20170058 Keonwoo Kim · April 19, 2020

Quiz 4

Problems

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#1. The linear equations can be transformed into the following matrix form: $A\mathbb{x} = \mathbb{b}$ where

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbb{b} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}.$$

Thus, the solution, provided by $\det A \neq 0$, is $\mathbb{x} = (x_1, x_2, x_3)^T$ where

$$x_1 = \frac{1}{\det A} \begin{vmatrix} 2 & 3 & -1 \\ 5 & 0 & -3 \\ 1 & 1 & -2 \end{vmatrix} = \frac{-38}{-8} = \frac{19}{4},$$

$$x_2 = \frac{1}{\det A} \begin{vmatrix} 2 & 2 & -1 \\ 2 & 5 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \frac{16}{-8} = -2, \text{ and}$$

$$x_3 = \frac{1}{\det A} \begin{vmatrix} 2 & 3 & 2 \\ 2 & 0 & 5 \\ 0 & 1 & 1 \end{vmatrix} = \frac{-12}{-8} = \frac{3}{2}.$$

Therefore, the solution is $\mathbb{x} = \left(\frac{19}{4}, -2, 3/2\right)^T$.

#2. (i) The characteristic polynomial of A is:

$$\det(tI - A) = \begin{vmatrix} t-2 & 0 & 0 \\ 10 & t+8 & -5 \\ 10 & 10 & t-7 \end{vmatrix}$$
$$= (t-2)\left[(t+8)(t-7) - (-5)\cdot 10\right]$$
$$= (t-2)^2(t+3).$$

Thus, the eigenvalues of A are 2 and -3.

(ii) $\lambda = 2$. With $\mathbf{x} = (x_1, x_2, x_3)^T$,

$$A\mathbf{x} = 2\mathbf{x} \iff \begin{cases} 2x_1 = 2x_1 \\ -10x_1 - 8x_2 + 5x_3 = 2x_2 \\ -10x_1 - 10x_2 + 7x_3 = 2x_3 \end{cases}$$

$$\iff -10x_1 - 10x_2 + 5x_3 = 0$$

$$\iff x_3 = 2x_1 + 2x_2.$$

~~Note~~ Thus, the eigenspace of A corresponding to the largest eigenvalue 2 of A is:

$$\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_3 = 2x_1 + 2x_2\}$$
$$= \{(x_1, x_2, 2x_1 + 2x_2) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\}.$$

Summary

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Apr 07. * Permutation : $\pi \in S_n$, $i \xmapsto{\text{bij}} \pi_i \in \{1, \dots, n\}$

- "Inversion" : a pair (i, j) w/ $i < j$ s.t. $\pi_j < \pi_i$.
- even permutation : a permutation having even # of inversions. $\to \operatorname{sgn}(\pi) = 1$
- o/w, it is called an odd permutation. $\to \operatorname{sgn}(\pi) = -1$.

* Determinant.

$$\det A := \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\, a_{1,\pi_1} \cdots a_{n,\pi_n}.$$

* Minor: the determinant of the matrix obtained from A by removing i^{th} row & j^{th} column. $=: M_{i,j}$.

* Cofactor $C_{ij} := (-1)^{i+j} M_{ij}$.

* Adjoint of A : $(\operatorname{adj} A)_{ij} = C_{ji}$.

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik}.$$

* Properties of det

- multilinearity.
- alternating : $\det(\cdots a_i \cdots a_j \cdots) = -\det(\cdots a_j \cdots a_i \cdots)$.
- └ equivalently, $\det(\cdots a_i \cdots a_i \cdots) = 0$. (same columns)
- $\exists A^{-1} \iff \det A \neq 0$.
- $\det A = \det(A^T)$.
- $\det(AB) = \det A \cdot \det B$.
- $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$.

* Cramer's rule. If A is invertible,

$$\mathbf{x} = A^{-1}\mathbf{b} = (x_i) \text{ where } x_i = \frac{\det(A_i)}{\det A},$$

(A_i: the matrix obtained from A by switching its i^{th} column by $\mathbf{b}$.)

* Determinants can be calculated using elem. row op's:

$$A = E_m E_{m-1} \cdots E_2 E_1. \qquad E_i : \text{elem row op's.}$$

$$\det(E(i;k)) = k, \quad \det(E(i,j;c)) = 1,$$
$$\det(E(i,j)) = -1.$$

Apr 09. * $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq 0$

$\Rightarrow$ $\mathbf{x}$: eigenvector corresponding to the eigenvalue λ.

\- $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$ $\cdots$ $\det(\lambda I - A) = 0$.

* Characteristic polynomial of A: $\det(tI - A)$.

* Eigenspace of A corresponding to ~~ξ~~ λ:

$$\{v \in \mathbb{R}^n : Av = \lambda v\} = \ker(\lambda I_n - A).$$

* Eigenvalue/-vectors for linear operators:

$$Tv = \lambda v, \quad v \neq 0.$$

This coincides w/ the ones of $[T]_{\mathcal{B}}$ for any basis $\mathcal{B}$ of $\mathbb{R}^n$.

* For an eigenvector v ~~corres~~ of A corresponding to λ, and a polynomial $p(x) = a_n x^n + \cdots + a_0$,

$$p(A)v = a_n A^n v + \cdots + a_0 v$$
$$= a_n \lambda^n v + \cdots + a_0 v = p(\lambda) \cdot v.$$