20170058 **Keonwoo Kim** · April 13, 2020
MAS365 Introduction to Numerical Analysis

Homework 3

1. Using the Theorem 3 in the lecture note for Chapter 3 could be a way to obtain a bound for the absolute error. Note $f(x) = x^{-2}$ is smooth on $[1, 2]$, so, for the Lagrange interpolating polynomial P, we have

$$\begin{aligned} |f(x) - P(x)| &= \left| \frac{f^{(3)}(\xi(x))}{3!} (x-1)\left(x - \frac{11}{8}\right)(x-2) \right| \\ &= \left| -4\,\xi(x)^{-5} (x-1)\left(x - \frac{11}{8}\right)(x-2) \right| \\ &\le 4 \cdot 1^{-5} \cdot \frac{9}{128} = \frac{9}{32} \approx 0.28125 \end{aligned}$$

because $\xi(x) \in [1, 2]$ and $(x-1)(x-11/8)(x-2)$ has two extrema on $[1, 2]$, which are at $x = 7/4$ and $x = 7/6$, where the former one gives the maximum absolute value of $(x-1)(x-11/8)(x-2)$ on $[1, 2]$.

(Idea for an alternative solution) We also can directly calculate the error term. Since $(f(x) - P(x))' = 0$ is a quartic equation, which can be always solved with an exact solution using radicals, the maximum of $|f(x) - P(x)|$ also can be calculated exactly, though it is difficult for a human. The maximum absolute error is approximately 0.041300, which is quite better than the answer above.

2. Suppose $x \neq x_0$, otherwise we have $f(x_0) - P_n(x_0) = 0$. Denote $R_n(x) = f(x) - P_n(x)$, and let

$$g(t) = R_n(t) - R_n(x)\,\frac{(t - x_0)^{n+1}}{(x - x_0)^{n+1}}.$$

I will use somewhat different generalization of the Rolle's theorem than the Theorem 1.10 in the textbook:

Suppose $f \in C^n[a, b]$ and $f^{(n+1)}$ exists at $x = a$, with $f(a) = \cdots = f^{(n)}(a) = 0$ and $f(b) = 0$. Then there is a number $c \in (a, b)$ with $f^{(n+1)}(c) = 0$.

This can be proved easily: find $\xi_1 \in (a, b)$ satisfying $f'(\xi_1) = 0$ using

Rolle's to f, find $\xi_2 \in (a, \xi_1)$ satisfying $f''(\xi_1) = 0$ using Rolle's to f', and so on.

Now the problem became easy: since

$$g^{(j)}(t) = R_n^{(j)}(t) - (n+1)\cdots(n+2-j)\frac{(t-x_0)^{n+1-j}}{(x-x_0)^{n+1}}$$

so that

$$g^{(j)}(x_0) = f^{(j)}(x_0) - P_n^{(j)}(x_0) = 0.$$

Also, $g(x) = 0$. Thus, applying the generalization of the Rolle's theorem above, we have a constant $c =: \xi(x)$ between x_0 and x, depending only on x. Since both x_0 and x are on $[a, b]$, we have $\xi(x) \in (a, b)$.

3. See the code below for the implementation.

(a) $\sqrt{3} = f(0.5) \approx 1.70833333$.

(b) $\sqrt{3} = f(3) \approx 1.73138614$.

```
% Neville's Method

% Define f(x)
f = @(x) 3^x;             % for (a)
% f = @(x) sqrt(x);       % for (b)

X = [-2, -1, 0, 1, 2];
N = length(X);

% Q{i, j} = Q_{i - 1, j - 1}
Q = cell(N, N);

for i = 1:N
    Q{i, 1} = [f(X(i))];
end

% Calculating Q{i, j}
for j = 2:N
    for i = j:N
        % (x - x_{i-j})Q_{i, j-1}
        first_term = sum_poly(     ...
            [Q{i, j-1} 0],         ...
            -X(i-j+1) * Q{i, j-1} ...
        );
        % -(x - x_{i})Q_{i-1, j-1}
        second_term = sum_poly(    ...
            -1 * [Q{i-1, j-1} 0], ...
```

```
                X(i) * Q{i-1, j-1}    ...
            );

            Q{i, j} =                               ...
                sum_poly(first_term, second_term) / ...
                (X(i) - X(i-j+1));
        end
    end

    % Evaluate the polynomial
    x = 0.5;    % for (a)
    % x = 3;    % for (b)
    fprintf('f(%f) = %.8f\n', x, polyval(Q{N, N}, x))

    % x * P = [P 0]
    % a * P = a * P
    % P + Q
    function s = sum_poly(a, b)
        N = max(length(a), length(b));
        pa = [zeros(1, N - length(a)) a];
        pb = [zeros(1, N - length(b)) b];
        s = pa + pb;
        return
    end
```

4. (a) Note $P_{1,2} = Q_{2,1}$ and $P_2 = Q_{2,0}$. With $x = 0.5$,

$$\frac{27}{7} = Q_{2,2} = \frac{(x-x_0)Q_{2,1} - (x-x_2)Q_{1,1}}{x_2 - x_0} = \frac{5}{7}Q_{2,1} + 1, \qquad Q_{2,1} = 4;$$
$$4 = Q_{2,1} = \frac{(x-x_1)Q_{2,0} - (x-x_2)Q_{1,0}}{x_2 - x_1} = \frac{Q_{2,0} + 5.6}{3}; \qquad Q_{2,0} = 6.4.$$

(b)

$$P_{0,1,2}(1.5) = Q_{2,2}(1.5) = \left.\frac{(x-x_0)Q_{2,1}(x) - (x-x_2)Q_{1,1}(x)}{x_2 - x_0}\right|_{x=1.5} = 3.25;$$
$$P_{0,1,2,3}(1.5) = Q_{3,3}(1.5) = \left.\frac{(x-x_0)Q_{3,2}(x) - (x-x_3)Q_{2,2}(x)}{x_3 - x_0}\right|_{x=1.5} = 3.625.$$

(b: alternative way) Since $P_{1,2,3}(x)$ is a quadratic polynomial with $P_{1,2,3}(1) = P_{1,2}(1) = 2$, $P_{1,2,3}(1.5) = 4$, and $P_{1,2,3}(2) = P_{1,2}(2) = 5$, we have $P_{1,2,3}(x) = -2x^2 + 9x - 5$, having $P_{0,1,2,3}(3) = 4$. So the graph of $P_{0,1,2,3}$ passes $(0,1), (1,2), (2,5)$, and $(3,4)$, we have

$$P_{0,1,2,3}(x) = -x^3 + 4x^2 - 2x + 1.$$

Thus $P_{0,1,2,3}(1.5) = 3.625$.

5. Let P be a polynomial of least degree, and show $P = H_{2n+1}$. In this way, we can simultaneously show that H_{2n+1} is such a polynomial of *least* degree and is actually unique. Note H_{2n+1} is a polynomial agreeing with f and f' at given points, though we do not yet know that H_{2n+1} is of least degree. Thus, by the minimality of the degree of P, we have $\deg P \le \deg H_{2n+1}$. Consequently, $\deg(H_{2n+1} - P) = \deg D \le \deg H_{2n+1} \le 2n+1$. Then, $\deg D \le 2n$. However, by the Rolle's theorem, we have $D'(\xi_i) = 0$ with some $x_i < \xi_i < x_{i+1}$ $(i = 0, \ldots, n-1)$ so that $D'(x) = 0$ for $x = x_0, \ldots, x_n, \xi_0, \ldots, \xi_{n-1}$. According to the fundamental theorem of algebra, $D' \equiv 0$. Hence D is a constant, which should be 0, because, say, $D(x_0) = 0$.

(Alternative solution) $D(x_i) = D'(x_i) = 0$ for $i = 0, \ldots, n$. Letting $D(x) = \sum_{i=0}^{2n+1} d_i x^i$ (d_i could be zero),

$$\begin{aligned}
&d_0 + d_1 x_0 + \cdots + d_{2n+1} x_0^{2n+1} = 0, \quad && d_1 + 2d_2 x_0 + \cdots + (2n+1) d_{2n+1} x_0^{2n} = 0,\\
&\qquad\vdots && \qquad\vdots\\
&d_0 + d_1 x_0 + \cdots + d_{2n+1} x_n^{2n+1} = 0, \quad && d_1 + 2d_2 x_n + \cdots + (2n+1) d_{2n+1} x_n^{2n} = 0,
\end{aligned}$$

i.e.,

$$\begin{pmatrix}
1 & x_0 & \cdots & x_0^{2n+1}\\
\vdots & \vdots & \ddots & \vdots\\
1 & x_n & \cdots & x_n^{2n+1}\\
0 & 1 & \cdots & (2n+1)x_0^{2n}\\
\vdots & \vdots & \ddots & \vdots\\
0 & 1 & \cdots & (2n+1)x_n^{2n}
\end{pmatrix}
\begin{pmatrix}
d_0\\ \vdots\\ d_n\\ d_{n+1}\\ \vdots\\ d_{2n+1}
\end{pmatrix} = \mathbf{0}.$$

Redefine n to be $n-1$ in the formulae above for the sake of simplicity. The determinant of the $2n \times 2n$ square matrix on the left side can be calculated by the induction on the number ℓ of rows with leading zero in the following form of matrix determinant: $(0 \le \ell \le m := 2n - \ell)$

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{m-1} & \cdots & x_{m-1}^{2n-1} \\ 0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2} \end{vmatrix} = (-1)^{\ell(\ell-1)/2} \prod_{0 \le i < j \le m-1} (x_j - x_i)^{\alpha_i \alpha_j}$$

where $\alpha_j = 2$ if $j < \ell$, and $\alpha_j = 1$ otherwise. When $\ell = 0$, we have nothing to do due to the Vandermonde determinant. Observe that expanding the determinant along m-th row, the other parts does not depend on x_{m-1}. As an induction step (assuming $0 \le \ell < n$ so $\alpha_{m-1} = 1$),

$$\begin{aligned} \frac{\partial}{\partial x_{m-1}} \begin{vmatrix} 1 & x_0 & \cdots & x_0^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{m-1} & \cdots & x_{m-1}^{2n-1} \\ 0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2} \end{vmatrix} &= \begin{vmatrix} 1 & x_0 & \cdots & x_0^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{m-2} & \cdots & x_{m-2}^{2n-1} \\ 0 & 1 & \cdots & (2n-1)x_{m-1}^{2n-2} \\ 0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2} \end{vmatrix} \\ &= (-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\alpha_i \alpha_j} \cdot \frac{\partial}{\partial x_{m-1}} \prod_{i=0}^{m-2} (x_{m-1} - x_i)^{\alpha_i}. \end{aligned}$$

By replacing x_{m-1} by x_ℓ, we have

$$\begin{aligned}
&\begin{vmatrix}
1 & x_0 & \cdots & x_0^{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_{m-2} & \cdots & x_{m-2}^{2n-1} \\
0 & 1 & \cdots & (2n-1)x_\ell^{2n-2} \\
0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2}
\end{vmatrix} \\
&= (-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\alpha_i \alpha_j} \cdot \frac{\partial}{\partial x_{m-1}} \prod_{i=0}^{m-2} (x_{m-1} - x_i)^{\alpha_i} \Big|_{x_{m-1}=x_\ell} \\
&= (-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\alpha_i \alpha_j} \cdot \left(\sum_{k=0}^{m-2} \alpha_k (x_{m-1} - x_k)^{\alpha_i - 1} \prod_{\substack{i=0 \\ i\neq k}}^{m-2} (x_{m-1} - x_i)^{\alpha_i} \Big|_{x_{m-1}=x_\ell} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(the summand is zero when } k \neq \ell) \\
&= (-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\alpha_i \alpha_j} \prod_{\substack{i=0 \\ i\neq \ell}}^{m-2} (x_\ell - x_i)^{\alpha_i} \qquad\qquad (\alpha_\ell = 1) \\
&= (-1)^{\ell(\ell-1)/2} \prod_{\substack{i<j<m-1 \\ i\neq\ell\neq j}} (x_j - x_i)^{\alpha_i \alpha_j} \prod_{i<\ell} (x_\ell - x_i)^{\alpha_i} \prod_{j>\ell} (x_j - x_\ell)^{\alpha_j} \prod_{\substack{i=0 \\ i\neq \ell}}^{m-2} (x_\ell - x_i)^{\alpha_i} \\
&= (-1)^{\ell(\ell-1)/2} \prod_{\substack{i<j<m-1 \\ i\neq\ell\neq j}} (x_j - x_i)^{\alpha_i \alpha_j} \left(\prod_{i<\ell} (x_\ell - x_i)^{\alpha_i} \prod_{j>\ell} (x_j - x_\ell)^{\alpha_j} \right)^2 \cdot (-1)^{m-1-(\ell+1)} \\
&= (-1)^{\ell(\ell-1)/2} \prod_{\substack{i<j<m-1 \\ i\neq\ell\neq j}} (x_j - x_i)^{\alpha_i \alpha_j} \prod_{\substack{i<j \\ i=\ell \text{ or } j=\ell \\ \alpha=\alpha_j \text{ if } i=\ell;\ \alpha_i \text{ o/w}}} (x_j - x_i)^{2\alpha} \\
&= (-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\tilde{\alpha}_i \tilde{\alpha}_j}
\end{aligned}$$

where $\tilde{\alpha}_i = \alpha_i$ if $i \neq \ell$ and $\tilde{\alpha}_\ell = 2$. (Note that $m - 1 - (\ell + 1) \equiv 0 \pmod 2$.)

Therefore,

$$
\begin{aligned}
&\begin{vmatrix}
1 & x_0 & \cdots & x_0^{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_{m-2} & \cdots & x_{m-2}^{2n-1} \\
0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2} \\
0 & 1 & \cdots & (2n-1)x_{\ell}^{2n-2}
\end{vmatrix} \\
&= (-1)^{\ell}
\begin{vmatrix}
1 & x_0 & \cdots & x_0^{2n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_{m-2} & \cdots & x_{m-2}^{2n-1} \\
0 & 1 & \cdots & (2n-1)x_{\ell}^{2n-2} \\
0 & 1 & \cdots & (2n-1)x_0^{2n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & (2n-1)x_{\ell-1}^{2n-2}
\end{vmatrix} \\
&= (-1)^{\ell}(-1)^{\ell(\ell-1)/2} \prod_{i<j<m-1} (x_j - x_i)^{\tilde{\alpha}_i \tilde{\alpha}_j} \\
&= (-1)^{(\ell+1)\ell/2} \prod_{i<j\le m-2} (x_j - x_i)^{\tilde{\alpha}_i \tilde{\alpha}_j}
\end{aligned}
$$

as desired. Thus, the determinant is zero iff there are duplicated points, but the problem asserted that all x_i's are distinct. So the linear equation above has only the trivial solution.