac{\mu_F}{\sigma_F}$ is the slope from the origin to portfolio F in the MV plane. Thus a necessary condition for the PSD dominance of F over G is that the slope from the origin to F is higher than the slope to G (see Figure A.1). For the specific F and G depicted in Figure A.1 there is no PSD dominance, because the slope of F is smaller than the slope of G (and G also cannot dominate because $\mu_F > \mu_G$). In contrast, F *may* dominate portfolio G′ (see Figure A.1), because F has both higher mean and higher slope [however, this dominance is not guaranteed because one still has to check whether Equation (A.2) holds]. In general, all the portfolios on the Markowitz efficient frontier to the right of the tangency point from the origin (point **a** in Figure A.1) are PSD efficient. Consider, for example, portfolio F: this portfolio is not dominated by any portfolio to its left, because F has a higher mean. It is also not dominated by any portfolio to its right, because F has a higher slope. Thus portfolio F, and all other portfolios to the right of point **a**, are PSD efficient. PSD relegates to the PSD inefficient set *at most* segment Oa of the efficient frontier. ■

Appendix B: The PSD- and MV-Efficient Sets in the Case of Lognormal Distributions

We show in this appendix that when the distributions are lognormal, the FSD- (and hence the PSD-) efficient set is a subset of the MV-efficient set, despite of the fact that the lognormal distributions are skewed. Let us elaborate.

In the text we analyze the relationship between MV analysis and PT for the case of normal distributions. It is well known, however, that for long investment horizons the return distributions are positively skewed, and the lognormal distribution is a better approximation for the empirical return distribution. A drawback of the lognormal distribution framework is that the sum of two random variables, which are lognormally distributed, is *not precisely* lognormal. However, Lintner (1972) shows that the sum of two such lognormal random variables is almost lognormal. He concludes,

> "the approximation to lognormally distributed portfolio outcomes of lognormally distributed stocks is sufficiently good that theoretical models based on these twin premises should be useful in a wide range of applications and empirical investigations."

[See also Ohlson and Ziemba (1976), and Dexter, Yu, and Ziemba (1980)].

In Theorem 3 we extend our main theoretical results given in the text to the case of lognormal distributions.

Theorem 3. *Consider n assets with lognormal return distributions and correlations $|R_{ij}| < 1$ for all i, j. Portfolios can be formed without restrictions, and all portfolio returns are also lognormally distributed. Objective probabilities may be subjectively distorted by any transformation which does not violate FSD. Then, the PSD-efficient set is a subset of the MV-efficient frontier, excluding at least the segment between the minimum variance portfolio and the point of tangency from coordinate* $(0, -1)$ *in the MV plane to the frontier (segment Oa′ in Figure A.2).*

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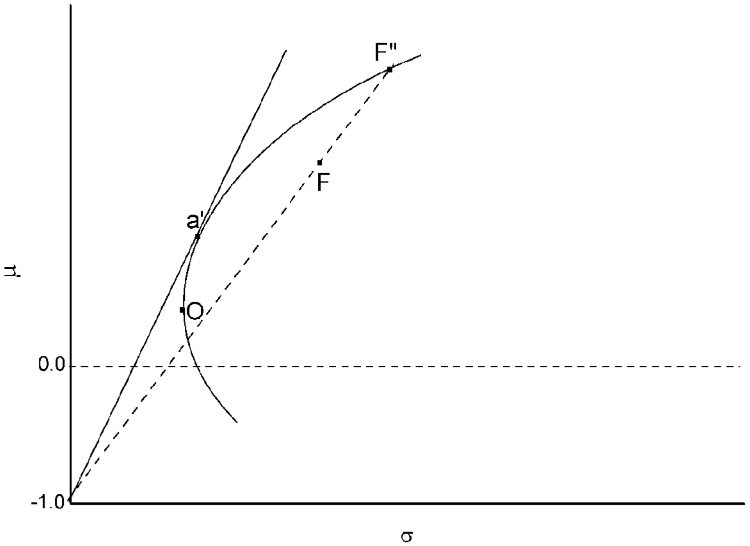

Figure A.2
The efficient frontier with lognormal distributions
F″ dominates F by FSD: it has a higher mean and both portfolios have the same slope from the point (−1, 0), that is, Equations (A.11) and (A.12) hold. Thus the FSD-efficient set is the segment of the MV frontier to the right of point a′. As the PSD-efficient set is a subset of the FSD-efficient set, it is a subset of the segment of the MV frontier to the right of a′.

Proof. We will first show that the FSD-efficient set coincides with the MV-efficient frontier, excluding the segment between the minimum variance portfolio and the point of tangency from the coordinate (0, −1) to the frontier (segment Oa′ in Figure A.1). As FSD ⇒ PSD, it then follows that the PSD efficient set is a subset of this segment.

Consider two investments F and G with lognormal distributions. F dominates G by FSD iff:

$$\mu_F > \mu_G \quad \text{and} \tag{A.6}$$

$$\sigma_{\log F} = \sigma_{\log G}, \tag{A.7}$$

where μ_F denotes the mean rate of return of F, σ_F its standard deviation, and $\sigma_{\log F}$ is the standard deviation of F′s log outcome, that is, the standard deviation of the variable which is normally distributed, and the parameters of G are defined in the same way [see Levy (1998)]. The standard deviation, $\sigma_{\log F}$, is related to μ_F and σ_F according to

$$\sigma_F^2 = (1+\mu_F)^2 \lfloor e^{\sigma_{\log F}} - 1 \rfloor \tag{A.8}$$

[see, e.g., Aitchison and Brown (1963), and note that the mean *total* return, to distinguish from the mean *rate* of return, is $1+\mu_F$]. This relationship can be rewritten as

$$\sigma_{\log F} = \ln\left[1 + \left(\frac{\sigma_F}{1+\mu_F}\right)^2\right]. \tag{A.9}$$

Thus

$$\sigma_{\log F} = \sigma_{\log G} \Leftrightarrow \frac{\sigma_F}{1+\mu_F} = \frac{\sigma_G}{1+\mu_G}, \tag{A.10}$$

and the FSD conditions of Equations (A.6) and (A.7) can therefore also be written as

$$\mu_F > \mu_G \quad \text{and} \tag{A.11}$$

$$\frac{\sigma_F}{1+\mu_F} = \frac{\sigma_G}{1+\mu_G}. \tag{A.12}$$

The advantage of Equation (A.12) over Equation (A.7) is that it is in terms of the returns, rather than the log-returns, hence it can be directly applied in the MV framework. The geometric interpretation of Equation (A.12) is that F and G have the same slope from the point located at coordinate (0, −1) in the mean standard deviation plane (see Figure A.2). Note that all portfolios on a straight line originating at (0, −1) have the same $\frac{\sigma}{1+\mu}$ ratio. This implies that the FSD-efficient set is exactly the segment of the MV frontier that is to the right of the point of tangency from coordinate (0, −1) to the frontier (point a′ in Figure A.2). Let us elaborate.

Any portfolio that is interior to the MV frontier, such as portfolio F in Figure A.2, is FSD dominated by a portfolio on the frontier with the same slope but a higher mean (portfolio F″ in Figure A.2). The same is true for any portfolio on segment Oa′ of the frontier. In contrast, any portfolio on the MV frontier to the right of point a′ is FSD efficient: simply, there are no two portfolios on this segment for which Equation (A.12) holds, and therefore all portfolios on the MV frontier to the right of a′ are FSD efficient.

Note that this result holds for the case where the objective probability distributions are employed, as well as for the case where the probability is subjectively transformed with any FSD maintaining transformation, such as the CPT transformation proposed by Tversky and Kahneman (1992).

As FSD ⇒ PSD, no MV interior portfolio can be PSD-efficient. As the PSD-efficient set is generally a subset of the FSD-efficient set, and as the FSD-efficient set is composed of all portfolios on the frontier to the right of point a′, the PSD-efficient set is also a *subset* of the MV-efficient frontier, and it is located to the right of point a′. However, the PSD-efficient set may generally be smaller than the entire range to the right of point a′. ■

An empirical analysis of the PSD- and MV-efficient sets when the return distributions are assumed to be lognormal reveals that only a small lower part of the MV-efficient frontier is relegated to the PSD-inefficient set, exactly as in the case of normal return distributions. When we repeat this analysis with the assumption that the cumulative probability distributions are transformed, as suggested by Tversky and Kahneman [see Equation (6)], we obtain very similar results. These empirical results are available upon request from the authors.

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