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MAS321 Introduction to Differential Geometry

Quiz #1

1. (Exercise 1.5)

Since the i-th coordinate of $\boldsymbol{\gamma}'(t)$ is $x_i'(t) = \lim_{h\to 0} \frac{x_i(t+h)-x_i(t)}{h}$, for any $\varepsilon > 0$, there is a positive real number $\delta_i(\varepsilon) > 0$ such that

$$\left| \frac{x_i(t+h) - x_i(t)}{h} - x_i'(t) \right| < \varepsilon$$

whenever $|h| < \delta_i(\varepsilon)$. Hence, for any $\varepsilon > 0$,

$$\begin{aligned} \left| \frac{\boldsymbol{\gamma}(t+h) - \boldsymbol{\gamma}(t)}{h} - \boldsymbol{\gamma}'(t) \right| &= \sqrt{\sum_{i=1}^{n} \left| \frac{x_i(t+h) - x_i(t)}{h} - x_i'(t) \right|^2} \\ &< \sqrt{\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}} \right)^2} = \varepsilon \end{aligned}$$

whenever $|h| < \min_{i=1,\cdots,n} \delta_i(\varepsilon/\sqrt{n})$. Therefore, we have

$$\boldsymbol{\gamma}'(t) = \lim_{h\to 0} \frac{\boldsymbol{\gamma}(t+h) - \boldsymbol{\gamma}(t)}{h}.$$

2. (Exercise 1.22)

When $\boldsymbol{\gamma}'(t)$ is orthogonal to $\boldsymbol{\gamma}(t)$ for any $t \in I$, we have $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}(t) \rangle = 0$ for any $t \in I$. Note that

$$\frac{d}{dt} \langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t) \rangle = 2 \left\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}(t) \right\rangle = 0,$$

hence $\langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t) \rangle$ is a constant on I. Since $\langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t) \rangle \geq 0$, there is a real number $C \geq 0$ such that $|\boldsymbol{\gamma}(t)|^2 = \langle \boldsymbol{\gamma}(t), \boldsymbol{\gamma}(t) \rangle = C^2$, which means $|\boldsymbol{\gamma}(t)| = C$ for any $t \in I$. Thus, the converse of the part (1) in Proposition 1.17 holds.

3. (Exercise 1.36)

Answer: f is not independent of parametrization.

Without loss of generality, consider the case that $\boldsymbol{\gamma}$ is a unit speed parametrized curve. Let $\tilde{\boldsymbol{\gamma}}$ be a (regular) reparametrization of $\boldsymbol{\gamma}$, that is, $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \circ \phi$ for some smooth bijection ϕ with nonvanishing derivative. Then,

$$\tilde{\boldsymbol{\gamma}}'(t) = \boldsymbol{\gamma}'(\phi(t))\phi'(t),$$

and

$$\tilde{\boldsymbol{\gamma}}''(t) = \boldsymbol{\gamma}''(\phi(t))(\phi'(t))^2 + \boldsymbol{\gamma}'(\phi(t))\phi''(t).$$

Since $\boldsymbol{\gamma}$ is a unit speed curve, $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle = 1$ so that $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}''(t) \rangle = 0$ by differentiating both sides. Now we have

$$\begin{aligned}
|\tilde{\boldsymbol{\gamma}}''(t)|^2 &= \left\langle \tilde{\boldsymbol{\gamma}}''(t), \tilde{\boldsymbol{\gamma}}''(t) \right\rangle \\
&= \left\langle \boldsymbol{\gamma}''(\phi(t))(\phi'(t))^2 + \boldsymbol{\gamma}'(\phi(t))\phi''(t), \boldsymbol{\gamma}''(\phi(t))(\phi'(t))^2 + \boldsymbol{\gamma}'(\phi(t))\phi''(t) \right\rangle \\
&= (\phi'(t))^4 \left\langle \boldsymbol{\gamma}''(\phi(t)), \boldsymbol{\gamma}''(\phi(t)) \right\rangle + (\phi''(t))^2 \left\langle \boldsymbol{\gamma}'(\phi(t)), \boldsymbol{\gamma}'(\phi(t)) \right\rangle \\
&= (\phi'(t))^4 |\boldsymbol{\gamma}''(\phi(t))|^2 + (\phi''(t))^2.
\end{aligned}$$

Since $|\tilde{\boldsymbol{\gamma}}'(t)| = |\phi'(t)|$, we have

$$\frac{|\tilde{\boldsymbol{\gamma}}''(t)|^2}{|\tilde{\boldsymbol{\gamma}}'(t)|^4} = |\boldsymbol{\gamma}''(\phi(t))|^2 + \frac{(\phi''(t))^2}{|\phi'(t)|^4} = f(\phi(t))^2 + \frac{(\phi''(t))^2}{|\phi'(t)|^4}.$$

Therefore, for a reparametrization such that $\phi''(t) \neq 0$, $f(t)$ is **not** independent of parametrization.

4. (Exercise 1.54)

By the definition of the signed curvature, we have

$$\kappa_s(t) = \frac{\langle \mathbf{a}(t), R_{90}(\mathbf{v}(t)) \rangle}{|\mathbf{v}(t)|^3}.$$

Within the polar coordinate system $(r, \theta)_\angle = (r\cos\theta, r\sin\theta)$, we have

$$R_{90}((r,\theta)_\angle) = (r, \theta + \pi/2)_\angle.$$

Also, $\frac{d}{d\theta}(\cos\theta, \sin\theta) = (-\sin\theta, \sin\theta) = R_{90}(\cos\theta, \sin\theta)$. Hence, we have

$$\begin{aligned}
\mathbf{v}(\theta) &= \frac{d}{d\theta}[r(\theta)(\cos\theta, \sin\theta)] \\
&= r'(\theta)(\cos\theta, \sin\theta) + r(\theta)(-\sin\theta, \cos\theta) \\
&= (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}(\theta) &= \frac{d}{d\theta}\mathbf{v}(\theta) \\
&= r''(\theta)(\cos\theta, \sin\theta) + 2r'(\theta)(-\sin\theta, \cos\theta) - r(\theta)(\cos\theta, \sin\theta). \\
&= (r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta, r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
R_{90}(\mathbf{v}(t)) &= R_{90}(r'(\theta)(\cos\theta, \sin\theta)) + R_{90}(r(\theta)(-\sin\theta, \cos\theta)) \\
&= R_{90}((r'(\theta), \theta)_{\angle}) + R_{90}((r(\theta), \theta + \pi/2)_{\angle}) \\
&= (r'(\theta), \theta + \pi/2)_{\angle} + (r(\theta), \theta + \pi)_{\angle} \\
&= r'(\theta)(-\sin\theta, \cos\theta) - r(\theta)(\cos\theta, \sin\theta) \\
&= (-r'(\theta)\sin\theta - r(\theta)\cos\theta, r'(\theta)\cos\theta - r(\theta)\sin\theta).
\end{aligned}$$

since R_{90} is linear. Therefore,

$$\begin{aligned}
&\langle \mathbf{a}(t), R_{90}(\mathbf{v}(t)) \rangle \\
&\quad = (r''(\theta)\cos\theta - 2r'(\theta)\sin\theta - r(\theta)\cos\theta)(-r'(\theta)\sin\theta - r(\theta)\cos\theta) \\
&\qquad + (r''(\theta)\sin\theta + 2r'(\theta)\cos\theta - r(\theta)\sin\theta)(r'(\theta)\cos\theta - r(\theta)\sin\theta) \\
&\quad = \cancel{-r''r'cs} + 2(r')^2 s^2 + \cancel{r'rcs} - r''rc^2 + \cancel{2r'rcs} + r^2c^2 \\
&\qquad + \cancel{r''r'cs} + 2(r')^2c^2 - \cancel{r'rcs} - r''rs^2 - \cancel{2r'rcs} + r^2s^2 \\
&\quad = 2(r'(\theta))^2 - r''(\theta)r(\theta) + r(\theta)^2
\end{aligned}$$

where $r = r(\theta), r' = r'(\theta), r'' = r''(\theta), c = \cos\theta$, and $s = \sin\theta$. Since

$$\begin{aligned}
|\mathbf{v}(\theta)|^2 &= (r'c - rs)^2 + (r's + rc)^2 \\
&= (r')^2c^2 - \cancel{2r'rcs} + r^2s^2 + (r')^2s^2 + \cancel{2r'rcs} + r^2c^2 \\
&= (r'(\theta))^2 + (r(\theta))^2,
\end{aligned}$$

we have

$$\kappa_s(\theta) = \frac{2(r'(\theta))^2 - r''(\theta)r(\theta) + r(\theta)^2}{((r'(\theta))^2 + (r(\theta))^2)^{3/2}}.$$