20170058 **Keonwoo Kim** · March 30, 2020
MAS365 Introduction to Numerical Analysis

Homework 1

1. (1) Let us define fl be the three-digit rounding function. Then the approximate arithmetic corresponding to $f(0.1)$ becomes $(e^{0.1} \ominus e^{-0.1}) \otimes 10$, where $a \ominus b = fl(fl(a) - fl(b))$ and $a \otimes b = fl(fl(a)fl(b))$. Thus,

$$\begin{aligned} f(0.1) &\approx fl\Big(fl\big(fl(e^{0.1}) - fl(e^{-0.1})\big) \cdot fl(10)\Big) \\ &= fl(fl(0.111 \times 10^1 - 0.905 \times 10^0) \cdot fl(10)) \\ &= fl(0.205 \cdot 10) = 2.05. \end{aligned}$$

(2) Since the third Taylor polynomial of e^x and e^{-x} at 0 are $1+x+\frac{x^2}{2}+\frac{x^3}{6}$ and $1-x+\frac{x^2}{2}-\frac{x^3}{6}$, respectively. Hence

$$f(0.1) \approx \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right) - \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}\right)}{x} = 2 + \frac{x^2}{3}$$

so that $f(0.1) \approx 2 \oplus \left((0.1 \otimes 0.1) \otimes \frac{1}{3}\right) = 2 \oplus (0.333 \times 10^{-2}) = 2.03$.

2. Let $y = \alpha \cdot 10^n$ where $n \in \mathbb{Z}$ and $0.1 \le |\alpha| < 1$, then

$$\begin{aligned} \left|\frac{y - fl(y)}{y}\right| &= \left|\frac{\alpha \cdot 10^n - fl(\alpha \cdot 10^n)}{\alpha \cdot 10^n}\right| \\ &= \left|\frac{\alpha \cdot 10^n - fl(\alpha) \cdot 10^n}{\alpha \cdot 10^n}\right| = \left|\frac{\alpha - fl(\alpha)}{\alpha}\right| \end{aligned}$$

so that we may assume $0.1 \le |y| < 1$. Since fl is the k-digit rounding, we have $|y - fl(y)| \le 5 \times 10^{-(k+1)}$. As $1 < 1/|y| \le 10$, we finally obtain $\left|\frac{y-fl(y)}{y}\right| \le 10 \cdot 5 \times 10^{-(k+1)} = 5 \times 10^{-k}$.

3. (1) Note $\ln(n+1) - \ln n = \ln\frac{n+1}{n} \to \ln 1 = 0$. Since

$$\big|(\ln(n+1) - \ln n) - 0\big| = \ln\left(1 + \frac{1}{n}\right) \le \frac{1}{n}.$$

Therefore, the rate of convergence of $\ln(n+1) - \ln n$ is $O(1/n)$, i.e., $p = 1$. (Smaller p is impossible since $\ln(1 + 1/n) \sim 1/n$ as $n \to \infty$.)

(2) Note $\lim\limits_{h\to 0} \frac{1-\cos h}{h} \underset{\text{L'H}}{=\!=} \lim\limits_{h\to 0} \sin h = 0$. Since

$$\left|\frac{1-\cos h}{h} - 0\right| = \frac{2}{h}\sin^2\left(\frac{h}{2}\right) \le \frac{h}{2}.$$

Therefore, the rate of convergence of $\frac{1-\cos h}{h}$ is $O(h)$, i.e., $p = 1$. (Smaller p is impossible since $\sin(h/2) \sim h/2$ as $h \to 0$.)

4. (1) $\sqrt[3]{31}$ is the unique real root of $f(x) := x^3 - 31 = 0$. Note that $f(2) = -23 < 0 < f(6) = 185$, so we can use the bisection method on $[2, 6]$.

(2) When we starts, the maximum error of the approximation is 2, and this becomes half when we proceed one step. So, after n iterations, the maximum error of the approximation is 2^{1-n}, which should be $\le 10^{-2}$; it is equivalent to $n \ge 8$. Thus, we need at least 8 iterations to guarantee the 10^{-2} accuracy.

5. (1) Since $f(x) := g(x) - x$ is strictly decreasing on $[\frac{1}{3}, 1]$, it has at most one fixed point. (Or by Theorem 2.3(ii), or by #6 below.) Since $f(\frac{1}{3}) = e^{-1/3} - \frac{1}{3} \ge (1 - \frac{1}{3}) - \frac{1}{3} = \frac{1}{3} > 0$ and $f(1) = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$, there is a real number $\xi \in [\frac{1}{3}, 1]$ satisfying $f(\xi) = 0$ by the intermediate value theorem.

(2) Since $g(x) = e^{-x}$ satisfies $|g'(x)| \le e^{-1/3}$, we have following two bounds ($p_0 = \frac{2}{3}$):

$$|p_n - p| \le \frac{1}{3}e^{-n/3} \quad \text{and} \quad |p_n - p| \le \frac{e^{-n/3}}{1 - e^{-1/3}}\left|e^{-2/3} - \frac{2}{3}\right|$$

Note that the first inequality is tighter than the second one. Using the first one,

$$\begin{aligned} &\frac{1}{3}e^{-n/3} \le 10^{-4} \\ \iff &-\frac{n}{3} - \log 3 \le -4\log 10 \\ \iff &n \ge 3\left(4\log 10 - \log 3\right) = 24.335\cdots \end{aligned}$$

Hence, we need at least 25 iterations.

6. With the same setting with Theorem 2.3(ii) except for the bound of g', we have

$$\frac{g(p) - g(q)}{p - q} = \frac{p - q}{p - q} = 1 = g'(\xi) < 1,$$

which is a contradiction.