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MAS555 Advanced Statistics

Homework 2

5.15. (a) $(n+1)\bar{X}_{n+1} = X_1 + \cdots + X_{n+1} = n\bar{X}_n + X_{n+1}$.

(b)

$$\begin{aligned}
nS_{n+1}^2 &= \sum_{i=1}^{n+1}(X_i - \bar{X}_{n+1})^2 \\
&= \sum_{i=1}^{n+1}\left(X_i - \frac{X_{n+1} + n\bar{X}_n}{n+1}\right)^2 + \left(X_{n+1} - \frac{X_{n+1} + n\bar{X}_n}{n+1}\right)^2 \\
&= \sum_{i=1}^{n}\left(X_i - \bar{X}_n - \frac{X_{n+1} - \bar{X}_n}{n+1}\right)^2 + \frac{n^2}{(n+1)^2}\left(X_{n+1} - \bar{X}_n\right)^2 \\
&= \sum_{i=1}^{n}\left[(X_i - \bar{X}_n)^2 + \left(\frac{X_{n+1} - \bar{X}_n}{n+1}\right)^2 \underbrace{- 2(X_i - \bar{X}_n)\left(\frac{X_{n+1} - \bar{X}_n}{n+1}\right)}_{\Sigma(\cdots)=0}\right] \\
&\quad + \frac{n^2}{(n+1)^2}\left(X_{n+1} - \bar{X}_n\right)^2 \\
&= \sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + \frac{n}{(n+1)^2}\left(X_{n+1} - \bar{X}_n\right)^2 + \frac{n^2}{(n+1)^2}\left(X_{n+1} - \bar{X}_n\right)^2 \\
&= (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \bar{X}_n)^2.
\end{aligned}$$

5.17. (a) Following the discussion just before Example 5.3.7 in the book, we can reduce this task to finding the pdf of $X = (U/p)/(V/q)$, where U and V are independent, $U \sim \chi_p^2$ and $V \sim \chi_q^2$. By transforming (u, v) into $(z, w) = (u+v, u/v)$, we have ((Z, W) is the image of (U, V) under the transformation)

$$\begin{aligned}
\mathrm{pdf}_{Z,W}(z, w) &= \mathrm{pdf}_{U,V}(u, v)\begin{vmatrix} \partial z/\partial u & \partial z/\partial v \\ \partial w/\partial u & \partial w/\partial v \end{vmatrix}^{-1} \\
&= C_{p,q}^{(1)}\, u^{p/2-1}v^{q/2-1}e^{-(u+v)/2}\,\frac{v^2}{u+v} \\
&= C_{p,q}^{(1)}\, z^{(p+q)/2-1}e^{-z/2}\left(\frac{w}{1+w}\right)^{p/2}\left(\frac{1}{1+w}\right)^{q/2} w^{-1}
\end{aligned}$$

where $C^{(1)}_{p,q} := (2^{(p+q)/2}\,\Gamma(p/2)\,\Gamma(q/2))^{-1}$. Thus, the marginal pdf for W is as follows:

$$\begin{aligned}
\mathrm{pdf}_W(w) &= \int_0^\infty \mathrm{pdf}_{Z,W}(z,w)\,dz \\
&= C^{(1)}_{p,q}\left[\int_0^\infty z^{(p+q)/2-1}e^{-z/2}\,dz\right]\left(\frac{w}{1+w}\right)^{p/2}\left(\frac{1}{1+w}\right)^{q/2} w^{-1} \\
&= C^{(1)}_{p,q}\,C^{(2)}_{p,q}\left(\frac{w}{1+w}\right)^{p/2}\left(\frac{1}{1+w}\right)^{q/2} w^{-1}
\end{aligned}$$

where $C^{(2)}_{p,q} = 2^{(p+q)/2}\,\Gamma((p+q)/2)$, which can be evaluated by substituting $z \mapsto z/2$ and using the definition of the Gamma function. Since $X = qW/p$, we have

$$\begin{aligned}
\mathrm{pdf}_X(x) &= (p/q)\ \mathrm{pdf}_W(px/q) \\
&= C^{(1)}_{p,q}\,C^{(2)}_{p,q}\left(\frac{px}{px+q}\right)^{p/2}\left(\frac{q}{px+q}\right)^{q/2} x^{-1} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\,\Gamma(\frac{q}{2})}\left(\frac{px}{px+q}\right)^{p/2}\left(\frac{q}{px+q}\right)^{q/2} x^{-1} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\,\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{p/2}\frac{x^{p/2-1}}{(1+px/q)^{(p+q)/2}}
\end{aligned}$$

where the support of x is $(0,\infty)$.

(b)

$$\begin{aligned}
\mathrm{E}X &= \int_0^\infty \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\,\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{p/2}\frac{x^{p/2-1}}{(1+px/q)^{(p+q)/2}}\cdot x\,dx \\
&= \frac{q}{p\,\mathrm{B}(\frac{p}{2},\frac{q}{2})}\int_0^\infty \frac{(px/q)^{p/2}}{(1+px/q)^{(p+q)/2}}\,d(px/q) \\
&= \frac{q}{p\,\mathrm{B}(\frac{p}{2},\frac{q}{2})}\int_0^\infty \left(\frac{u}{1+u}\right)^{p/2}\left(\frac{1}{1+u}\right)^{q/2} du \qquad (u := px/q) \\
&= \frac{q}{p\,\mathrm{B}(\frac{p}{2},\frac{q}{2})}\int_0^1 (1-t)^{p/2}t^{q/2-2}\,dt \qquad (t := 1/(1+u)) \\
&= \frac{q\,\mathrm{B}(\frac{p}{2}+1,\frac{q}{2}-1)}{p\,\mathrm{B}(\frac{p}{2},\frac{q}{2})} = \frac{q}{q-2} \qquad (q>2)
\end{aligned}$$

where $\mathrm{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta function, and the integral $\int_0^1 (1-t)^{p/2}t^{q/2-2}\,dt$ converges only if

$\frac{q}{2} - 1 > 0$, that is, $q > 2$.

$$\begin{aligned}
\mathrm{E}[X^2] &= \int_0^\infty \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\,\Gamma(\frac{q}{2})} \left(\frac{p}{q}\right)^{p/2} \frac{x^{p/2-1}}{(1+px/q)^{(p+q)/2}} \cdot x^2\,dx \\
&= \frac{q^2}{p^2\,\mathrm{B}(\frac{p}{2},\frac{q}{2})} \int_0^\infty \frac{(px/q)^{p/2+1}}{(1+px/q)^{(p+q)/2}}\,d(px/q) \\
&= \frac{q^2}{p^2\,\mathrm{B}(\frac{p}{2},\frac{q}{2})} \int_0^\infty \left(\frac{u}{1+u}\right)^{p/2+1} \left(\frac{1}{1+u}\right)^{q/2-1} du \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u \coloneqq px/q) \\
&= \frac{q^2}{p^2\,\mathrm{B}(\frac{p}{2},\frac{q}{2})} \int_0^1 (1-t)^{p/2+1} t^{q/2-3}\,dt \qquad (t \coloneqq 1/(1+u)) \\
&= \frac{q^2\,\mathrm{B}(\frac{p}{2}+2,\frac{q}{2}-2)}{p^2\,\mathrm{B}(\frac{p}{2},\frac{q}{2})} = \frac{q^2(p+2)}{p(q-2)(q-4)} \qquad (q > 4), \\
\therefore \operatorname{Var} X &= \frac{q^2(p+2)}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} = \frac{2q^2(p+q-2)}{p(q-2)^2(q-4)} \qquad (q > 4).
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{pdf}_{1/X}(y) &= \mathrm{pdf}_X(1/y) \cdot y^{-2} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \left(\frac{p}{q}\right)^{p/2} \frac{y^{-(p/2+1)}}{(1+p/(qy))^{(p+q)/2}} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \left(\frac{p}{q}\right)^{p/2} \frac{(\frac{q}{p})^{(p+q)/2} y^{q/2-1}}{(1+qy/p)^{(p+q)/2}} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{q}{2})\Gamma(\frac{p}{2})} \left(\frac{q}{p}\right)^{q/2} \frac{y^{q/2-1}}{(1+qy/p)^{(p+q)/2}},
\end{aligned}$$

hence $1/X \sim F_{q,p}$.

(d) Define $Y = (p/q)X/[1+(p/q)X]$. Then,

$$\begin{aligned}
\mathrm{pdf}_Y(y) &= \mathrm{pdf}_X\left(\frac{qy}{p(1-y)}\right) \left|\frac{\partial}{\partial y}\frac{qy}{p(1-y)}\right| \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{q}{2})\Gamma(\frac{p}{2})} \left(\frac{p}{q}\right)^{p/2} \left(\frac{qy}{p(1-y)}\right)^{p/2-1} (1-y)^{(p+q)/2} \cdot \frac{q}{p(1-y)^2} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{q}{2})\Gamma(\frac{p}{2})} \left(\frac{y}{1-y}\right)^{p/2} (1-y)^{(p+q)/2} \cdot \frac{1}{y(1-y)} \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{q}{2})\Gamma(\frac{p}{2})} y^{p/2-1}(1-y)^{q/2-1}
\end{aligned}$$

so that $Y = (p/q)X/[1 + (p/q)X]$ has a beta distribution with parameters $p/2$ and $q/2$.

5.25. Define $Y_1 = \frac{X_{(1)}}{X_{(2)}}, \cdots, Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}}, Y_n = X_{(n)}$ and let $(x_1, \ldots, x_n) = (y_1 \cdots y_n, \ldots, y_{n-1}y_n, y_n)$. Then,

$$\begin{aligned}
&\mathrm{pdf}_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) \\
&= \mathrm{pdf}_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) \cdot \left| \left(\frac{\partial x_i}{\partial y_j} \right)_{ij} \right| \\
&= \mathrm{pdf}_{X_{(1)},\ldots,X_{(n)}}(y_1 \cdots y_n, \ldots, y_n) \cdot y_2 y_3^2 \cdots y_n^{n-1} \\
&= n!\, I(0 < y_1 < 1, \ldots, 0 < y_{n-1} < 1) \prod_{j=1}^{n} f(y_j \cdots y_n) \cdot y_2 y_3^2 \cdots y_n^{n-1} \\
&= \left(\prod_{j=1}^{n-1} I(0 < y_j < 1) \cdot ja\, y_j^{ja-1} \right) \left(I(0 < y_n < \theta)\, \frac{na}{\theta^{na}}\, y_n^{na-1} \right).
\end{aligned}$$

This proves that $Y_j = X_{(j)}/X_{(j+1)}$ has a pdf $ja\, y_j^{ja-1}$ $(0 < y_j < 1)$ for $j = 1, \ldots, n-1$ and $na\theta^{-na} y_n^{na-1}$ $(0 < y_n < \theta)$ for $j = n$ and all the Y_j's are independent.

5.36. Seeing the mgf of Y,

$$\begin{aligned}
\mathrm{mgf}_Y(t) &= \mathrm{E}[\mathrm{E}(e^{tY}|N)] = \mathrm{E}[\mathrm{mgf}_{Y|N}(t)] = \mathrm{E}[(1-2t)^{-N}] \\
&= \sum_{n=0}^{\infty} (1-2t)^{-n}\, \frac{\theta^{-n} e^{-\theta}}{n!} = e^{-\theta}\, e^{\theta/(1-2t)},
\end{aligned}$$

we have

$$\begin{aligned}
\text{(a)} \quad \mathrm{E}Y &= \frac{d}{dt}\, \mathrm{mgf}_Y(t) \bigg|_{t=0} = e^{-\theta} e^{\theta/(1-2t)}\, \frac{2\theta}{(1-2t)^2} \bigg|_{t=0} = 2\theta, \\
\mathrm{E}Y^2 &= \frac{d^2}{dt^2}\, \mathrm{mgf}_Y(t) \bigg|_{t=0} \\
&= e^{-\theta} e^{\theta/(1-2t)} \left(\left(\frac{2\theta}{(1-2t)^2} \right)^2 + \frac{8\theta}{(1-2t)^3} \right) \bigg|_{t=0} = 4\theta^2 + 8\theta, \\
\mathrm{Var}\, Y &= (4\theta^2 + 8\theta) - (2\theta)^2 = 8\theta.
\end{aligned}$$

And since

$$\mathrm{mgf}_{(Y - \mathrm{E}Y)/\sqrt{\mathrm{Var}\, Y}}(t) = e^{-t\sqrt{\theta/2}} \mathrm{E}[e^{Y \cdot t/\sqrt{8\theta}}]$$

$$
\begin{aligned}
&= e^{-t\sqrt{\theta/2}} e^{-\theta} e^{\theta/(1-(t/\sqrt{2\theta}))} \\
&= \exp\left(\frac{\theta}{1-\frac{t}{\sqrt{2\theta}}} - \theta - \frac{\theta t}{\sqrt{2\theta}}\right) \\
&= \exp\left(\theta \sum_{n\geq 0}\left(\frac{t}{\sqrt{2\theta}}\right)^n - \theta - \frac{\theta t}{\sqrt{2\theta}}\right) \\
&= \exp\left(\sum_{n\geq 0}\left(\frac{t^{n+2}}{2^{n/2+1}\theta^{n/2}}\right)\right) \\
&\xrightarrow[\theta\to\infty]{} \exp(t^2/2) = \mathrm{mgf}_{\mathrm{n}(0,1)}(t),
\end{aligned}
$$

$(Y - \mathrm{E}Y)/\sqrt{\operatorname{Var} Y}$ converges to a standard normal random variable in distribution.

5.38. (a) For $0 < t < h$, $P\left(S_n > a\right) = P\left(e^{tS_n} > e^{ta}\right) \leq \mathrm{E}e^{-at}e^{tS_n} = e^{-at}[M_X(t)]^n$. Similarly, for $-h < t < 0$, $P\left(S_n \leq a\right) = P\left(e^{tS_n} \geq e^{ta}\right) \leq \mathrm{E}e^{-at}e^{tS_n} = e^{-at}[M_X(t)]^n$.

(b) Since $M_X'(0) = \mathrm{E}X < 0$, we have $M_X(t) = 1 + \mathrm{E}X\,t + O(t^2) \leq 1 + \frac{\mathrm{E}X}{2}\,t < 1 + \frac{\mathrm{E}X}{4}\,t$ for $0 < t < h'$, where $h'(\leq h)$ is some sufficiently small positive real number. Thus, fixing any $0 < t < h'$, for $c = 1 + \frac{\mathrm{E}X}{4}\,t$, we have

$$\frac{e^{-at}M_X(t)^n}{c^n} \to 0 \text{ as } n \to \infty.$$

Therefore, $P\left(S_n > a\right) \leq e^{-at}M_X(t)^n \leq c^n$ for sufficiently large n.

Similarly, when $\mathrm{E}X = M_X'(0) > 0$, we have $M_X(t) = 1 + \mathrm{E}X\,t + O(t^2) \leq 1 + \frac{\mathrm{E}X}{2}\,t < 1 + \frac{\mathrm{E}X}{4}\,t$ for $-h' < t < 0$ for some $h'(\leq h)$. Analogously, we have $P\left(S_n \leq a\right) \leq c^n$ where $c = 1 + \frac{\mathrm{E}X}{4}\,t$ and n is sufficiently large.

(c) Note that $\mathrm{E}Y_i = \mathrm{E}(X_i - \mu - \varepsilon) = -\varepsilon < 0$. Thus,

$$P\left(\bar{X}_n - \mu > \varepsilon\right) = P\left(\sum_{i=1}^{n} Y_i > 0\right) \leq c^n$$

for some $0 < c < 1$ and every sufficiently large n.

(d) Note that $\mathrm{E}Y_i = \mathrm{E}(-X_i + \mu - \varepsilon) = -\varepsilon < 0$. Thus,

$$P\left(-X_i + \mu > \varepsilon\right) = P\left(\sum_{i=1}^{n} Y_i > 0\right) \le (c')^n$$

for some $0 < c' < 1$ and every sufficiently large n. Combining this, with $0 < c'' := \max(c, c') < 1$ where c is from (c) and c' is from the above, we have

$$P\left(|X_i - \mu| > \varepsilon\right) = P\left(\bar{X}_n - \mu > \varepsilon\right) + P\left(-X_i + \mu > \varepsilon\right) \le 2(c'')^n,$$

which completes the proof.

p.8 (2). The characteristic function of X_n is as follows:

$$\mathrm{chf}_{X_n}(t) = \mathrm{E}[e^{itX_n}] = \sum_{k=0}^{n} \binom{n}{k} (e^{it}p_n)^k (1-p_n)^{n-k} = (1 - p_n + e^{it}p_n)^n.$$

Thus,

$$\lim_{n\to\infty} \mathrm{chf}_{X_n}(t) = \left(1 + \frac{1}{n} np_n(e^{it} - 1)\right)^n \to e^{\lambda(e^{it}-1)}.$$

Note that the Poisson distribution has the same chf: for $P \sim \mathrm{Poisson}(\lambda)$,

$$\mathrm{chf}_P(t) = \sum_{n\ge 0} \frac{(e^{it}\lambda)^n e^{-\lambda}}{n!} = e^{e^{it}\lambda} e^{-\lambda} = e^{\lambda(e^{it}-1)}.$$

Hence, by the continuity theorem, $X_n \xrightarrow[n\to\infty]{d} P(\lambda)$.

extra credit. Find the limiting distribution of the intercept estimate and check the Lindeberg–Feller condition.

Since $\bar{Y} = \hat{\alpha}_n^{LSE} + \hat{\beta}_n^{LSE}\bar{x}$,

$$\begin{aligned} \hat{\alpha}_n^{LSE} &= \bar{Y} - \hat{\beta}_n^{LSE}\bar{x} \\ &= \alpha + \beta\bar{x} + \frac{1}{n}\sum_{j=1}^{n} e_j - \left(\beta + \sum_{j=1}^{n} \frac{x_j - \bar{x}}{S_{xx}} e_j\right)\bar{x} \end{aligned}$$

$$= \alpha + \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right) e_j.$$

Denote $X_{nj} = \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right) e_j$ and $S_n = \sum_{j=1}^{n} X_{nj}$. Then,

$$\begin{aligned} \mathrm{E} X_{nj} = 0, \qquad \operatorname{Var} X_{nj} &= \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 \sigma^2, \\ \therefore \operatorname{Var} S_n = \sum_{j=1}^{n} \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 \sigma^2 &= \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) \sigma^2. \end{aligned}$$

Then the corresponding Lindeberg–Feller condition is as follows: for every $\varepsilon > 0$,

$$\begin{aligned} &\frac{1}{\operatorname{Var}(S_n)} \sum_{j=1}^{n} \mathrm{E} \left[X_{nj}^2 \cdot I \left(|X_{nj}| \geq \varepsilon \sqrt{\operatorname{Var}(S_n)} \right) \right] \\ &= \frac{1}{\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) \sigma^2} \sum_{j=1}^{n} \left\{ \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 \cdot \right. \\ &\qquad \left. \mathrm{E} \left[e_j^2 \cdot I \left(\left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 e_j^2 \geq \varepsilon^2 \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) \right) \right] \right\} \\ &\leq \frac{1}{\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) \sigma^2} \left[\sum_{j=1}^{n} \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 \right] \cdot \\ &\qquad \mathrm{E} \left[e_1^2 \cdot I \left(e_1^2 \geq \varepsilon^2 \sigma^2 \frac{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}}{\max_{1 \leq j \leq n} \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2} \right) \right] \xrightarrow{n \to \infty} 0 \end{aligned}$$

if $\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) \Big/ \max_{1 \leq j \leq n} \left(\frac{1}{n} - \frac{(x_j - \bar{x})\bar{x}}{S_{xx}} \right)^2 \xrightarrow{n \to \infty} 0$. If so, we have

$$\frac{S_n}{\sqrt{\operatorname{Var} S_n}} = \frac{\hat{\alpha}_n - \alpha}{\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}}\, \sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$