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MAS321 Introduction to Differential Geometry

Quiz #2

1. (Exercise 3.40)

When f is a constant function, then $df_p(v) = (f \circ \gamma)'(0)$ for some curve γ where $\gamma(0) = p$ and $\gamma'(0) = v$. Since f is a constant function, so is $f \circ \gamma$, which implies $df_p(v) = 0$.

Conversely, suppose that $df_p(v) = 0$ for any $p \in S$ and $v \in T_pS$. Suppose f is not a constant, meaning that there are two points p_1 and p_2 on S where $f(p_1) \neq f(p_2)$. Since S is path-connected and regular, there is a smooth curve γ passing through p_1 and p_2, that is, $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Note that $df_{\gamma(t)}(\gamma'(t)) = (f \circ \gamma)'(t)$ for $t \in (0, 1)$ so that

$$\int_0^1 df_{\gamma(t)}(\gamma'(t))\, dt = (f \circ \gamma)(1) - (f \circ \gamma)(0) = q - p \neq \mathbf{0}.$$

Therefore, there is a point $p = \gamma(t)$ and a vector $v = \gamma'(t) \in T_pS$ where $df_p(v) \neq 0$. This completes the proof.

2. (Exercise 3.55)

f preserves the orientation.

$v_1 = (1,0,0), v_2 = (0,1,0), B = \{v_1, v_2\}$ is a positively oriented ordered basis of $\mathcal{V}$, since $(v_1 \times v_2)/|v_1 \times v_2| = (0,0,1)$ agrees with the orientation of $\mathcal{V}$. By f, $f(v_1) = (0,0,1)$ and $f(v_2) = (0,1,0)$, which yield a positively oriented ordered basis of $\tilde{\mathcal{V}}$: $(f(v_1) \times f(v_2))/|f(v_1) \times f(v_2)| = (-1,0,0)$. Therefore, f preserves the orientation.

3. (Exercise 3.81)

When f is an isometry, the length of $f \circ \gamma$ is as follows:

$$\begin{aligned}
\text{arc length}(f \circ \gamma) &= \int_a^b \sqrt{|(f \circ \gamma)'(t)|^2}\, dt \\
&= \int_a^b \sqrt{|(f \circ \gamma)'(t)|^2}\, dt \\
&= \int_a^b \sqrt{\left\langle df_{\gamma(t)}(\gamma'(t)), df_{\gamma(t)}(\gamma'(t))\right\rangle}\, dt \\
&= \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t)\rangle}\, dt \\
&= \text{arc length}(\gamma)
\end{aligned}$$

To prove the converse, suppose f is not an isometry. Then, there are $p \in S_1$ and $x \in T_pS_1$ where $|x|_p^2 \neq |df_p(x)|_{f(p)}^2$. Pick a regular curve $\gamma\colon [a,b] \to S_1$, where $\gamma(a) = p$ and $\gamma'(a) = x$. Choose $\delta > 0$ small enough to satisfy the following: $|\gamma'(a+\eta)|_p^2 - |df_{\gamma(a+\eta)}(\gamma'(a+\eta))|_{f(\gamma(a+\eta))}$ has the same sign with $|x|_p^2 - |df_p(x)|_{f(p)}^2$ for every $0 < \eta < \delta$. It is possible since γ is a regular curve and f is a diffeomorphism.
(For instance, choose $\delta_1 > 0$ to make

$$\left||\gamma'(a+\eta)|_{\gamma(a+\eta)}^2 - |\gamma'(a)|_{\gamma(a)}^2\right| < \epsilon/3,$$

and choose $\delta_2 > 0$ to make

$$\left||df_{\gamma(a+\eta)}(\gamma'(a+\eta))|_{(f\circ\gamma)(a+\eta)}^2 - |df_{\gamma(a)}(\gamma'(a))|_{(f\circ\gamma)(a)}^2\right| < \epsilon/3,$$

where $\epsilon = \left||x|_p^2 - |df_p(x)|_{f(p)}^2\right|$. Then, letting $\delta = \min(\delta_1, \delta_2)$, we can accomplish the desired procedure.)
After that, reparametrize $\gamma|_{[a,a+\delta]}$ with $t\colon [a, a+\delta] \to [a,b]$, $t(x) = \frac{b-a}{\delta}(x-a) + a$. Then we get a regular curve $\tilde{\gamma}\colon [a,b] \to S_1$, where the property mentioned above is preserved. Now, observe that

$$\begin{aligned}
\text{arc length}(\tilde{\gamma}) - \text{arc length}(f \circ \tilde{\gamma}) &= \text{arc length}(\gamma|_{[a,a+\delta]}) - \text{arc length}(f \circ \tilde{\gamma}|_{[a,a+\delta]}) \\
&= \int_0^\delta |\gamma'(a+\eta)|_p^2 - |df_{\gamma(a+\eta)}(\gamma'(a+\eta))|_{f(\gamma(a+\eta))}\, d\eta
\end{aligned}$$

has the same sign with $|x|_p^2 - |df_p(x)|_{f(p)}^2$, which is nonzero. Therefore, f does not preserve the arc length of $\tilde{\gamma}$, which completes the proof.

4. (Exercise 4.28)

Let the orientation of the monkey saddle at $(u, v, u^3 - 3v^2u)$ have the same direction with $(1, 0, 3u^2 - 3v^2) \times (0, 1, -6uv) = (3v^2 - 3u^2, 6uv, 1)$. Now, the Gauss map is as follows:

$$N(p) = \frac{(3v^2 - 3u^2, 6uv, 1)}{\sqrt{1 + 9(u^2 + v^2)^2}}.$$

Then, we can calculate the Weingarten map at the origin is as follows: pick a vector $v = (x, y, 0) \in T_{\mathbf{0}}S$ with $x^2 + y^2 = 1$ and let $\gamma(t) = t\mathbf{v}$, then we have $\gamma(0) = \mathbf{0}$ and $\gamma'(0) = \mathbf{v}$ Now, we have

$$\begin{aligned}
dN_{\mathbf{0}}(v) &= (M \circ \gamma)'(0) \\
&= \frac{d(N(tx, ty, 0))}{dt}\bigg|_{t=0} \\
&= \frac{d}{dt} \frac{(3(ty)^2 - 3(tx)^2, 6t^2xy, 1)}{\sqrt{1 + 9t^2(x^2 + y^2)^2}}\bigg|_{t=0} \\
&= \frac{d}{dt} \frac{(3(y^2 - x^2)t^2, 6xyt^2, 1)}{\sqrt{1 + 9t^2}}\bigg|_{t=0} \\
&= \left(\frac{3(y^2 - x^2)t(2 + 9t^2)}{\sqrt{1 + 9t^2}^3}, \frac{6xyt(2 + 9t^2)}{\left(\sqrt{1 + 9t^2}\right)^3}, -\frac{9t}{\left(\sqrt{1 + 9t^2}\right)^3} \right)\bigg|_{t=0} \\
&= (0, 0, 0).
\end{aligned}$$

Therefore, dN_p is the zero map and hence $\mathcal{W}_{\mathbf{0}} = 0$. Finally, we have $k_1 = k_2 = 0$ so the Gaussian curvature of the monkey saddle at the origin $\mathbf{0}$ equals zero.

Using Proposition 4.14., we can found the following proposition: If $K(p) = 0$, for any (open) neighborhood U of p in S, $U \setminus \{p\}$ cannot lie entirely in one side of the plane $p + T_pS$. The proof is simple: if not, either $K(p) > 0$ or $K(p) < 0$, where the Proposition 4.14. says both cases are impossible. In case of the monkey saddle, the above proposition (clearly) works, seen from the picture.