

The expression  $\chi$  is (similar to) the variance between outputs of Liam's code,  $\alpha_i$ , and Jane's code,  $\epsilon_i$ . Our goal is to minimize  $\chi$ , so we want to solve the system:  $\frac{\partial \chi}{\partial f} = \frac{\partial \chi}{\partial \sigma} = 0$  for some optimal values of  $f_{wn}$  and  $\sigma_{sat}$ . We will use Newton's Method. Note: I have abbreviated  $f_{wn}$  and  $\sigma_{sat}$  as  $f$  and  $\sigma$ . The purpose of this document is to use as a reference while coding.

$$\chi(f, \sigma) = \sum_i (\alpha_i - \epsilon_i(f, \sigma))^2$$

$$\frac{\partial \chi}{\partial f}(f, \sigma) = -2 \sum_i (\alpha_i - \epsilon_i(f, \sigma)) \left( \frac{\partial \epsilon_i}{\partial f}(f, \sigma) \right)$$

$$\frac{\partial \chi}{\partial \sigma}(f, \sigma) = -2 \sum_i (\alpha_i - \epsilon_i(f, \sigma)) \left( \frac{\partial \epsilon_i}{\partial \sigma}(f, \sigma) \right)$$

$$\frac{\partial^2 \chi}{\partial f^2}(f, \sigma) = 2 \sum_i \left( \frac{\partial \epsilon_i}{\partial f}(f, \sigma) \right)^2$$

$$\frac{\partial^2 \chi}{\partial \sigma^2}(f, \sigma) = 2 \sum_i \left[ \left( \frac{\partial \epsilon_i}{\partial \sigma}(f, \sigma) \right)^2 - (\alpha_i - \epsilon_i(f, \sigma)) \left( \frac{\partial^2 \epsilon_i}{\partial \sigma^2}(f, \sigma) \right) \right]$$

$$\frac{\partial^2 \chi}{\partial \sigma \partial f}(f, \sigma) = 2 \sum_i \left[ \left( \frac{\partial \epsilon_i}{\partial \sigma}(f, \sigma) \right) \left( \frac{\partial \epsilon_i}{\partial f}(f, \sigma) \right) - (\alpha_i - \epsilon_i(f, \sigma)) \left( \frac{\partial^2 \epsilon_i}{\partial \sigma \partial f}(f, \sigma) \right) \right]$$

$$\epsilon_i(f, \sigma) = f \epsilon_{wn_i}(\sigma) + (1 - f) \epsilon_{sat_i}(\sigma)$$

$$\frac{\partial \epsilon_i}{\partial f}(\sigma) = \epsilon_{wn_i}(\sigma) - \epsilon_{sat_i}(\sigma)$$

$$\frac{\partial \epsilon_i}{\partial \sigma}(f, \sigma) = f \frac{\partial \epsilon_{wn_i}}{\partial \sigma}(\sigma) + (1 - f) \frac{\partial \epsilon_{sat_i}}{\partial \sigma}(\sigma)$$

$$\frac{\partial^2 \epsilon_i}{\partial \sigma^2} = f \frac{\partial^2 \epsilon_{wn_i}}{\partial \sigma^2}(\sigma) + (1 - f) \frac{\partial^2 \epsilon_{sat_i}}{\partial \sigma^2}(\sigma)$$

$$\frac{\partial^2 \epsilon_i}{\partial \sigma \partial f} = \frac{\partial^2 \epsilon_i}{\partial f \partial \sigma} = \frac{\partial \epsilon_{wn_i}}{\partial \sigma}(\sigma) - \frac{\partial \epsilon_{sat_i}}{\partial \sigma}(\sigma)$$

$$\varepsilon_{wn_i}(\sigma) = \frac{N}{2\sigma} [T_{1_i}(1 - e^{-T_{2_i}\sigma}) + T_{2_i}(1 - e^{-T_{1_i}\sigma})]$$

$$\begin{aligned} \frac{\partial \varepsilon_{wn_i}}{\partial \sigma}(\sigma) &= \frac{N}{2\sigma} T_{1_i} T_{2_i} (e^{-T_{1_i}\sigma} + e^{-T_{2_i}\sigma}) - \frac{N}{2\sigma^2} [T_{1_i}(1 - e^{-T_{2_i}\sigma}) + T_{2_i}(1 - e^{-T_{1_i}\sigma})] \\ \frac{\partial^2 \varepsilon_{wn_i}}{\partial \sigma^2}(\sigma) &= \frac{N}{\sigma^3} [T_{1_i}(1 - e^{-T_{2_i}\sigma}) + T_{2_i}(1 - e^{-T_{1_i}\sigma})] - \frac{N}{\sigma^2} T_{1_i} T_{2_i} (e^{-T_{1_i}\sigma} + e^{-T_{2_i}\sigma}) - \frac{N}{2\sigma} (T_{1_i}^2 T_{2_i} e^{-T_{1_i}\sigma} + T_{1_i} T_{2_i}^2 e^{-T_{2_i}\sigma}) \end{aligned}$$

$$\varepsilon_{sat_i}(\sigma) = \frac{N}{\sigma} T_{min} (1 - e^{-T_{max}\sigma})$$

$$\begin{aligned} \frac{\partial \varepsilon_{sat_i}}{\partial \sigma}(\sigma) &= \frac{N}{\sigma} T_{min_i} T_{max_i} e^{-T_{max_i}\sigma} - \frac{N}{\sigma^2} T_{min_i} (1 - e^{-T_{max_i}\sigma}) \\ \frac{\partial^2 \varepsilon_{sat_i}}{\partial \sigma^2}(\sigma) &= \frac{2N}{\sigma^3} T_{min_i} (1 - e^{-T_{max_i}\sigma}) - \frac{2N}{\sigma^2} T_{min_i} T_{max_i} e^{-T_{max_i}\sigma} - \frac{N}{\sigma} T_{min_i} T_{max_i}^2 e^{-T_{max_i}\sigma} \end{aligned}$$