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MAS365 Introduction to Numerical Analysis

Homework 6

1. (a) $f(t, y) = t^{-2}(\sin(2t) - 2ty)$ is continuous on $D = [1, 2] \times \mathbb{R}$, and in fact, it is linear (of degree 1 as a polynomial) in y so that it is obviously Lipschitz in y on D. Therefore, the given IVP has a unique solution $y(t)$ for $1 \le t \le 2$, by Theorem 2 in the lecture slides.

(b) $f(t, y) = \cos(yt)$ is continuous on $D = [0, 1] \times \mathbb{R}$ and Lipschitz in y because its (partial) derivative w.r.t. y is $-t\sin(yt)$ is bounded so that by the mean value theorem, for any $t \in [0, 1]$ and $y_1 \ne y_2$,

$$\frac{|f(t, y_1) - f(t, y_2)|}{|y_1 - y_2|} = |\partial_y f(t, \xi)| \le |t| \le 1$$

for some ξ. Hence, by Theorem 3 in the lecture slides, the given IVP is well-posed.

2. (a) $w_0 = y(0) = 1$,

$$w_{i+1} = w_i + hf(t_i, w_i) = w_i + \frac{1}{2}\exp\left[\frac{i}{2} - w_i\right].$$

t_i	w_i
0.0	1.0000000
0.5	1.3678794
1.0	1.7877203

(b) Since $f(t, y) = e^{t-y}$ is not Lipschitz in y on $D = [0, 1] \times \mathbb{R}$, we cannot use Theorem 4 directly. However, we know that $y_i = y(t_i) \ge 1$ since $y(t_i) = y(0) + \int_0^t y_i e^{t-y(t)}\, dt \ge y(0)$. Similarly, $w_i \ge 1$. Therefore,

$$\left|\frac{e^{t_i}(e^{-y_i} - e^{-w_i})}{y_i - w_i}\right| = e^{t-\xi_i}, \qquad \xi_i \text{ is between } y_i \text{ and } w_i$$

implying

$$\begin{aligned}
|y_{i+1} - w_{i+1}| &\le |y_i - w_i|(1 + h\, e^{t_i - \xi_i}) + \frac{h^2}{2}|y''(\eta_i)| \\
&\le |y_i - w_i|(1 + he^{t_i - 1}) + \frac{h^2}{2} \sup_{t\in[0,1]} |y''(t)| \\
&\le |y_i - w_i|(1 + h) + \frac{h^2}{2} \sup_{t\in[0,1]} |y''(t)| \qquad (t_i \in [0,1])
\end{aligned}$$

for some ξ_i between y_i and w_i, so that $\xi_i \ge 1$, and $\eta_i \in [0,1]$. Since

$$y''(t) = \frac{(e-1)e^t}{(e^t + e - 1)^2}$$

attains its maximum at $t = \log(e-1)$ with the value $1/4$, we have the following recursive error bound:

$$|y_{i+1} - w_{i+1}| \le |y_i - w_i|\,(1+h) + \frac{h^2}{8}$$

yielding

$$|y(t_i) - w_i| \le \frac{h}{8}\,(e^{ih} - 1) = \frac{h}{8}\,(e^{t_i} - 1)$$

by Lemma 1. $(t_i = ih)$

3. $f(t, y) = 1 + t\sin(ty)$,

$$\begin{aligned}
f'(t, y(t)) &= \sin(ty) + t\cos(ty)(y + ty'(t)) \\
&= \sin(ty) + t\cos(ty)(y + t + t^2\sin(ty))
\end{aligned}$$

$w_0 = y(0) = 0,$

$$\begin{aligned}
w_{i+1} &= w_i + h\, f(t_i, w_i) + \frac{h^2}{2}\, f'(t_i, w_i) \\
&= w_i + h\,(1 + t_i\sin(t_i w_i)) + \frac{h^2}{2}\big(\sin(t_i w_i) + t_i\cos(t_i w_i)(w_i + t_i + t_i^2\sin(t_i w_i))\big)
\end{aligned}$$

t_i	w_i
0.0	0.000000000000000
0.2	0.200000000000000
0.4	0.404004473397373
0.6	0.626645905992322
0.8	0.893219651185996
1.0	1.236705986258755
1.2	1.665406315361360
1.4	2.060419988396295
1.6	2.229474504336433
1.8	2.208300083165663
2.0	2.088054524845546

4. (a) Since $y(t_{i+1}) = y_i + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$, for an approximation w_i of $y(t_i)$, we have

$$w_{i+1} = w_i + \int_{t_i}^{t_{i+1}} P(t)\,dt$$

where P is the interpolating polynomial determined by $(t_{i-1}, f(t_{i-1}, w_{i-1}))$ and $(t_i, f(t_i, w_i))$: a polynomial approximation of $y'(t) = f(t, y(t))$. As

$$P(t) = \frac{t - t_i}{t_{i-1} - t_i} f(t_{i-1}, w_{i-1}) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f(t_i, w_i),$$

we have

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} P(t)\,dt &= \frac{f(t_{i-1}, w_{i-1})}{t_{i-1} - t_i} \int_{t_i}^{t_{i+1}} (t - t_i)\,dt + \frac{f(t_i, w_i)}{t_i - t_{i-1}} \int_{t_i}^{t_{i+1}} (t - t_{i-1})\,dt \\
&= \frac{f(t_{i-1}, w_{i-1})}{-h} \frac{h^2}{2} + \frac{f(t_i, w_i)}{h} \frac{3h^2}{2} \\
&= \frac{3h}{2} f(t_i, w_i) - \frac{h}{2} f(t_{i-1}, w_{i-1}).
\end{aligned}$$

Thus,

$$w_{i+1} = w_i + \frac{3h}{2} f(t_i, w_i) - \frac{h}{2} f(t_{i-1}, w_{i-1}),$$

which is exactly the Adams–Bashforth two–step method.

(b) Simpson's rule implies ($h = t_{i+1} - t_i$)

$$\int_{t_{i-1}}^{t_{i+1}} f(t, y(t))\,dt \approx \frac{h}{3}[f(t_{i-1}, y(t_{i-1})) + 4f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))],$$

which gives the approximation $w_i \approx y(t_i)$ where

$$w_{i+1} = w_{i-1} + \frac{h}{3}[f(t_{i-1}, w_{i-1}) + 4f(t_i, w_i) + f(t_{i+1}, w_{i+1})].$$

Its local truncation error at $(i+1)$-st step is

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_{i-1})}{h} - \frac{1}{3}\left[f(t_{i-1}, y(t_{i-1})) + 4f(t_i, y(t_i)) + f(t_{i+1}, y(t_{i+1}))\right]$$

for $i \geq 1$.