**20170058** **Keonwoo Kim** · April 19, 2020
MAS365 Introduction to Numerical Analysis

# Homework 4

**1.** (a) Note that 1 mph is $5280/3600 = 22/15$ feet per seconds. So, we have (every quantity is in feets and seconds)

$$d(0) = 0, \quad d'(0) = 418/3, \quad d(0.4) = 55.5, \quad d'(0.4) = 2024/15.$$

Then, the Hermite polynomial of the distance function is as follows:

$$\begin{aligned}
H_{1,1}(t) &= \frac{25}{4}(3-5t)t^2, \\
\hat{H}_{1,0}(t) &= \frac{25}{4}\,t\left(t-\frac{2}{5}\right)^2 \\
\hat{H}_{1,1}(t) &= \frac{25}{4}\,t^2\left(t-\frac{2}{5}\right), \\
\therefore H_3(t) &= d(0.4)H_{1,1}(t) + d'(0)\hat{H}_{1,0}(t) + d'(0.4)\hat{H}_{1,1}(t) \\
&= \frac{111}{2}\cdot\frac{25}{4}(3-5t)t^2 + \frac{418}{3}\cdot\frac{25}{4}\,t\left(t-\frac{2}{5}\right)^2 + \frac{2024}{15}\cdot\frac{25}{4}\,t^2\left(t-\frac{2}{5}\right) \\
&= \frac{418}{3}\,t + \frac{53}{8}\,t^2 - \frac{485}{24}\,t^3.
\end{aligned}$$

(b) $H_3'(0.2) = \frac{16747}{120}$ feet/s $= \frac{16747}{176}$ mph $\approx 95.1534$ mph.

(c) The maximum of the approximated velocity $H_3'$ is achieved when $t = 53/485$ (seconds) with the value $H_3'(53/485) = \frac{1630267}{11640}$ feet/s $= \frac{1630267}{17072}$ mph $\approx 95.4936$ mph.

**2.** (a) We have the following four equations:

$$\begin{aligned}
S_0(2) &= 1 + B - D = 1, \\
S_1(3) &= 1 + b - \frac{3}{4} + d = 0, \\
S_0'(2) &= B - 3D = S_1'(2) = b, \\
S_1''(3) &= -\frac{3}{2} + 6d = 0.
\end{aligned}$$

$\therefore d = 1/4,\ b = -1/2,\ B = D = 1/4.$

(b) $s_0(1) = 1 + B = s_1(1) = 1$, thus $B = 0$. $s_0'(1) = -2 = s_1'(1) = b$ implies $b = -2$. So $f'(0) = s_0'(0) = 0$ and $f'(2) = s_1'(2) = 11$.

**3.** (a) Let $M := \max_{a \le x \le b} f'(x)$. For some $\xi_i \in (x_i, x_{i+1})$ satisfying $f'(\xi_i) = \big(f(x_{i+1}) - f(x_i)\big)/(x_{i+1} - x_i)$,

$$\begin{aligned}
&\max_{a \le x \le b} |f(x) - F(x)| \\
&= \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} |f(x) - F(x)| \\
&= \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} \left| \int_{x_i}^{x} \big(f'(t) - F'(t)\big)\, dt \right| \\
&= \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} \left| \int_{x_i}^{x} \left(f'(t) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right) dt \right| \\
&= \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} \left| \int_{x_i}^{x} \big(f'(t) - f'(\xi_i)\big)\ dt \right| \\
&\le \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} \int_{x_i}^{x} \big(|f'(t)| + |f'(\xi_i)|\big)\, dt \\
&\le \max_{i=0,\cdots,n-1} \max_{x_i \le x \le x_{i+1}} 2M(x - x_i) \\
&\le 2M \max_{i=0,\cdots,n-1} (x_{i+1} - x_i).
\end{aligned}$$

(b)

$$F(x) = \begin{cases} e^0 + \dfrac{e^{0.1} - e^0}{0.05 - 0} x = 1 + 20(e^{0.1} - 1)x, & \text{if } 0 \le x \le 0.05, \\ e^{0.1} + \dfrac{e^{0.2} - e^{0.1}}{0.10 - 0.05}(x - 0.05) & \\ \quad = (2e^{0.1} - e^{0.2}) + 20(e^{0.2} - e^{0.1})x, & \text{if } 0.05 \le x \le 0.1 \end{cases}.$$

Then $\int_0^{0.1} F(x)\, dx = \frac{1}{2}(e^0 + e^{0.1} + e^{0.1} + e^{0.2}) \cdot 0.05 = (1 + 2e^{0.1} + e^{0.2})/40 \approx 0.1107936$. Note the actual value is $\int_0^{0.1} e^{2x}\, dx = \frac{1}{2}(e^{0.2} - 1) \approx 0.1107014$. The approximated integral is slightly larger, because the given function $f$ is convex. The absolute error is about $9.22 \times 10^{-5}$.

(c) $M = \max_{0 \le x \le 0.1} f'(x) = 2e^{0.2} \approx 2.4428055$.

$$\max_{0 \le x \le 0.1} |f(x) - F(x)| \le 2M \cdot \max\{0.05, 0.05\} = \frac{1}{5} e^{0.2}.$$

$$
\begin{aligned}
\left|\int_0^{0.1} f(x)\,dx - \int_0^{0.1} F(x)\,dx\right| &= \int_0^{0.1} |f(x) - F(x)|\,dx \\
&\le 0.1 \max_{0\le x\le 0.1} |f(x) - F(x)| \\
&\le \frac{e^{0.2}}{50} \approx 0.0244281.
\end{aligned}
$$

***4.*** (a)

$$
s(x) = \begin{cases} 1 + ax + bx^2 + cx^3, & \text{if } 0 \le x \le .05 \\ e^{0.1} + A(x - .05) + B(x - .05)^2 + C(x - .05)^3, & \text{if } .05 \le x \le .1 \end{cases}.
$$

We have the following conditions:

- $s(0.05) = 1 + 0.05a + (0.05)^2 b + (0.05)^3 c = e^{0.1}$,
- $s(0.1) = e^{0.1} + 0.05A + (0.05)^2 B + (0.05)^3 C = e^{0.2}$,
- $s'(0.05) = a + 2 \cdot 0.05\, b + 3(0.05)^2 c = A$,
- $s''(0.05) = 2b + 6 \cdot 0.05\, c = 2B$,
- $s'(0) = a = f'(0) = 2$,
- $s'(0.1) = A + 2B \cdot 0.05 + 3C \cdot 0.05^2 = f'(0.1) = 2e^{0.2}$.

These yield the following matrix representation:

$$
\begin{pmatrix}
0.05 & 0.05^2 & 0.05^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0.05^2 & 0.05^3 \\
1 & 2 \cdot 0.05 & 3 \cdot 0.05^2 & -1 & 0 & 0 \\
0 & 2 & 6 \cdot 0.05 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 \cdot 0.05 & 3 \cdot 0.05^2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ A \\ B \\ C \end{pmatrix}
=
\begin{pmatrix} e^{0.1} - 1 \\ e^{0.2} - e^{0.1} \\ 0 \\ 0 \\ 2 \\ 2e^{0.2} \end{pmatrix}
$$

so that

$$
\begin{pmatrix} a \\ b \\ c \\ A \\ B \\ C \end{pmatrix}
\approx
\begin{pmatrix} 2.0000000 \\ 1.9983018 \\ 1.4013081 \\ 2.2103399 \\ 2.2084980 \\ 1.5487625 \end{pmatrix}.
$$

Therefore,

$$\begin{aligned}\int_0^{0.1} s(x)\,dx &= \int_0^{0.05} \big((1+ax+bx^2+cx^3)+(e^{0.1}+Ax+Bx^2+Cx^3)\big)\,dx \\ &\approx 0.110701363708553.\end{aligned}$$

The true value $\int_0^{0.1} f(x)\,dx \approx 0.110701379080085$ is very close to the approximated integral. The absolute error is about $1.5371532 \times 10^{-8}$.

(b) Since $M = \max_{0\le x\le 0.1} f^{(4)}(x) = 16e^{0.2}$ so that

$$\begin{aligned}\max_{0\le x\le 0.1} |f(x)-s(x)| &\le \frac{5\cdot 16e^{0.2}}{384}\cdot \max\{0.05, 0.05\}^4 \\ &= \frac{5e^{0.2}\cdot 0.05^4}{24} \approx 1.590368\times 10^{-6}, \\ \left|\int_0^{0.1} f(x)\,dx - \int_0^{0.1} s(x)\,dx\right| &= \int_0^{0.1} |f(x)-s(x)|\,dx \\ &\le 0.1 \max_{0\le x\le 0.1} |f(x)-s(x)| \\ &\le \frac{e^{0.2}\cdot 0.05^4}{48} \approx 1.590368\times 10^{-7}.\end{aligned}$$