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Numerical Linear Algebra
Homework 7
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1. Lecture 20

20.3 Suppose an $m \times m$ matrix A is written in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$. Assume that $A \in \mathbb{C}^{m \times m}$ is nonsingular.

(a) Verify the formula

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the Schur complement of A_{11} in A.

Verify:

$$\begin{aligned} LHS &= \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ -A_{21}A_{11}^{-1}A_{11} + A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ -A_{21} + A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \end{aligned}$$

(b) Suppose A_{21} is eliminated row by row by needs of n steps of Gaussian elimination. Show that the bottom-right $(m-n) \times (m-n)$ block of the result is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Let L_n denote the lower-triangular matrix that performs the first n steps of Gaussian elimination. After these n steps, the submatrix A_{21} vanishes and A_{11} is fully reduced to upper triangular form. Thus L_n must have the form

$$L_n = \begin{bmatrix} L_{11}^{-1} & 0 \\ X & I \end{bmatrix},$$

where $A_{11} = L_{11}U_{11}$ is the LU decomposition of the submatrix A_{11} and X is some undermined $(m-n)\times n$ matrix that must be chosen so that A_{21} vanishes. Observe:

$$L_n A = \begin{bmatrix} L_{11}^{-1} & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11}^{-1}A_{11} & L_{11}^{-1}A_{12} \\ XA_{11} + A_{21} & XA_{12} + A_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & (L_nA)_{12} \\ 0 & (L_nA)_{22} \end{bmatrix}.$$

For the last equality to hold, we must have

$$XA_{11} + A_{21} = 0 \implies X = -A_{21}A_{11}^{-1}.$$

Therefore, the bottom-right $(m-n)\times(m-n)$ block is given by

$$(L_nA)_{22} = XA_{12} + A_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12},$$

which is the same as the block given in part (a).

20.4 Like most of the algorithms in this book, Gaussian elimination involves a triply nested loop. In Algorithm 20.1, there are two explicit for loops, and the third loop is implicit in the vectors $u_{j,k:m}$ and $u_{k,k:m}$. Rewrite this algorithm with just one explicit for loop indexed by k. Inside this loop, U will be updated at each step by a certain rank-one outer product. This "outer product" form of Gaussian elimination may be a better starting point than Algorithm 20.1 if one wants to optimize computer performance.

The "outer product" form of the algorithm is as follows:

$$\begin{aligned}
&U = A,\ L = I \\
&\textbf{for } k = 1 \textbf{ to } m-1 \\
&\qquad \ell_{(k+1):m,k} = u_{(k+1):m,k}/u_{kk} \\
&\qquad u_{(k+1):m,k:m} = u_{(k+1):m,k:m} - \ell_{(k+1):m,k} \otimes u_{k,k:m}
\end{aligned}$$

where the last line includes a rank-one outer product resulting in a matrix of size $(m-(k+1))\times(m-k)$.

2. Lecture 23

23.1 Let A be a nonsingular square matrix and let $A = QR$ and $A^*A = U^*U$ by QR and Cholesky factorizations, respectively, with the usual normalizations $r_{jj}, u_{jj} > 0$. Is it true or false that $R = U$?

Proof. Since A is nonsingular, A has a unique QR factorization with $r_{jj} > 0$, by Theorem 7.2. Then we have that

$$A^*A = (QR)^*(QR) = R^*Q^*QR = R^*R,$$

where we have used the unitarity of Q in the last equality. The product A^*A is hermitian because

$$(A^*A)^* = A^*(A^*)^* = A^*A.$$

Again, since A is nonsingular, $Ax \neq 0$ for all $x \neq 0 \in \mathbb{C}^m$. Then we have that

$$x^*(A^*A)x = (Ax)^*(Ax) = \|Ax\|_2^2 > 0,$$

so A^*A is positive definite. By Theorem 23.1, every hermitian positive definite matrix has a unique Cholesky factorization $A^*A = U^*U$. Therefore, $A^*A = R^*R = U^*U$ implies that $R = U$.

3. Code up the Gaussian elimination without pivoting to solve the system

$$H(n)\mathbf{x} = \mathbf{b},$$

where $H(n)$ is the $n \times n$ Hilbert matrix with $H_{ij} = \frac{1}{i+j-1}$ $(1 \leq i, j \leq n)$, and $\mathbf{b} = (1, \cdots, 1)^T$. The condition number of the $n \times n$ Hilbert matrix grows as $O\left((1+\sqrt{2})^{4n}/\sqrt{n}\right)$. To see the deteriorating accuracy, you will nee to take $n = 2, 4, 6, 8, 10$.

4. Code up the Cholesky decomposition to solve the same linear system as above.

5. You may compare the results obtained by the Gaussian elimination and the Cholesky decomposition with those obtained by QR decompositions. Please make pertinent comments on these methods.

Based on the numerical results from Homework 3, I decided to implement the modified GramSchmidt QR factorization method. This method is simple to write and provided identical results to the Householder method for system dimensions $n = 10 - 30$. This is sufficient for the problem at hand.

In Table 1, the solutions for $n = 2, 4, 6, 8, 10$ are shown along with the corresponding condition number $\kappa(H(n))$ for the matrix $H(n)$. The quantity $\|H(n)\mathbf{x} = \mathbf{b}\|$ is also shown for each case to give a measure of the accuracy of the methods. It is clear that the condition number increases drastically with the dimension of system. Of the three methods, LU decomposition obtained results closest to the true solution. The Cholesky decomposition method had comparable results, though it was consistently outperformed by LU decomposition. The method with the most instability was QR decomposition, particularly in the $n = 10$ case. The value of $\|H(n)\mathbf{x} = \mathbf{b}\|$ for QR was two, four, and six orders of magnitude larger than the other two methods for the $n = 6, 8, 10$ cases, respectively.

Table 1: Comparison of solutions to the Hilbert system, $H(n)\mathbf{x} = \mathbf{b}$, using three different decomposition methods.

		LU		Cholesky		QR	
n	$\kappa(H(n))$	$\|H(n)\mathbf{x}-\mathbf{b}\|$	$\mathbf{x}$	$\|H(n)\mathbf{x}-\mathbf{b}\|$	$\mathbf{x}$	$\|H(n)\mathbf{x}-\mathbf{b}\|$	$\mathbf{x}$
2	19.33	0	$\begin{bmatrix}-2\\6\end{bmatrix}$	2.220e-16	$\begin{bmatrix}-2\\6\end{bmatrix}$	2.264e-15	$\begin{bmatrix}-2\\6\end{bmatrix}$
4	1.561e4	3.553e-15	$\begin{bmatrix}-4\\60\\-180\\140\end{bmatrix}$	3.553e-15	$\begin{bmatrix}-4\\60\\-180\\140\end{bmatrix}$	4.730e-13	$\begin{bmatrix}-4\\60\\-180\\140\end{bmatrix}$
6	1.512e7	1.172e-13	$\begin{bmatrix}-6\\210\\-1680\\5040\\-6300\\2772\end{bmatrix}$	1.137e-13	$\begin{bmatrix}-6\\210\\-1680\\5040\\-6300\\2772\end{bmatrix}$	4.513e-10	$\begin{bmatrix}-6\\210\\-1680\\5040\\-6300\\2772\end{bmatrix}$
8	1.549e10	3.183e-12	$\begin{bmatrix}-8\\504\\-7560\\46200\\-138600\\216216\\-168168\\51480\end{bmatrix}$	5.264e-12	$\begin{bmatrix}-8\\504\\-7560\\46200\\-138600\\216216\\-168168\\51480\end{bmatrix}$	4.609e-08	$\begin{bmatrix}-8\\504\\-7560\\46200\\-138600\\216216\\-168168\\51480\end{bmatrix}$
10	1.633e13	1.899e-10	$\begin{bmatrix}-9.998\\989.9\\-23757\\240212\\-1261126\\3783412\\-6726118\\7000699\\-3937915\\923713\end{bmatrix}$	2.152e-10	$\begin{bmatrix}-9.998\\989.8\\-23756\\240204\\-1261090\\3783313\\-6725954\\7000540\\-3937831\\923694\end{bmatrix}$	4.667e-4	$\begin{bmatrix}-7158\\622487\\-13326251\\121642613\\-582142950\\1604739227\\-2.639e9\\2.555e9\\-1.344e9\\2.960e8\end{bmatrix}$