- Module 2 -
Applied Math
(Linear Algebra) Review

Outline

- Linear Algebra
  - Scalars, Vectors and Matrices
  - Matrix Operations
  - Special Kinds of Matrices and Vectors

Scalars, Vectors and Matrices (1)

- **Scalar:** a single number s
  - written in italics and lowercase
  - rational number, irrational number, etc.
    (e.g., 2, $\sqrt{2}$, π)
  - types of numbers
    - **integer** number ($\mathbb{Z}$): positive and negative numbers that can be written without a fractional component, including zeros $\{\dots, -2, -1, 0, 1, 2, \dots\}$
    - **natural** number ($\mathbb{N}$): counting numbers (positive integers) $\{1, 2, 3, \dots\}$
    - **real** number ($\mathbb{R}$): numbers of a continuous quantity that can have decimal representations with a finite or infinite sequence of digits to the right of the decimal point
  - when a scalar is introduced, its type should be specified
    (e.g., "Let $s \in \mathbb{R}$ be the slope of the line", where s is a real-valued scalar)

Scalars, Vectors and Matrices (2)

- **Vector:** a one-dimensional array of numbers

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

  - written in bold (when it has N elements) lowercase
  - **elements** of a vector are identified by writing its name with a subscript

    (e.g., first element of $\boldsymbol{x}$ is x_1, etc.)
  - when a vector is introduced, its element type and size should be specified

    (e.g., $\boldsymbol{x} \in \mathbb{R}^N$, where $\boldsymbol{x}$ has N elements and each element is in $\mathbb{R}$)
  - a vector can be thought of as a point in space, with each element giving the coordinate along a different axis

Scalars, Vectors and Matrices (3)

- **Matrix:** a two-dimensional array of numbers

Rows

$$\boldsymbol{X}_{N\times p} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{N,1} & x_{N,2} & \cdots & x_{N,p} \end{bmatrix}$$

N rows by p columns

Columns

  - written in bold uppercase letters
  - each element of a matrix is identified by two indices: **row** by the first index and **column** by the second index

    (e.g., $x_{i,j}$ denotes the element in the i-th row, j-th column)
  - when a matrix is introduced, its element type and dimensions and should be specified

    (e.g., $\boldsymbol{X} \in \mathbb{R}^{N\times p}$, where $\boldsymbol{X}$ has N rows, p columns and each element is in $\mathbb{R}$)

Matrix Operations (1)

- **Addition** of two matrices (same dimensions) is done by adding the corresponding elements

$$\boldsymbol{C}_{N\times p} = \boldsymbol{A}_{N\times p} + \boldsymbol{B}_{N\times p}, \text{ where } c_{i,j} = a_{i,j} + b_{i,j}$$

- **Subtraction** of two matrices (same dimensions) is done by subtracting the corresponding elements

$$\boldsymbol{C}_{N\times p} = \boldsymbol{A}_{N\times p} - \boldsymbol{B}_{N\times p}, \text{ where } c_{i,j} = a_{i,j} - b_{i,j}$$

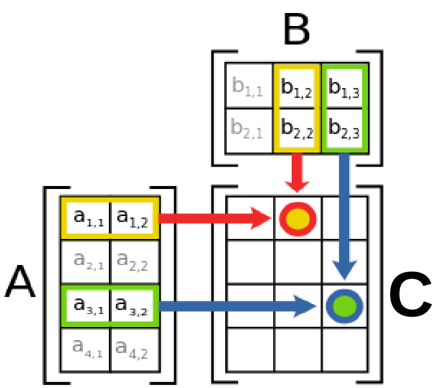
- **Scalar multiplication** of a matrix is done by multiplying each element of the matrix by a scalar

$$\boldsymbol{B}_{N\times p} = c\boldsymbol{A}_{N\times p}, \text{ where } b_{i,j} = c \cdot a_{i,j}$$

Matrix Operations (2)

- **Multiplication** of two matrices results in a third matrix
  - in order for this multiplication to be defined
    ➔ $\boldsymbol{A}$ must have the same number of columns as $\boldsymbol{B}$ have rows

$$\boldsymbol{A}_{m\times n}=\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}, \boldsymbol{B}_{n\times p}=\begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,p} \\ b_{2,1} & b_{2,2} & \dots & b_{2,p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,p} \end{bmatrix},$$



where $\boldsymbol{C}_{m\times p}=\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times p}$ and $c_{i,j}=\sum_{k=1}^{n}a_{i,k}b_{k,j}$

- Properties:
  - Distributive: $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$
  - Associative: $\boldsymbol{A}(\boldsymbol{BC})=(\boldsymbol{AB})\boldsymbol{C}$
  - **NOT** Commutative: $\boldsymbol{AB}\neq\boldsymbol{BA}$

Matrix Operations (3)

- **Transpose** of a matrix is the mirror image across the main diagonal

$$\boldsymbol{X} = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{X}^T = \begin{bmatrix} x_{1,1} & x_{2,1} & x_{3,1} \\ x_{1,2} & x_{2,2} & x_{3,2} \end{bmatrix}$$

  - Transpose of a matrix $\boldsymbol{X}$ is denoted by $\boldsymbol{X}^T$ and is defined as

$$(\boldsymbol{X}^T)_{i,j} = x_{j,i}$$

- Identities:
  - $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
  - $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$
  - if $\boldsymbol{A} = \boldsymbol{A}^T$, then $\boldsymbol{A}$ is a **symmetric** matrix

Matrix Operations (4)

- **Identity** matrix is a matrix that does not change any vector when we multiply that vector by that matrix

$$\boldsymbol{I}_n \in \mathbb{R}^{n \times n}, \text{ and } \forall \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{I}_n \boldsymbol{x} = \boldsymbol{x}$$

  - The structure of the identity matrix is a **square** matrix where all the entries along the main diagonal are 1, while all other entries are 0

    (e.g., $\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$)

Matrix Operations (5)

- **Inverse** matrix of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$ and it is defined as

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n,$$ where $\boldsymbol{I}_n$ is the identity matrix

  - **Invertibility:** the inverse matrix does not exist for all matrices and requires that
    - $\boldsymbol{A}$ is a square matrix
    - all columns of $\boldsymbol{A}$ are linearly independent, i.e., none of the columns of $\boldsymbol{A}$ can be expressed as a linear combination of other columns
  - Properties:
    - $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
    - if $\det(\boldsymbol{A}) = 0$, $\boldsymbol{A}$ does not have an inverse
    - for any invertible matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Matrix Operations (6)

- If $\boldsymbol{A}$ is a 2x2 invertible matrix, then

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix},$$

where $\det(\boldsymbol{A}) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$

Matrix Operations (7)

- A system of **linear equations** written as

$$
\begin{gathered}
a_{1,1}x_1 + \cdots + a_{1,p}x_p = b_1 \\
a_{2,1}x_1 + \cdots + a_{2,p}x_p = b_2 \\
\vdots \\
a_{N,1}x_1 + \cdots + a_{N,p}x_p = b_N
\end{gathered}
$$

can be rewritten as $\boldsymbol{A}x = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,p} \\ a_{2,1} & a_{2,2} & \dots & a_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \dots & a_{N,p} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \quad \text{and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

- To solve the system of linear equations for $\boldsymbol{x}$:

$$\boldsymbol{A}x = \boldsymbol{b}$$

$$\boldsymbol{A}^{-1}\boldsymbol{A}x = \boldsymbol{I}_n x = \boldsymbol{A}^{-1}\boldsymbol{b}$$

$$x = \boldsymbol{A}^{-1}\boldsymbol{b}$$

Matrix Operations (8)

- **Norm** of a vector $\boldsymbol{x}$ measures the distance from the origin to the point $\boldsymbol{x}$. The L^p norm is defined as

$$\|\boldsymbol{x}\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$$

  - L^1 norm, with $p = 1$, is known as the **Manhattan norm**, increases by $\boldsymbol{\epsilon}$ every time an element of $\boldsymbol{x}$ moves away from $\mathbf{0}$ by $\boldsymbol{\epsilon}$

$$\|\boldsymbol{x}\|_1 = \sum_i |x_i|$$

  - L^2 norm, with $p = 2$, is known as the **Euclidean norm**, is the Euclidean distance from the origin to the point identified by $\boldsymbol{x}$ and can be calculated simply as $\sqrt{\boldsymbol{x}^T\boldsymbol{x}}$

$$\|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots x_N^2}$$

Special Kinds of Matrices and Vectors

- **Diagonal** matrix has nonzero entries only along the main diagonal
  ➔ $\boldsymbol{D}$ is diagonal if and only if (iff) $d_{i,j} = 0$ for all $i \neq j$

- **Symmetric** matrix is a matrix that is equal to its own transpose

$$\boldsymbol{A} = \boldsymbol{A}^T$$

- **Orthonormal** vectors: two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal ($\boldsymbol{x}^T\boldsymbol{y} = 0$) to each other and both have **unit norm** ($\|\boldsymbol{x}\|_2 = 1$)

$$\boldsymbol{x}^T\boldsymbol{y} = 0 \text{ and } \|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2 = 1$$

- **Orthogonal** matrix is a square matrix whose rows are mutually orthonormal and whose columns are also mutually orthonormal

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}_n, \text{ which implies that } \boldsymbol{A}^{-1} = \boldsymbol{A}^T$$
$$(\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n)$$