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MAS321 Introduction to Differential Geometry

Midterm

1. [Exercise 1.43]

Without loss of generality, suppose γ is parametrized by arc length, since the curvature does not depend on reparametrization of the curve. Since $|\gamma(t)|^2 = \langle \gamma(t), \gamma(t) \rangle$ has a local maximum value of r^2 at time t_0, we have $\frac{d^2}{dt} \langle \gamma(t), \gamma(t) \rangle |_{t_0} \leq 0$, that is,

$$\frac{1}{2}\frac{d^2}{dt} \langle \gamma(t), \gamma(t) \rangle |_{t_0} = \frac{d}{dt} \langle \gamma(t), \gamma'(t) \rangle |_{t_0} = \langle \gamma(t_0), \gamma''(t_0) \rangle + \langle \gamma'(t_0), \gamma'(t_0) \rangle = \langle \gamma(t_0), \gamma''(t_0) \rangle + 1 \leq 0.$$

Therefore, $\langle \gamma(t_0), \gamma''(t_0) \rangle \leq -1$, $| \langle \gamma(t_0), \gamma''(t_0) \rangle | \geq 1$, so that

$$\kappa(t_0) = |\gamma''(t_0)| \geq 1/|\gamma(t_0)| = 1/r.$$

There is no upper bound for $\kappa(t_0)$, as a circle centered at $(r - \epsilon, 0)$ with radius ϵ yields a local maximum value of r of a function $t \mapsto |\gamma(t)|$ while the curvature at that point is $1/\epsilon$, which can be arbitrarily large. So there is no upper bound on κ. □

2. [Exercise 1.62]

$\gamma' = (1, 2t, 3t^2)$, $\gamma'' = (0, 2, 6t)$, so

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{|(6t^2, -6t, 2)|}{|(1, 2t, 3t^2)|^3} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}.$$

$\gamma''' = (0, 0, 6)$, so

$$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2} = \frac{12}{4(1 + 9t^2 + 9t^4)} = \frac{3}{1 + 9t^2 + 9t^4}.$$

This formula comes from the Exercise 1.65. □

3. [Exercise 1.76]

Let κ be the curvature of γ and $\hat{\kappa}$ be the curvature of $\hat{\gamma}$. Similarly, Then, let τ be the torsion of γ and $\hat{\tau}$ be the torsion of $\hat{\gamma}$.

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = |\hat{\gamma}'(t)|,$$

$$\gamma' \times \gamma'' = (y'z'' - y''z', z'x'' - z''x', x'y'' - x''y'),$$
$$\hat{\gamma}' \times \hat{\gamma}'' = (-x'y'' + x''y', -y'z'' + y''z', z'x'' - z''x')$$

so that $|\gamma' \times \gamma''| = |\hat{\gamma}' \times \hat{\gamma}''|$ and hence

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{|\hat{\gamma}' \times \hat{\gamma}''|}{|\hat{\gamma}'|^3} = \hat{\kappa}.$$

Similarly, for the torsion,

$$(\gamma' \times \gamma'') \cdot \gamma''' = \det(\gamma', \gamma'', \gamma''') = \begin{vmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{vmatrix} = \begin{vmatrix} z & z' & z'' \\ x & x' & x'' \\ y & y' & y'' \end{vmatrix} = -\begin{vmatrix} z & z' & z'' \\ x & x' & x'' \\ -y & -y' & -y'' \end{vmatrix} = -(\hat{\gamma}' \times \hat{\gamma}'') \cdot \hat{\gamma}'''$$

so that
$$\hat{\tau} = \frac{(\hat{\gamma}' \times \hat{\gamma}'') \cdot \hat{\gamma}'''}{|\hat{\gamma}' \times \hat{\gamma}''|^2} = \frac{-(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2} = -\tau.$$

In summary, $\kappa = \hat{\kappa}$ and $\tau = -\hat{\tau}$. $\square$

4. [Exercise 3.5]

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

$$\begin{aligned}
(f \circ \gamma)'(t) &= \nabla f(\gamma(t)) \cdot \gamma'(t) = f_u(\gamma(t))\gamma_1'(t) + f_v(\gamma(t))\gamma_2'(t), \\
(f \circ \gamma)''(t) &= \gamma_1'(t)\nabla f_u(\gamma(t)) \cdot \gamma'(t) + f_u(\gamma(t))\gamma_1''(t) + \gamma_2'(t)\nabla f_v(\gamma(t)) \cdot \gamma'(t) + f_v(\gamma(t))\gamma_2''(t), \\
(f \circ \gamma)''(0) &= a(f_{uu}(q)a + f_{uv}(q)b) + \cancelto{0}{f_u(q)\gamma_1''(0)} + b(f_{vu}(q)a + f_{vv}(q)b) + \cancelto{0}{f_v(q)\gamma_2''(0)} \\
&= a^2 f_{uu}(q) + 2ab f_{uv}(q) + b^2 f_{vv}(q).
\end{aligned}$$

$\square$

5. [Exercise 3.7]

By stretching E along the axes, we may find a *linear* diffeomorphism f from S^2 to E, which is the unit sphere in $\mathbb{R}^3$ as in Example 3.14. Let $L \in O(3)$, that is a linear transformation on $\mathbb{R}^3$ whose image of S^2 is again S^2. ($O(3)$ is the set of orthogonal transformations on $\mathbb{R}^3$.) Then, we have a linear diffeomorphism between E and E itself:

$$\varphi_L \colon E \to E, \quad \varphi_L = f \circ L \circ f^{-1}.$$

[Reason: Note that there is a 1-1 correspondence between L's and φ_L's, since f is invertible. Let φ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ that restrict to a diffeomorphism from E to E, then, since $f^{-1} \circ \varphi \circ f$ is linear and maps S^2 to S^2, it should be a linear transformation in $O(3)$.] Therefore, the set of such linear transformations is

$$\{f \circ L \circ f^{-1} : L \in O(3)\}; \qquad f \colon S^2 \to E, \quad f(x, y, z) = (ax, by, cz).$$

There is no difference on whether a, b, c are distinct or not. In particular when $a = b = c$, the above set coinsides with $O(3)$.

6. [Exercise 3.12]

Extend γ to a periodic function on $\mathbb{R}$, by defining $\gamma_{\text{new}}(t) = \gamma(\tilde{t})$, where $\tilde{t} = t + k(a, b)$ for some integer k and satisfying $\tilde{t} \in [a, b)$. Write γ_{new} as just γ with an abusing of notation.

(1) Note $d\varphi_{(t,x,y)} = ((1 - x\kappa)\mathfrak{t} - y\tau\mathfrak{n} + x\tau\mathfrak{b}, \mathfrak{n}, \mathfrak{b})$, from Frenet–Serret formula. So, the Jacobian matrix of φ does not vanish whenever $|x| < 1/\kappa$. So, by the inverse function theorem, we can find $\epsilon_t > 0$ and $U_t \ni t$, an open set in $\mathbb{R}$, such that φ is a diffeomorphism onto $U_t \times B_{\epsilon_t}$ from the image of this under φ. Since $[a, b]$ is compact, we can find a finite numbers of t, namely $t_1, \ldots, d_k$ such that $\{U_{t_i} : i = 1, \ldots, k\}$ covers $[a, b]$ entirely. Then, letting $\epsilon = \min(\epsilon_1, \ldots, \epsilon_k, \delta/2)/2$ where δ denote the Lebesgue number of the open covering $\{U_t : t \in [a, b]\}$, any open ball of radius ϵ is entirely contained in one of U_t's, so by gluing those open neighborhoods, we can find a smooth inverse of φ on $U \times B_{2\epsilon}$, where U is an open set containing $[a, b]$ and the smoothness comes from the inverse function theorem.

(2) Since $\varphi \colon U \times B_{2\epsilon}$ is diffeomorphic, by restricting, we have ϕ is a diffeomorphism.

(3) Suppose not, then there is either a point $p \in S_\epsilon$ where $\operatorname{dist}(p, C) \neq \epsilon$, or there is a point satisfying $\operatorname{dist}(p, C) = \epsilon$ while $p \notin S_\epsilon$.

Assume the first. Then $p = \phi(t, \theta)$ for some $t \in [a, b]$ and $\theta \in [0, 2\pi]$. Since $\operatorname{dist}(p, \gamma(t)) = \epsilon$, we obtain $\operatorname{dist}(p, C) < \epsilon$, as it cannot be greater than ϵ. Let $\gamma(t_0)$ be the nearest point on C from p, then we have $p = \varphi(t_0, x, y)$ for some $t_0 \in [a, b]$ and $(x, y) \in B_\epsilon$. But this contradicts to that p is the boundary point of the image of φ_ϵ.

Assume the second: there is a point satisfying $\operatorname{dist}(p, C) = \epsilon$ while $p \notin S_\epsilon$. Then, letting $\gamma(t)$ be the nearest point on C from p, $\mathfrak{t}(t)$ should be perpendicular to $p - \gamma(t)$, since $\frac{d}{dt}\langle p-\gamma(t), p-\gamma(t)\rangle = \langle p-\gamma(t), \mathfrak{t}(t)\rangle = 0$. Therefore, there exists θ such that $p-\gamma(t) = (\epsilon\cos\theta)\mathfrak{n} + (\epsilon\sin\theta)\mathfrak{b}$, which is a contradiction.

Therefore, $S_\epsilon = \{p \in \mathbb{R}^3 : \operatorname{dist}(p, C) = \epsilon\}$.

7. [Exercise 3.14]

Let us define $\varphi\colon U \to \{(x,y,z) : x^2+y^2=1\}$ as

$$\varphi(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \log\sqrt{x^2+y^2}\right).$$

Note that this is well-defined as $\sqrt{x^2+y^2} \neq 0$, and the square sum of the first and the second coordinate of $\varphi(x,y)$ is 1. Let us show φ is actually a diffeomorphism.

First, φ is clearly differentiable, since it is a composition of differentiable functions. Second, φ has an inverse function, namely

$$\psi\colon \{(x,y,z) : x^2+y^2=1\} \to U, \quad \psi(x,y,z) = (xe^z, ye^z),$$

which is also differentiable. Therefore, φ is a diffeomorphism between $U = \mathbb{R}^2 - \{(0,0)\}$ and $\{(x,y,z) : x^2+y^2=1\}$, indicating that the given cylinder can be covered by a single surface patch from U. $\square$

8. [Exercise 3.18]

(1) Following the problem, assume γ is parametrized by arc length, n is a unit vector, and by applying a rigid motion, assume P equals the xy-plane. Consider an tubular ϵ-neighborhood of the curve γ:

$$E = \{\gamma(t) + \eta N(t) : \eta \in (-\epsilon, \epsilon), N(t) = R_{90}(\gamma'(t)))\}$$

where φ being a diffeomorphism from $(-\epsilon,\epsilon)\times(a,b)$ onto its image and being injective on $(-\epsilon,\epsilon)\times[a,b)$, for some $0<\epsilon\leq 1$.

Let D be the standard cylinder, and let

$$F = \{(x,y,z) : 1-\epsilon < \sqrt{x^2+y^2} < 1+\epsilon, z \in \mathbb{R}\}$$

be a tubular neighborhood of D, as in Exercise 3.11. Define a map as follows:

$$f\colon F \to E, \qquad f((1+\eta)\cos\theta, (1+\eta)\sin\theta, z) = \gamma(t) + \eta N(t) + zn,$$

where $t = a + (b - a)\theta/(2\pi)$ and $\eta \in (-\epsilon, \epsilon)$. Then f becomes a diffeomorphism, as $(t, \eta) \mapsto \gamma(t) + \eta N(t)$ is a diffeomorphism, $z \mapsto zn$ is also a diffeomorphism from a straight line to a straight line,and $(t, \eta) \mapsto \gamma(t) + \eta N(t)$ runs on xy-plane while $z \mapsto zn$ runs over the z-axis. Therefore, f is a diffeomorphism from F to E. Therefore, by restricting f on C, we get a diffeomorphism from D to C, the standard cylinder.

(2) The projection of $\tilde{C}$ should not self-intersect. If so, let p and q be those points, then there is a real number k satisfying $p - q = kn$, which violates the bijectivity. Otherwise, construct a generalized cylinder $C = \{\gamma(t) + sn : t \in [a, b], s \in \mathbb{R}\}$ where $\gamma(t) = \tilde{\gamma}(t) - \langle \tilde{\gamma}(t), n \rangle n$ is the orthogonal projection of $\tilde{\gamma}(t)$ onto a plane orthogonal to n passing $(0, 0, 0)$. Since γ is a plane curve, it is well-defined by (1). Now, observe that $\tilde{C} = C$ as in (3) below. Thus, $\tilde{C}$ will be a regular surface diffeomorphic to the standard cylinder if the projection of $\tilde{C}$ should not self-intersect.

(3)

$$\begin{aligned}\tilde{C} = \{\tilde{\gamma}(t) + sn : t \in [a, b], s \in \mathbb{R}\} &= \{\tilde{\gamma}(t) + (s - \langle \tilde{\gamma}(t), n \rangle)n : t \in [a, b], s \in \mathbb{R}\} \\ &= \{\gamma(t) + sn : t \in [a, b], s \in \mathbb{R}\} = C.\end{aligned}$$

Since $\langle \gamma(t), n \rangle = \langle \tilde{\gamma}(t) \rangle - \langle \tilde{\gamma}(t) \rangle \langle n, n \rangle = 0$, the trace of γ lies in the plane P containing $(0, 0, 0)$ with normal vector n. This completes the proof.

9. [Exercise 3.21]

It is enough to follow the proof of Theorem 3.27. By hypothesis, one of the two partial derivative of f at p is nonzero. Assume without loss of generality, that $\frac{\partial f}{\partial y}(p) \neq 0$. Otherwise, rotate the domain in order to make the assumption hold. Define $\psi \colon U \to \mathbb{R}^2$ as $\psi(x, y) = (x, f(x, y))$. The Jacobian matrix for ψ at p is

$$d\psi_p = \begin{pmatrix} 1 & 0 \\ f_x(p) & f_y(p) \end{pmatrix},$$

where $\det(d\psi_p) = f_y(p) \neq 0$. By the inverse function theorem, there exists a neighborhood $\tilde{U}$ of p in the preimage of $C = f^{-1}(\lambda)$ and a neighborhood W of $\psi(p)$ where $\psi \colon \tilde{U} \to W$ is invertible with smooth inverse $\psi^{-1} \colon W \to \tilde{U}$. When $\psi(x, y_0) = (x, w_0)$ where $(x, y_0) \in \tilde{U}$ and $(x, w_0) \in W$, define $\psi_y^{-1}(w_0) = y_0$. Let $\mathcal{U} = \{x : (x, y) \in \tilde{U} \text{ for some } y\}$, then $\mathcal{U} \subseteq \mathbb{R}$ is an open set since it is the projection of an open set $\tilde{U}$ onto the x-axis. Now, define

$$h \colon \mathcal{U} \to \mathbb{R}, \qquad h(x) = \psi_y^{-1}(x, y),$$

which is the height of the unique point in $\tilde{U}$ above x, where the temperature equals λ. Note that $C \cap \tilde{U} = \{(x, h(x)) : x \in \mathcal{U}\}$, which is a graph of h. Therefore, it is diffeomorphic to the domain $\mathcal{U}$, by the diffeomorphism h, by Lemma 3.17. Therefore, $C \cap \tilde{U}$ is an open neighborhood of p in C, which is the trace of a regular plane curve $(x, h(x)) : x \in \mathcal{U}$. This completes the proof. □

10. [Exercise 3.24]

Let $q = (s, t) \in U$. Now,

$$d\sigma_q = \begin{pmatrix} (\sigma_1)_s(q) & (\sigma_1)_t(q) \\ (\sigma_2)_s(q) & (\sigma_2)_t(q) \\ (\sigma_3)_s(q) & (\sigma_3)_t(q) \end{pmatrix} = \begin{pmatrix} | & | \\ \gamma'(t) & \gamma'(t) + s\gamma''(t) \\ | & | \end{pmatrix}.$$

This matrix has the same rank as $\begin{pmatrix} \gamma'(t) & \gamma''(t) \end{pmatrix}$ by an elementary row operation, as $s \neq 0$. Suppose it has rank less than 2, then $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent so that $\gamma' \times \gamma'' = \mathbf{0}$, which contradicts to the assumption that the curvature of γ is nowhere-vanishing. Therefore, $\begin{pmatrix} \gamma'(t) & \gamma''(t) \end{pmatrix}$ should have rank 2, hence so is $d\sigma_q$. □