

Homework 1

1. Prove that $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E} \langle \mu, x^k \rangle)^2 \rightarrow 0$ as $N \rightarrow \infty$.

As we did in the calculation of $\mathbb{E} \langle \mu, x^k \rangle$, we can rephrase $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E} \langle \mu, x^k \rangle)^2$ as follows:

$$\begin{aligned}
 & \mathbb{E} [\langle \mu, x^k \rangle^2] - (\mathbb{E} \langle \mu, x^k \rangle)^2 \\
 &= \mathbb{E} \left[\left(\frac{1}{N} \sum_{j=1}^N \lambda_j^k \right)^2 \right] - \left(\frac{1}{N} \mathbb{E} \sum_{j=1}^N \lambda_j^k \right)^2 \\
 &= \frac{1}{N^2} \left(\mathbb{E} \left[\left(\sum_{j=1}^N \lambda_j^k \right)^2 \right] - \left(\mathbb{E} \sum_{j=1}^N \lambda_j^k \right)^2 \right) \\
 &= \frac{1}{N^2} \left(\mathbb{E} [(\text{tr } H^k)^2] - \left(\mathbb{E} [\text{tr } H^k] \right)^2 \right) \\
 &= \frac{1}{N^2} \left(\mathbb{E} \left[\sum_{i_1, \dots, i_k=1}^N \sum_{i'_1, \dots, i'_k=1}^N H_{i_1, i_2} \cdots H_{i_k, i_1} H_{i'_1, i'_2} \cdots H_{i'_k, i'_1} \right] \right. \\
 &\quad \left. - \left(\mathbb{E} \sum_{i_1, \dots, i_k=1}^N H_{i_1, i_2} \cdots H_{i_k, i_1} \right)^2 \right) \\
 &= \frac{1}{N^2} \sum_{\mathbf{i}, \mathbf{i}' \in \{1, 2, \dots, N\}^k} (\mathbb{E} [T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E} T_{\mathbf{i}} \mathbb{E} T_{\mathbf{i}'})
 \end{aligned}$$

where $T_{(i_1, \dots, i_k)} = H_{i_1, i_2} H_{i_2, i_3} \cdots H_{i_k, i_1}$.

Like a way we defined a graph for each $\mathbf{i} \in \{1, \dots, N\}^k$, we may associate a pair $(\mathbf{i}, \mathbf{i}')$ to a graph as follows:

Let $\mathbf{i} = (i_1, \dots, i_k), \mathbf{i}' = (i'_1, \dots, i'_k) \in \{1, \dots, N\}^k$. Define the graph $G_{\mathbf{i}} = (V_{\mathbf{i}}, E_{\mathbf{i}})$ associated with \mathbf{i} where $V_{\mathbf{i}} = \{i_j : j \in \{1, \dots, k\}\}$ and $E_{\mathbf{i}} = \{\{i_j, i_{j+1}\} : j \in \{1, \dots, k\}\}$ with $i_{k+1} := i_1$. Also, define the graph $G_{\mathbf{i}, \mathbf{i}'} = (V_{\mathbf{i}, \mathbf{i}'}, E_{\mathbf{i}, \mathbf{i}'})$ associated with $\{\mathbf{i}, \mathbf{i}'\}$ where $V_{\mathbf{i}, \mathbf{i}'} = V_{\mathbf{i}} \cup V_{\mathbf{i}'}$

and $E_{\mathbf{i}, \mathbf{i}'} = E_{\mathbf{i}} \cup E_{\mathbf{i}'}$.

Traversing the graph $G_{\mathbf{i}}$ ($i_1 \rightarrow \cdots \rightarrow i_k \rightarrow i_1$) or $G_{\mathbf{i}, \mathbf{i}'}$ ($i_1 \rightarrow \cdots \rightarrow i_k \rightarrow i_1$ and $i'_1 \rightarrow \cdots \rightarrow i'_k \rightarrow i'_1$), let $N_{\mathbf{i}}$ or $N_{\mathbf{i}, \mathbf{i}'}(e)$ ($e \in E_{\mathbf{i}}$ or $e \in E_{\mathbf{i}, \mathbf{i}'}$) be the number of times the traverse passes e (in any direction), respectively.

With these definitions, we obtain

$$\begin{aligned} \mathbb{E}[T_{\mathbf{i}} T_{\mathbf{i}'}] &= \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}[H_e^{N_{\mathbf{i}, \mathbf{i}'}(e)}] \cdot \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}[H_e^{N_{\mathbf{i}, \mathbf{i}'}(e)}] \\ &= \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}[H_{11}^{N_{\mathbf{i}, \mathbf{i}'}(e)}] \cdot \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}[H_{12}^{N_{\mathbf{i}, \mathbf{i}'}(e)}], \end{aligned}$$

where $H_e = H_{ij}$ if $e = \{i, j\}$, due to the identical distribution conditions. Similarly,

$$\begin{aligned} \mathbb{E}[T_{\mathbf{i}}] &= \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}[H_e^{N_{\mathbf{i}}(e)}] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}[H_e^{N_{\mathbf{i}}(e)}] \\ &= \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}[H_{11}^{N_{\mathbf{i}}(e)}] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}[H_{12}^{N_{\mathbf{i}}(e)}]. \end{aligned}$$

Because $\mathbb{E}H_{11} = \mathbb{E}H_{12} = 0$, unless $N_{\mathbf{i}, \mathbf{i}'}(e) = N_{\mathbf{i}}(e) + N_{\mathbf{i}'}(e) \geq 2$ for all $e \in E_{\mathbf{i}, \mathbf{i}'}$, we have $\mathbb{E}[T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E}T_{\mathbf{i}} \mathbb{E}T_{\mathbf{i}'} = 0$. Also when $E_{\mathbf{i}} \cap E_{\mathbf{i}'} = \emptyset$, we have $\mathbb{E}[T_{\mathbf{i}} T_{\mathbf{i}'}] = \mathbb{E}[T_{\mathbf{i}}] \mathbb{E}[T_{\mathbf{i}'}]$ due to the independence conditions. Moreover, when there is a bijection on $\{1, \dots, N\}$ which maps \mathbf{i} to \mathbf{j} and \mathbf{i}' to \mathbf{j}' , then we have

$$\mathbb{E}[T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E}T_{\mathbf{i}} \mathbb{E}T_{\mathbf{i}'} = \mathbb{E}[T_{\mathbf{j}} T_{\mathbf{j}'}] - \mathbb{E}T_{\mathbf{j}} \mathbb{E}T_{\mathbf{j}'}$$

due to the identical distribution (by applying the bijection on the product). So, this defines an equivalence relation on $\left(\{1, \dots, N\}^k\right)^2$.

Now, we will count those equivalence classes (of $(\mathbf{i}, \mathbf{i}')$'s) by $|V_{\mathbf{i}, \mathbf{i}'}| (\leq 2k)$. Let us \mathcal{G}_v denote the set of all representatives for equivalence classes of $a_{\mathbf{i}, \mathbf{i}'}$'s (defined by the bijection on $\{1, \dots, N\}$) with $|V_{\mathbf{i}, \mathbf{i}'}| = v$, $N_{\mathbf{i}, \mathbf{i}'}(e) \geq 2$ for every $e \in E_{\mathbf{i}, \mathbf{i}'}$, and $E_{\mathbf{i}} \cap E_{\mathbf{i}'} \neq \emptyset$. Note that the cardinality of an equivalence class is exactly $v! \binom{N}{v}$, if N is sufficiently large, that is, $N \geq v$.

Using this observation, we have (when $N \geq 2k$)

$$\begin{aligned}
 & \mathbb{E} \left[\langle \mu, x^k \rangle^2 \right] - \left(\mathbb{E} \langle \mu, x^k \rangle \right)^2 \\
 &= \frac{1}{N^2} \sum_{\mathbf{i}, \mathbf{i}' \in \{1, 2, \dots, N\}^k} (\mathbb{E} [T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E} T_{\mathbf{i}} \mathbb{E} T_{\mathbf{i}'}) \\
 &= \frac{1}{N^2} \sum_{\mathbf{i}, \mathbf{i}' \in \{1, 2, \dots, N\}^k} (\mathbb{E} [T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E} T_{\mathbf{i}} \mathbb{E} T_{\mathbf{i}'}) \\
 &= \frac{1}{N^2} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v} (\mathbb{E} [T_{\mathbf{i}} T_{\mathbf{i}'}] - \mathbb{E} T_{\mathbf{i}} \mathbb{E} T_{\mathbf{i}'}) \\
 &= \frac{1}{N^2} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v} \left(\prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E} \left[H_{11}^{N_{\mathbf{i}, \mathbf{i}'(e)}} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E} \left[H_{12}^{N_{\mathbf{i}, \mathbf{i}'(e)}} \right] \right. \\
 &\quad \left. - \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E} \left[H_{11}^{N_{\mathbf{i}}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E} \left[H_{12}^{N_{\mathbf{i}}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E} \left[H_{11}^{N_{\mathbf{i}'}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E} \left[H_{12}^{N_{\mathbf{i}'}(e)} \right] \right) \\
 &= \frac{1}{N^{k+2}} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v} \left(\prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E} \left[\hat{H}_{11}^{N_{\mathbf{i}, \mathbf{i}'(e)}} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}, \mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E} \left[\hat{H}_{12}^{N_{\mathbf{i}, \mathbf{i}'(e)}} \right] \right. \\
 &\quad \left. - \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E} \left[\hat{H}_{11}^{N_{\mathbf{i}}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E} \left[\hat{H}_{12}^{N_{\mathbf{i}}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E} \left[\hat{H}_{11}^{N_{\mathbf{i}'}(e)} \right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E} \left[\hat{H}_{12}^{N_{\mathbf{i}'}(e)} \right] \right)
 \end{aligned}$$

where $\hat{H}_{ij} := N^{-1/2} H_{ij} \sim \mathcal{N}(0, 1)$.

Moreover, we can observe that \mathcal{G}_v does not depend on N : as we can pick a representative whose set of vertices is contained in $\{1, \dots, v\}$. Since any k -th moment of a standard normal random variable are finitely well-defined, $\sum_{(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v} (\dots)$ in the last line of the equation above does not depend on N . Denoting those terms (independent of N) as C_v , we have

$$\mathbb{E} \left[\langle \mu, x^k \rangle^2 \right] - \left(\mathbb{E} \langle \mu, x^k \rangle \right)^2 = \sum_{v=1}^{2k} C_v \cdot v! \cdot N^{-(k+2)} \binom{N}{v}.$$

Therefore, it suffices to prove that $\mathcal{G}_v = \emptyset$ so that $C_v = 0$ for $v \geq k+2$, since other (lower degree) terms disappears as $N \rightarrow \infty$, since $\binom{N}{v} \sim N^v$.

Suppose $(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v$. Since $E_{\mathbf{i}} \cap E_{\mathbf{i}'} \neq \emptyset$, $G_{\mathbf{i}, \mathbf{i}'}$ is connected, with v vertices and $\leq k$ edges, as every edge should be passed more than once during traverse. Since $v = |V(G_{\mathbf{i}, \mathbf{i}'})| \leq |E(G_{\mathbf{i}, \mathbf{i}'})| + 1 \leq k + 1$ for a connected

graph $G_{i,i'}$, we have $\mathcal{G}_v = \emptyset$ when $v \geq k + 2$. This completes the proof. \square