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MAS555 Advanced Statistics

Homework 3

7.10. (a) Note that the pdf of the common distribution of $X_1, \ldots, X_n$ is

$$\mathrm{pdf}_{X_i}(x|\alpha,\beta) = \alpha\beta^{-\alpha}x^{\alpha-1}\mathbf{1}_{x\geq 0}\mathbf{1}_{x\leq\beta}$$

so that the joint pdf is

$$\begin{aligned}
&\mathrm{pdf}_{X_1,\ldots,X_n}(x_1,\ldots,x_n|\alpha,\beta)\\
&= \alpha^n\beta^{-n\alpha}\Big(\prod_{i=1}^n x_i\Big)^{\alpha-1}\mathbf{1}_{\min_i x_i\geq 0}\mathbf{1}_{\max_i x_i\leq\beta}\\
&= \exp\left[(\alpha-1)\sum_{i=1}^n \log x_i + \log \mathbf{1}_{\beta\geq\max_i x_i} + n\log\alpha - n\alpha\log\beta\right]\cdot\mathbf{1}_{\min x_i\geq 0}.
\end{aligned}$$

Thus, by the factorization theorem, for $X = (X_1, \ldots, X_n)$,

$$T(X) = \begin{pmatrix}\sum_{i=1}^n \log X_i\\ \max_{i=1,\ldots,n} X_i\end{pmatrix}$$

is a two-dimensional sufficient statistic for (α, β).

(b) To maximize the likelihood $\ell(\alpha,\beta) = \mathrm{pdf}_{X_1,\ldots,X_n}(x_1,\ldots,x_n|\alpha,\beta)$, the exponent in the pdf should be maximized:

$$\text{maximize}\quad (\alpha-1)\sum_{i=1}^n \log x_i + \log\mathbf{1}_{\beta\geq\max_i x_i} + n\log\alpha - n\alpha\log\beta$$

under the assumption that $x_i \geq 0$ for any i. Since the likelihood is zero unless $\beta \geq \max_i x_i$ and since $-n\alpha\log\beta$ is decreasing in β, the MLE of β is obtained exactly at $\beta = \max_i x_i$. Thus, $\hat\beta_{\mathrm{MLE}} = X_{(1)} = \max_{1\leq i\leq n} X_i$. Putting this into the likelihood, when $x_i \geq 0$ for any i,

$$\frac{\partial}{\partial\alpha}\log\ell(\alpha,\hat\beta_{\mathrm{MLE}}) = \sum_{i=1}^n \log x_i + \frac{n}{\alpha} - n\log\hat\beta_{\mathrm{MLE}} = 0,$$

i.e., $\hat\alpha_{\mathrm{MLE}} = \left(\log\hat\beta_{\mathrm{MLE}} - \frac{1}{n}\sum_{i=1}^n \log X_i\right)^{-1} = \left(\log X_{(1)} - \frac{1}{n}\sum_{i=1}^n \log X_i\right)^{-1}$.

(c) With the given data, we have $x_{(1)} = 25.0$ and $\sum \log x_i = 43.95\cdots$. Therefore, with $n = 14$,

$$\hat{\beta}_{\text{MLE}} = 25.0, \qquad \hat{\alpha}_{\text{MLE}} = (\log 25.0 - 43.95\cdots/14)^{-1} = 12.6.$$

7.13. The joint pdf of $X_1, \dots, X_n$ is

$$\operatorname{pdf}_{X_1,\dots,X_n}(x_1, \dots, x_n|\theta) = \exp\left[-\sum_{i=1}^{n} |\theta - x_i| - n\log 2\right].$$

Thus, the MLE of θ is the minimizer of $\sum_{i=1}^{n} |\theta - x_i|$. When n is odd, we have $\hat{\theta}_{\text{MLE}} = X_{(\frac{n+1}{2})}$; and when n is even, any value in the interval $[X_{(\frac{n}{2})}, X_{(\frac{n}{2}+1)}]$ is an MLE of θ.

7.52. (a) For a iid Poisson(λ) random sample $X = (X_1, \dots, X_n)$, $T(X) = \sum_{i=1}^{n} X_i$ is a CSS for λ. Actually, the joint pdf of $X_1, \dots, X_n$ at $x = (x_1, \dots, x_n)$ is

$$\exp\left[T(x)\log\lambda - n\lambda\right] \prod_{i=1}^{n} x_i!$$

so that $T(X)$ is sufficient, and it is also complete since the above is a full-rank exponential family with the natural parameter $\eta = \log\lambda$. Since $\bar{X}$ is an unbiased estimator for $\lambda = \mathrm{E}X = \mathrm{E}\bar{X}$, by Lehmann–Scheffé, $\bar{X}$ is the UMVUE for λ.

7.58. (a) The joint pdf of a iid random sample $X_1, \dots, X_n$ from the given pdf is

$$\operatorname{pdf}_{X_1,\dots,X_n}(x_1, \dots, x_n|\theta) = \left(\frac{\theta}{2}\right)^{\sum_i |x_i|} (1-\theta)^{n-\sum_i |x_i|}$$

(b) The expected value of $T(X)$ is

$$\sum_{x=-1,0,1} T(x) f(x|\theta) = 2 \cdot \frac{\theta}{2} = \theta.$$

Hence $T(X)$ is an unbiased estimator of θ.

(c) Since $f(x|\theta)$ is a function of θ and $|x|$, by the factorization theorem, $|X|$ is a sufficient statistic of θ. Thus, the Rao–Blackwellization of $T(X)$

conditioned on $|X|$ gives a (probably) better unbiased estimator:

$$T_1(x) = \mathrm{E}\big[T(X) \,\big|\, |X| = |x|\big] = \begin{cases} 0, & x = 0 \\ 1, & |x| = 1 \end{cases}.$$

T_1 is actually better than T because $\mathrm{E}T_1(X) = \mathrm{E}T(X) = \theta$, and the variance of $T(X)$ is

$$\operatorname{Var} T(X) = \sum_{x=-1,0,1} T(x)^2 f(x|\theta) - (\mathrm{E}T(X))^2 = 2\theta - \theta^2$$

but the variance of $T_1(X)$ is

$$\operatorname{Var} T_1(X) = \sum_{x=-1,0,1} T_1(x)^2 f(x|\theta) - (\mathrm{E}T_1(X))^2 = \theta - \theta^2 \le 2\theta - \theta^2.$$

7.62. (a) The squared error loss is the sum of the variance and the square of the bias. Therefore,

$$\begin{aligned} R(\theta,\delta) &= \operatorname{Var} \delta(X) + [\operatorname{Bias} \delta(X)]^2 \\ &= a^2 \operatorname{Var}(\bar{X}) + (b + a\mathrm{E}\bar{X} - \theta)^2 \\ &= a^2 \frac{\sigma^2}{n} + (b - (1-a)\theta)^2. \end{aligned}$$

(b) The posterior distribution is

$$\begin{aligned} p(\theta|x) &\propto \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\sigma}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{\theta-\mu}{\tau}\right)^2\right] \\ &= \exp\left[-\frac{1}{2}\left(\theta^2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right) - 2\theta\left(\frac{\sum_{i=1}^n x_i}{\sigma^2} + \frac{\mu}{\tau^2}\right)\right)\right] \\ &= \exp\left[-\frac{1}{2}\left(\theta^2\eta^{-1}\tau^{-2} - 2\theta\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu}{\tau^2}\right)\right)\right], \end{aligned}$$

which is a pdf of

$$\mathrm{n}\left(\eta\tau^2\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu}{\tau^2}\right),\ \mu\tau^2\right).$$

Thus, the Bayes estimator is

$$\delta^\pi = \eta\tau^2\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu}{\tau^2}\right) = \frac{n\tau^2\bar{x}+\sigma^2\mu}{n\tau^2+\sigma^2} = (1-\eta)\bar{x} + \eta\mu,$$

and the risk for the Bayes estimator is $(1-\eta)^2\frac{\sigma^2}{n}+\eta^2(\theta-\mu)^2$, which can be obtained by putting $a=1-\eta$, $b=\eta\mu$ into the formula in (a).

(c)

$$\begin{aligned}
B(\pi,\delta^\pi) &= \mathrm{E}_{\theta\sim\mathrm{n}(\mu,\tau^2)}\left[(1-\eta)^2\frac{\sigma^2}{n}+\eta^2(\theta-\mu)^2\right] \\
&= (1-\eta)^2\frac{\sigma^2}{n}+\eta^2\tau^2 \\
&= \left(\frac{n\tau^2}{n\tau^2+\sigma^2}\right)^2\frac{\sigma^2}{n}+\left(\frac{\sigma^2}{n\tau^2+\sigma^2}\right)^2\frac{n\tau^2}{n} \\
&= \frac{\tau^2\sigma^2}{n\tau^2+\sigma^2}=\tau^2\eta.
\end{aligned}$$