The expression χ is (similar to) the variance between outputs of Liam's code, α_i, and Jane's code, ϵ_i. Our goal is to minimize χ, so we want to solve the system: $\frac{\partial\chi}{\partial f} = \frac{\partial\chi}{\partial\sigma} = 0$ for some optimal values of $f_w n$ and σ_{sat}. We will use Newton's Method. Note: I have abbreviated f_{wn} and σ_{sat} as f and σ. The purpose of this document is to use as a reference while coding.

$$\chi(f,\sigma) = \sum_i (\alpha_i - \varepsilon_i(f,\sigma))^2$$

$$\frac{\partial\chi}{\partial f}(f,\sigma) = -2\sum_i (\alpha_i - \varepsilon_i(f,\sigma))\left(\frac{\partial\varepsilon_i}{\partial f}(f,\sigma)\right)$$
$$\frac{\partial\chi}{\partial\sigma}(f,\sigma) = -2\sum_i (\alpha_i - \varepsilon_i(f,\sigma))\left(\frac{\partial\varepsilon_i}{\partial\sigma}(f,\sigma)\right)$$

$$\frac{\partial^2\chi}{\partial f^2}(f,\sigma) = 2\sum_i \left(\frac{\partial\varepsilon_i}{\partial f}(f,\sigma)\right)^2$$
$$\frac{\partial^2\chi}{\partial\sigma^2}(f,\sigma) = \sum_i \left[\left(\frac{\partial\varepsilon_i}{\partial\sigma}(f,\sigma)\right)^2 - (\alpha_i - \varepsilon_i(f,\sigma))\left(\frac{\partial^2\varepsilon_i}{\partial\sigma^2}(f,\sigma)\right)\right]$$
$$\frac{\partial^2\chi}{\partial\sigma\partial f}(f,\sigma) = 2\sum_i \left[\left(\frac{\partial\varepsilon_i}{\partial\sigma}(f,\sigma)\right)\left(\frac{\partial\varepsilon_i}{\partial f}(f,\sigma)\right) - (\alpha_i - \varepsilon_i(f,\sigma))\left(\frac{\partial^2\varepsilon_i}{\partial\sigma\partial f}(f,\sigma)\right)\right]$$
$$\frac{\partial^2\chi}{\partial f\partial\sigma}(f,\sigma) = 2\sum_i \left[\left(\frac{\partial\varepsilon_i}{\partial f}(f,\sigma)\right)\left(\frac{\partial\varepsilon_i}{\partial\sigma}(f,\sigma)\right) - (\alpha_i - \varepsilon_i(f,\sigma))\left(\frac{\partial^2\varepsilon_i}{\partial f\partial\sigma}(f,\sigma)\right)\right]$$

$$\varepsilon_i(f,\sigma) = f\varepsilon_{wn_i}(\sigma) + (1-f)\varepsilon_{sat_i}(\sigma)$$

$$\frac{\partial\varepsilon_i}{\partial f}(\sigma) = \varepsilon_{wn_i}(\sigma) - \varepsilon_{sat_i}(\sigma)$$
$$\frac{\partial\varepsilon_i}{\partial\sigma}(f,\sigma) = f\frac{\partial\varepsilon_{wn_i}}{\partial\sigma}(\sigma) + (1-f)\frac{\partial\varepsilon_{sat_i}}{\partial\sigma}(\sigma)$$

$$\frac{\partial^2\varepsilon_i}{\partial\sigma^2} = f\frac{\partial^2\varepsilon_{wn_i}}{\partial\sigma^2}(\sigma) + (1-f)\frac{\partial^2\varepsilon_{sat_i}}{\partial\sigma^2}(\sigma)$$
$$\frac{\partial^2\varepsilon_i}{\partial\sigma\partial f} = \frac{\partial^2\varepsilon_i}{\partial f\partial\sigma} = \frac{\partial\varepsilon_{wn_i}}{\partial\sigma}(\sigma) - \frac{\partial\varepsilon_{sat_i}}{\partial\sigma}(\sigma)$$

$$\varepsilon_{wn_i}(\sigma) = \frac{N}{2\sigma}\left[T_{1_i}(1-e^{-T_{2_i}\sigma}) + T_{2_i}(1-e^{-T_{1_i}\sigma})\right]$$

$$\frac{\partial\varepsilon_{wn_i}}{\partial\sigma}(\sigma) = \frac{N}{2\sigma}T_{1_i}T_{2_i}(e^{-T_{1_i}\sigma}+e^{-T_{2_i}\sigma}) - \frac{N}{2\sigma^2}[T_{1_i}(1-e^{-T2\sigma}) + T_{2_i}(1-e^{-T1\sigma})]$$

$$\frac{\partial^2\varepsilon_{wn_i}}{\partial\sigma^2}(\sigma) = \frac{N}{\sigma^3}[T_{1_i}(1-e^{-T_{2_i}\sigma}) + T_{2_i}(1-e^{-T_{1_i}\sigma})] - \frac{N}{\sigma^2}T_{1_i}T_{2_i}(e^{-T_{1_i}\sigma}+e^{-T_{2_i}\sigma}) - \frac{N}{2\sigma}(T_{1_i}^2T_{2_i}e^{-T_{1_i}\sigma} + T_{1_i}T_{2_i}^2e^{-T_{2_i}\sigma})$$

$$\varepsilon_{sat_i}(\sigma) = \frac{N}{\sigma}T_{min}(1-e^{-T_{max}\sigma})$$

$$\frac{\partial\varepsilon_{sat_i}}{\partial\sigma}(\sigma) = \frac{N}{\sigma}T_{min_i}T_{max_i}e^{-T_{max_i}\sigma} - \frac{N}{\sigma^2}T_{min_i}(1-e^{-T_{max_i}\sigma})$$

$$\frac{\partial^2\varepsilon_{sat_i}}{\partial\sigma^2}(\sigma) = \frac{2N}{\sigma^3}T_{min_i}(1-e^{-T_{max_i}\sigma}) - \frac{2N}{\sigma^2}T_{min_i}T_{max_i}e^{-T_{max_i}\sigma} - \frac{N}{\sigma}T_{min_i}T_{max_i}^2e^{-T_{max_i}\sigma}$$