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MAS109(E,F) Introduction to Linear Algebra

Quiz 5

Problems

#1. The eigenvalues of A are the roots of

$$\det(tI-A) = \begin{vmatrix} t+1 & 2 & -2 \\ -4 & t-3 & 4 \\ 0 & 2 & t-1 \end{vmatrix}$$

$$= (t+1)\Big((t-3)(t-1)-8\Big)+4\Big(2(t-1)+4\Big)$$

$$= (t+1)(t-3)(t-1) = 0.$$

Thus, A has three distinct eigenvalues, ± 1 and 3. Solving the linear equation $A\mathbf{x}=\lambda\mathbf{x}$ for $\lambda=\pm1, 3$, we have the following eigenvectors of A:

* eigenvectors corresponding to $\lambda=3$: $\mathbf{x} = c\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}$.
* ——— " ——— $\lambda=1$: $\mathbf{x} = c\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}$.
* ——— " ——— $\lambda=-1$: $\mathbf{x} = c\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}$.

Therefore, $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix},\quad D = \begin{pmatrix} 3 & & \\ & 1 & \\ & & -1 \end{pmatrix},\quad P^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

#2. Since A and AA^T has the same rank and the dimension of AA^T is $m\times m$, by the dimension theorem (aka rank-nullity theorem),

$$\text{nullity}(AA^T) = m - \cancel{\text{null}}\ \text{rank}(AA^T) = m-k$$

Summary

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* equiv. rel'n. $R \subseteq X \times X$ s.t.
  - reflexive: $\forall x \in X$, xRx.
  - symmetric: $\forall x, y \in X$, $xRy \Rightarrow yRx$
  - transitive: $\forall x, y, z \in X$, $xRyRz \Rightarrow xRz$.
* Similarity: $A \sim B \iff \exists P$: invertible s.t. $B = P^{-1}AP$.
  ↳ same with the row-equivalence for the square matrices.
  - Two square matrices are similar iff there are bases which the matrices represent the same operator w.r.t.
* Similarity invariants: det, tr, char. poly., dim'n of eigenspaces
* Multiplicity of an eigenvalue:
  - algebraic multi: multiplicity of λ in $p_A(t)$ as a zero.
  - geometric multi: dimension of the eigenspace.
* Diagonalization: $D = P^{-1}AP$, D: diagonal & P: invertible.
* TFAE:
  ① A is diagonalizable
  ② A has n linearly indep. eigenvectors.
  ③ The eigenvectors of A form a basis of $\mathbb{R}^n$.
  ④ geom. mult. = alge. mult. for all eigenvalues.
  $\Rightarrow$ If A has n distinct eigenvalues, it is diagonalizable.
* Orthogonal diagonalizability: $D = P^{-1}AP$ where $\underbrace{P^{-1} = P^T}_{\text{orthogonal}}$.
  - orthogonal diagonalizability = real symmetricity.
    (due to the spectral decomposition)

    $P = [v_1 \cdots v_n]$, v_i: eigenvector corr. to λ_i.

    $\Rightarrow A = P\,\mathrm{diag}(\lambda_1 \cdots \lambda_n)P^T = \sum \lambda_i v_i v_i^T$,

    $\langle \lambda_1 v_1, v_2 \rangle = \langle Av_1, v_2 \rangle \underset{\text{real symm}}{=} \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$

    A is self-adjoint $\Rightarrow \langle v_1, v_2 \rangle = 0$.
* Cayley-Hamilton: $p_A(A) = O$.

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* Minimal poly: monic, smallest degree m_A s.t. $m_A(A)=0$.
  - By the C-H thm, $m_A \mid p_A$.
  - Every irreducible factor of p_A ~~is also a~~ divides m_A.
* For diagonalizable A, $A = PDP^{-1}$, $D = \mathrm{diag}(\lambda_1 \cdots \lambda_n)$

$$e^A = e^{PDP^{-1}} = \sum \frac{1}{k!} PD^kP^{-1} = Pe^DP^{-1}$$
$$= P \cdot \mathrm{diag}(e^{\lambda_1} \cdots e^{\lambda_n}) P^{-1}.$$

* How to find ~~a~~ the spectral orthogonalizat° of a real symmetric matrix A?
  - The power method: to find the ~~largest~~ eigenvalue of the largest modulus

$$x_n = \frac{Ax_{n-1}}{\|Ax_{n-1}\|}$$

converges to a unit dominant eigenvector and

$$Ax_n \cdot x_n \longrightarrow \lambda : \text{dominant eigenvalue}.$$

* rank A = dim row A, ~~null(A) =~~
  nullity A = dim null A.
* Theorems:
  - $W \leq \mathbb{R}^n \Rightarrow W^\perp \cap W = 0$ & $W^{\perp\perp} = W$
  - $S \neq \emptyset$. $S \subseteq \mathbb{R}^n \Rightarrow S^\perp = \mathrm{span}(S)^\perp$.
  - $\mathrm{row}(A)^\perp = \mathrm{null}(A) = \mathrm{col}(A^T)^\perp$,
  - ~~row~~ $A \underset{r}{\sim} B \Rightarrow \mathrm{row}(A) = \mathrm{row}(B)$ & $\mathrm{null}(A) = \mathrm{null}(B)$.
    If $A \underset{r}{\sim} B$ where B is a row echelon form, then nonzero rows of B form a basis of $\mathrm{row}(A)$ $(= \mathrm{row}(B))$.
* Dimension thm: $A: \mathbb{R}^n \to \mathbb{R}^m$, $W \leq \mathbb{R}^n$,
  $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$. $\dim W + \dim W^\perp = n$.
* Rank thm: $\dim \mathrm{row}(A) = \dim \mathrm{col}(A) = \mathrm{rank}\, A$.
* A & A^TA & AA^T have the same rank.