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Numerical Linear Algebra
Homework 3
7 February 2019

1. Lecture 6 problems:

6.1 If P is an orthogonal projector, then $I-2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Proof. If P is a projector, then it is idempotent by definition, $P^2 = P$. By Theorem 6.1, P is also orthogonal if and only if $P = P^*$. So we have that $P^2 = PP = PP^* = P$. Then,

$$(I-2P)(1-2P)^* = (I-2P)(1-2P^*) = I - 2P - 2P^* + 4PP^* = 1 - 2P - 2P + 4P = I.$$

Therefore, $I - 2P$ is unitary.

To visualize a geometric interpretation, consider the rank-one orthogonal projector

$$P_a = \frac{aa^*}{a^*a},$$

where a is some arbitrary non-zero vector in $\mathbb{C}^m$. This projector isolates the component of any vector $v \in \mathbb{C}^m$ in the direction of a. Then the rank $m-1$ complement

$$P_{\perp a} = I - P_a = I - \frac{aa^*}{a^*a}$$

eliminates the components of v in the direction of a. The set of vectors unaffected by $P_{\perp a}$ form the kernel of P_a

$$\ker(P_a) = \left\{ v \in \mathbb{C}^n \middle| Pv = 0 \right\}.$$

So if $P_{\perp a}$ removes the component of $v \in \mathbb{C}^n$ in the direction of a, the vector $P_{\perp a}v$ lives on the $m-1$ dimensional "surface" $\ker(P_a)$. Therefore, $P_{\perp a} - P_a = I - 2P_a$ can be viewed as a mirror reflection through $\ker(P_a)$.

6.2 Let E be the $m \times m$ matrix that extracts the "even part" of an m-vector: $Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, ..., x_m)^T$ to $(x_m, ..., x_1)^T$. Is E an orthogonal projector, an oblique projector, or not a projector at all? What are its entries?

Proof. The matrix F has entries 1 along the secondary diagonal, and 0 elsewhere. Observe:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \implies Fx = \begin{bmatrix} x_m \\ x_{m-1} \\ \vdots \\ x_1 \end{bmatrix} \implies Ex = \frac{1}{2} \begin{bmatrix} x_1 + x_m \\ x_2 + x_{m-1} \\ \vdots \\ x_m + x_1 \end{bmatrix}.$$

Therefore, E has entries $\frac{1}{2}$ along the primary and secondary diagonals:

$$E = \frac{1}{2} \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & ⋰ & \vdots & \vdots \\ \vdots & \vdots & ⋰ & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

Clearly $E^* = E$, so by Theorem 6.1, E is an orthogonal projector.

6.3 Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Proof. A is $m \times n$, so A^*A is $n \times n$. By Theorem 1.3, A^*A is nonsingular if and only if A^*A has n nonzero eigenvalues. Then by Theorem 5.4, the nonzero singular values of A are the square roots of the nonzero eigenvalues of A^*A. So A^*A has n nonzero singular values. Therefore,A^*A is nonsingular if and only if A has full rank.

6.4 (a) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

What is the orthogonal projector P onto range(A), and what is the image under P of the vector $v = (1, 2, 3)^*$.

Proof. The orthogonal projector onto range(A) is given by equation (6.13):

$$\begin{aligned} P &= A(A^*A)^{-1}A^* \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{aligned}$$

The image of v under P is

$$Pv = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+3 \\ 4 \\ 1+3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

2. Read Lectures 7-10.

3. Given an arbitrary matrix $A \in \mathbb{C}^{m\times n}$, construct the QR decomposition by using the following three different procedures:

   (a) the classical Gram-Schmidt method

   (b) the modified Gram-Schmidt method

   (c) the Householder transform based method

   Each method should return Q and R matrices in a suitable format, where $Q \in \mathbb{C}^{m\times n}$ is a matrix with orthonormal columns and $R \in \mathbb{C}^{n\times n}$ is an upper triangular matrix.

   Case 1. Let

$$Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

   The QR decompositions of Z using the Gram-Schmidt methods were exactly the same:

$$Q = \begin{bmatrix} 0.10101525 & 0.31617307 & 0.54199690 \\ 0.40406102 & 0.35336990 & 0.51618752 \\ 0.70710678 & 0.39056673 & -0.52479065 \\ 0.40406102 & -0.55795248 & 0.38714064 \\ 0.40406102 & -0.55795248 & -0.12044376 \end{bmatrix}, \quad R = \begin{bmatrix} 9.89949494 & 9.49543392 & 9.69746443 \\ 0.0 & 3.29191961 & 3.01294337 \\ 0.0 & 0.0 & 1.97011472 \end{bmatrix}.$$

   The Householder transform based method, on the other hand, gave the same matrices as above with the signs of Q and R flipped. All three methods produced an orthogonal Q (with an absolute tolerance of 1E-10).

   Case 2. Let

$$A = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

   Again, all three methods returned the same QR decomposition of A, except for the flipped signs of Q and R for the Householder method:

$$Q = \begin{bmatrix} 0.70710173 & 0.70711183 \\ 0.70711183 & -0.70710173 \end{bmatrix}, \quad R = \begin{bmatrix} 0.98995656 & 1.00000455 \\ 0.0 & 0.00000714 \end{bmatrix}$$

   All three methods produced an orthogonal Q.

4. Turn in codes to Jian Song (songji12@msu.edu).