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MAS583(A) Random Matrix Theory and Its Application

Homework 1

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1. Prove that $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E}\langle \mu, x^k \rangle)^2 \to 0$ as $N \to \infty$.

As we did in the calculation of $\mathbb{E}\langle \mu, x^k \rangle$, we can rephrase $\mathbb{E}[\langle \mu, x^k \rangle^2] - (\mathbb{E}\langle \mu, x^k \rangle)^2$ as follows:

$$
\begin{aligned}
&\mathbb{E}\left[\langle \mu, x^k \rangle^2\right] - \left(\mathbb{E}\langle \mu, x^k \rangle\right)^2 \\
&= \mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^N \lambda_j^k\right)^2\right] - \left(\frac{1}{N}\mathbb{E}\sum_{j=1}^N \lambda_j^k\right)^2 \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[\left(\sum_{j=1}^N \lambda_j^k\right)^2\right] - \left(\mathbb{E}\sum_{j=1}^N \lambda_j^k\right)^2\right) \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[(\operatorname{tr} H^k)^2\right] - \left(\mathbb{E}\left[\operatorname{tr} H^k\right]\right)^2\right) \\
&= \frac{1}{N^2}\left(\mathbb{E}\left[\sum_{i_1,\ldots,i_k=1}^N \sum_{i'_1,\ldots,i'_k=1}^N H_{i_1,i_2}\cdots H_{i_k,i_1} H_{i'_1,i'_2}\cdots H_{i'_k,i'_1}\right]\right. \\
&\qquad \left. - \left(\mathbb{E}\sum_{i_1,\ldots,i_k=1}^N H_{i_1,i_2}\cdots H_{i_k,i_1}\right)^2\right) \\
&= \frac{1}{N^2}\sum_{\mathbf{i},\mathbf{i}'\in\{1,2,\ldots,N\}^k} \left(\mathbb{E}\left[T_{\mathbf{i}}T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\,\mathbb{E}T_{\mathbf{i}'}\right)
\end{aligned}
$$

where $T_{(i_1,\ldots,i_k)} = H_{i_1,i_2}H_{i_2,i_3}\cdots H_{i_k,i_1}$.

Like a way we defined a graph for each $\mathbf{i} \in \{1,\ldots,N\}^k$, we may associate a pair $(\mathbf{i},\mathbf{i}')$ to a graph as follows:

> Let $\mathbf{i} = (i_1,\ldots,i_k), \mathbf{i}' = (i'_1,\ldots,i'_k) \in \{1,\ldots,N\}^k$. Define the graph $G_{\mathbf{i}} = (V_{\mathbf{i}}, E_{\mathbf{i}})$ associated with $\mathbf{i}$ where $V_{\mathbf{i}} = \{i_j : j \in \{1,\ldots,k\}\}$ and $E_{\mathbf{i}} = \{\{i_j, i_{j+1}\} : j \in \{1,\ldots,k\}\}$ with $i_{k+1} := i_1$. Also, define the graph $G_{\mathbf{i},\mathbf{i}'} = (V_{\mathbf{i},\mathbf{i}'}, E_{\mathbf{i},\mathbf{i}'})$ associated with $\{\mathbf{i},\mathbf{i}'\}$ where $V_{\mathbf{i},\mathbf{i}'} = V_{\mathbf{i}} \cup V_{\mathbf{i}'}$

and $E_{\mathbf{i},\mathbf{i}'} = E_{\mathbf{i}} \cup E_{\mathbf{i}'}$.

Traversing the graph $G_{\mathbf{i}}$ ($i_1 \to \cdots \to i_k \to i_1$) or $G_{\mathbf{i},\mathbf{i}'}$ ($i_1 \to \cdots \to i_k \to i_1$ and $i'_1 \to \cdots \to i'_k \to i'_1$), let $N_{\mathbf{i}}$ or $N_{\mathbf{i},\mathbf{i}'}(e)$ ($e \in E_{\mathbf{i}}$ or $e \in E_{\mathbf{i},\mathbf{i}'}$) be the number of times the traverse passes e (in any direction), respectively.

With these definitions, we obtain

$$\begin{aligned}\mathbb{E}\left[T_{\mathbf{i}}T_{\mathbf{i}'}\right] &= \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[H_e^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_e^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \\ &= \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[H_{11}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_{12}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right],\end{aligned}$$

where $H_e = H_{ij}$ if $e = \{i, j\}$, due to the identical distribution conditions. Similarly,

$$\begin{aligned}\mathbb{E}\left[T_{\mathbf{i}}\right] &= \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}\left[H_e^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_e^{N_{\mathbf{i}}(e)}\right] \\ &= \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}\left[H_{11}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_{12}^{N_{\mathbf{i}}(e)}\right].\end{aligned}$$

Because $\mathbb{E}H_{11} = \mathbb{E}H_{12} = 0$, unless $N_{\mathbf{i},\mathbf{i}'}(e) = N_{\mathbf{i}}(e) + N_{\mathbf{i}'}(e) \geq 2$ for all $e \in E_{\mathbf{i},\mathbf{i}'}$, we have $\mathbb{E}\left[T_{\mathbf{i}}T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\,\mathbb{E}T_{\mathbf{i}'} = 0$. Also when $E_{\mathbf{i}} \cap E_{\mathbf{i}'} = \emptyset$, we have $\mathbb{E}[T_{\mathbf{i}}T_{\mathbf{i}'}] = \mathbb{E}[T_{\mathbf{i}}]\mathbb{E}[T_{\mathbf{i}'}]$ due to the independence conditions. Moreover, when there is a bijection on $\{1, \ldots, N\}$ which maps $\mathbf{i}$ to $\mathbf{j}$ and $\mathbf{i}'$ to $\mathbf{j}'$, then we have

$$\mathbb{E}\left[T_{\mathbf{i}}T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\,\mathbb{E}T_{\mathbf{i}'} = \mathbb{E}\left[T_{\mathbf{j}}T_{\mathbf{j}'}\right] - \mathbb{E}T_{\mathbf{j}}\,\mathbb{E}T_{\mathbf{j}'}$$

due to the identical distribution (by applying the bijection on the product). So, this defines an equivalence relation on $\left(\{1, \ldots, N\}^k\right)^2$.

Now, we will count those equivalence classes (of $(\mathbf{i}, \mathbf{i}')$'s) by $|V_{\mathbf{i},\mathbf{i}'}| (\leq 2k)$. Let us $\mathcal{G}_v$ denote the set of all representatives for equivalence classes of $a_{\mathbf{i},\mathbf{i}'}$'s (defined by the bijection on $\{1, \ldots, N\}$) with $|V_{\mathbf{i},\mathbf{i}'}| = v$, $N_{\mathbf{i},\mathbf{i}'}(e) \geq 2$ for every $e \in E_{\mathbf{i},\mathbf{i}'}$, and $E_{\mathbf{i}} \cap E_{\mathbf{i}'} \neq \emptyset$. Note that the cardinality of an equivalence class is exactly $v!\binom{N}{v}$, if N is sufficiently large, that is, $N \geq v$.

Using this observation, we have (when $N \geq 2k$)

$$
\begin{aligned}
&\mathbb{E}\left[\langle \mu, x^k\rangle^2\right] - \left(\mathbb{E}\langle \mu, x^k\rangle\right)^2 \\
&= \frac{1}{N^2} \sum_{\mathbf{i},\mathbf{i}' \in \{1,2,\ldots,N\}^k} \left(\mathbb{E}\left[T_{\mathbf{i}} T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\, \mathbb{E}T_{\mathbf{i}'}\right) \\
&= \frac{1}{N^2} \sum_{\mathbf{i},\mathbf{i}' \in \{1,2,\ldots,N\}^k} \left(\mathbb{E}\left[T_{\mathbf{i}} T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\, \mathbb{E}T_{\mathbf{i}'}\right) \\
&= \frac{1}{N^2} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i},\mathbf{i}') \in \mathcal{G}_v} \left(\mathbb{E}\left[T_{\mathbf{i}} T_{\mathbf{i}'}\right] - \mathbb{E}T_{\mathbf{i}}\, \mathbb{E}T_{\mathbf{i}'}\right) \\
&= \frac{1}{N^2} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i},\mathbf{i}') \in \mathcal{G}_v} \Bigg(\prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[H_{11}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_{12}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \\
&- \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}\left[H_{11}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_{12}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[H_{11}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[H_{12}^{N_{\mathbf{i}}(e)}\right] \Bigg) \\
&= \frac{1}{N^{k+2}} \sum_{v=1}^{2k} v! \binom{N}{v} \sum_{(\mathbf{i},\mathbf{i}') \in \mathcal{G}_v} \Bigg(\prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[\hat{H}_{11}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i},\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[\hat{H}_{12}^{N_{\mathbf{i},\mathbf{i}'}(e)}\right] \\
&- \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a loop}}} \mathbb{E}\left[\hat{H}_{11}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}} \\ e \text{ is a non-loop}}} \mathbb{E}\left[\hat{H}_{12}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a loop}}} \mathbb{E}\left[\hat{H}_{11}^{N_{\mathbf{i}}(e)}\right] \cdot \prod_{\substack{e \in E_{\mathbf{i}'} \\ e \text{ is a non-loop}}} \mathbb{E}\left[\hat{H}_{12}^{N_{\mathbf{i}}(e)}\right] \Bigg)
\end{aligned}
$$

where $\hat{H}_{ij} \coloneqq N^{-1/2} H_{ij} \sim \mathcal{N}(0,1)$ if $i \neq j$, $\quad \hat{H}_{ij} \sim \mathcal{N}(0,2)$ if $i = j$.

Moreover, we can observe that $\mathcal{G}_v$ does not depend on N: as we can pick a representative whose set of vertices is contained in $\{1, \ldots, v\}$. Since any k-th moment of a normal random variable are finitely well-defined, $\sum_{(\mathbf{i},\mathbf{i}') \in \mathcal{G}_v} (\cdots)$ in the last line of the equation above does not depend on N. Denoting those terms (independent of N) as C_v, we have

$$
\mathbb{E}\left[\langle \mu, x^k\rangle^2\right] - \left(\mathbb{E}\langle \mu, x^k\rangle\right)^2 = \sum_{v=1}^{2k} C_v \cdot v! \cdot N^{-(k+2)} \binom{N}{v}.
$$

Therefore, it suffices to prove that $\mathcal{G}_v = \emptyset$ so that $C_v = 0$ for $v \geq k+2$, since other (lower degree) terms disappears as $N \to \infty$, since $\binom{N}{v} \sim N^v$.

Suppose $(\mathbf{i}, \mathbf{i}') \in \mathcal{G}_v$. Since $E_{\mathbf{i}} \cap E_{\mathbf{i}'} \neq \emptyset$, $G_{\mathbf{i},\mathbf{i}'}$ is connected, with v vertices and $\leq k$ edges, as every edge should be passed more than once during traverse. Since $v = |V(G_{\mathbf{i},\mathbf{i}'})| \leq |E(G_{\mathbf{i},\mathbf{i}'})| + 1 \leq k+1$ for a connected graph $G_{\mathbf{i},\mathbf{i}'}$, we have $\mathcal{G}_v = \emptyset$ when $v \geq k+2$. This completes the proof. $\square$