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MAS365 Introduction to Numerical Analysis

Homework 9

1. Denote $\langle f, g\rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx$.

(a) Choose $\tilde{L}_1(x) = x$, then

$$\langle L_0, L_0\rangle = \int_0^\infty e^{-x}\,dx = 1, \qquad \left\langle L_0, \tilde{L}_1\right\rangle = \int_0^\infty xe^{-x}\,dx = 1$$

so that $L_1 = \tilde{L}_1 - \frac{\left\langle L_0, \tilde{L}_1\right\rangle}{\langle L_0, L_0\rangle} L_0 = x - 1$ is orthogonal to L_0. Note that

$$\langle L_1, L_1\rangle = \int_0^\infty (x-1)^2 e^{-x}\,dx = 1.$$

Choosing $\tilde{L}_2(x) = x^2$,

$$\left\langle L_0, \tilde{L}_2\right\rangle = \int_0^\infty x^2 e^{-x}\,dx = 2, \qquad \left\langle L_1, \tilde{L}_2\right\rangle = \int_0^\infty (x-1)x^2 e^{-x}\,dx = 4$$

so that

$$L_2 = \tilde{L}_2 - \frac{\left\langle L_0, \tilde{L}_2\right\rangle}{\langle L_0, L_0\rangle} L_0 - \frac{\left\langle L_1, \tilde{L}_2\right\rangle}{\langle L_1, L_1\rangle} L_1 = x^2 - 2 - 4(x-1) = x^2 - 4x + 2.$$

Note that $\langle L_2, L_2\rangle = 4$.

(b) We need to find

$$P(x) = \sum_{k=0}^{2} a_k L_k(x)$$

where the coefficients satisfy the following:

$$\int_0^\infty f(x)L_j(x)e^{-x}\,dx = \sum_{k=0}^{2} a_k \int_0^\infty L_j(x)L_k(x)e^{-x}\,dx, \qquad j = 0, 1, 2,$$

that is,

$$\langle f, L_j\rangle = \sum_{k=0}^{2} a_k \langle L_j, L_k\rangle = a_j \langle L_j, L_j\rangle, \quad \therefore a_j = \frac{\langle f, L_j\rangle}{\langle L_j, L_j\rangle}, \quad j = 0, 1, 2$$

by the orthogonality. Since we have

$$\langle f, L_0 \rangle = 6, \qquad \langle f, L_1 \rangle = 18, \qquad \langle f, L_2 \rangle = 36,$$

$a_0 = 6$, $a_1 = 18$, $a_2 = 9$ so that

$$P(x) = 6 + 18(x-1) + 9(x^2 - 4x + 2) = 9x^2 - 18x + 6.$$