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MAS583(A) Random Matrix Theory and Its Application

Homework 2

1. Define $I_{ij} \coloneqq \{(k,\ell) : 1 \le k \le \ell \le n, (k,\ell) \ne (i,j)\}$.

$$\begin{aligned}
\mathbb{E}[H_{ij}G_{ji}] &= \mathbb{E}\Big[\mathbb{E}\big[H_{ij}G_{ji}\big|H_{k\ell} \text{ for any } (k,\ell) \in I_{ij}\big]\Big] \\
&= \mathbb{E}\left[\operatorname{Var}\big[H_{ij}^2\big|H_{k\ell} \text{ for any } (k,\ell)\in I_{ij}\big] \cdot \mathbb{E}\left[\frac{\mathrm{d}G_{ji}}{\mathrm{d}H_{ij}}\middle|H_{k\ell} \text{ for any } (k,\ell) \in I_{ij}\right]\right] \\
&= \operatorname{Var}\big[H_{ij}^2\big] \cdot \mathbb{E}\left[\mathbb{E}\left[\frac{\mathrm{d}G_{ji}}{\mathrm{d}H_{ij}}\middle|H_{k\ell} \text{ for any } (k,\ell) \in I_{ij}\right]\right].
\end{aligned}$$

Here, the Stein's lemma is used. I'll justify the validity of the lemma applied here at the end of the solution.

For a function $f \colon \mathbb{R}^{n^2} \to \mathbb{C},\ (A_{ij})_{1\le i,j\le n} \mapsto f((A_{ij})_{1\le i,j\le n})$ and a square matrix $\mathbf{A}$ (with a special structure such as symmetricity), we have

$$\frac{\mathrm{d}f}{\mathrm{d}A_{ij}} = \sum_{k,\ell} \frac{\partial f}{\partial A_{k\ell}} \frac{\partial A_{k\ell}}{\partial A_{ij}} = \operatorname{tr}\left[\left(\frac{\partial f}{\partial \mathbf{A}}\right)\frac{\partial \mathbf{A}}{\partial A_{ij}}\right].$$

In our case, $f(H) = G_{ji} = \big((H - zI)^{-1}\big)_{ji}$ and H is symmetric so that

$$\left(\frac{\partial H}{\partial H_{ij}}\right)_{k\ell} = \frac{\partial H_{k\ell}}{\partial H_{ij}} = \delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i} - \delta_{ij}\delta_{ki}\delta_{\ell j} = \begin{cases} \delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i} & \text{if } i \ne j \\ \delta_{ki}\delta_{\ell j} & \text{if } i = j \end{cases}.$$

Hence, we have

$$\begin{aligned}
\frac{\mathrm{d}G_{ji}}{\mathrm{d}H_{ij}} &= \sum_{k,\ell} \frac{\partial G_{ji}}{\partial H_{k\ell}} \frac{\partial H_{k\ell}}{\partial H_{ij}} \\
&= \begin{cases} \dfrac{\partial G_{ji}}{\partial H_{ij}} + \dfrac{\partial G_{ji}}{\partial H_{ji}} & \text{if } i \ne j \\ \dfrac{\partial G_{ji}}{\partial H_{ij}} & \text{if } i = j \end{cases}.
\end{aligned}$$

Moreover, the following is a basic matrix identity:

$$\frac{\partial \mathbf{A}^{-1}}{\partial x} = -\mathbf{A}^{-1}\frac{\partial A}{\partial x}\mathbf{A}^{-1}$$

Therefore, with $\mathbf{A} = H - zI$ and $x = H_{ij}$,

$$\frac{\partial G_{k\ell}}{\partial H_{ij}} = -G_{ki}G_{j\ell},$$

yielding

$$\frac{\mathrm{d}G_{ji}}{\mathrm{d}H_{ij}} = \begin{cases} \dfrac{\partial G_{ji}}{\partial H_{ij}} + \dfrac{\partial G_{ji}}{\partial H_{ji}} = -G_{ji}^2 - G_{ii}G_{jj} & \text{if } i \neq j \\ \dfrac{\partial G_{ji}}{\partial H_{ij}} = -G_{ji}^2 = -G_{ii}G_{jj} & \text{if } i = j \end{cases}. \tag{1}$$

Consequently,

$$\begin{aligned} \mathbb{E}[H_{ij}G_{ji}] &= \mathrm{Var}\big[H_{ij}^2\big] \cdot \mathbb{E}\left[\mathbb{E}\left[\frac{\mathrm{d}G_{ji}}{\mathrm{d}H_{ij}}\middle| H_{k\ell} \text{ for any } (k,\ell) \in I_{ij}\right]\right] \\ &= \begin{cases} \dfrac{1}{N}\mathbb{E}\left[-G_{ji}^2 - G_{ii}G_{jj}\right] & \text{if } i \neq j \\ \dfrac{2}{N}\mathbb{E}\left[-G_{ji}^2\right] = \dfrac{1}{N}\mathbb{E}\left[-G_{ji}^2 - G_{ii}G_{jj}\right] & \text{if } i = j \end{cases} \\ &= \frac{1}{N}\mathbb{E}\left[-G_{ji}^2 - G_{ii}G_{jj}\right] \end{aligned}$$

as desired.

Now, let us justify the application of the Stein's lemma appeared above. Since the inner expectation is conditioned given every $H_{k\ell}$'s except H_{ij}, H_{ij} is the only variable which is not fixed. So, it suffices to show that $f(H_{ij}) = G_{ji}$ is good enough; $\mathbb{E}\,|f'(H_{ij})| < \infty$. Using the formula (1) of the derivative of G_{ji} and the analyticity of the resolvent, $\mathbb{E}\,|f'(H_{ij})| \leq \mathbb{E}|G_{ji}^2| + \mathbb{E}|G_{ii}G_{jj}| < \infty$. This completes the proof.