20170058 **Keonwoo Kim** · June 13, 2020
MAS365 Introduction to Numerical Analysis

Homework 8

1. (a)

(i) $\|A\|_F \geq 0$ since it is the 1/2-th power of a sum of squares, a nonnegative number.

(ii) $\|A\|_F = 0$ iff $\sum_{1\leq i,j\leq n} |a_{ij}|^2 = 0$, which is equivalent to $a_{ij} = 0$ for every i and j, i.e., A is the zero matrix.

(iii) $\|\alpha A\|_F = \left(\sum_{1\leq i,j\leq n} |\alpha a_{ij}|^2\right)^{1/2} = \left(|\alpha|^2 \sum_{1\leq i,j\leq n} |a_{ij}|^2\right)^{1/2} = |\alpha|\|A\|_F$.

(iv) Let A and B be two $n \times n$ matrices. Transforming each matrix A (and B) into an n^2-dimensional vector $\mathbf{a}$ (and $\mathbf{b}$), $\|A\|_F = \|\mathbf{a}\|_2$ (the ℓ^2-norm) and similar results hold for B and $A+B$. Consequently,

$$\|A+B\|_F = \|\mathbf{a}+\mathbf{b}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2 = \|A\|_F + \|B\|_F$$

by the triangle inequality.

For a direct proof,

$$\begin{aligned}
\|A+B\|_F^2 &= \sum_{1\leq i,j\leq n} |a_{ij}+b_{ij}|^2 \\
&= \sum_{1\leq i,j\leq n} |a_{ij}|^2 + \sum_{1\leq i,j\leq n} |b_{ij}|^2 + 2\sum_{1\leq i,j\leq n} |a_{ij}b_{ij}| \\
&= \sum_{1\leq i,j\leq n} |a_{ij}|^2 + \sum_{1\leq i,j\leq n} |b_{ij}|^2 + 2\sum_{1\leq i,j,i',j'\leq n} |a_{ij}b_{i'j'}|\delta_{ii'}\delta_{jj'} \\
&\leq \sum_{1\leq i,j\leq n} |a_{ij}|^2 + \sum_{1\leq i,j\leq n} |b_{ij}|^2 + 2\sum_{1\leq i,j,i',j'\leq n} |a_{ij}b_{i'j'}| \\
&\leq \sum_{1\leq i,j\leq n} |a_{ij}|^2 + \sum_{1\leq i,j\leq n} |b_{ij}|^2 + 2\left(\sum_{1\leq i,j\leq n} |a_{ij}|^2\right)^{1/2}\left(\sum_{1\leq i,j\leq n} |b_{ij}|^2\right)^{1/2} \\
&\qquad \text{(Cauchy–Schwarz)} \\
&= (\|A\|_F + \|B\|_F)^2.
\end{aligned}$$

(v)

$$\begin{aligned}
\|AB\|_F^2 &= \sum_{1\le i,j\le n} \left| \sum_{1\le k\le n} a_{ik}b_{kj} \right|^2 \\
&\le \sum_{1\le i,j\le n} \left(\sum_{1\le k\le n} |a_{ik}|^2 \right) \left(\sum_{1\le k'\le n} |b_{k'j}|^2 \right) \\
&\qquad \text{(Cauchy–Schwarz)} \\
&= \sum_{1\le i,j,k,k'\le n} |a_{ik}|^2 |b_{k'j}|^2 \\
&= \left(\sum_{1\le i,k\le n} |a_{ik}|^2 \right) \left(\sum_{1\le k',j\le n} |b_{k'j}|^2 \right) \\
&= \|A\|_F^2 \|B\|_F^2.
\end{aligned}$$

(b) When $\|\mathbf{x}\|_2 = 1$,

$$\begin{aligned}
\|A\mathbf{x}\|_2^2 &= \sum_{i=1}^n \left(\sum_{j=1}^n A_{ij}x_j \right)^2 \\
&\le \sum_{i=1}^n \left(\sum_{j=1}^n |A_{ij}|^2 \right) \left(\sum_{j=1}^n |x_j|^2 \right) && \text{(Cauchy–Schwarz)} \\
&\le \sum_{i=1}^n \left(\sum_{j=1}^n |A_{ij}|^2 \right) && (\|\mathbf{x}\|_2 = 1) \\
&= \|A\|_F^2.
\end{aligned}$$

Therefore, $\|A\|_2 = \max_{\|\mathbf{x}\|_2=1} \|A\mathbf{x}\|_2 \le \|A\|_F$.

2. (a) $\|A\|_2 = \rho(A^tA)^{1/2} = \rho(A^2)^{1/2}$ since $A = A^t$ by Theorem 5(1) in the lecture note. Here, when λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, we have $A^2\mathbf{v} = \lambda^2\mathbf{v}$ so that A^2 has λ^2 as an eigenvalue. By counting them, we notice that every eigenvalue of A^2 is of such form, considering the multiplicities. (For example, if A has ± 3 as eigenvalues, then A^2 has 9 as an eigenvalue of multiplicity 2.) Therefore,

$$\rho(A^2) = \max_{\lambda:\ \text{eigenvalue of } A^2} |\lambda| = \max_{\lambda:\ \text{eigenvalue of } A} |\lambda^2| = \rho(A)^2.$$

Consequently, we have $\|A\|_2 = \rho(A)$.

(b) First, $\rho(A) \leq \|A\|$, Theorem 5(2) in the lecture note, implies the second inequality.

Now, observe that when λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, we have $A^{-1}\mathbf{v} = \lambda^{-1}A^{-1}\lambda\mathbf{v} = \lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}\mathbf{v}$ so that A^{-1} has λ^{-1} as an eigenvalue. Note that the first inequality is equivalent to that $|\lambda|^{-1} \leq \|A^{-1}\|$ for any eigenvalue λ of A by the observation above, which is again equivalent to that $\rho(A^{-1}) \leq \|A^{-1}\|$. Therefore, Theorem 5(2) in the lecture note again proves this.