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POW 2020-14

Q. Say there are n points. For each pair of points, we add an edge with probability $1/3$. Let P_n be the probability of the resulting graph to be connected (meaning any two vertices can be joined by an edge path). What can you say about the limit of P_n as n tends to infinity?

Sol. $\underline{P_n \to 1}$; in other words, the resulting random graph is asymptotically almost surely connected.

In order to be disconnected for a graph with n vertices, it should be divided into two sets of vertices which have no edges between them. Letting they, say sets A and B which partition the vertex set, have k and $n-k$ vertices respectively, the probability that there are no edges between A and B is $(2/3)^{k(n-k)}$. Since there are $\binom{n}{k}$ ways to choose (A, B) for a fixed k, $1 - P_n$ is at most

$$\sum_{k=1}^{n-1} \binom{n}{k} \left(\frac{2}{3}\right)^{k(n-k)}.$$

Let $K < n/2$ be a positive integer constant chosen later. Note that we can observe $k(n-k) \geq n-1$ for $1 \leq k \leq n-1$, and $k(n-k) \geq K(n-K)$ for $K \leq k \leq n-K$, due to the concavity of the quadratic polynomial $x(n-x)$. Moreover, by an application of the Stirling's formula, we have

$$\binom{n}{\lfloor n/2 \rfloor} \sim \frac{2^n}{\sqrt{\pi n/2}} \implies \binom{n}{\lfloor n/2 \rfloor} \leq 2^n \quad \text{for } n \text{ large enough}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n-1} \binom{n}{k} \left(\frac{2}{3}\right)^{k(n-k)} &= 2\sum_{k=1}^{K-1} \binom{n}{k} \left(\frac{2}{3}\right)^{k(n-k)} + \sum_{k=K}^{n-K} \binom{n}{k} \left(\frac{2}{3}\right)^{k(n-k)} \\
&\leq 2\sum_{k=1}^{K-1} n^{K-1} \left(\frac{2}{3}\right)^{n-1} + \sum_{k=K}^{n-K} \binom{n}{\lfloor n/2 \rfloor} \left(\frac{2}{3}\right)^{K(n-K)} \\
&= 2(K-1)n^{K-1} \left(\frac{2}{3}\right)^{n-1} + (n-2K+1)2^n \left(\frac{2}{3}\right)^{K(n-K)}
\end{aligned}$$

Here $2(K-1)n^{K-1}\left(\frac{2}{3}\right)^{n-1} \xrightarrow{n\to\infty} 0$, and

$$(n-2K+1)2^n\left(\frac{2}{3}\right)^{K(n-K)} = (n-2K+1)\left(\frac{2}{3}\right)^{-K^2}\left(2\cdot\left(\frac{2}{3}\right)^K\right)^n \xrightarrow{n\to\infty} 0$$

whenever $2\cdot\left(\frac{2}{3}\right)^K < 1$, that is, $K \geq 2$. Hence,

$$1-P_n \leq \sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{2}{3}\right)^{k(n-k)} \xrightarrow{n\to\infty} 0$$

proving $P_n \to 1$.