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MAS365 Introduction to Numerical Analysis

Homework 5

1. (a) Three-point midpoint formula yields an error no greater than $\frac{h^2}{6}|f^{(3)}(\xi)|$, and five-point endpoint formula yields an error no greater than $\frac{h^4}{5}|f^{(5)}(\xi)|$. With an assumption that $f^{(3)}$ and $f^{(5)}$ are of the same scale, the five-point endpoint formula is better than the three-point midpoint formula in general. Using the five-point midpoint formula if it is possible and the five-point endpoint formula otherwise, we have ($h = 0.2$)

- $f'(-3.0) \approx \frac{1}{12h}[-25f(-3.0) + 48f(-2.8) - 36f(-2.6) +16f(-2.4) - 3f(-2.2)] = -\mathbf{19.08087}$, (endpoint)
- $f'(-2.8) \approx \frac{1}{12h}[-25f(-2.8) + 48f(-2.6) - 36f(-2.4) +16f(-2.2) - 3f(-2.0)] = -\mathbf{15.44088}$, (endpoint)
- $f'(-2.6) \approx \frac{1}{12h}[f(-3.0) - 8f(-2.8) + 8f(-2.4) - f(2.2)] = -\mathbf{12.46303}$, (midpoint)
- $f'(-2.4) \approx \frac{1}{12h}[f(-2.8) - 8f(-2.6) + 8f(-2.2) - f(2.0)] = -\mathbf{10.02259}$, (midpoint)
- $f'(-2.2) \approx \frac{1}{-12h}[-25f(-2.2) + 48f(-2.4) - 36f(-2.6) +16f(-2.8) - 3f(-3.0)] = -\mathbf{8.020973}$, (endpoint, reversed)
- $f'(-2.0) \approx \frac{1}{-12h}[-25f(-2.0) + 48f(-2.2) - 36f(-2.4) +16f(-2.6) - 3f(-2.8)] = -\mathbf{6.385728}$, (endpoint, reversed)

(b)

x	$f(x)$	$f'(x)$ (approx)	$f'(x)$ (true)	$f'_{approx}(x) - f'_{true}(x)$
-3.0	16.08554	-19.08087	-19.08554	0.00467
-2.8	12.64465	-15.44088	-15.44465	0.00377
-2.6	9.863738	-12.46303	-12.46374	0.00071
-2.4	7.623176	-10.02259	-10.02318	0.00059
-2.2	5.825013	-8.020973	-8.025013	0.004040
-2.0	4.389056	-6.385728	-6.389056	0.003328

For those obtained by the midpoint formula, the error bound is ($h = 1/5, f^{(5)}(x) = -e^{-x}$)

$$|\text{absolute error}| \leq \frac{h^4}{30} \max_{x\in[-3,-2]} |f^{(5)}(x)| = e^3/(30 \cdot 5^4) \approx 0.0010712286.$$

For those obtained by the endpoint formula, the error bound is ($h = 1/5, f^{(5)}(x) = -e^{-x}$)

$$|\text{absolute error}| \le \frac{h^4}{5} \max_{x\in[-3,-2]} |f^{(5)}(x)| = e^3/5^5 \approx 0.0064273718.$$

Both bounds are admissible, seeing the actual data.

2.

$$\begin{array}{c|lllll}
- & M = & N(h) + & K_1h^2 + & K_2h^4 + & K_3h^6 + \cdots \\
+ & \big[M = & N(\frac{h}{3}) + & K_1(\frac{h}{3})^2 + & K_2(\frac{h}{3})^4 + & K_3(\frac{h}{3})^6 + \cdots \ \big] \times 3^2 \\
\hline
= & (3^2-1)M = & [3^2N(\frac{h}{3}) - N(h)] & & + K_2(\frac{1}{3^2}-1)h^4 + & K_3(\frac{1}{3^4}-1)h^6 + \cdots
\end{array}$$

Now define $\tilde{N}(h) = (3^2-1)^{-1}[3^2N(\frac{h}{3}) - N(h)]$ and

$$\tilde{K}_j = \frac{1}{3^2-1}(3^{-2(j-1)}-1)K_j.$$

In a similar fashion, we have

$$\begin{array}{c|llll}
- & M = & \tilde{N}(h) + & \tilde{K}_2h^4 + & \tilde{K}_3h^6 + \cdots \\
+ & \big[M = & \tilde{N}(\frac{h}{3}) + & \tilde{K}_2(\frac{h}{3})^4 + & \tilde{K}_3(\frac{h}{3})^6 + \cdots \ \big] \times 3^4 \\
\hline
= & (3^4-1)M = & [3^4\tilde{N}(\frac{h}{3}) - \tilde{N}(h)] & & + K_3(\frac{1}{3^2}-1)h^6 + \cdots
\end{array}$$

Therefore,

$$\begin{aligned}
\approx\!\!\!\!\!N(h) &:= \frac{1}{3^4-1}\left(3^4\tilde{N}\left(\frac{h}{3}\right) - \tilde{N}(h)\right) \\
&= \frac{3^6}{(3^4-1)(3^2-1)} N\left(\frac{h}{9}\right) \\
&\qquad - \frac{3^4+3^2}{(3^4-1)(3^2-1)} N\left(\frac{h}{3}\right) + \frac{1}{(3^4-1)(3^2-1)} N(h) \\
&= \frac{1}{640}\left[729N\left(\frac{h}{9}\right) - 90N\left(\frac{h}{3}\right) + N(h)\right]
\end{aligned}$$

is an $O(h^6)$ approximation to M.

3. (a)

h	$N(h)$
0.01	2.718304481241747
0.02	2.718372444800622
0.04	2.718644377221238

An $O(h^j)$ approximation of e can be obtained by the following recurrence formula: $N_1(h) = N(h) = e + O(h)$,

$$N_j(h) = N_{j-1}(h/2) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{2^{j-1} - 1},$$

and it gives

$$\begin{aligned} N_2(h) &= 2N(h/2) - N(h) = 2\left(\frac{4+h}{4-h}\right)^{2/h} - \left(\frac{2+h}{2-h}\right)^{1/h}, \\ N_3(h) &= N_2(h/2) + \frac{N_2(h/2) - N_2(h)}{3} = \frac{4}{3}N_2(h/2) - \frac{1}{3}N_2(h) \\ &= \frac{8}{3}N(h/4) - 2N(h/2) + \frac{1}{3}N(h) = e + O(h^3). \end{aligned}$$

Evaluating at $h = 0.4$, we have $N_3(0.4) = 2.718281852783827$.

(b) Since $e - N(h) = K_1 h + K_2 h^2 + \cdots$ is an even function, every coefficient of an odd degree term should be vanished. Therefore, $K_j = 0$ for every odd j. Then an $O(h^{2j})$ approximation $\tilde{N}_j$ to e can be made as follows: $\tilde{N}_1(h) = N(h) = e + O(h^2)$,

$$\begin{array}{c|lllll} - & e = & \tilde{N}_{j-1}(h) + & \tilde{K}_{2(j-1)} h^{2(j-1)} + & \tilde{K}_{2j} h^{2j} + \cdots & \\ + & [e = & \tilde{N}_{j-1}(\frac{h}{2}) + & \tilde{K}_{2(j-1)} (\frac{h}{2})^{2(j-1)} + & \tilde{K}_{2j} (\frac{h}{2})^{2j} + \cdots &] \times 2^{2(j-1)} \\ \hline = & (2^{2(j-1)} - 1)M = & [2^{2(j-1)} \tilde{N}_{j-1}(\frac{h}{2}) - \tilde{N}_{j-1}(h)] & & + \tilde{K}_{2j}(\frac{1}{2^2} - 1) h^{2j} + \cdots & \end{array},$$

i.e.,

$$\tilde{N}_j(h) = \tilde{N}_{j-1}(h/2) + \frac{\tilde{N}_{j-1}(h/2) - \tilde{N}_{j-1}(h)}{2^{2(j-1)} - 1} = e + O(h^{2j}).$$

We need to find $\tilde{N}_3$:

$$\begin{aligned} \tilde{N}_2(h) &= N(h/2) + \frac{N(h/2) - N(h)}{3} = \frac{4}{3}N(h/2) - \frac{1}{3}N(h), \\ \tilde{N}_3(h) &= \tilde{N}_2(h/2) + \frac{\tilde{N}_2(h/2) - \tilde{N}_2(h)}{15} \end{aligned}$$

$$= \frac{1}{45}\left[64N\left(\frac{h}{4}\right) - 20N\left(\frac{h}{2}\right) + N(h)\right].$$

Evaluating at $h = 0.4$, we have $\tilde{N}_3(0.4) = 2.718281828459570$.

4. With $f(x) = x^k$ $(k = 0, 1, 2, 3, 4)$,

$$\int_{-1}^{1} dx = 2 = a + b + c,$$
$$\int_{-1}^{1} x^k\, dx = \frac{1}{k+1}(1 - (-1)^{k+1}) = a \cdot (-1)^k + c + dk(-1)^{k-1} + ek. \quad (k > 0)$$

Thus, we have

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 2 \\ -1 & 0 & 1 & 3 & 3 \\ 1 & 0 & 1 & -4 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2/3 \\ 0 \\ 2/5 \end{pmatrix},$$

which yields $(a, b, c, d, e) = \frac{1}{15}(7, 16, 7, 1, -1)$.