Lecture 2

COMP 3760 – Fall 2019

Textbook: 2.1, 2.2, 2.3

Please write a function on a sticky note.

Use one of these or invent your own.

Then keep it. We'll use it later.

8675309

$4n^2+10$

3^n

$\log_2 n$

$50n^3+20n+4$

$4n\log_2 n$

2^n-1

$1+\log_2 6$

$3\log_2 n+1$

$3737n$

$n(2n+1)$

$3\log_2 n+n$

$5!+3^2$

n^2+3n^3

$\log_2 n+9n!$

Today's highlights

- Review last time
- Common functions in algorithm analysis
- Order of growth
- Big-O notation

Quiz next week!

- Our first quiz
- During lab time
  - Attendance
  - Quiz at approximately X:00 (30 minutes into lab)
  - Length probably 15-20 minutes
- Coverage: Lectures 1&2
- Be on time! We might start early!

My billiard-ball trick

- Seems like I can never remember:

$$\sum_{i=0}^{n} i = ??$$

$$? \quad \frac{n(n+1)}{2} \quad ? \quad \frac{n(n-1)}{2} \quad ?$$

- ... so this helps:

n=5,
15 = 30/2
so it's 5(5+1)

Today’s objectives

- Recognize commonly-seen functions from the world of algorithm analysis
- Rank functions by “order of growth”
- Recite the definition of *Big-O notation*
- Express the order of growth of an algorithm (expressed in pseudocode) in Big-O notation
- Explain the difference between *best case, worst case,* and *average case*.
- Given a function, provide an example of an algorithm (either by common name or expressed in pseudocode) whose order of growth is Big-O of that function

What did we learn last time?

1. Efficiency of an algorithm depends on **input size**
2. Efficiency of an algorithm also depends on **basic operation**
3. Efficiency can be expressed by **counting** the basic operation

```
Loops(A[0..n-1])
 for i ← 1 to n-1 do
   v ← A[i]
   j ← i-1
   while j≥0 and A[j]>v do
     A[j+1] ← A[j]
     j ← j-1
   A[j+1] ← v
```

$$C(n) = \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} 1 = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \approx \frac{n^2}{2}$$

Example 1

- Problem: find the max element in a list
- Input size measure:
  - *Number of list items, i.e. n*
- Basic operation:
  - *Comparison*

```
ALGORITHM MaxElement(A[0..n − 1])
    maxval ← A[0]
    for i ← 1 to n − 1 do
        if A[i] > maxval
            maxval ← A[i]
    return maxval
```

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1$$

Example 2

- Problem: Multiplication of two matrices
- Input size measure:
  - *Matrix* dimension *(elements per row/col)*
- Basic operation:
  - *Multiplication of two numbers*

ALGORITHM $MatrixMultiplication(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$
    for $i \leftarrow 0$ **to** $n-1$ **do**
        for $j \leftarrow 0$ **to** $n-1$ **do**
            $C[i, j] \leftarrow 0.0$
            for $k \leftarrow 0$ **to** $n-1$ **do**
                $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$
    return C

$$C(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1 = n^3$$

Example 3

- Problem: calculating an unusual sum
- Input size measure:
  - *Number n*
- Basic operation:
  - *Addition*

```
Example3(n)
 sum ← 0
 i ← n
 while i ≥ 1
   sum ← sum + 1
   i ← i/2
 return sum
```

$$C(n) = \log n$$

Example 4

- Problem: Searching for key in a list of n items
- Input size measure:
  - *Number of list items, i.e. n*
- Basic operation:
  - *Key comparison*

ALGORITHM *SequentialSearch*$(A[0..n-1], K)$

$i \leftarrow 0$
while $i < n$ **and** $A[i] \neq K$ **do**
    $i \leftarrow i + 1$
if $i < n$ **return** i
else return -1

Depends on order of input

$C_{worst}(n) = n$
$C_{best}(n) = 1$

Worst case, average case, best case

- Worst case:
  - Most possible number of steps needed by an algorithm
- Average case:
  - Number of steps needed “on average”
- Best case:
  - Number of steps needed if you “get lucky” with the input or processing

- Consider the problem of finding an element in an unsorted list

Which to use: best, worst, average?

- We will focus on worst-case analysis in this course
  - Unless otherwise specified, you should always analyze the worst case

- There are many situations where best case = worst case
  - Example: find the *largest* element in an unsorted list

Running time efficiency can be many different functions

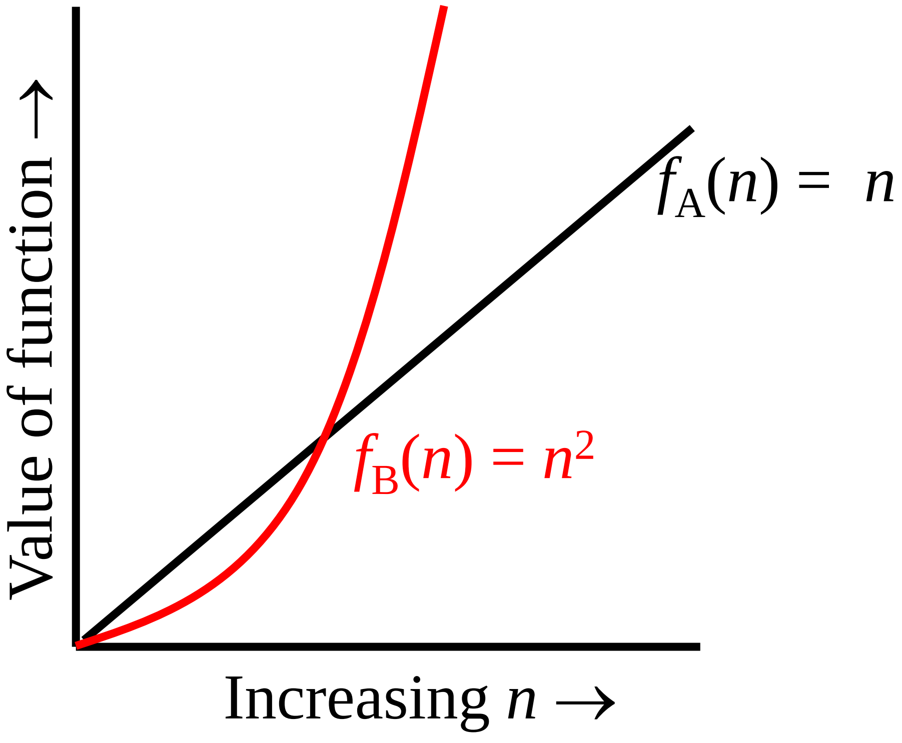
- $C(n) = n(n-1)/2$
- $C(n) \approx 0.5n^2$
- $C(n) = \log n + 5$
- $C(n) = n!$

- Which one is the better algorithm?

Let’s look at some functions

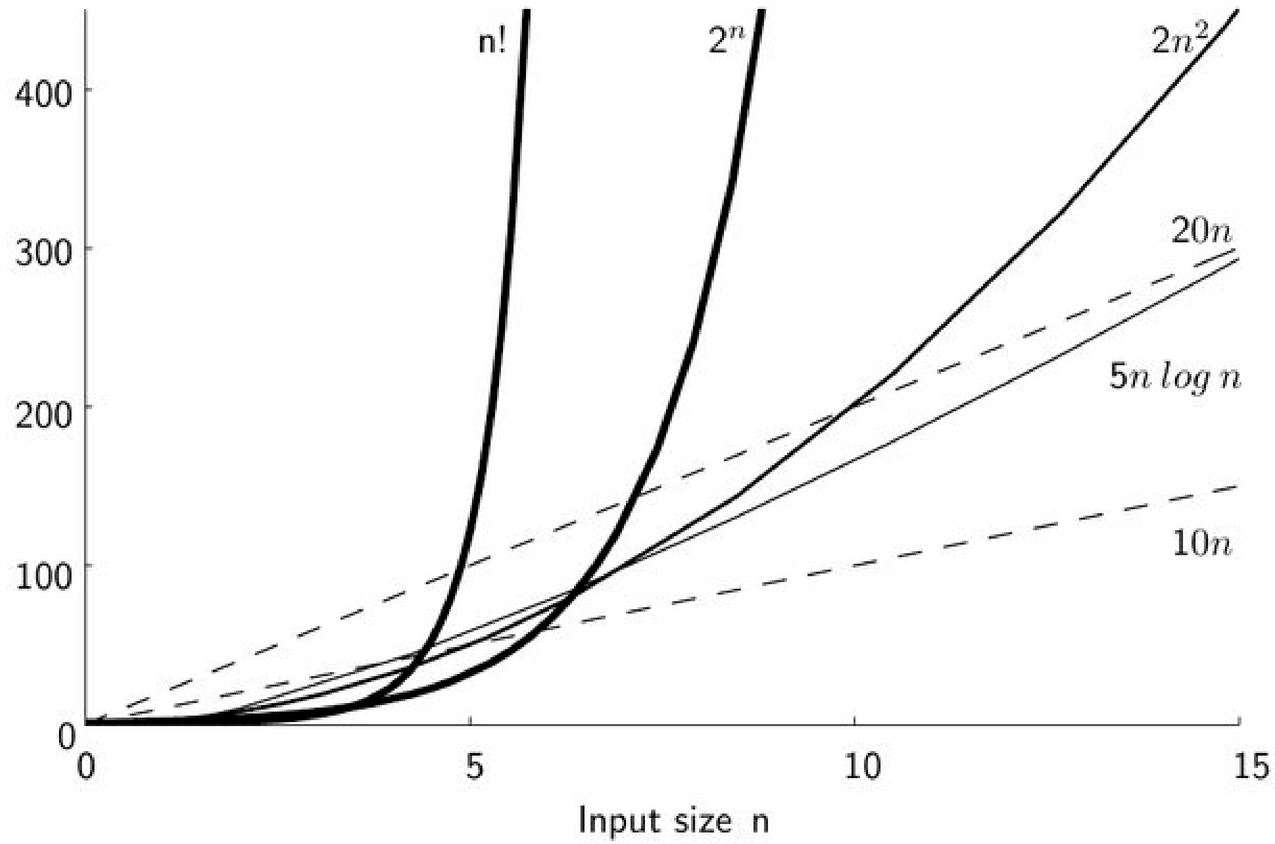
- DESMOS

Order of growth



Order of growth

- What we really care about:
  - Order of growth as $n \rightarrow \infty$



Orders of growth

these represent possible functions that classify basic ops counts

TABLE 2.1 Values (some approximate) of several functions important for analysis of algorithms

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

$1.5\text{x}10^{133}$ years on the world's fastest supercomputer

Orders of growth

$$\log_2 n \quad < \quad n \quad < \quad n\log_2 n \quad < \quad n^2 \quad < \quad n^3 \quad < \quad 2^n \quad < \quad n!$$

Common efficiency classes (part 1)

Class	Name	Comments
1	*constant*	Short of best-case efficiencies, very few reasonable examples can be given since an algorithm's running time typically goes to infinity when its input size grows infinitely large.
$\log n$	*logarithmic*	Typically, a result of cutting a problem's size by a constant factor on each iteration of the algorithm (see Section 5.5). Note that a logarithmic algorithm cannot take into account all its input (or even a fixed fraction of it): any algorithm that does so will have at least linear running time.
n	*linear*	Algorithms that scan a list of size n (e.g., sequential search) belong to this class.
$n \log n$	"*n-log-n*"	Many divide-and-conquer algorithms (see Chapter 4), including mergesort and quicksort in the average case, fall into this category.

Common efficiency classes (part 2)

n^2	*quadratic*	Typically, characterizes efficiency of algorithms with two embedded loops (see the next section). Elementary sorting algorithms and certain operations on n-by-n matrices are standard examples.
n^3	*cubic*	Typically, characterizes efficiency of algorithms with three embedded loops (see the next section). Several nontrivial algorithms from linear algebra fall into this class.
2^n	*exponential*	Typical for algorithms that generate all subsets of an n-element set. Often, the term "exponential" is used in a broader sense to include this and larger orders of growth as well.
$n!$	*factorial*	Typical for algorithms that generate all permutations of an n-element set.

General strategy for analysis of non-recursive algorithms

This strategy is taken from page 62 of your textbook:

1. Decide on a parameter indicating the input's size.
2. Identify the algorithm's basic operation.
3. Check whether the number of times the basic operation is executed depends only on the size of the input.
   - If it depends on some other property, the best/worst/average case efficiencies must be investigated separately
4. Set up a sum expressing the number of times the basic operation is executed.
5. Use standard formulas and rules of sum manipulation to find a closed form formula c(n) for the sum from step 4 above.
6. Determine the efficiency class of the algorithm using asymptotic notations

Asymptotic order of growth

A way of comparing functions

- Big O (Pronounced “big oh”)
- Big Ω
- Big Θ

3^n

$2^n - 1$

$50n^3+20n+4$ $\quad$ n^2+3n^3

$4n^2+10$ $\quad$ $n(2n+1)$

$\log_2 n+9n!$

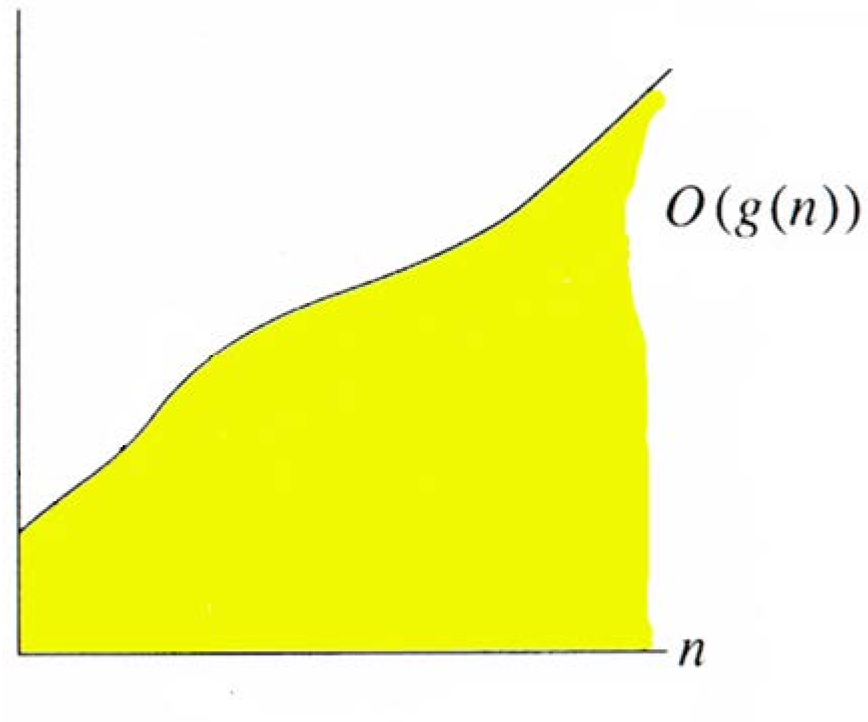
$5!+3^2$ $\quad$ 8675309 $\quad$ $1+\log_2 6$

$4n\log_2 n$

$3\log_2 n+1$ $\quad$ $\log_2 n$

$3737n$ $\quad$ $3\log_2 n+n$

Big-O in pictures

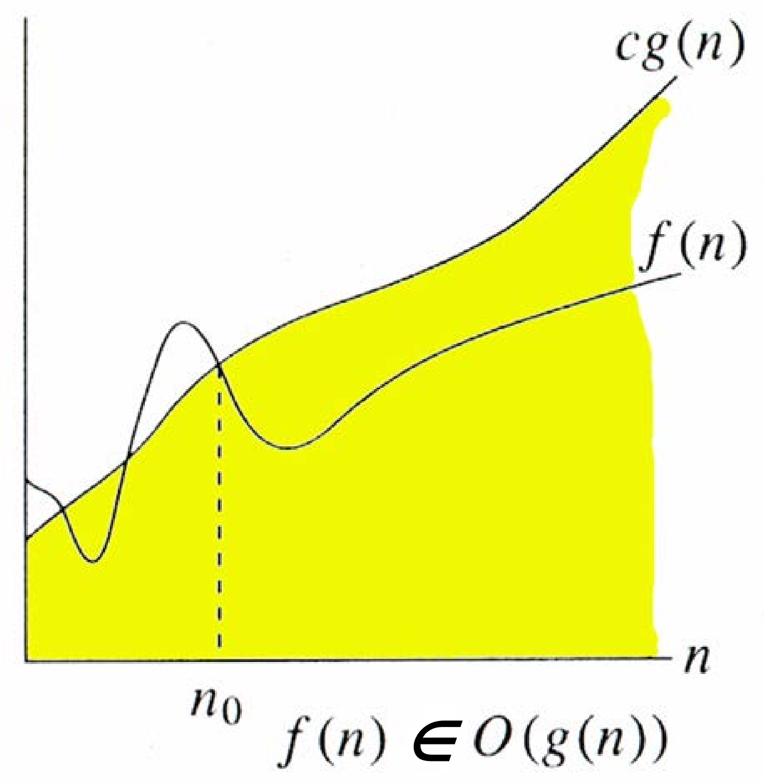


Set of all functions whose *rate of growth* is the same as or lower than that of *g*(*n*).

We also say “f(n) is *bounded above* by a constant multiple of g(n)”

or (carelessly) just “f(n) is bounded by g(n)”

Big-O in pictures



$f(n) \le c * g(n)$, for all $n \ge n_0$

Big-O (formal definition)

<u>Definition:</u>

- a function **f(n)** is in the set **O(g(n))** [*denoted: $f(n) \in O(g(n))$*] if there is a constant **c** and a positive integer $\mathbf{n_0}$ such that

$$f(n) \le c * g(n) \text{ , for all } n \ge n_0$$

i.e. f(n) is bounded above by some constant multiple of g(n)

Example

- Is $f(n) = 2n+6 \in O(n)$?
- By the definition:
  - Need to find a constant c and a constant n_0 such that $f(n) \le cg(n)$ for all $n > n_0$
- Many will work
  - Use $c = 4$ and $n_0 = 3$
- $\rightarrow$ $f(n)$ is $\in O(n)$

n	f(n)	c*g(n)
1	8	4
2	10	8
3	12	12
4	14	16
5	16	20
6	18	24
…	…	…

Looks good from here down

Big-O

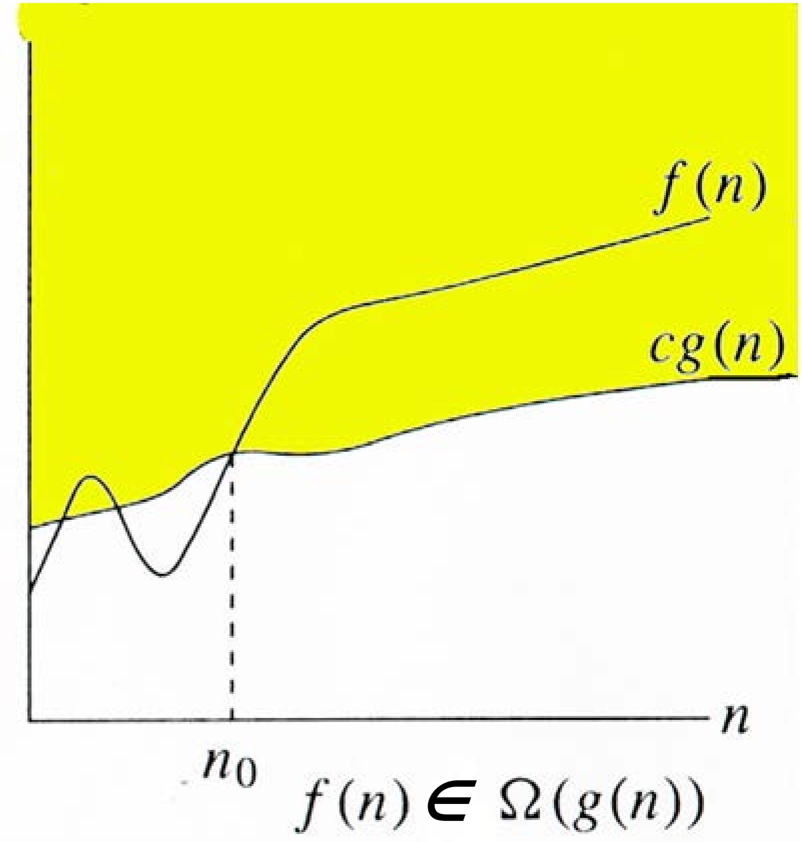
- Simple Rule: Drop lower order terms and constant factors

1. $50n^3 + 20n + 4 \quad \in O(n^3)$
2. $4n^2 + 10 \quad \in O(n^2)$
3. $n(2n + 1) \quad \in O(n^2)$
4. $3\log n + 1 \quad \in O(\log n)$
5. $3\log n + n \quad \in O(n)$
6. $1 + \log 6 \quad \in O(1)$
7. $5! + 3^2 \quad \in O(1)$

Big Omega

Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.

Big Omega



$f(n) \geq c * g(n)$, for all $n \geq n_0$

Big Omega

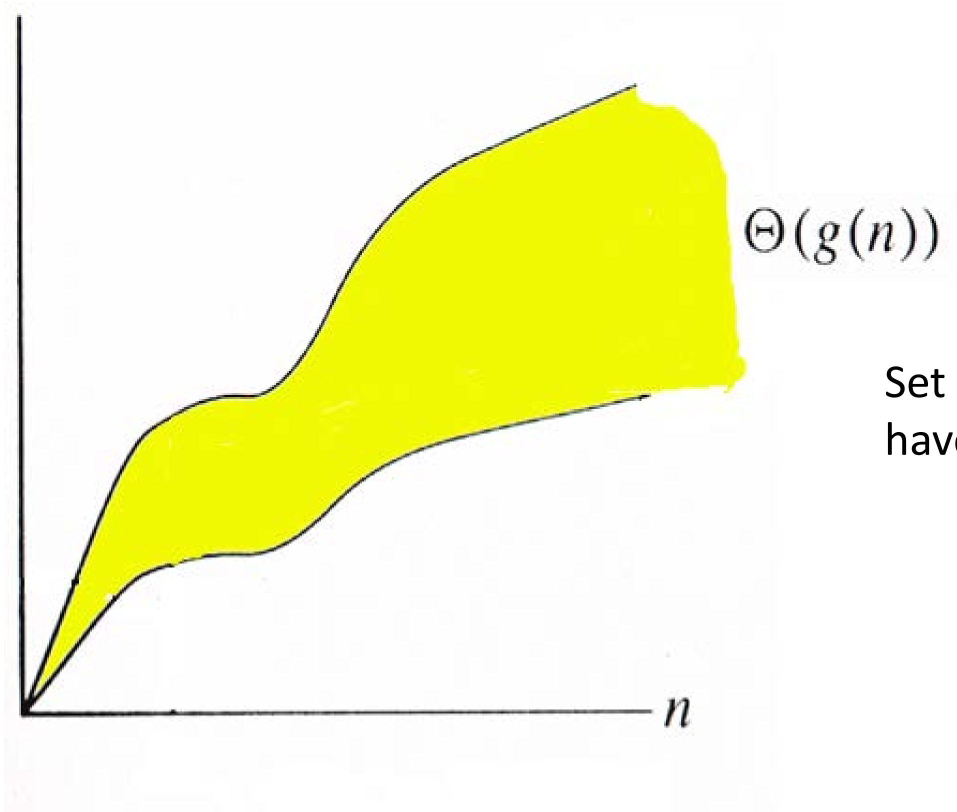
Definition:

- a function **$f(n)$** is in the set **$\Omega(g(n))$** [*denoted: $f(n) \in \Omega(g(n))$*] if there is a constant **c** and a positive integer **n_0** such that

$$f(n) \geq c * g(n) \text{ , for all } n \geq n_0$$

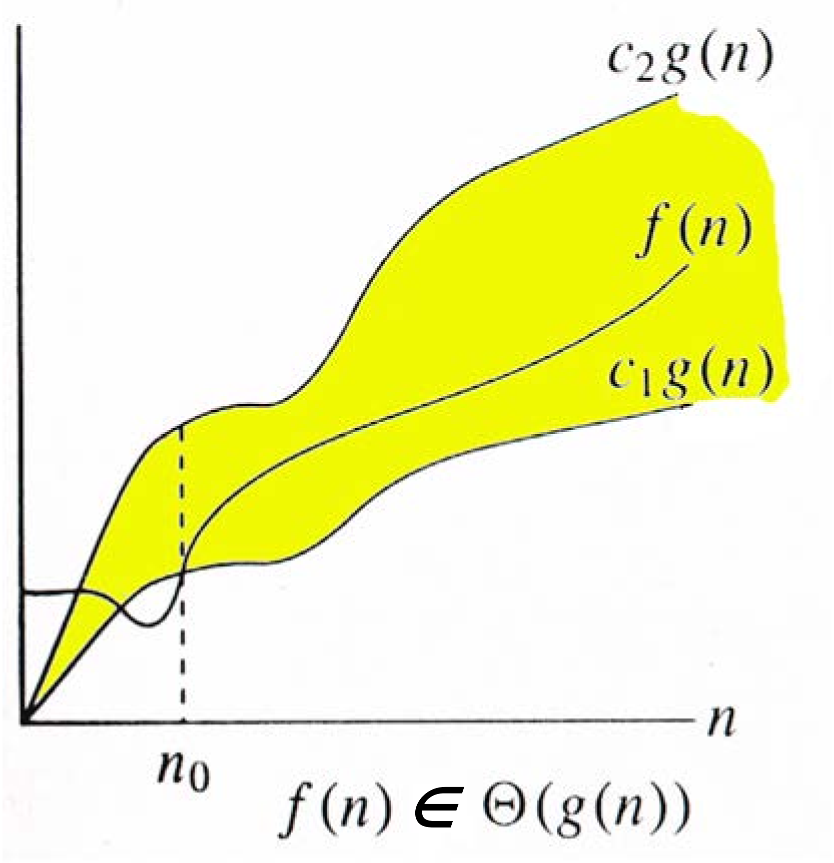
- *i.e.* f(n) is bounded below by some constant multiple of g(n)

Big Theta



Set of all functions that
have the same *rate of growth* as $g(n)$.

Big Theta



$c_2\, g(n) \le f(n) \le c_1\, g(n)$, for all $n \ge n_0$

Big Theta

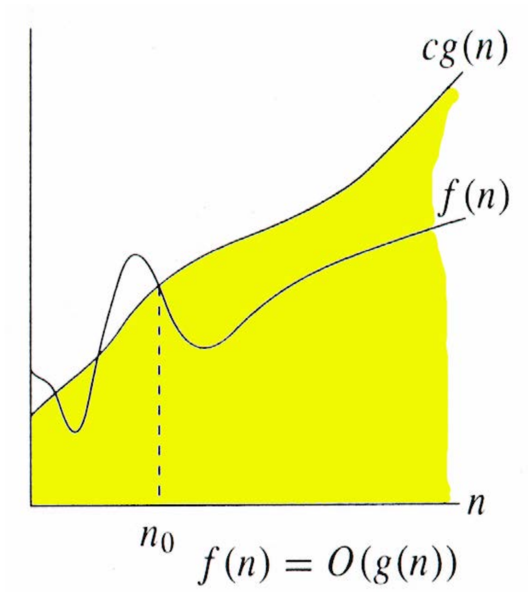
Definition:

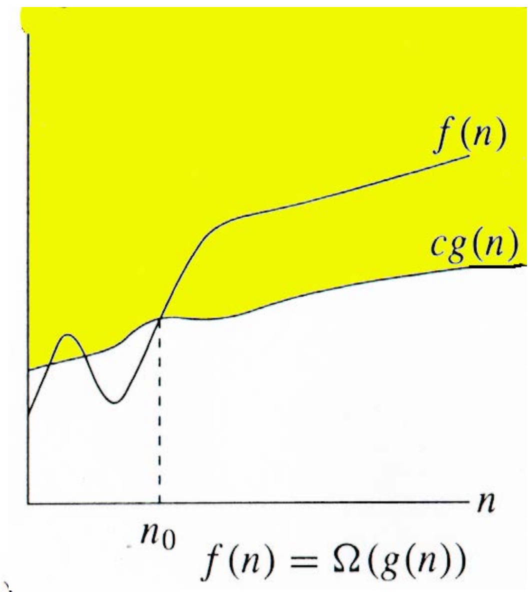
- a function **f(n)** is in the set **Θ(g(n))** [*denoted: f(n) ∈ Θ(g(n))*] if there are constants $\mathbf{c_1}$ **and** $\mathbf{c_2}$**,** and a positive integer $\mathbf{n_0}$ such that

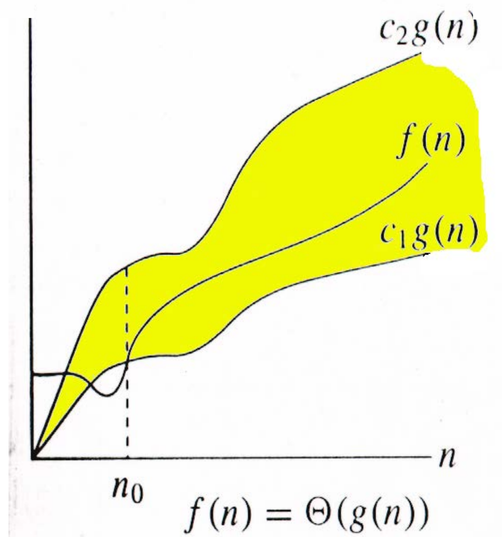
$$c_2\, g(n) \le f(n) \le c_1\, g(n)\ ,\ \text{for all } n \ge n_0$$

- *i.e.* f(n) is bounded both above and below by constant multiples of g(n)

Summary of notations - pictorial







Summary of notations - intuition

- Big-O → execution will take *at MOST that long*
- Big-Ω → execution will take *at LEAST that long*
- Big-Θ → execution will take *at least AND at most* that long

In general…

- We will usually focus on Big-O
- Why?
  - Focuses on worst case efficiency
  - Most common when people talk about algorithms

Examples

What is the efficiency class of the following functions?

- $10n$ O(n)
- $5n^2 + 20$ $O(n^2)$
- $10000n + 2^n$ $O(2^n)$
- log(n) * (1 + n) O(nlog(n))

Example 1

- Problem: find the max element in a list
- Input size measure:
  - *Number of list items, i.e. n*
- Basic operation:
  - *Comparison*

```
ALGORITHM MaxElement(A[0..n − 1])
    maxval ← A[0]
    for i ← 1 to n − 1 do
        if A[i] > maxval
            maxval ← A[i]
    return maxval
```

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 \in O(n)$$

Example 2

- Problem: *Multiplication of two matrices*
- Input size measure:
  - *Matrix dimensions or total number of elements*
- Basic operation:
  - *Multiplication of two numbers*

ALGORITHM $MatrixMultiplication(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

for $i \leftarrow 0$ **to** $n-1$ **do**

  for $j \leftarrow 0$ **to** $n-1$ **do**

    $C[i, j] \leftarrow 0.0$

    for $k \leftarrow 0$ **to** $n-1$ **do**

      $C[i, j] \leftarrow C[i, j] + A[i, k] * B[k, j]$

return C

$$C(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1 = n^3 \in O(n^3)$$

Example 3: Element uniqueness problem

ALGORITHM *UniqueElements*$(A[0..n-1])$
//Determines whether all the elements in a given array are distinct
//Input: An array $A[0..n-1]$
//Output: Returns "true" if all the elements in A are distinct
// and "false" otherwise
for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
        if $A[i] = A[j]$ **return false**
return true

Example 3

```
ALGORITHM  UniqueElements(A[0..n − 1])
    //Determines whether all the elements in a given array are distinct
    //Input: An array A[0..n − 1]
    //Output: Returns "true" if all the elements in A are distinct
    //           and "false" otherwise
    for i ← 0 to n − 2 do
        for j ← i + 1 to n − 1 do
            if A[i] = A[j] return false
    return true
```

- Parameter for input size:

  n, the size of the array

- Basic operation:

  Comparison in the innermost loop

- Worst case efficiency count… nested loop:

$$\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i - 1 + 1) = \sum_{i=0}^{n-2} (n - 1 - i)$$

$$= \sum_{i=0}^{n-2} n - \sum_{i=0}^{n-2} 1 - \sum_{i=0}^{n-2} i = n(n-1) - (n-1) - (n-2)(n-1)/2$$

$$= n^2 - n - n + 1 - n^2/2 + 3n/2 - 1$$

$$= n^2/2 - n/2 \in O(n^2)$$

Practice problems

1. Chapter 2.1, page 50, question 2
2. Chapter 2.2, page 60, question 5
3. Chapter 2.3, page 68, questions 5, 6

Practice problems

For each of the following algorithms determine:

a) its basic operation
b) basic operation count
c) if basic op count depends on input form

1. Computing the sum of a set of numbers
2. Computing n! (n factorial)
3. Checking whether all elements in a given array are distinct