20170058 **Keonwoo Kim** · April 7, 2020
MAS365 Introduction to Numerical Analysis

Homework 2

1. (a) Since the $\lim_{n\to\infty} p_n = 0$, $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^k = 1$ implies that p_n converges sublinearly to 0.

(b) Since the $\lim_{n\to\infty} p_n = 0$, $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1 \in (0,\infty)$ implies that p_n converges quadratically to 0.

2. Suppose $m \geq 2$, since the case of $m = 1$ is covered by the class material. Letting $f(x) = (x-p)^m h(x)$ with $h(p) \neq 0$, $h^{(m)}(x) = (f(x)/(x-p)^m)^{(m)}$ exists and is continuous, and

$$\begin{aligned}
g'(x) &= 1 - \frac{mf'(x)^2 - mf(x)f''(x)}{f'(x)^2} \\
&= 1 - m + \frac{mf(x)f''(x)}{f'(x)^2} \\
&= 1 - m + m(x-p)^m h(x) \times \\
&\quad \frac{(x-p)^m h''(x) + 2m(x-p)^{m-1}h'(x) + m(m-1)(x-p)^{m-2}h(x)}{((x-p)^m h'(x) + m(x-p)^{m-1}h(x))^2} \\
&= 1 - m + mh(x)\,\frac{(x-p)^2 h''(x) + 2m(x-p)h'(x) + m(m-1)h(x)}{((x-p)h'(x) + mh(x))^2}
\end{aligned}$$

so that

$$g'(p) = 1 - m + mh(p)\,\frac{m(m-1)h(p)}{(mh(p))^2} = 0.$$

3. (a) Note that $p = e^{-x}$ with $p_n - p = p_{n+1} - p - \frac{(-x)^{n+1}}{(n+1)!}$ and we have

$$|p_{n+1} - p| \leq \frac{x^{n+2}}{(n+2)!}$$

by the estimation for the error term in $(n+1)^{\text{th}}$ Taylor polynomial. So,

$$\lim_{n\to\infty} \left| \frac{p_{n+1} - p}{(-x)^{n+1}/(n+1)!} \right| \leq \lim_{n\to\infty} \left| \frac{x^{n+2}/(n+2)!}{(-x)^{n+1}/(n+1)!} \right| = \lim_{n\to\infty} \frac{|x|}{n+2} = 0.$$

This implies that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}\right| &= \lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_{n+1}-p-\frac{(-x)^{n+1}}{(n+1)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\frac{p_{n+1}-p}{(-x)^{n+1}/(n+1)!}}{\frac{p_{n+1}-p}{(-x)^{n+1}/(n+1)!}-1}\right| = 0\end{aligned}$$

whence $\lim_{n\to\infty}\frac{p_{n+1}-p}{p_n-p}=0<1$.

(b)

$$\begin{aligned}\hat{p}_0 &= 0.33333, \quad \hat{p}_1 = 0.37500, \quad \hat{p}_2 = 0.36667 \\ \hat{p}_3 &= 0.36806, \quad \hat{p}_4 = 0.36786, \quad \hat{p}_5 = 0.36788.\end{aligned}$$

```
% MATLAB code
x = 1;
N = 5;

p = zeros(1, 1 + N + 2);
p(1) = 1;

% p_j
for j=1:N+2
    p(j+1) = p(j) + (-x)^(j) / factorial(j);
end

% \hat p_j
hp = zeros(1, 1 + N);
for j=1:1+N
    hp(j) = ...
        p(j) - ((p(j+1) - p(j))^2) / ...
        (p(j+2) - 2 * p(j+1) + p(j));
    fprintf('\\hat p_{%d} = %.5f\n', j-1, hp(j))
end

fprintf('true value: %.5f\n', exp(-1))
```

4.

$$\begin{aligned}P(x) &= \sum_{k=0}^{2} f(x_k)L_{n,k}(x) \\ &= 1\cdot\frac{(x-\frac{11}{8})(x-2)}{(1-\frac{11}{8})(1-2)} + \frac{8^2}{11^2}\cdot\frac{(x-1)(x-2)}{(\frac{11}{8}-1)(\frac{11}{8}-2)} + \frac{1}{2^2}\cdot\frac{(x-1)(x-\frac{11}{8})}{(2-1)(2-\frac{11}{8})}.\end{aligned}$$