20170058 Keonwoo Kim · May 2, 2020
MAS583(A) Random Matrix Theory and Its Application

Homework 3

1. (Reproducing kernel property)

$$\begin{aligned}\int K_N(x,y)K_N(y,z)\,\mathrm{d}y &= \int \sum_{k,\ell=1}^{N} \psi_{k-1}(x)\psi_{k-1}(y)\psi_{\ell-1}(y)\psi_{\ell-1}(z)\,\mathrm{d}y \\ &= \sum_{k,\ell=1}^{N} \psi_{k-1}(x)\left[\int \psi_{k-1}(y)\psi_{\ell-1}(y)\,\mathrm{d}y\right]\psi_{\ell-1}(z) \\ &= \sum_{k,\ell=1}^{N} \psi_{k-1}(x)\delta_{k\ell}\psi_{\ell-1}(z) \\ &= \sum_{k=1}^{N} \psi_{k-1}(x)\psi_{\ell-1}(z) \\ &= K_N(x,z).\end{aligned}$$

2. (Christoffel–Darboux formula)

The given equation is meaningless when $N=0$. For $N \geq 1$, we have

$$\begin{aligned}&\sqrt{N}\left[\frac{\psi_N(x)\psi_{N-1}(y)-\psi_N(y)\psi_{N-1}(x)}{x-y}\right] \\ &= \frac{e^{-(x^2+y^2)/4}}{(N-1)!\sqrt{2\pi}(x-y)}\big[H_N(x)H_{N-1}(y)-H_N(y)H_{N-1}(x)\big]. \qquad (\star)\end{aligned}$$

It is easy to see that the equation holds when $N=1$:

$$\begin{aligned}\frac{\psi_1(x)\psi_0(y)-\psi_1(y)\psi_0(x)}{x-y} &= \frac{e^{-(x^2+y^2)/4}}{\sqrt{2\pi}(x-y)}\big[H_1(x)H_(y)-H_1(y)H_1(x)\big] \\ &= \frac{e^{-(x^2+y^2)/4}}{\sqrt{2\pi}(x-y)}(x-y) \\ &= \frac{e^{-(x^2+y^2)/4}}{0!\sqrt{2\pi}} \\ &= \psi_0(x)\psi_0(y) = K_1(x,y).\end{aligned}$$

Moreover, the Hermite polynomials satisfy the following recurrence formula ($n \geq 1$):

$$\begin{aligned} H_{n+1}(x) &= (-1)^{n+1} e^{x^2/2} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} e^{-x^2/2} \\ &= (-1)^{n+1} e^{x^2/2} \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} (-x e^{-x^2/2}) \\ &= (-1)^{n+1} e^{x^2/2} \left[-x \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} e^{-x^2/2} - n \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} e^{-x^2/2} \right] \\ &= x H_n(x) - n H_{n-1}(x) \end{aligned}$$

due to the generalized Leibniz rule $(fg)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} f^{(r)} g^{(n-r)}$. Using this, whenever $N \geq 2$,

$$\begin{aligned} &H_N(x)H_{N-1}(y) - H_N(y)H_{N-1}(x) \\ &= (x-y)H_{N-1}(x)H_{N-1}(y) \\ &\qquad + (N-1)\big[H_{N-1}(x)H_{N-2}(y) - H_{N-1}(y)H_{N-2}(x)\big] \qquad (\dagger) \end{aligned}$$

so that

$$\begin{aligned} &\sqrt{N}\left[\frac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{x-y}\right] \\ &\overset{(\star)}{=} \frac{e^{-(x^2+y^2)/4}}{(N-1)!\sqrt{2\pi}(x-y)} \big[H_N(x)H_{N-1}(y) - H_N(y)H_{N-1}(x)\big] \\ &\overset{(\dagger)}{=} \frac{e^{-(x^2+y^2)/4}}{(N-1)!\sqrt{2\pi}} H_{N-1}(x)H_{N-1}(y) \\ &\quad + \frac{e^{-(x^2+y^2)/4}}{(N-2)!\sqrt{2\pi}(x-y)} \big[H_{N-1}(x)H_{N-2}(y) - H_{N-1}(y)H_{N-2}(x)\big] \\ &\overset{\text{(IH)}}{=} \psi_{N-1}(x)\psi_{N-1}(y) + K_{N-1}(x,y) = K_N(x,y) \end{aligned}$$

because of the induction hypothesis.

3. (Asymptotic behaviour)

Before going into the main proof, let me summarize some properties of the Hermite polynomials. First, the Hermite polynomial H_n is a polynomial of degree n, which can be proved by the recurrence formula above. Second, H_n has odd-degree terms only when n is odd, and has even-degree terms only when n is even, which we can also easily derive from the recurrence formula. Third, $H_n'(x) = nH_{n-1}(x)$, since

$$\begin{aligned} H_n'(x) &= (-1)^n \left[e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2} \right]' \\ &= (-1)^n \left[xe^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2} + e^{x^2/2} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} e^{-x^2/2} \right] \\ &= xH_n(x) - H_{n+1}(x) = nH_{n-1}(x) \end{aligned}$$

by the recurrence formula, again. Finally, $y = y_n(x) \coloneqq e^{-x^2/4} H_n(x)$ satisfies the following differential equation:

$$y'' + \left(n + \frac{1}{2}\right) y = \frac{x^2}{4} y,$$

because

$$\begin{aligned} y'' &= \left[e^{-x^2/4} H_n(x) \right]'' \\ &= (e^{-x^2/4})'' H_n(x) + 2(e^{-x^2/4})' H_n'(x) + e^{-x^2/4} H_n''(x) \\ &= \left(\frac{x^2}{4} - \frac{1}{2} \right) e^{-x^2/4} H_n(x) - xe^{-x^2/4} nH_{n-1}(x) + e^{-x^2/4} n(n-1) H_{n-2}(x) \\ &= \left(\frac{x^2}{4} - \frac{1}{2} \right) e^{-x^2/4} H_n(x) - ne^{-x^2/4} H_n(x) \\ &= \frac{x^2}{4} y - \left(n + \frac{1}{2}\right) y. \end{aligned}$$

An arbitrary solution of this differential equation can be represented by $y = y_h + y_p$, where the homogeneous solution is $y_h = c_1 y_1 + c_2 y_2$ with

$$y_1 = \cos\left(\sqrt{n + \frac{1}{2}}\, x \right), \quad \text{and} \quad y_2 = \sin\left(\sqrt{n + \frac{1}{2}}\, x \right),$$

and the particular solution y_p is given by

$$\begin{aligned} y_p &= \int_0^x \frac{y_1(x) y_2(t) - y_2(x) y_1(t)}{y_1'(t) y_2(t) - y_2'(t) y_1(t)} \cdot \frac{t^2}{4} y(t) \,\mathrm{d}t \\ &= \int_0^x \left[\cos\left(\sqrt{n + \frac{1}{2}}\, t \right) \sin\left(\sqrt{n + \frac{1}{2}}\, x \right) \right. \\ &\qquad \left. - \sin\left(\sqrt{n + \frac{1}{2}}\, t \right) \cos\left(\sqrt{n + \frac{1}{2}}\, x \right) \right] \frac{t^2}{4} e^{-t^2/4} H_n(t) \,\mathrm{d}t \end{aligned}$$

Fix a particular n. As $x \to 0$, $y \to y(0) = H_n(0)$, and $y_p = O(x^4)$ so that

$$y = c_1 y_1 + c_2 y_2 + y_p = c_1(1 + O(x^2)) + c_2 \left(\sqrt{n + \frac{1}{2}}\, x + O(x^3) \right) + O(x^4).$$

For an odd integer n, $y \to c_1 = H_n(0) = 0$. Also, $y'(0)$, the coefficient of the term of degree 1, is $y'(0) = 0 + H_n'(0) = nH_{n-1}(0) = c_2(n + \frac{1}{2})^{1/2}$, therefore $c_2 = nH_{n-1}(0)(n + \frac{1}{2})^{-1/2}$. Analogously for an even integer n, we have $y \to c_1 = H_n(0)$ as $x \to 0$ and $c_2 = nH_{n-1}(0)(n + \frac{1}{2})^{-1/2} = 0$. To summarize, we obtained

$$y = e^{-x^2/4} H_n(x) = \begin{cases} \frac{nH_{n-1}(0)}{\sqrt{n+\frac{1}{2}}} \sin\left(\sqrt{n + \frac{1}{2}}\, x \right) + y_p, & n \text{ is odd} \\ H_n(0) \cos\left(\sqrt{n + \frac{1}{2}}\, x \right) + y_p, & n \text{ is even} \end{cases}.$$

Actually, we can calculate the value of $H_n(0)$, using the recurrence formula: $H_n(0) = -(n-1)H_{n-2}(0)$, with the value of the base cases $H_0(0) = 1$ and $H_1(0) = 0$. This gives the following result:

$$H_n(0) = \begin{cases} 0, & n \text{ is odd} \\ (-1)^{n/2}\,(n-1)!! = \frac{(-1)^{n/2}\, n!}{2^{n/2}\, \frac{n}{2}!}, & n \text{ is even} \end{cases}.$$

Note that $\psi_n(x) = (n!\sqrt{2\pi})^{-1/2} y$. Therefore, with $n = N - \ell$,

$$\begin{aligned} & N^{1/4}\psi_n(N^{-1/2}t) \\ & = N^{1/4}(n!\sqrt{2\pi})^{-1/2} y_n(N^{-1/2}t) \\ & = N^{1/4}(n!\sqrt{2\pi})^{-1/2} y_h(N^{-1/2}t) + N^{1/4}(n!\sqrt{2\pi})^{-1/2} y_p(N^{-1/2}t)). \end{aligned}$$

When n is even,

$$\begin{aligned} y_p(N^{-1/2}t) &= \int_0^{N^{-1/2}t} O(u^2) \cdot H_n(u)\, \mathrm{d}u \\ &= \int_0^{N^{-1/2}t} O(u^2) \cdot (H_n(0) + O(u))\, \mathrm{d}u \\ &= H_n(0) O((N^{-1/2}t)^3) + O((N^{-1/2}t)^4) \\ &= H_n(0) O_t(N^{-3/2}). \end{aligned}$$

(Here, the convergence is locally uniform on t, since the coefficients for N is polynomially growing: $O(t^3)$.) Thus,

$$
\begin{aligned}
&N^{1/4}\psi_n(N^{-1/2}t) \\
&= N^{1/4}(n!\sqrt{2\pi})^{-1/2} y_n(N^{-1/2}t) \\
&= N^{1/4}(n!\sqrt{2\pi})^{-1/2} H_n(0) \left[\cos\left(\sqrt{\frac{n+1/2}{N}}\, t\right) + O_t(N^{-3/2})\right] \\
&= \frac{(-1)^{n/2}}{\sqrt{\pi}} \cos t + o_t(1) \\
&= \frac{1}{\sqrt{\pi}} \cos\left(t - \frac{n\pi}{2}\right) + o_t(1)
\end{aligned}
$$

as

$$
\begin{aligned}
N^{1/4}(n!\sqrt{2\pi})^{-1/2} H_n(0) &= (-1)^{n/2} N^{1/4}(n!\sqrt{2\pi})^{-1/2} \frac{n!}{2^{n/2}(n/2)!} \\
&= (-1)^{n/2} N^{1/4}(2\pi)^{-1/4} \frac{(n!)^{1/2}}{2^{n/2}(n/2)!} \\
&\sim (-1)^{n/2} N^{1/4}(2\pi)^{-1/4} \frac{(2\pi n)^{1/4}(n/e)^{n/2}}{2^{n/2}\sqrt{\pi n}(n/(2e))^{n/2}} \\
&= \frac{(-1)^{n/2}}{\sqrt{\pi}}
\end{aligned}
$$

(Here, $\sim$ means the ratio tends to 1 as n grows.) And note that the convergence of $o_t(1)$ term is also locally uniform since $\cos(t\,(n+1/2)^{-1/2}N^{-1/2}) \to \cos t$ locally uniformly.

In a similar fashion, when n is odd, $y_p(N^{-1/2}t) = O((N^{-1/2}t)^4) = O_t(N^{-2})$ with local uniform convergence since $H_n(0) = 0$. Therefore, we have

$$
\begin{aligned}
&N^{1/4}\psi_n(N^{-1/2}t) \\
&= N^{1/4}(n!\sqrt{2\pi})^{-1/2} y_n(N^{-1/2}t) \\
&= N^{1/4}(n!\sqrt{2\pi})^{-1/2} \frac{nH_{n-1}(0)}{\sqrt{n+\frac{1}{2}}} \sin\left(\sqrt{\frac{n+1/2}{N}}\, t\right) + O_t(N^{-2}) \\
&= \frac{(-1)^{(n-1)/2}}{\sqrt{\pi}} \sin t + o_t(1) \\
&= \frac{1}{\sqrt{\pi}} \cos\left(t - \frac{n\pi}{2}\right) + o_t(1)
\end{aligned}
$$

as

$$
\begin{aligned}
&N^{1/4}(n!\sqrt{2\pi})^{-1/2}\,\frac{nH_{n-1}(0)}{\sqrt{n+\frac{1}{2}}} \\
&= (-1)^{(n-1)/2}N^{1/4}(n!\sqrt{2\pi})^{-1/2}n\left(n+\frac{1}{2}\right)^{-1/2}\frac{(n-1)!}{2^{(n-1)/2}((n-1)/2)!} \\
&= (-1)^{(n-1)/2}N^{1/4}(\sqrt{2\pi})^{-1/2}n^{1/2}\left(n+\frac{1}{2}\right)^{-1/2}\frac{((n-1)!)^{1/2}}{2^{(n-1)/2}((n-1)/2)!} \\
&= (-1)^{(n-1)/2}N^{1/4}(\sqrt{2\pi})^{-1/2}n^{1/2}\left(n+\frac{1}{2}\right)^{-1/2}\frac{(2\pi(n-1))^{1/4}(\frac{n-1}{e})^{(n-1)/2}}{2^{(n-1)/2}\sqrt{\pi(n-1)}(\frac{n-1}{2e})^{(n-1)/2}} \\
&= \frac{(-1)^{(n-1)/2}}{\sqrt{\pi}}.
\end{aligned}
$$

This completes the proof of the asymptotic of ψ_n.