20170058 **Keonwoo Kim** · April 3, 2020
MAS109(E,F) Introduction to Linear Algebra

Quiz 2

Problems

MAS109EF Quiz 2 ID: 20170058 Name: Keonwoo Kim.

#1. $\begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[E(2,3)]{} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \xrightarrow[E(1,3;\,-1)]{} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U.$

○: pivot positions

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{L^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U.}$$

$\Rightarrow$ $PA = LU$ where $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$A\mathbf{x} = \mathbf{b} \iff LU\mathbf{x} = P\mathbf{b}$

$\iff L\mathbf{y} = P\mathbf{b}$ and $\mathbf{y} = U\mathbf{x}$.

Here, $P\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$ so that

$$L\mathbf{y} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \iff \mathbf{y} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \quad \left(\left[L \,\middle|\, \begin{matrix} -2 \\ 4 \\ 1 \end{matrix} \right] \xrightarrow{E(1,3,-1)} \left[I_3 \,\middle|\, \begin{matrix} -2 \\ 4 \\ 3 \end{matrix} \right] \right)$$

and $U\mathbf{x} = \mathbf{y}$ yields

$$U\mathbf{x} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \overset{\text{1-1 corr.}}{\iff} \left[\begin{array}{ccc|c} 3 & -1 & 0 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{E(3,2;-1)} \left[\begin{array}{ccc|c} 3 & -1 & 0 & -2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{E(2;\frac{1}{2})} \left[\begin{array}{ccc|c} 3 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{E(2,1;\,1)} \left[\begin{array}{ccc|c} 3 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{E(1;\,1/3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

$$\iff \mathbf{x} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}.$$

Note $\begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ by a direct calculation also.

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#2. (a) A^{-1} can be found by elem. row op's: $[A|I_n] \xrightarrow{\text{elem row op's}} [I_n|A^{-1}]$.

$$\left[\begin{array}{cccc|cccc} 1 & -1 & & & 1 & & & \\ & 1 & -2 & & & 1 & & \\ & & 1 & -1 & & & 1 & \\ & & & 1 & & & & 1 \end{array}\right] \xrightarrow{E(4,3;1)} \left[\begin{array}{cccc|cccc} 1 & -1 & & & 1 & & & \\ & 1 & -2 & & & 1 & & \\ & & 1 & 0 & & & 1 & 1 \\ & & & 1 & & & & 1 \end{array}\right]$$

Empty positions have the value 0
as the corresponding component.

$$\xrightarrow{E(3,2;2)} \left[\begin{array}{cccc|cccc} 1 & -1 & & & 1 & & & \\ & 1 & & & & 1 & 2 & 2 \\ & & 1 & & & & 1 & 1 \\ & & & 1 & & & & 1 \end{array}\right]$$

$$\xrightarrow{E(2,1;1)} \left[\begin{array}{cccc|cccc} 1 & & & & 1 & 1 & 2 & 2 \\ & 1 & & & & 1 & 2 & 2 \\ & & 1 & & & & 1 & 1 \\ & & & 1 & & & & 1 \end{array}\right]$$

Therefore, $\begin{bmatrix} 1 & -1 & & 0 \\ & 1 & -2 & \\ & & 1 & -1 \\ 0 & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ & 1 & 2 & 2 \\ & & 1 & 1 \\ 0 & & & 1 \end{bmatrix}$.
zeroes

Ans: $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $tr(B^TB) = \sum_{i=1}^{4} (B^TB)_{ii} = \sum_{i=1}^{4}\sum_{j=1}^{4} \cancel{B} (B^T)_{ij} \cdot B_{ji} = \sum_{i=1}^{4}\sum_{j=1}^{4} (B_{ji})^2$

is the sum of all components in the matrix B. Thus the answer is:

$$tr(B^TB) = 1^2 \cdot 6 + 2^2 \cdot 4 = 6 + 16 = \underline{22}$$

*the sum of all components $\rightarrow$ the square sum of all components

Summary

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* $\psi(A)$ = the set of positions of leading 1's in the RREF of A.
  - → $\psi(A)$ is well-defined.
  - → $|\{\psi(A) : A \text{ is an } m\times n \text{ matrix}\}| = \operatorname{rk} A$
  - → row echelon form is not unique, but RREF is unique.
* Elementary row op. = multiplying (row) elem. matrices.
* $[A \mid I_n] \xrightarrow{E_1} [E_1 A \mid E_1]$ (Gauss-Jordan elimination)
  $\xrightarrow{\cdots} \xrightarrow{E_k} [E_k E_{k-1} \cdots E_1 A = I_n \mid \underbrace{E_k \cdots E_1}_{= A^{-1}}]$
  $\Rightarrow A^{-1} = (E_k E_{k-1} \cdots E_1)$
* Thm. TFAE for an $n\times n$ matrix A:
  ① the RREF of A is I_n.
  ② A is a product of elem matrices.
  ③ A is invertible
  ④ $Ax = 0 \iff x = 0$
  ⑤ $Ax = b$ has a solution, $\forall b \in \mathbb{R}^n$.
  ⑥ $Ax = b$ has exactly one solution, $\forall b \in \mathbb{R}^n$.
* $A = LU$ — LU decomposition
  Not every square matrix has an LU decomposition, but $\exists P$ permutation matrix s.t. $PA = LU$ for some L (lower triangular matrix) and U (upper triangular matrix)
* During the elem. row op, exchanging two rows make the need of "P".

* LU decomp. reduces the # of operations required in solving $Ax=b$.
  → using LU decomp: $\underbrace{\frac{2}{3}n^3}_{LU} + \underbrace{n^2}_{Ly=b} + \underbrace{n^2}_{Ux=y} = \frac{2}{3}n^3 + O(n^2)$
  → using A^{-1}: $\underbrace{2n^3}_{A^{-1}} + \underbrace{n^2}_{A^{-1}b} = 2n^3 + O(n^2)$.

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* $v_i \in \mathbb{R}^n$ $(i=1,\dots,s)$, $c_i \in \mathbb{R}$.
  → $\sum c_i v_i$: linear combinat°
* $W = \{x \in \mathbb{R}^n : Ax = 0\} = \ker A$: solut° space.
* $\{v_1,\dots,v_s\}$ is linearly indep. if $\sum_{i=1}^{s} c_i v_i = 0 \Rightarrow \forall i,\ c_i = 0$.
* A is invertible iff columns of A are lin. indep.
* $\mathrm{span}\{v_1,\dots,v_s\} = \{\sum c_i v_i ;\ c_i \in \mathbb{R}\}$
* $S \subseteq V$ is a basis of V if S is lin. indep. and spans V.
* $\{e_1,\dots,e_n\}$: standard basis of $\mathbb{R}^n$
* Every basis has the same cardinality.
  → called "dimension" of V.
* There is a unique "coordinate" of a vector w.r.t. a basis.
* S spans V $\Longleftrightarrow$ $\exists B \subseteq S$ s.t. B is a basis of V.
* S is lin. indep in V $\Longleftrightarrow$ $\exists B \supseteq S$ s.t. — " —
* The space of $m \times n$ matrices can be viewed as an mn-dimensional vector space. $\cong \mathbb{R}^{mn}$