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MAS109(E,F) Introduction to Linear Algebra

# Midterm Homework

***1.*** (a)

$$\begin{bmatrix}2&2&4&4&6\\3&3&6&4&7\\0&2&2&2&4\\1&2&3&3&5\\2&2&4&4&6\end{bmatrix} \xrightarrow{E(2,3)} \begin{bmatrix}2&2&4&4&6\\0&2&2&2&4\\3&3&6&4&7\\1&2&3&3&5\\2&2&4&4&6\end{bmatrix} \xrightarrow{E(1;1/2)} \begin{bmatrix}1&1&2&2&3\\0&2&2&2&4\\3&3&6&4&7\\1&2&3&3&5\\2&2&4&4&6\end{bmatrix}$$

$$\begin{array}{c}\xrightarrow{E(1,3;-3)}\\\xrightarrow{E(1,4;-1)}\\\xrightarrow{E(1,5;-2)}\end{array} \begin{bmatrix}1&1&2&2&3\\0&2&2&2&4\\0&0&0&-2&-2\\0&1&1&1&2\\0&0&0&0&0\end{bmatrix} \xrightarrow{E(2;1/2)} \begin{bmatrix}1&1&2&2&3\\0&1&1&1&2\\0&0&0&-2&-2\\0&1&1&1&2\\0&0&0&0&0\end{bmatrix}$$

$$\xrightarrow{E(2,4;-1)} \begin{bmatrix}1&1&2&2&3\\0&1&1&1&2\\0&0&0&-2&-2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} \xrightarrow{E(3;-1/2)} \begin{bmatrix}1&1&2&2&3\\0&1&1&1&2\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} = U$$

$$\begin{array}{c}\xrightarrow{E(3,1;-2)}\\\xrightarrow{E(3,2;-1)}\end{array} \begin{bmatrix}1&1&2&0&1\\0&1&1&0&1\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix} \xrightarrow{E(2,1;-1)} \begin{bmatrix}1&0&1&0&0\\0&1&1&0&1\\0&0&0&1&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

(b) $PA = LU$ where

$$P = E(2,3) = \begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}, \qquad U = \begin{bmatrix}1&1&2&2&3\\&1&1&1&2\\&&0&1&1\\&&&0&0\\&&&&0\end{bmatrix},$$

$$L = [E(3;-1/2)E(2,4;-1)E(2;1/2)E(1,5;-2)E(1,4;-1)E(1,3;-3)E(1;1/2)]^{-1}$$

$$= \begin{bmatrix} 2 & & & & \\ 0 & 2 & & & \\ 3 & 0 & -2 & & \\ 1 & 1 & 0 & 1 & \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Empty spots are filled with zeroes.)

(c) Note that $B = \mathbf{b}_1\mathbf{v}_1^T + \mathbf{b}_2\mathbf{v}_2^T + \mathbf{b}_4\mathbf{v}_3^T$, where

$$\begin{aligned} \mathbf{b}_1 &= [1,0,0,0]^T, & \mathbf{b}_2 &= [0,1,0,0]^T, & \mathbf{b}_4 &= [0,0,1,0]^T \\ \mathbf{v}_1 &= [1,0,2,0,3]^T, & \mathbf{v}_2 &= [0,1,4,0,5]^T, & \mathbf{v}_3 &= [0,0,0,1,6]^T. \end{aligned}$$

Letting a composition of elementary row operations $E = E_n \cdots E_1$ be operated on $M$ to produce the RREF $B = EM$, we have $EM = E[\mathbf{c}_1, \ldots, \mathbf{c}_5] = B = [\mathbf{b}_1, \ldots, \mathbf{b}_5]$. Thus,

$$M = E^{-1}B = E^{-1}\mathbf{b}_1\mathbf{v}_1^T + E^{-1}\mathbf{b}_2\mathbf{v}_2^T + E^{-1}\mathbf{b}_4\mathbf{v}_3^T = \mathbf{c}_1\mathbf{v}_1^T + \mathbf{c}_2\mathbf{v}_2^T + \mathbf{c}_4\mathbf{v}_3^T.$$

Hence, $\mathbf{u}_1 = \mathbf{c}_1$, $\mathbf{u}_2 = \mathbf{c}_2$, $\mathbf{u}_3 = \mathbf{c}_4$, and $\mathbf{v}_i$'s as above make the given equality true.

**2.** Since $A$ is of rank $k$, the RREF $R = EA$ ($E$: a product of elementary matrices) of $A$ has exactly $k$ nonzero rows. Then, as in the problem 1(c),

$$R =: [\mathbf{v}_1, \cdots, \mathbf{v}_k, \mathbf{0} \cdots, \mathbf{0}]^T, \qquad R = \sum_{i=1}^{k} \mathbf{e}_i\mathbf{v}_i^T.$$

Here, $\mathbf{v}_i$'s are linearly independent since they are the rows of the RREF $R$. Therefore, $A = E^{-1}R = \sum_{i=1}^{k}(E^{-1}\mathbf{e}_i)\mathbf{v}_i^T$. Note that the one-hot vectors $\mathbf{e}_i$ ($i = 1, \ldots, k$) are linearly independent, and the multiplication of $E^{-1}$, an invertible matrix, does not affect to the linear independence. Therefore, $\mathbf{u}_i \coloneqq E^{-1}\mathbf{e}_i$ ($i = 1, \ldots, k$) form a linearly independent set. This completes the proof.

**3.** (a) Suppose they are linearly independent. Then there are $a, b, c, d \in \mathbb{R}$, where not all of them are zero, such that

$$axe^x \cos x + bxe^x \sin x + ce^x \cos x + de^x \sin x = 0$$

for any $x \in \mathbb{R}$. Equivalently,

$$ax\cos x + bx\sin x + c\cos x + d\sin x = 0$$

for any $x \in \mathbb{R}$. Evaluating it at $x = 0$, we get $c = 0$. Evaluating it at $x = 2\pi$, we have $a \cdot 2\pi = 0$, that is, $a = 0$. Then, $bx\sin x + d\sin x = 0$ for any $x \in \mathbb{R}$. Dividing the both side by $\sin x$ assuming $\sin x$ is nonzero, we have $bx + d = 0$, at least for two values of $x$. This implies $b = d = 0$. This contradicts to the initial assumption.

(b)

$$\begin{aligned}
\frac{d}{dx} xe^x\cos x &= xe^x\cos x - xe^x\sin x + e^x\cos x, \\
\frac{d}{dx} xe^x\sin x &= xe^x\sin x + xe^x\cos x + e^x\sin x, \\
\frac{d}{dx} e^x\cos x &= e^x\cos x - e^x\sin x, \\
\frac{d}{dx} e^x\sin x &= e^x\sin x + e^x\cos x.
\end{aligned}$$

Therefore,

$$[D]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

(c) The characteristic polynomial of $D$ is

$$p_{[D]_{\mathcal{B}}}(t) = t^4 - 4t^3 + 8t^2 - 8t + 4 = (t^2 - 2t + 2)^2.$$

So the eigenvalues are $1 \pm i$. By a simple calculation,

$$\begin{aligned}
[D]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = (1+i)[\mathbf{v}]_{\mathcal{B}} &\implies [\mathbf{v}]_{\mathcal{B}} = c(0,0,1,i)^T, \quad \text{and} \\
[D]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = (1-i)[\mathbf{v}]_{\mathcal{B}} &\implies [\mathbf{v}]_{\mathcal{B}} = c(0,0,i,1)^T
\end{aligned}$$

for an arbitrary $c \in \mathbb{C}$. Therefore, the eigenvectors corresponding to $\lambda = 1 + i$ is $ce^x(\cos x + i\sin x) = ce^{(1+i)x}$; and the eigenvectors corresponding to $\lambda = 1 - i$ is $ce^x(\cos x - i\sin x) = ce^{(1-i)x}$, for $c \in \mathbb{C}$.

**4.** Let us solve (b) first and then (a).

(b) Take $\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1,1,1)^T$, and a (unit) vector $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1,0,-1)^T$ in the orthogonal complement of $\mathbf{v}_1$. Finally, let $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2 = \frac{1}{\sqrt{6}}(-1, 2, -1)^T$. Then, $T$ becomes a rotation following the right-hand direction with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, that is, the direction of other fingers when the thumb is pointing the vector $(1, 1, 1)^T$. Thus, $T$ is a CCW rotation on a plane orthogonal to $\mathbf{v}_1$, so that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(30°) & -\sin(30°) \\ 0 & \sin(30°) & \cos(30°) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

(a) Letting

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix},$$

we have $[T]_{\mathcal{E}} = P[T]_{\mathcal{B}}P^{-1}$ where $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$. By a simple calculation, using $P^{-1} = P^T$, we get

$$[T]_{\mathcal{E}} = \frac{1}{3}\begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{bmatrix}.$$

***5.*** (a) Note $W = \mathrm{col}(A)$ where $A = [\mathbf{u}_1, \mathbf{u}_2]$, and

$$[T]_{\mathcal{E}} = A(A^TA)^{-1}A^T.$$

(b) Taking an orthogonal basis $\mathcal{B}$ extending $\{\mathbf{u}_1, \mathbf{u}_2\}$ which exists due to, say, Gram–Schmidt orthogonalization, we have

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_2 & O_{2\times(n-2)} \\ O_{(n-2)\times 2} & O_{(n-2)\times(n-2)} \end{bmatrix}.$$

***6.*** (a) The characteristic polynomial of $A$ is

$$\det(tI - A) = t^4 - 16t^3 - 20t^2 = t^2(t^2 - 16t - 20).$$

Thus, the eigenvalues are $\lambda_1 = 8 + 2\sqrt{21}$, $\lambda_2 = 8 - 2\sqrt{21}$, and $\lambda_3 = \lambda_4 = 0$. Corresponding eigenvectors are:

- $\mathbf{v}_1 = (13 + 3\sqrt{21}, 18 + 4\sqrt{21}, 23 + 5\sqrt{21}, 28 + 6\sqrt{21})^T$,
- $\mathbf{v}_2 = (-13 + 3\sqrt{21}, -18 + 4\sqrt{21}, -23 + 5\sqrt{21}, -28 + 6\sqrt{21})^T$,
- $\mathbf{v}_3 = (2, -3, 0, 1)^T$,
- $\mathbf{v}_4 = (1, 2, -7, 4)^T$.

Normalizing them, we obtain

- $\mathbf{u}_1 = \frac{1}{2\sqrt{903+197\sqrt{21}}}\mathbf{v}_1$,
- $\mathbf{u}_2 = \frac{1}{2\sqrt{903-197\sqrt{21}}}\mathbf{v}_2$,
- $\mathbf{u}_3 = \frac{1}{\sqrt{14}}\mathbf{v}_3$,
- $\mathbf{u}_4 = \frac{1}{\sqrt{70}}\mathbf{v}_4$.

Let $P = [\mathbf{u}_1, \cdots, \mathbf{u}_4]$. Then the orthogonal diagonalization of $A$ is

$$P^T AP = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \mathrm{diag}(8 + 2\sqrt{21}, 8 - 2\sqrt{21}, 0, 0).$$

(b) The spectral decomposition of $A$ is

$$\begin{aligned} A &= (8 + 2\sqrt{21})\mathbf{u}_1\mathbf{u}_1^T + (8 - 2\sqrt{21})\mathbf{u}_2\mathbf{u}_2^T \\ &\left(= \frac{8 + 2\sqrt{21}}{4(903 + 197\sqrt{21})}\mathbf{v}_1\mathbf{v}_1^T + \frac{8 - 2\sqrt{21}}{4(903 - 197\sqrt{21})}\mathbf{v}_2\mathbf{v}_2^T\right) \end{aligned}$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ are as in (a).

(c) Singular values of $A^T A = A^2$ are $\lambda_i^2$ $(i = 1, 2)$, which are $(8 \pm 2\sqrt{21})^2$.

(d)

$$\begin{aligned} e^A &= \sum_{k\geq 0} \frac{1}{k!} A^k \\ &= \sum_{k\geq 0} \frac{1}{k!}(\lambda_1^k \mathbf{u}_1\mathbf{u}_1^T + \lambda_2^k \mathbf{u}_2\mathbf{u}_2^T) \\ &= \sum_{k\geq 0} \frac{1}{k!}((8 + 2\sqrt{21})^k \mathbf{u}_1\mathbf{u}_1^T + (8 - 2\sqrt{21})^k \mathbf{u}_2\mathbf{u}_2^T) \end{aligned}$$

where $\mathbf{u}_1, \mathbf{u}_2$ are as in (a).

(e) Let $\mathbf{w}_1 = (1,2,3,4)^T$ and $\mathbf{w}_2 = (2,3,4,5)^T$, then an orthogonal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ for $\operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$ is given by

$$\mathbf{v}_1 = \mathbf{w}_1, \qquad \mathbf{q}_1 = \frac{1}{\sqrt{30}}(1,2,3,4)^T,$$

and

$$\mathbf{v}_2 = \mathbf{w}_2 - \operatorname{proj}_{\mathbf{w}_1} \mathbf{w}_2 = (2,3,4,5) - \frac{4}{3}(1,2,3,4)^T = \frac{1}{3}(2,1,0,-1)^T$$

$$\mathbf{q}_2 = \frac{1}{\sqrt{6}}(2,1,0,-1)^T.$$

Let us extend $\{\mathbf{q}_1, \mathbf{q}_2\}$ into an orthonormal basis of $\mathbb{R}^4$. Letting $\mathbf{w}_3 = (1,0,0,0)^T$, we have

$$\mathbf{v}_3 = \mathbf{w}_3 - (\mathbf{w}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{w}_3 \cdot \mathbf{q}_2)\mathbf{q}_2 = \frac{1}{10}(3,-4,-1,2)^T,$$

$$\mathbf{q}_3 = \frac{1}{\sqrt{30}}(3,-4,-1,2)^T.$$

Letting $\mathbf{w}_4 = (0,1,0,0)^T$, we have

$$\mathbf{v}_4 = \mathbf{w}_4 - (\mathbf{w}_4 \cdot \mathbf{q}_1)\mathbf{q}_1 - (\mathbf{w}_4 \cdot \mathbf{q}_2)\mathbf{q}_2 - (\mathbf{w}_4 \cdot \mathbf{q}_3)\mathbf{q}_3 = \frac{1}{6}(0,1,-2,1)^T,$$

$$\mathbf{q}_4 = \frac{1}{\sqrt{6}}(0,1,-2,1)^T.$$

Thus, $Q = [\mathbf{q}_1, \cdots, \mathbf{q}_4]$ is an orthogonal matrix. With $A = [\mathbf{a}_1, \cdots, \mathbf{a}_4]$,

$$R = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 & \mathbf{a}_3 \cdot \mathbf{q}_1 & \mathbf{a}_4 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 & \mathbf{a}_3 \cdot \mathbf{q}_2 & \mathbf{a}_4 \cdot \mathbf{q}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{30} & \frac{4}{3}\sqrt{3} & \frac{5}{3}\sqrt{3} & 2\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{6} & \frac{2}{3}\sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $A = QR$.

(f) The SVD of $A$ is given by $A = U\Sigma V^T$, where $A^TA$ has eigenvalues $\sigma_1^2 = (8 + 2\sqrt{21})^2$, $\sigma_2^2 = (8 - 2\sqrt{21})^2$ and $\lambda_3 = \lambda_4 = 0$ (with multiplicity 2), and the (unit) eigenvectors

$$V = [\mathbf{v}_1, \cdots, \mathbf{v}_4] = \begin{bmatrix} -0.3147 & 0.7752 & 0.3041 & -0.4556 \\ -0.4275 & 0.3424 & -0.0659 & 0.8341 \\ -0.5402 & -0.0903 & -0.7805 & -0.3015 \\ -0.6530 & -0.5231 & 0.5423 & -0.0770 \end{bmatrix}$$

(where $\mathbf{v}_i$ corresponds to $\sigma_i^2$.) And $\mathbf{u}_i = A\mathbf{v}_i/\sigma_i$ for $i = 1, 2$, yielding

$$\begin{aligned} \mathbf{u}_1 &= (-0.3147, -0.4275, -0.5402, -0.6530)^T, \\ \mathbf{u}_2 &= (-0.7752, -0.3424, 0.0903, 0.5231)^T. \end{aligned}$$

Extending $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthogonal basis of $\mathbb{R}^4$, we have

$$U = [\mathbf{u}_1, \dots, \mathbf{u}_4] = \begin{bmatrix} -0.3147 & -0.7752 & 0.5442 & 0.0619 \\ -0.4275 & -0.3424 & -0.7718 & 0.3231 \\ -0.5402 & 0.0903 & -0.0891 & -0.8319 \\ -0.6530 & 0.5231 & 0.3167 & 0.4469 \end{bmatrix}.$$

With $\Sigma = \operatorname{diag}(8 + 2\sqrt{21}, 8 - 2\sqrt{21}, 0, 0)$, we have $A = U\Sigma V^T$. Then, the following is the polar decomposition of $A$: $A = QS$ where

$$\begin{aligned} Q = UV^T &= \begin{bmatrix} -0.3646 & -0.1152 & -0.2034 & 0.9014 \\ -0.5128 & 0.3858 & 0.7668 & 0.0149 \\ 0.5919 & -0.4261 & 0.6040 & 0.3213 \\ 0.5037 & 0.8101 & -0.0764 & 0.2900 \end{bmatrix}, \\ S = V\Sigma V^T &= \begin{bmatrix} 2.4004 & 2.6186 & 2.8368 & 3.0551 \\ 2.6186 & 3.2733 & 3.9279 & 4.5826 \\ 2.8368 & 3.9279 & 5.0190 & 6.1101 \\ 3.0551 & 4.5826 & 6.1101 & 7.6376 \end{bmatrix}. \end{aligned}$$

**7.** (a) The characteristic polynomial of $A$ is

$$\begin{aligned} \det(tI - A) &= \det \begin{bmatrix} t-1 & -1 \\ -1 & t-1 \end{bmatrix} \cdot \det \begin{bmatrix} t-2 & -2 & 0 \\ 0 & t-2 & -1 \\ 0 & 0 & t-2 \end{bmatrix} \cdot \det \begin{bmatrix} t & 0 \\ -1 & t \end{bmatrix} \\ &= [(t-1)^2 - 1](t-2)^3 t^2 = t^3(t-2)^4. \end{aligned}$$

(b) Eigenvalues are 0 and 2. Eigenvectors corresponding to 0 are

$$(\underline{-1,1},0,0,0,0,0)^T, \quad (0,0,0,0,0,\underline{0,1})^T,$$

and eigenvectors corresponding to 2 are

$$(\underline{1,1},0,0,0,0,0)^T, \quad (0,0,\underline{1,0,0},0,0)^T.$$

(c) Algebraic multiplicities of eigenvalues 0 and 2 are $\underline{\text{3 and 4}}$, respectively. Geometric multiplicities of both eigenvalues are $\underline{2}$.

(d) The minimal polynomial of $A$ has the factor $t$ and $t-2$. Since $A(A-2I)$, $A(A-2I)^2$, $A(A-2I)^3$, $A^2(A-2I)$, $A^2(A-2I)^2$ are nonzero and $A^2(A-2I)^3 = O_{7\times 7}$, the minimal polynomial of $A$ is $m_A(t) = t^2(t-2)^3$.

**8.** Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$T(X) = \begin{bmatrix} -3b+2c & -2a-3b+2d \\ 3a+3c-3d & 3b-2c \end{bmatrix}.$$

(a) The kernel of $T$ is

$$\ker T = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 3b = 2c,\ d = a+c \right\} = \left\{ \begin{bmatrix} a & 2c/3 \\ c & a+c \end{bmatrix} : a,\ c \in \mathbb{R}^4 \right\}.$$

(b) With the standard basis $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_4\}$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \mathbf{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \mathbf{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we have

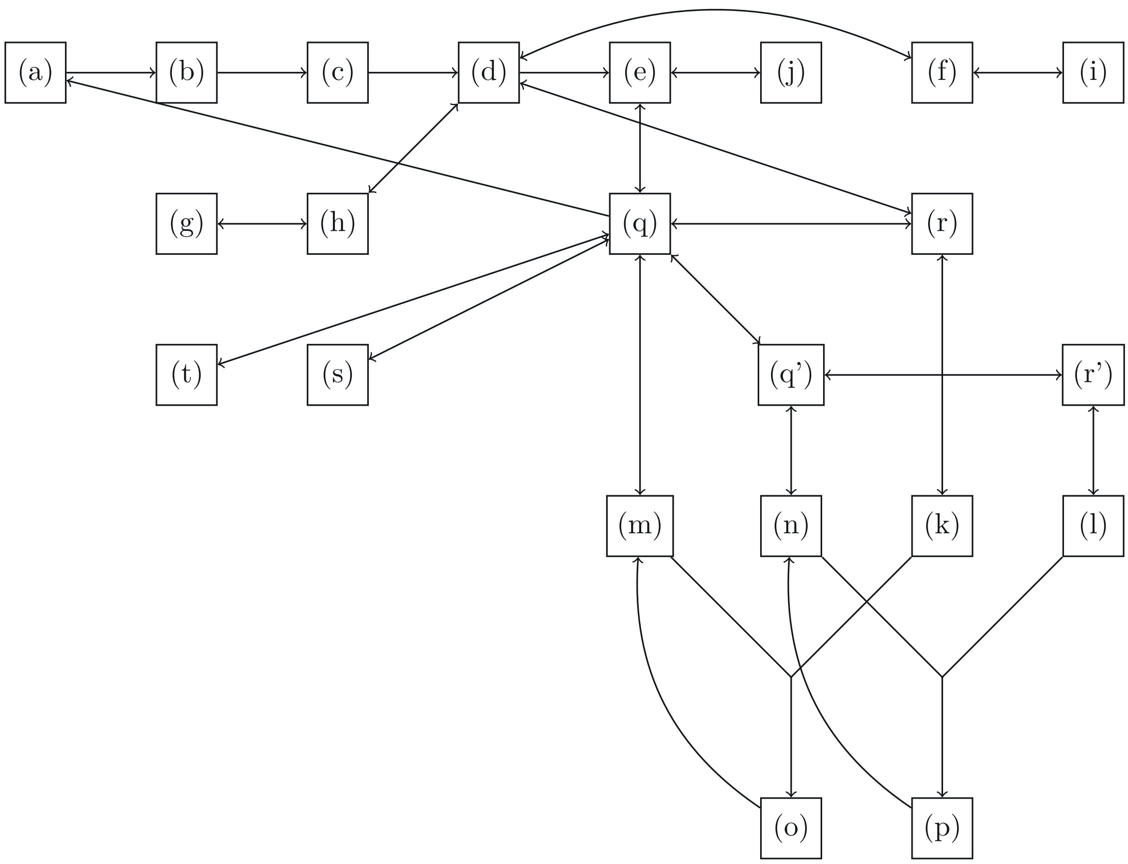
$$\begin{aligned} T\mathbf{e}_1 &= -2\mathbf{e}_2 + 3\mathbf{e}_3, \\ T\mathbf{e}_2 &= -3\mathbf{e}_1 - 3\mathbf{e}_2 + 3\mathbf{e}_4, \\ T\mathbf{e}_3 &= 2\mathbf{e}_1 + 3\mathbf{e}_3 - 2\mathbf{e}_4, \\ T\mathbf{e}_4 &= 2\mathbf{e}_2 - 3\mathbf{e}_3, \end{aligned}$$

so that

$$[T]_{\mathcal{E}} = \begin{bmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 3 & -3 \\ 0 & 3 & -2 & 0 \end{bmatrix}.$$

Thus, the characteristic polynomial of $T$ is the same with $\det(tI - [T]_{\mathcal{E}}) = t^4 - 33t^2$.

***9.*** I will show the following: (the arrows mean the implications)




where (q') $\text{rank}(A^T) = n$ and (r') $\text{nullity}(A^T) = 0$.

(a) $\Longrightarrow$ (b): Since $A$ is row-equivalent to $I_n$, there is matrix $E = E_n \cdots E_1$ which is a product of elementary matrices such that $EA = I_n$. Then $A = E_1^{-1} \cdots E_n^{-1}$, where each $E_i^{-1}$ is again an elementary matrix. Thus, $A$ is expressible as a product of elementary matrices.

(b) $\Longrightarrow$ (c): If $A = E_n \cdots E_1$, $A^{-1} = E_1^{-1} \cdots E_n^{-1}$.

(c) $\Longrightarrow$ (d): $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = A^{-1}A\mathbf{x} = A\mathbf{0} = \mathbf{0}$.

(d) $\Longrightarrow$ (e): There is a solution $\mathbf{b} = A^{-1}\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$.

(e) $\Longleftrightarrow$ (j): By definition.

(d) $\Longleftrightarrow$ (h): $\lambda = 0$ is an eigenvalue of $A$ iff there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. The contrapositive of this statement is exactly (d) $\Longleftrightarrow$ (h).

(h) $\Longleftrightarrow$ (g): $\lambda = 0$ is an eigenvalue of $A$ iff $\det(tI - A)|_{t=0} = (-1)^n \det(A) = 0$ iff $\det(A) = 0$. The contrapositive of this statement is exactly (h) $\Longleftrightarrow$ (g).

(e) $\Longleftrightarrow$ (q): (e) iff $T_A(\mathbb{R}^n) = \mathbb{R}^n$ iff $T_A(\mathbb{R}^n)$, which is a subspace of $\mathbb{R}^n$, has the dimension $n$ iff $\operatorname{rank}(A) = n$.

(d) $\Longleftrightarrow$ (f): ($\Longleftarrow$) is clear. ($\Longrightarrow$) If $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$, then $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ so that $\mathbf{x}_1 - \mathbf{x}_2$, a solution of $A\mathbf{x} = \mathbf{0}$, should be $\mathbf{0}$.

(d) $\Longleftrightarrow$ (r): By definition.

(f) $\Longleftrightarrow$ (i): By definition.

(q) $\Longrightarrow$ (a): Since the rank of $A$ and the rref of $A$ are the same, the rref of $A$ has rank $n$. Since $I_n$ is the only possible candidate of rref having the rank $n$, the rref of $A$ should be $I_n$.

(q) $\Longleftrightarrow$ (r): The rank–nullity theorem implies this.

(q) $\Longleftrightarrow$ (q'): $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

(q') $\Longleftrightarrow$ (r'): The rank–nullity theorem implies this.

(q) $\Longleftrightarrow$ (r): The rank–nullity theorem implies this.

(q) $\Longrightarrow$ (m): The columns of $A$ span the subspace of the dimension $n$ in $\mathbb{R}^n$, which is $\mathbb{R}^n$.

(q') $\Longrightarrow$ (n): The columns of $A^T$, i.e. the (transposes of) rows of $A$, span the subspace of the dimension $n$ in $\mathbb{R}^n$, which is $\mathbb{R}^n$.

(q) $\Longrightarrow$ (k): Suppose the contrary. Then the columns, $A\mathbf{e}_i$, of $A$ has a linear dependency for some $a_i \in \mathbb{R}$ which are not simultaneously zero:

$$\sum_{i=1}^{n} a_i A\mathbf{e}_i = A \sum_{i=1}^{n} a_i \mathbf{e}_i = \mathbf{0}.$$

This implies $\mathbf{0} \neq \sum_{i=1}^{n} a_i \mathbf{e}_i \in \mathrm{null}(T_A)$, which leads to a contradiction that $\mathrm{nullity}(A) > 0$.

(r') $\Longrightarrow$ (l): The analogous argument of the above works.

(m) $\wedge$ (k) $\Longrightarrow$ (o): By definition.

(n) $\wedge$ (l) $\Longrightarrow$ (p): By definition.

(o) $\Longrightarrow$ (m): By definition.

(p) $\Longrightarrow$ (n): By definition. (The columns of $A^T$ is the transposes of the rows of $A$.)

(q) $\Longleftrightarrow$ (s): $A$ has $n$ singular values iff $A^TA$ does not have $\sigma^2 = 0$ as an eigenvalue, which is equivalent, by (h) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (r) $\Longleftrightarrow$ (q), to that $\mathrm{rank}(A^TA) = n$. Therefore, (s) $\Longleftrightarrow \mathrm{rank}(A^TA) = n$. Since $\mathrm{rank}(A^TA) = \mathrm{rank}(A)$, we have (q) $\Longleftrightarrow$ (s).

(q) $\Longleftrightarrow$ (t): $A$ is of rank $< n$ iff there are vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbb{R}^n$ such that for the $i$-th column $A_i$ of $A$,

$$A_i = \sum_{j=1}^{n-1} c_{ij}\mathbf{v}_j$$

for some $c_{ij}$'s. That is,

$$A = \sum_{j=1}^{n-1} [c_{1j}, \cdots, c_{nj}]\mathbf{v}_j.$$

for some $c_{ij}$'s. This is equivalent to that $A$ is a sum of $< n$ rank-one matrices $[c_{1j}, \cdots, c_{nj}]\mathbf{v}_j$ $(j = 1, \ldots, n-1)$.

***10.*** (a) When $n = 2$,

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad p_{A^TA}(t) = \det(tI - A^TA) = t^2 - 3t + 1,$$

$$\text{singular values: } \sqrt{\frac{3 \pm \sqrt{5}}{2}} = \frac{\sqrt{5} \pm 1}{2} \approx 0.6180,\ 1.6180.$$

When $n = 3$,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad p_{A^T A}(t) = t^3 - 6t^2 + 5t - 1,$$

singular values: 0.554958, 0.801938, 2.246980.

When $n = 4$,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$p_{A^T A}(t) = t^4 - 10t^3 + 15t^2 - 7t + 1 = (t-1)(t^3 - 9t^2 + 6t - 1),$$

singular values: 0.532089, 0.652704, 1, 2.879385.

(b)

- $n = 10$: $(6.6907, 2.2470, 1.3686, 1.0000, 0.8019, 0.6821, 0.6052, 0.5550, 0.5232, 0.5056)$
- $n = 20$: $(13.0539, 4.3598, 2.6262, 1.8869, 1.4792, 1.2223, 1.0466, 0.9198, 0.8248, 0.7515, \dots)$
- $n = 30$: $(19.4190, 6.4787, 3.8941, 2.7889, 2.1769, 1.7890, 1.5219, 1.3272, 1.1795, 1.0639, \dots)$
- $n = 40$: $(25.7847, 8.5992, 5.1647, 3.6946, 2.8794, 2.3618, 2.0045, 1.7434, 1.5445, 1.3882, \dots)$
- $n = 50$: $(32.1506, 10.7203, 6.4363, 4.6018, 3.5838, 2.9370, 2.4900, 2.1629, 1.9133, 1.7169, \dots)$
- $n = 60$: $(38.5166, 12.8417, 7.7085, 5.5098, 4.2893, 3.5133, 2.9768, 2.5840, 2.2841, 2.0478, \dots)$
- $n = 70$: $(44.8826, 14.9634, 8.9810, 6.4182, 4.9952, 4.0904, 3.4645, 3.0061, 2.6559, 2.3799, \dots)$
- $n = 80$: $(51.2487, 17.0851, 10.2536, 7.3268, 5.7015, 4.6679, 3.9527, 3.4288, 3.0284, 2.7128, \dots)$
- $n = 90$: $(57.6148, 19.2069, 11.5264, 8.2356, 6.4081, 5.2456, 4.4413, 3.8518, 3.4014, 3.0461, \dots)$
- $n = 100$: $(63.9809, 21.3287, 12.7993, 9.1446, 7.1148, 5.8236, 4.9300, 4.2751, 3.7746, 3.3798, \dots)$

(c)

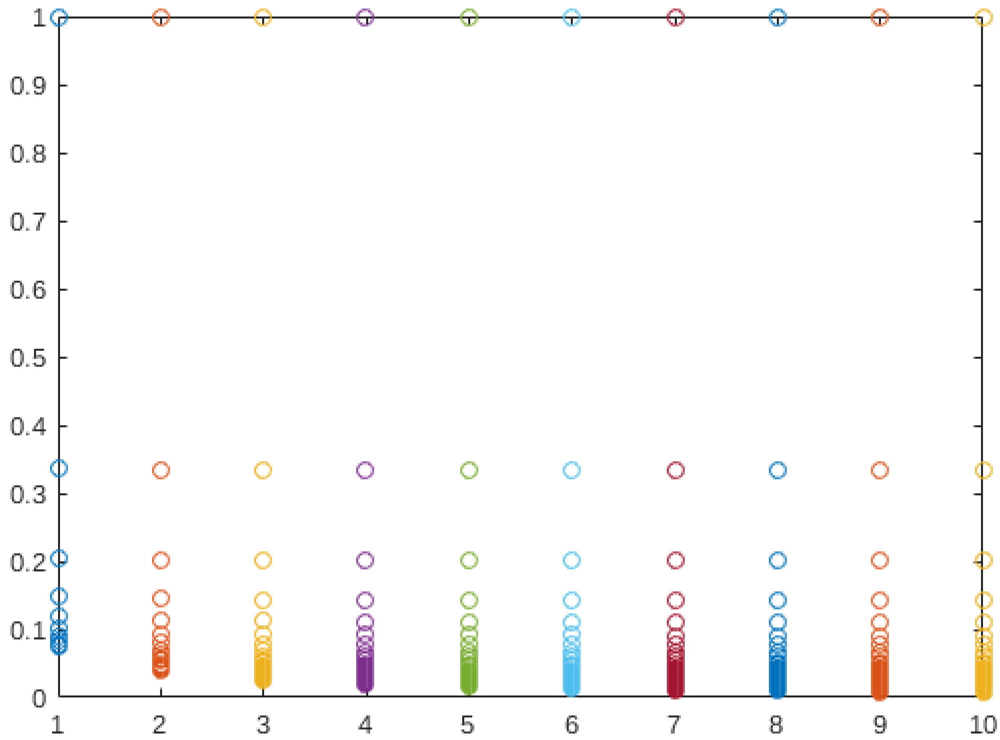



Figure 1: Figure for #10 (c)

(d) Using the SVD $A = \sigma_1 \mathbf{u}_1^T \mathbf{v}_1^T + \cdots$ of $A$, the closest rank-one matrix to $A$ is

$$\sigma_1 \mathbf{u}_1^T \mathbf{v}_1^T =$$

$$\begin{bmatrix} 8.4793 & 8.2898 & 7.9152 & 7.3638 & 6.6479 & 5.7835 & 4.7899 & 3.6893 & 2.5063 & 1.2673 \\ 8.2898 & 8.1047 & 7.7384 & 7.1993 & 6.4994 & 5.6543 & 4.6829 & 3.6069 & 2.4503 & 1.2390 \\ 7.9152 & 7.7384 & 7.3888 & 6.8740 & 6.2057 & 5.3988 & 4.4713 & 3.4439 & 2.3396 & 1.1830 \\ 7.3638 & 7.1993 & 6.8740 & 6.3952 & 5.7734 & 5.0227 & 4.1598 & 3.2040 & 2.1766 & 1.1006 \\ 6.6479 & 6.4994 & 6.2057 & 5.7734 & 5.2121 & 4.5344 & 3.7554 & 2.8925 & 1.9650 & 0.9936 \\ 5.7835 & 5.6543 & 5.3988 & 5.0227 & 4.5344 & 3.9448 & 3.2671 & 2.5164 & 1.7095 & 0.8644 \\ 4.7899 & 4.6829 & 4.4713 & 4.1598 & 3.7554 & 3.2671 & 2.7058 & 2.0841 & 1.4158 & 0.7159 \\ 3.6893 & 3.6069 & 3.4439 & 3.2040 & 2.8925 & 2.5164 & 2.0841 & 1.6052 & 1.0905 & 0.5514 \\ 2.5063 & 2.4503 & 2.3396 & 2.1766 & 1.9650 & 1.7095 & 1.4158 & 1.0905 & 0.7408 & 0.3746 \\ 1.2673 & 1.2390 & 1.1830 & 1.1006 & 0.9936 & 0.8644 & 0.7159 & 0.5514 & 0.3746 & 0.1894 \end{bmatrix}.$$

(d) Using the SVD $A = \sigma_1 \mathbf{u}_1^T \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2^T \mathbf{v}_2^T + \cdots$ of $A$, the closest rank-two matrix to $A$ is

$$\sigma_1 \mathbf{u}_1^T \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2^T \mathbf{v}_2^T =$$

$$\begin{bmatrix} 9.3933 & 9.0229 & 8.3220 & 7.3638 & 6.2411 & 5.0505 & 3.8758 & 2.7752 & 1.7733 & 0.8605 \\ 9.0229 & 8.6925 & 8.0647 & 7.1993 & 6.1732 & 5.0665 & 3.9499 & 2.8739 & 1.8625 & 0.9128 \\ 8.3220 & 8.0647 & 7.5698 & 6.8740 & 6.0247 & 5.0726 & 4.0645 & 3.0371 & 2.0134 & 1.0020 \\ 7.3638 & 7.1993 & 6.8740 & 6.3952 & 5.7734 & 5.0227 & 4.1598 & 3.2040 & 2.1766 & 1.1006 \\ 6.2411 & 6.1732 & 6.0247 & 5.7734 & 5.3932 & 4.8607 & 4.1622 & 3.2993 & 2.2912 & 1.1746 \\ 5.0505 & 5.0665 & 5.0726 & 5.0227 & 4.8607 & 4.5327 & 4.0001 & 3.2494 & 2.2974 & 1.1906 \\ 3.8758 & 3.9499 & 4.0645 & 4.1598 & 4.1622 & 4.0001 & 3.6199 & 2.9982 & 2.1488 & 1.1227 \\ 2.7752 & 2.8739 & 3.0371 & 3.2040 & 3.2993 & 3.2494 & 2.9982 & 2.5193 & 1.8235 & 0.9582 \\ 1.7733 & 1.8625 & 2.0134 & 2.1766 & 2.2912 & 2.2974 & 2.1488 & 1.8235 & 1.3287 & 0.7008 \\ 0.8605 & 0.9128 & 1.0020 & 1.1006 & 1.1746 & 1.1906 & 1.1227 & 0.9582 & 0.7008 & 0.3705 \end{bmatrix}.$$