20170058 Keonwoo Kim · March 30, 2020
MAS583 Random Matrix Theory and Its Application

Homework 1

---

1. Prove that $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E}\langle \mu, x^k \rangle)^2 \to 0$ as $N \to \infty$.

As we did in the calculation of $\mathbb{E}\langle \mu, x^k \rangle$, we can rephrase $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E}\langle \mu, x^k \rangle)^2$ as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\langle \mu, x^k \rangle^2\right] - \left(\mathbb{E}\langle \mu, x^k \rangle\right)^2 \\
&= \mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{N} \lambda_j^k\right)^2\right] - \left(\frac{1}{N}\mathbb{E}\sum_{j=1}^{N} \lambda_j^k\right)^2 \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[\left(\sum_{j=1}^{N} \lambda_j^k\right)^2\right] - \left(\mathbb{E}\sum_{j=1}^{N} \lambda_j^k\right)^2\right) \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[(\operatorname{tr} H^k)^2\right] - \left(\mathbb{E}\left[\operatorname{tr} H^k\right]\right)^2\right) \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[\sum_{i_1,\ldots,i_k=1}^{N} \sum_{i'_1,\ldots,i'_k=1}^{N} H_{i_1,i_2}\cdots H_{i_k,i_1} H_{i'_1,i'_2}\cdots H_{i'_k,i'_1}\right]\right. \\
&\qquad\left. - \left(\mathbb{E}\sum_{i_1,\ldots,i_k=1}^{N} H_{i_1,i_2}\cdots H_{i_k,i_1}\right)^2\right) \\
&= \frac{1}{N^2}\sum_{\mathbf{i},\mathbf{i}'\in\{1,2,\ldots,N\}^k} \mathbb{E}\left[T_{\mathbf{i}}T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\,\mathbb{E}T_{\mathbf{i}'}
\end{aligned}
$$

where $T_{(i_1,\ldots,i_k)} = H_{i_1,i_2}H_{i_2,i_3}\cdots H_{i_k,i_1}$.

Like a way we defined a graph for each $\mathbf{i} \in \{1,\ldots,N\}^k$, we may associate a pair $(\mathbf{i},\mathbf{i}')$ to a graph as follows: