20170058 Keonwoo Kim · March 24, 2020
MAS555 Advanced Statistics

Homework 1

1.45. (1) Nonnegativity: $P(X = x_i) = P(\{s_j \in S : X(s_j) = x_i\}) \geq 0$;

(2) total probability:

$$\begin{aligned} P(X \in \mathcal{X}) &= \sum_{x_i \in \mathcal{X}} P(X = x_i) \\ &= \sum_{x_i \in \mathcal{X}} P(\{s_j \in S : X(s_j) = x_i\}) \\ &= P\left(\bigcup_{x_i \in \mathcal{X}} \{s_j \in S : X(s_j) = x_i\}\right) \\ &= P(\{s_j \in S : X(s_j) \in \mathcal{X}\}) = 1; \end{aligned}$$

(3) countable additivity: if $A_i \subseteq \mathcal{X}$ are pairwise disjoint,

$$\begin{aligned} P\Bigg(X \in \bigcup_{i \geq A_i} A_i\Bigg) &= P\Bigg(\Bigg\{s \in S : X(s) \in \bigcup_{i \geq 1} A_i\Bigg\}\Bigg) \\ &= P\Bigg(\bigcup_{i \geq 1} \{s \in S : X(s) \in A_i\}\Bigg) \\ &= \sum_{i \geq 1} P(\{s \in S : X(s) \in A_i\}) \\ &= \sum_{i \geq 1} P(X \in A_i). \end{aligned}$$

2.10. (a) Let F_X has a jump at x_0, $y_0 = F_X(x_0)$. Let x_1 be the right next to x_0, that is, $\lim_{x \to x_1 -} F_X(x) = F_X(x_0) < F_X(x_1)$. Now assume $y = y_0 + \epsilon$ for some $0 \leq \epsilon < F_X(x_1) - F_X(x_0)$. Then $P(X < x_1) = \lim_{x \to x_1 -} F_X(x) = P(X \leq x_0)$ so that

$$\begin{aligned} P(Y \leq y) &= P(Y \leq y_0) \\ &= P(X < x_1) = P(X \leq x_0) = F_X(x_0) = y_0 \leq y \end{aligned}$$

and the inequality becomes strict whenever ϵ is nonzero.

(b) Since $F_Y(y) = 1 - P(Y > y)$, the result is analogous to (a).

2.38. (a)

$$
\begin{aligned}
M_X(t) = \mathrm{E}e^{tX} &= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x \\
&= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \big((1-p)e^t\big)^x \\
&= \frac{p^r}{\big(1-(1-p)e^t\big)^r} \underbrace{\sum_{x=0}^{\infty} \binom{r+x-1}{x} \big(1-(1-p)e^t\big)^r \big((1-p)e^t\big)^x}_{=1} \\
&= \frac{p^r}{\big(1-(1-p)e^t\big)^r}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
\lim_{p\downarrow 0} M_Y(t) = \lim_{p\downarrow 0} \mathrm{E}e^{2ptX} &\\
&= \left(\lim_{p\downarrow 0} \frac{p}{1-(1-p)e^{2pt}}\right)^r \\
&\overset{\text{L'H}}{=\!=} \left(\lim_{p\downarrow 0} \frac{1}{e^{2pt}-2t(1-p)e^{2pt}}\right)^r = \left(\frac{1}{1-2t}\right)^r, \qquad |t| < 1/2.
\end{aligned}
$$

3.20. (a)

$$
\begin{aligned}
\mathrm{E}X &= \int_0^\infty x\, f(x)\, dx \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty x\, e^{-x^2/2}\, dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^0 e^u\, du \qquad (u = -x^2/2) \\
&= \frac{2}{\sqrt{2\pi}}, \\
\mathrm{Var}\, X &= \int_0^\infty x^2\, f(x)\, dx - (\mathrm{E}X)^2 \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2\, e^{-x^2/2}\, dx - \frac{2}{\pi} \\
&= \frac{2}{\sqrt{2\pi}} \left(-x\, e^{-x^2/2}\Big|_0^\infty + \int_0^\infty e^{-x^2/2}\, dx \right) - \frac{2}{\pi} \\
&= 1 - \frac{2}{\pi}.
\end{aligned}
$$

(b) Assume g to be strictly increasing on $[0, \infty)$ and both g and g^{-1} to be smooth. Then we have

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}\, P\left(Y \le y\right) = \frac{d}{dy}\, P\left(X \le g^{-1}(y)\right) \\ &= \frac{d}{dy}\, F_X(g^{-1}(y)) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}. \end{aligned}$$

When $g(x) = x^2$ $(x \ge 0)$, we have

$$f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}}\, e^{-y/2}, \qquad Y \sim \text{gamma}(1/2,\, 2).$$

4.32. (a) • Marginal pdf:

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{Y|\Lambda}(y|\lambda) f_\Lambda(\lambda)\, d\lambda \\ &= \frac{1}{y!\,\Gamma(\alpha)\,\beta^\alpha} \int_0^\infty \lambda^y e^{-\lambda} \cdot \lambda^{\alpha-1} e^{-\lambda/\beta}\, d\lambda \\ &= \frac{\Gamma(y+\alpha)\,(\beta')^{y+\alpha}}{y!\,\Gamma(\alpha)\,\beta^\alpha} \underbrace{\frac{1}{\Gamma(y+\alpha)\,(\beta')^{y+\alpha}} \int_0^\infty \lambda^{y+\alpha-1} e^{-\lambda/\beta'}\, d\lambda}_{=1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\beta' := \beta/(\beta+1)) \\ &= \frac{\Gamma(y+\alpha)\,\beta^y}{y!\,\Gamma(\alpha)\,(\beta+1)^{y+\alpha}} \\ &= \frac{\Gamma(y+\alpha)}{y!\,\Gamma(\alpha)} \left(\frac{1}{\beta+1}\right)^\alpha \left(\frac{\beta}{\beta+1}\right)^y \end{aligned}$$

When α is a positive integer, $Y \sim \text{NB}(\alpha, (\beta+1)^{-1})$.

• Mean: $\mathrm{E}Y = \mathrm{E}\left[\mathrm{E}\left(Y|\Lambda\right)\right] = \mathrm{E}\Lambda = \alpha\beta$.

• Variance: By Theorem 4.4.7 (Conditional variance identity, or the law of total variance),

$$\operatorname{Var} Y = \mathrm{E}\left[\operatorname{Var}(Y|\Lambda)\right] + \operatorname{Var}\left[E(Y|\Lambda)\right] = \mathrm{E}\Lambda + \operatorname{Var}\Lambda = \alpha\beta + \alpha\beta^2.$$

4.58. (a)

$$\begin{aligned} \operatorname{Cov}(X, \mathrm{E}(Y|X)) &= \mathrm{E}\left[X \cdot \mathrm{E}(Y|X)\right] - \mathrm{E}X \cdot \mathrm{E}\left[\mathrm{E}(Y|X)\right] \\ &= \mathrm{E}\left[\mathrm{E}(XY|X)\right] - \mathrm{E}X \cdot \mathrm{E}\left[\mathrm{E}(Y|X)\right] \\ &= \mathrm{E}(XY) - \mathrm{E}X \cdot \mathrm{E}Y = \operatorname{Cov}(X, Y). \end{aligned}$$

(b) Since Cov is bilinear, by (a), $\mathrm{Cov}(X, Y - \mathrm{E}(Y|X)) = \mathrm{Cov}(X, Y) - \mathrm{Cov}(X, \mathrm{E}(Y|X)) = 0$.

(c) $\mathrm{Var}\,[Y - \mathrm{E}(Y|X)] = \mathrm{Var}\,Y + \mathrm{Var}\,[\mathrm{E}(Y|X)] - 2\,\mathrm{Cov}(Y, \mathrm{E}(Y|X))$, where

$$\mathrm{E}\,[Y\,\mathrm{E}(Y|X)] = \mathrm{E}\big(\mathrm{E}\,[Y\,\mathrm{E}(Y|X) \mid X]\big) = \mathrm{E}\left[\mathrm{E}(Y|X)^2\right]$$

so that

$$\mathrm{Cov}(Y, \mathrm{E}(Y|X)) = \mathrm{E}\,[Y\,\mathrm{E}(Y|X)] - \mathrm{E}Y \cdot \mathrm{E}\,[\mathrm{E}(Y|X)] = \mathrm{Var}\,[\mathrm{E}(Y|X)]\,,$$

and $\mathrm{Var}\,Y = \mathrm{E}\,[\mathrm{Var}(Y|X)] + \mathrm{Var}\,[E(Y|X)]$. Thus

$$\mathrm{Var}\,[Y - \mathrm{E}(Y|X)] = \mathrm{Var}\,Y - \mathrm{Var}\,[\mathrm{E}(Y|X)] = \mathrm{E}\,[\mathrm{Var}(Y|X)]\,.$$