Multi-Agent Systems

Homework Assignment 2
MScAl, VU

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2 Game Theory: Nash equilibrium

2.1 Dining out

Alice and Bob are going to dinner and plan to split the bill evenly no matter who orders what. There are two meals, a cheap one (C) priced at 10 Euro, which gives each of them 12 Euro's worth of pleasure, and an expensive dinner (E) dinner priced at 20 Euro, which gives them each 18 Euro's worth of pleasure.

1. Write down the pay-off matrix.

   Here C stands for cheap and E stands for expensive.

	Alice (C)	Alice (E)
Bob (C)	$2, 2$	$-3, 3$
Bob (E)	$3, -3$	$-2, -2$

2. Assuming that they both order simultaneously and without coordinating, what will they order and why?

   They will both order the expensive meal, since C is strictly dominated.

3. Alice is quite the romantic type and gets an additional s Euro's worth of pleasure if they happen to pick the same meal (either both cheap or both expensive). Bob, on the other hand, is a bit of a contrarian and gets an additional amount of pleasure (also equivalent to s Euro) when they happen to favour different meal choices. Assume that $0 < s \leq 2$. How does this change the pay-off matrix and the Nash equilibrium (or equilibria) of this game?

   Given this information, the new pay-off matrix will be:

	Alice (C)	Alice (E)
Bob (C)	$2 + s, 2$	$-3, 3 + s$
Bob (E)	$3, s - 3$	$s - 2, -2$

There are three cases, if 0 < s < 1, the NE will remain the same. For s = 1, E still weakly dominates C, as can be seen in figure 1. For s between 1 and 2, there is pure NE, as can be seen in figure 1.

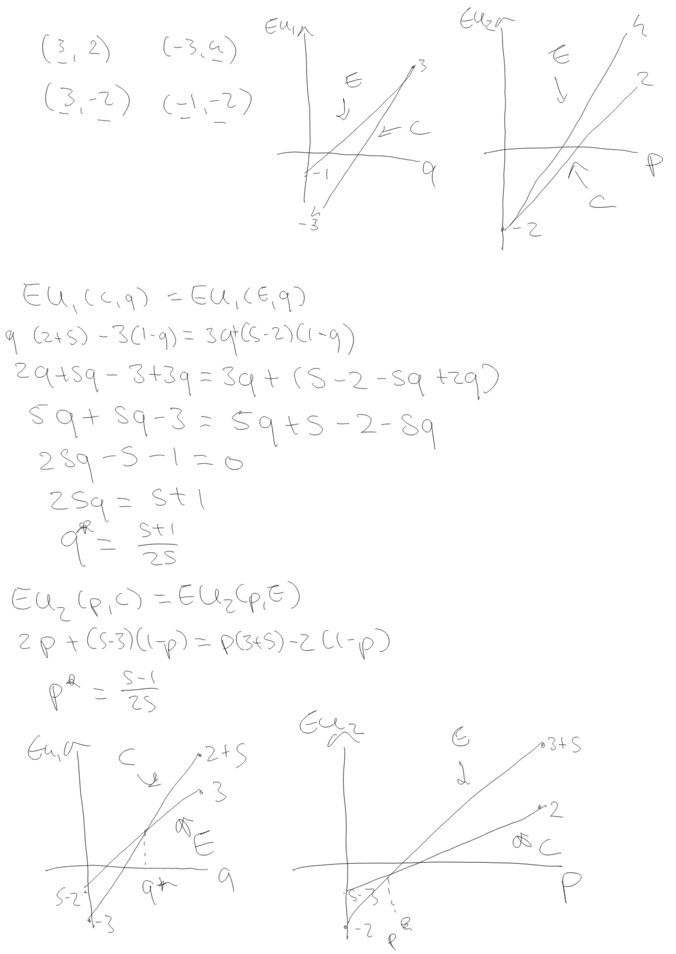

Figure 1: Pay-off for v

2.2 Hawk versus Dove

Two animals are in conflict over some resource worth $v > 0$. Simultaneously, they choose whether to behave like hawks (H) or doves (D). Hawks are willing to fight over the good, whereas doves are not. So if one animal chooses hawk and the other dove, the hawk gets everything leaving nothing for the dove. If both behave like doves, they split the resource equally. If however, both adopt an hawk strategy, they fight and on average get half of the food. The fighting however comes at a cost c to both of them.

Questions

- Write down the pay-off matrix for this game.

	Hawk	Dove
Hawk	$0.5v - c, 0.5v - c$	$v, 0$
Dove	$0, v$	$0.5v, 0.5v$

- Determine the Nash equilibria for this game and discuss how they change as the cost of aggression (c) increases. Do your results make sense?

  The Nash equilibrium is when both choose hawk, since a higher pay-off is not possible. However, if c $>$ 0.5, then the reward becomes negative, meaning that the new equilibrium is mixed and is everything except where both choose dove. If c is exactly v/2, then H weakly dominates D, since it is possible the pay-out for hawk is 0.

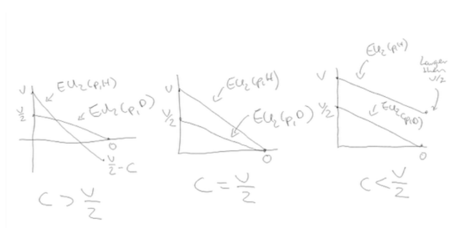

Figure 2: Pay-off for all three costs

2.3 Investment in recycling

Two neighbouring countries, $i = 1, 2$, simultaneously choose how many resources r_i (in hours) to spend in recycling activities. The average benefit (for country i) per hour spent equals

$$b_i(r_i, r_j) = 10 - r_i + \frac{r_j}{2}$$

Notice that country i 's average benefit is increasing in the resources that neighbouring country j spends on his recycling because a clean environment produces positive external effects on other countries. The (opportunity) cost per hour for each country is 4 . Hence, the expected utility for country i equals:

$$u_i(r_i, r_j) = b_i(r_i, r_j) r_i - 4r_i$$

Questions

1. Determine each country's best-response function.

   For Country $i = 1$:

   $$\frac{\partial u_1}{\partial r_1} = 10 - 2r_1 + \frac{r_2}{2} - -r_1 - 4 = 0$$

   $$r_1 = 3 + \frac{r_2}{4}$$

   For Country $i = 2$ this is exactly the same:

   $$r_2 = 3 + \frac{r_1}{4}$$

   The NE in this case is (4,4).

2. Indicate the pure strategy Nash Equilibrium (r_1^*, r_2^*) on the graph;

   The nash equilibrium is displayed in figure 3.

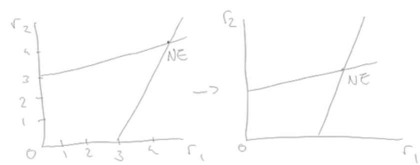

Figure 3: Nash equilibirium and its shift

3. On your previous figure, show how the equilibrium would change if the intercept of one of the countries' average benefit functions b_i fell from 10 to some smaller number. What would this mean for the recycling efforts of both countries?

   See figure 3. Same slope, but lower intercept. Effort from 1 reduces slightly, however, for 2 reduces more.

2.4 Tragedy of the Commons

- n players sharing some common resource (of total size 1)
- E.g., village green, bandwidth in network, etc.
- Each player i would like to have as big a share ($0 \le x_i \le 1$) as possible!
- However, each player's utility (pay-off) depends on what the others do:

$$u_i\left(x_i, x_{-i}\right) = \begin{cases} x_i\left(1-\sum_{j=1}^{n} x_j\right) & \text{if } \sum_j x_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

One way to interpret this utility is that in order to have maximum utility there has to be sufficient (unused) "slack" to accommodate small fluctuations. Think of a highway: initially more cars means more throughput, but at some point the increase in density starts to hamper throughput.

Questions

- Consider the special case where there are only two players (i.e. $n = 2$). Determine the individual shares x_1 and x_2 in the Nash equilibrium for this game.

  To find the Nash equilibrium, we analyze the conditions for the players' utility functions when $n = 2$:

  Player 1's utility function:

  $$u_1(x_1, x_2) = \begin{cases} x_1(1-x_1-x_2) & \text{if } x_1 + x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

  Player 2's utility function:

  $$u_2(x_1, x_2) = \begin{cases} x_2(1-x_1-x_2) & \text{if } x_1 + x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

  In the Nash equilibrium, neither player has an incentive to change their strategy given the other player's strategy. Therefore, both players' shares should be such that any unilateral change by one player would not increase their utility.

  We calculate the partial derivative as:

  $$\frac{\partial SW}{\partial x_1} = \frac{\partial}{\partial x_1} x_1(1-x_1-x_2)$$

  Simplifying this gives:

  $$(1-x_2) - 2x_1$$

  $$x_1^* = \mathrm{BR}_1(x_2) = \frac{1-x_2}{2}$$

  Similarly:

  $$x_2^* = \mathrm{BR}_2(x_1) = \frac{1-x_1}{2}$$

  Substituting this into each other gives NE of (1/3, 1/3)

- Does this Nash equilibrium optimise social welfare which is the aggregated utility of all players (i.e. $u_1(x) + u_2(x)$?

  No, filling this in gives 2/9.

- Can you generalise this result to arbitrary n ? The social welfare function $SW(x)$, considering the utility function for each player i, can be written as the sum of all players' utilities:

  Again we can use parts of the previous functions.

$$\frac{\partial}{\partial x_1} u_i(x_i, x_{-i}) = \frac{\partial}{\partial x_1}\left(x_i \cdot \left(1 - \sum_{j=1}^{n} x_j\right)\right)$$

  Since we have multiple agents, we use i as main and all other agents we will just call A, this gives:

$$\frac{\partial}{\partial x_1} u_i(x_i, x_{-i}) = \frac{\partial}{\partial x_1}\left(x_i \cdot (1 - (x_i + A))\right)$$

  Which returns:

$$1 - 2x_i - A$$

  The best response is then:

$$x_i^* = \frac{1 - A}{2}$$

  Because of symmetry, we can say:

$$A = (n-1)x_i^*$$

  Substituting this into the previous equation returns:

$$x_i^* = \frac{1}{1+n}$$

  For social welfare:

$$SW(x) = \sum_{i=1}^{n} u_i(x_i, x_{-i})$$

  The individual utility function for each player i in this scenario is given by:

$$u_i(x_i, x_{-i}) = \begin{cases} x_i \left(1 - \sum_{j=1}^{n} x_j\right) & \text{if } \sum_j x_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

Substituting the individual utility function into the social welfare function, we get:

$$SW(x) = \sum_{i=1}^{n} x_i \cdot \left(1 - \sum_{j=1}^{n} x_j\right)$$

Substituting the summation parts with A, gives A(1 - A). Optimising this will return A = 1/2.

The maximum social welfare is thus 1/2n, which is smaller then NE. Hence, the 'tragedy'.