20170058 **Keonwoo Kim** · December 15, 2020
MAS321 Introduction to Differential Geometry

Final Exam

1. [Exercise 3.63] Let N be an orientation for S, and $p \in S$. Let v_1, v_2 be a positively oriented basis on T_pS. Then $df_p \colon T_pS \to T_{f(p)}\tilde{S}$ is an isomorphism between two tangent spaces. Define $\tilde{N}(f(p))$ as the unit vector orthogonal to $T_{f(p)}\tilde{S}$ at $f(p)$, which is parallel to $df_p(v_1) \times df_p(v_2)$. Note that v_1 and v_2 can be chosen smoothly for p, so $df_p(v_1) \times df_p(v_2)$ is also smooth with this choice. Thus, $\tilde{N}(f(p)) = \frac{df_p(v_1) \times df_p(v_2)}{|df_p(v_1) \times df_p(v_2)|}$ becomes a well-defined orientation for $\tilde{S}$.

2. [Exercise 3.74] The surface can be parametrized with $\sigma(u,v) = (u, e^{-u}\cos v, e^{-u}\sin v)$ for $u \in (0,\infty)$ and $v \in [0, 2\pi)$.

$$\sigma_u(u,v) = (1, -e^{-u}\cos v, e^{-u}\sin v)$$

and

$$\sigma_v(u,v) = (0, -e^{-u}\sin v, e^{-u}\cos v)$$

so that $\|d\sigma_{(u,v)}\| = e^{-u}\sqrt{2}$. Therefore,

$$Area = \int_0^\infty \int_0^{2\pi} \sqrt{2}e^{-u}\, dv\, du = 2\sqrt{2}\pi,$$

which is finite.

3. [Exercise 3.106] Let $\sigma(t,s) = s \cdot \gamma(t)$. Then,

$$\begin{aligned} E &= s^2|\gamma'(t)|^2, \\ F &= s\left\langle \gamma'(t), \gamma(t) \right\rangle, \\ G &= |\gamma(t)|^2. \end{aligned}$$

So, the first fundamental form of the generalized cone is

$$\mathcal{F}_1 = s^2|\gamma'(t)|^2\,dt^2 + 2s\left\langle \gamma'(t), \gamma(t) \right\rangle\,dt\,ds + |\gamma(t)|^2\,ds^2.$$

4. [Exercise 4.9] Since S is an oriented regular surface, we can think of principal curvatures of S at p, say k_1 and k_2. Then

$$H^2 = \left(\frac{k_1 + k_2}{2}\right)^2 = \left(\frac{k_1 - k_2}{2}\right)^2 + k_1 k_2 \geq K.$$

5. [Exercise 4.11] Since γ is a unit-speed curve in S^2, we have $N(\gamma(t)) = \gamma(t)$ (since N is chosen to point outward). Therefore,

$$\begin{aligned}\big\langle \gamma(t) \times \gamma'(t), \gamma''(t) \big\rangle &= \big\langle \gamma(t) \times \gamma'(t), \kappa_n(t) \cdot \gamma(t) + \kappa_g(t) \cdot R_{90}(\gamma'(t)) \big\rangle \\ &= \kappa_g(t) \big\langle \gamma(t) \times \gamma'(t), R_{90}(\gamma'(t)) \big\rangle\end{aligned}$$

as $\gamma(t)$ and $\gamma(t) \times \gamma'(t)$ are orthogonal. Since $\gamma'(t), R_{90}(\gamma'(t))$, and $\gamma(t) = N(\gamma(t))$ form an (positively oriented) orthonormal basis of $\mathbb{R}^3$, we have $\gamma(t) \times \gamma'(t) = R_{90}(\gamma'(t))$. Therefore, $\langle \gamma(t) \times \gamma'(t), R_{90}(\gamma'(t)) \rangle = 1$ and hence we have $\langle \gamma(t) \times \gamma'(t), \gamma''(t) \rangle = \kappa_g(t)$.

6. [Exercise 4.32] Let v_1, v_2 be an orthonormal basis of T_pS with respect to which $\mathcal{W}_p$ is represented by a diagonal matrix. Note that II_p is the quadratic form associated to $\mathcal{W}_p$. Then, taking a point on the circle described in the problem, $v = (\cos t)v_1 + (\sin t)v_2 \in T_pS$ with $0 \leq t < 2\pi$, we have

$$II_p(v) = \langle \mathcal{W}_p(v), v \rangle = k_1 \cos^2 t + k_2 \sin^2 t.$$

Therefore, we the average of the normal curvature of S at p is

$$\frac{1}{2\pi} \int_0^{2\pi} (k_1 \cos^2 t + k_2 \sin^2 t)\, dt = \frac{k_1 + k_2}{2} = H(p).$$

7. [Exercise 5.6] Let γ and $\tilde{\gamma}$ be two geodesics and $\gamma(t_0) = \tilde{\gamma}(t_0)$. We will show first that γ and $\tilde{\gamma}$ are the same on $[t_0, \infty) \cap I$. Define $t_1 = \inf\{t \in I : t \geq t_0,\ \gamma(t) \neq \tilde{\gamma}(t)\}$. Note that t_1 is well-defined since there is an intersection of γ and $\tilde{\gamma}$. If $t_1 \notin I$, then γ and $\tilde{\gamma}$ are the same on $[t_0, t_1) = [t_0, \infty) \cap I$. If $t_1 = \sup I$ and $t_1 \in I$, then γ and $\tilde{\gamma}$ are the same on $[t_0, t_1] = [t_0, \infty) \cap I$. Otherwise, still $\gamma(t_1) = \tilde{\gamma}(t_1)$ because of the continuity of two curves. (Consider an increasing sequence $s_n \nearrow t_1$ in $[t_0, \infty) \cap I$, then $\gamma(s_n) = \tilde{\gamma}(s_n)$ so that $\gamma(t_1) = \lim_{n\to\infty} \gamma(s_n) = \lim_{n\to\infty} \tilde{\gamma}(s_n) = \tilde{\gamma}(t_1)$.) Then, using Proposition 5.3 at $\gamma(t_1) = \tilde{\gamma}(t_1)$, any two geodesics with the domain $(t_1 - \epsilon, t_1 + \epsilon)$ should be equal where $\epsilon = \epsilon(\gamma(t_1), 1)$ is provided by the proposition. So the proposition implies that $\gamma(t)$ and $\tilde{\gamma}(t)$ coincide for $t \in [t_1, t_1 + \epsilon)$, which contradicts to the definition of t_1. Therefore, γ and $\tilde{\gamma}$ coincide on $[t_0, \infty) \cap I$.

Analogously, they coincide on $(-\infty, t_0] \cap I$ vice versa. This completes the proof.

8. [Exercise 5.34] Proposition 5.28 says that for each fixed θ_0, $g(r) = r - \frac{r^3}{6}K(p) + o(r^3)$ where $o(r^3)$ means the error term in little-o notation. Thus,

$$\begin{aligned}
area(C_r(p)) &= \int_0^r \int_0^{2\pi} \sqrt{EG - F^2}\, d\theta\, d\tilde{r} \\
&= \int_0^r \int_0^{2\pi} g(\tilde{r})\, d\theta\, d\tilde{r} \\
&= \int_0^r \int_0^{2\pi} \left(\tilde{r} - \frac{\tilde{r}^3}{6}K(p) + o(\tilde{r}^3)\right)\, d\theta\, d\tilde{r} \\
&= \int_0^r \int_0^{2\pi} \left(\tilde{r} - \frac{\tilde{r}^3}{6}K(p) + o(\tilde{r}^3)\right)\, d\theta\, d\tilde{r} \\
&= 2\pi\frac{r^2}{2} - 2\pi\frac{r^4}{24}K(p) + o(r^4) = \pi r^2 - \pi\frac{r^4}{12}K(p) + o(r^4).
\end{aligned}$$

as $r \to 0$. As $o(r^4)/r^4 \to 0$, we have the desired result when we express the formula above with the $K(p)$ alone on the LHS:

$$\begin{aligned}
K(p) &= \frac{12}{\pi}\frac{\pi r^2 - area(C_r(p)) + o(r^4)}{r^4} \\
&= \frac{12}{\pi}\frac{\pi r^2 - area(C_r(p))}{r^4} + o(1) \\
&\longrightarrow \lim_{r\to 0}\frac{12}{\pi}\frac{\pi r^2 - area(C_r(p))}{r^4}.
\end{aligned}$$

For the part that $\int_0^r \int_0^{2\pi} f(\tilde{r}, \theta)\, d\theta\, d\tilde{r} = o(r^4)$ if $f(r, \theta) = o(r^3)$, we can observe that for any $\epsilon > 0$ there is $\delta > 0$ so that $f(r, \theta) < \epsilon r^3$ for any $0 \le r < \delta$, so

$$\int_0^r \int_0^{2\pi} f(\tilde{r}, \theta)\, d\theta\, d\tilde{r} \le \int_0^r \int_0^{2\pi} \epsilon\tilde{r}^3\, d\theta\, d\tilde{r} = 2\pi\epsilon r^4$$

and hence $\int_0^r \int_0^{2\pi} f(\tilde{r}, \theta)\, d\theta\, d\tilde{r}$ is $o(r^4)$ since ϵ is arbitrary.

9. [Exercise 5.67] Let us show that $\sigma(s,t) = \gamma(t) + s \cdot \mathfrak{b}(t)$ makes $\gamma(t)$ a geodesic. By Lemma 5.52, $\gamma(t) = \sigma(0,t)$ is a geodesic in the surface generated by σ if and only if

$$u'' + (u')^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + (v')^2\Gamma^1_{22} = 0$$

and

$$v'' + (u')^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + (v')^2\Gamma^2_{22} = 0$$

where $u(t) = 0$ and $v(t) = t$. That is,

$$\Gamma^1_{22} = 0 \quad \text{and} \quad \Gamma^2_{22} = 0.$$

Observe that

$$\sigma_{tt} = \gamma''(t) + s \cdot \mathfrak{b}''(t).$$

By Frenet–Serret formulas, we have $\gamma''(t) = \mathfrak{t}'(t) = \kappa\mathfrak{n}(t)$. Therefore, along $\gamma(t)$, i.e., when $s = 0$, we have $\sigma_{tt} = \gamma''(t) = \kappa\mathfrak{n}(t)$. Note that $\{\sigma_s = \mathfrak{b}(t), \sigma_t = \gamma'(t) = \mathfrak{t}(t), \mathfrak{n}(t)\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. Therefore, we have $N = \mathfrak{n}(t)$ and hence $\sigma_{tt} = \kappa N$, which is parallel to N. This means $\Gamma^1_{22} = \Gamma^2_{22} = 0$, i.e., γ is a geodesic in the surface induced by σ.

10. [Exercise 6.9] Suppose that there are two simple closed geodesics γ_1 and γ_2 in S that do not intersect. Then, they induce a region S' bounded by γ_1 and γ_2 on S. Let f be a diffeomorphism between S and the unit sphere S^2, provided by Corollary 6.17. Then $f(\gamma_1)$ and $f(\gamma_2)$ are two curves on S^2 which do not intersect on S^2, so they induce a region which is homeomorphic to a cylinder. Therefore, $\chi(S') = 0$. Now, by the global Gauss-Bonnet theorem, we have

$$\iint_{S'} K\,dA + \int_{\partial S'} \kappa_g(t)\,dt + \sum \alpha_i = 0.$$

Note that S' have no vertices on its boundary (two regular curves, which are images of γ_1 and γ_2 under f). Also, since γ_i are geodesics, the second term in the LHS of the equation above is zero. Therefore, we meet $\iint_{S'} K\,dA = 0$, which is impossible since $K > 0$. Therefore, any two simple closed geodesics should intersect.