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* Orthogonal projection of $\mathbb{x}$ along $\mathbb{a}$

$$\mathrm{proj}_{\mathbb{a}}\mathbb{x} = \frac{\mathbb{x}\cdot\mathbb{a}}{\|\mathbb{a}\|^2}\mathbb{a}$$

* $W \leq \mathbb{R}^n \Rightarrow^{\forall \mathbb{x}\in\mathbb{R}^n} \exists!\ \mathbb{x}_1 \in W,\ \mathbb{x}_2 \in W^\perp$ s.t. $\mathbb{x} = \mathbb{x}_1 + \mathbb{x}_2$.

  $\Rightarrow \mathbb{R}^n = W \oplus W^\perp$.

  Take an $n\times k$ matrix A s.t. $k = \dim W$, $W = \mathrm{col}(A)$.

  $\Rightarrow \mathrm{proj}_W \mathbb{x} := A(A^TA)^{-1}A^T\mathbb{x}$: best approx to $\mathbb{x}$ from W.

* Least squares solution: best approx $A\mathbb{x}$ to $\mathbb{b}$ from $W = \mathrm{col}(A)$.

  $\rightsquigarrow$ Normal equation associated w/ $A\mathbb{x} = \mathbb{b}$.

$$A^TA\mathbb{x} = A^T\mathbb{b}.$$

  When A has ~~a~~ full column rank, $\exists!\ \hat{\mathbb{x}} = (A^TA)^{-1}A^T\mathbb{b}$.

* If $\{\mathbb{w}_1, \cdots, \mathbb{w}_k\}$ is an orthogonal basis for $W \leq \mathbb{R}^n$,

$$\mathrm{proj}_W \mathbb{x} = \sum_{i=1}^{k} \frac{\mathbb{x}\cdot\mathbb{v}_i}{\|\mathbb{v}_i\|^2}\mathbb{v}_i.$$

* (Gram-Schmidt) Every nonzero subspace of $\mathbb{R}^n$ yields an orthogonal basis.

  pf) $W_j := $ ~~span~~ $\mathrm{span}\{\mathbb{w}_1, \cdots, \mathbb{w}_k\}$,

  $\{\mathbb{w}_1, \cdots, \mathbb{w}_k\}$: basis of W_j.

  $\Rightarrow$ ~~$\mathbb{v}_i$~~ $\mathbb{v}_i :=$ ~~$\mathbb{w}_i$~~ $\mathbb{w}_i - \mathrm{proj}_{\mathrm{span}(\mathbb{w}_1, \cdots, \mathbb{w}_{i-1})}\mathbb{w}_i$.

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* $\{w_i\} \xrightarrow{G-S} \{v_i\} \xrightarrow{\text{normalization}} \{q_i\}$.

$$\Rightarrow w_j := \sum_{i=1}^{j} (w_j \cdot q_i)\, q_i.$$

$$\therefore [w_1 \cdots w_k] = [q_1 \cdots q_k] \begin{bmatrix} w_1 \cdot q_1 & \cdots & w_k \cdot q_1 \\ & \ddots & \vdots \\ 0 & & w_k \cdot q_k \end{bmatrix}$$

$$\cancel{W = QA}$$

$$A = Q \cdot R \qquad (\text{col}(A) = W)$$

Since Q is orthogonal, $R = Q^T A$. $\Downarrow$ R is invertible

* $A = QR$, $A^TAx = A^Tb \Rightarrow Rx = Q^Tb$.

$$\Rightarrow \hat{x} = R^{-1}Q^Tb.$$

* Reflection about a hyperplane $a^\perp$:

$$Tx = x - 2\,\text{proj}_{a^\perp} x = x - 2\frac{x \cdot a}{\|a\|^2}a.$$

* $\exists$ orthogonal $\underbrace{U, V}_{n\times n}$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ ($k = \text{rank } A$)

s.t. $A = U\Sigma V^T$

(Use the symmetricity of A^TA.)

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Quiz 6.

#1. The given linear system can be transformed into the following matrix form:

$$\underbrace{\begin{bmatrix} 0 & -2 \\ 1 & 4 \\ -1 & 6 \\ 4 & 2 \end{bmatrix}}_{=:A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{=:\mathbb{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 3 \\ 15 \end{bmatrix}}_{=:\mathbb{b}}$$

Then the least square solution is given by

$$A^T A \mathbb{x} = A^T \mathbb{b},$$

where

$$A^T A = \begin{bmatrix} 0 & 1 & -1 & 4 \\ -2 & 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 4 \\ -1 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 6 & 60 \end{bmatrix} = 6 \begin{bmatrix} 3 & 1 \\ 1 & 10 \end{bmatrix}$$

$$\Rightarrow (A^T A)^{-1} = \cancel{6} \frac{1}{6^2 \cdot 29} \cdot 6 \begin{bmatrix} 10 & -1 \\ -1 & 3 \end{bmatrix} \cancel{= \frac{1}{6 \times 29}}$$

$$= \frac{1}{174} \begin{bmatrix} 10 & -1 \\ -1 & 3 \end{bmatrix},$$

$$A^T \mathbb{b} = \begin{bmatrix} 0 & 1 & -1 & 4 \\ -2 & 4 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 15 \end{bmatrix} = \begin{bmatrix} 57 \\ 48 \end{bmatrix}$$

$$\Rightarrow \mathbb{x} = \frac{1}{174} \begin{bmatrix} 10 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 57 \\ 48 \end{bmatrix} = \frac{1}{174} \begin{bmatrix} 522 \\ 87 \end{bmatrix} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}.$$

Thus the solution is $[3, 1/2]^T$.

#2.

$$A^TA = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Rightarrow \text{char. poly of } A^TA : \det(tI - A) = t(t-2)(t-8),$$

Eigenvalues & Eigenvectors:

$$\lambda = 8 : \quad (1, 2, 1)^T =: \sqrt{6}\, \mathbb{w}_1$$

$$\lambda = 2 \quad (1, -1, 1)^T =: \sqrt{3}\, \mathbb{w}_2$$

$$\lambda = 0 \quad (1, 0, -1)^T =: \sqrt{2}\, \mathbb{w}_3 .$$

Note that

$$A\mathbb{w}_1 = \frac{1}{\sqrt{6}} \cdot A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2\sqrt{2} \\ 6 \\ 2 \end{pmatrix},$$

$$A\mathbb{w}_2 = \frac{1}{\sqrt{3}} A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2} \\ 0 \\ 2 \end{pmatrix}.$$

$$A\mathbb{w}_3 = \frac{1}{\sqrt{2}} A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (0, 0, 0)^T.$$

$$\therefore \mathbb{u}_1 = A\mathbb{w}_1 / \sqrt{8} = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 3 \\ 1 \end{pmatrix}.$$

$$\mathbb{u}_2 = A\mathbb{w}_2 / \sqrt{2} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ \sqrt{2} \end{pmatrix}.$$

$$\mathbb{u}_3 = \mathbb{u}_1 \times \mathbb{u}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, 1 \right)$$

$$\therefore U = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ \sqrt{3}/2 & 0 & -1/2 \\ 1/2\sqrt{3} & \sqrt{2}/\sqrt{3} & 1 \end{pmatrix} \leftarrow \text{positive entries.}$$

$$\Sigma = \operatorname{diag}(\sqrt{8}, \sqrt{2}, 0),$$

$$V = \begin{pmatrix} \mathbb{w}_1 & \mathbb{w}_2 & \mathbb{w}_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} \leftarrow \text{positive entries.}$$

$$\rightsquigarrow A = U \Sigma V^T$$