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Preface

This lecture note is based on *Linear Algebra, 2nd edition* by K. Hoffman and R. Kunze.

Chapter 1

Vector Space

1.1 Matrix

Remind. A **field** is a *good* algebraic structure, which has the addition and the multiplication. Formally, a field $(F, +, \cdot)$, or simply F , is a pair of a set and two operations which is from $F \times F$ to F satisfying the following:

- $(F, +)$ is an **abelian group**, that is, $+$ is commutative, associative, and there is an additional identity 0 and the inverse element $-a$ of a for all $a \in F$.
- (F^\times, \cdot) is also an abelian group, that is, \cdot is commutative, associative, and there is an multiplicative identity 1 and the inverse element a^{-1} of a for all $a \in F^\times$, where $F^\times = F - \{0\}$.
- $+$ and \cdot are *compatible*, which means \cdot is distributing over $+$.

We simply write $a - b := a + (-b)$ and $a/b = ab^{-1}$.

A **matrix** over a field F is a rectangular arrangement of *scalars*, elements of the field F . The space of m by n matrices is denoted as $\mathfrak{M}_{m,n}(F)$.

Example 1.1.1.

- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are well-known(?) fields.
- $\mathfrak{M}_{m,n}(F) \approx F^{mn}$, without the product.
- Matrices do not form a field.

1.1.1 Transpose and Trace

Definition 1.1.1. For every m by n matrix $A \in \mathfrak{M}_{m,n}(F)$, the **transpose** of A is defined as follows:

$$A^T = (a_{ji})_{n,m}.$$

For every n by n square matrix $A \in \mathfrak{M}_{n,n}(F)$, the **trace** of A is defined as follows:

$$\operatorname{tr} A = \sum_{i=1}^n a_{ii}.$$

Proposition 1.1.1 (Linearity of transpose and trace). *For every pair of m by n matrices A and $B \in \mathfrak{M}_{m,n}(F)$ and every pair of scalars $a, b \in F$,*

$$(aA + bB)^\top = aA^\top + bB^\top.$$

For every pair of n by n square matrices A and $B \in \mathfrak{M}_{n,n}(F)$ and every pair of scalars $a, b \in F$,

$$\operatorname{tr}(aA + bB) = a \operatorname{tr} A + b \operatorname{tr} B.$$

Proof. ☞ ☞. □

Proposition 1.1.2 (Behaviour of transpose and trace).

$$(AB)^\top = B^\top A^\top, \quad \operatorname{tr} A = \operatorname{tr} A^\top, \quad \operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Proof. Try it! □

1.1.2 Inverse Matrix

Definition 1.1.2. For a square matrix $A \in \mathfrak{M}_{n,n}(F)$, if there is another square matrix $B \in \mathfrak{M}_{n,n}(F)$ such that

$$AB = I = BA,$$

then we call $B = A^{-1}$ the(?) **inverse matrix** of A .

Proposition 1.1.3 (Uniqueness of inverse matrix). *The inverse matrix of a matrix A is unique (if exists). This justifies the occurrence of ‘**the**’ above.*

Proof. Let those be A^{-1} and \tilde{A}^{-1} , then

$$\tilde{A}^{-1} = (A^{-1}A)\tilde{A}^{-1} = A^{-1}(A\tilde{A}^{-1}) = A^{-1}.$$

□

Question. If $AB = I$ for two square matrices A and B , what can we say about the invertibility of them? We will solve this problem using a ‘function,’ which is from and to some vector spaces, defined below.

1.2 Vector Space

Definition 1.2.1. A **vector space** V over F , or simply an **F -vector space** V , is a *good* algebraic structure, which has the addition $+$: $V \times V \rightarrow V$ and the F -scalar multiplication SM_F : $F \times V \rightarrow V$. Formally, a field $(V, F, +, \text{SM}_F)$, or simply V , is a pair of a set, a field and two operations satisfying the following:

- $(V, +)$ is an abelian group.
- SM_F and \cdot_F are compatible: $(ab)v = a(bv)$ for every $a, b \in F$ and $v \in V$, and $1v = v$ for all $v \in V$.
- $+$ and SM_F are compatible: $(a+b)v = av + bv$, $a(v+w) = av + aw$ for every $a, b \in F$ and $v, w \in V$.

We simply write $v - w := v + (-w)$.

Example 1.2.1. The following structures are examples of vector space.

- $\{0\}$ is a vector space over *arbitrary field*, and is called the **trivial space**.
- \mathbb{R}^n and \mathbb{C}^n are vector spaces. In fact, for any field F , F^n is an F -vector space, trivially.
- Hence, a matrix space $\mathfrak{M}_{m,n}$ of m by n matrices over F is a vector space since is *the same with* F^{mn} , and the polynomial space of n -th degree $\mathbf{P}_n[t] = \{\sum_{i=0}^n a_i t^i : a_i \in F\}$ is also a vector space, *the same with* F^{n+1} .
- (Field extension) If there are two fields which one is a subfield of another, namely $E \geq F$, then E is a F -vector space, with its addition and multiplication (as a scalar multiplication.)
- (**dual space!**) A **linear functional** f on V is a *linear* map from V to F , that is,

$$f(av + b) = af(v) + b$$

for every $a, b \in F$ and $v \in V$. The **dual space** of V is the space of linear functionals on V , and it forms a F -vector space, with the following operations:

$$(f + g)(v) = f(v) + g(v), \quad (af)(v) = af(v).$$

The dual space is one of the most interesting things not only in linear algebra, but in abstract algebra, or even in *any* branches which use the term *dual* (e.g., in projective geometry).

Proposition 1.2.1.

- $0v = 0$ and $(-1)v = -v$.

Proof. $0v = (0+0)v = 0v + 0v$ implies $0v = 0$ and $v + (-v) = 0 = 0v = (1+(-1))v = 1v + (-1)v = v + (-1)v$ implies $-v = (-1)v$. \square

- Every **linear combination** $\sum_i a_i v_i$ of vectors $v_i \in V$ is in V .

Proof. Easy induction on the number of summands(terms). \square

Definition 1.2.2. A subset $W \subseteq V$ is called a **subspace** of V if it forms a vector space itself with the *inherited* operations from V . We denote it $W \leq V$.

Proposition 1.2.2.

$$W \leq V \iff \forall c \in F, \forall v, w \in W, \quad cv + w \in W.$$

Proof. (\Leftarrow) The other axioms of vector space is satisfied by the fact that the operations are inherited by V , and hence it suffices to show that the operations are closed. First, let $c = -1$ and $v = w$. Then we have $(-1)w + w = 0 \in W$ by the proposition above. Then, the addition is closed if we let $c = 1$; and so is the scalar multiplication if we let $w = 0$, which is in W , as we proved. Hence W forms a vector space.

(\Rightarrow) $cv + w$ is a linear combination. $\Rightarrow \Rightarrow$. \square

Example 1.2.2.

- Removing a coordinate(**projection**):

$$F^2 = \{(a, b) : a, b \in F\} \leq \{(a, b, c) : a, b, c \in F\} = F^3.$$

- **Symmetric, alternating** matrices over F :

$$\text{Sym}_n(F) = \{A \in \mathfrak{M}_{n,n}(F) : A = A^\top\} \leq M_{n,n}(F),$$

$$\text{Alt}_n(F) = \{A \in \mathfrak{M}_{n,n}(F) : A = -A^\top\} \leq M_{n,n}(F).$$

- **Hermitian** matrices:

$$\text{Her}_n = \{A \in \mathfrak{M}_{n,n}(\mathbb{C}) : A = \overline{A^\top}\} \leq M_{n,n}(\mathbb{C}).$$

- The solution space of a system of homogeneous linear (differential) equations.

1.2.1 Spanned subspace

Definition 1.2.3. For a subset S of a vector space V , the **subspace spanned by S** is the following set:

$$\langle S \rangle = \left\{ \sum_{\text{finite}} a_i v_i : a_i \in F, v_i \in S \right\},$$

where $\sum_{\text{finite}} a_i v_i$ means $a_i \neq 0$ for *only* some finitely many indices i , and $a_i = 0$ for others; i.e., we only add finitely many vectors.[†]

And we call a element of $\langle S \rangle$ a **linear combination** of S .

[†]We take finitely many vectors since it is not sufficient to define ‘convergence’ of an infinite series with just axioms of vector space.

Example 1.2.3. Let \mathbb{R}^∞ be the space of all sequences which are **eventually zero**, that is, there are only *finitely many* nonzero terms. Then,

$$\mathcal{E}^\infty = \left\{ \mathbf{e}_i = \left(0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots \right) : i \in \mathbb{N} \right\}$$

spans \mathbb{R}^∞ . Hence, for example, a sequence

$$(1, 1, 1, \dots)$$

is not a linear combination of \mathcal{E}^∞ .

Proposition 1.2.3. For every subset $S \in V$, $\langle S \rangle$ is a subspace of V .

Proof. Use **Proposition 1.2.2**. □

Proposition 1.2.4. For every subset $S \in V$, $\langle S \rangle$ is the smallest subspace of V which contains S , namely,

$$\langle S \rangle = \bigcap_{\substack{S \subseteq W \\ W: \text{vector space}}} W.$$

Proof. (\subseteq) Since W is a vector space containing S , it must contain other linear combinations of S also. Therefore $\langle S \rangle \subseteq W$ for every W satisfying the condition whence

$$\langle S \rangle \subseteq \bigcap_{\substack{S \subseteq W \\ W: \text{vector space}}} W.$$

(\supseteq) $\langle S \rangle$ is a vector space containing S . □

1.2.2 Basis

Definition 1.2.4. A subset $S = \{v_i : i \in I\}$ is **linearly independent** if every vector of S cannot be represented by a linear combination other vectors; i.e.,

$$\forall i \in I, \forall a_j \in F, \quad v_i \neq \sum_{\substack{j \neq i \\ \text{finite}}} a_j v_j.$$

Equivalently, if every linear combination whose coefficients are not all zero is non-zero, the subset is linearly independent. If not, S is **linearly dependent**.

Definition 1.2.5. A **(Hamel) basis** \mathfrak{B} of V is a subset of V which satisfies the followings:

$$\langle \mathfrak{B} \rangle = V$$

and

\mathfrak{B} is linearly independent.

We usually fix the order of elements of \mathfrak{B} , which is called an **ordered basis**. Hereafter, *every basis is an ordered basis*. For example, a basis $\{(1, 0), (0, 1)\}$ and $\{(0, 1), (1, 0)\}$ are different bases.

Example 1.2.4.

- (standard basis)

$$\mathcal{E} = \left\{ \mathbf{e}_i = \left(0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots, 0 \right) : 1 \leq i \leq n \right\}$$

is a basis for F^n , for arbitrary field F .

- $\{(1, 0), (1, 1)\}$ is a basis for \mathbb{R}^2 (over the field \mathbb{R}).

1.2.3 Dimension

Definition 1.2.6. ‘The’ **dimension** $\dim V$ of given vector space V is the **cardinality** (the number of elements of given set for finite set) of a basis.

Definition 1.2.7. A **finite dimensional vector space** is a vector space whose bases are all finite.[†]

We consider *finite dimensional vector spaces only* unless there is an additory description.

Lemma 1.2.1. Let $\mathfrak{B} = \{v_i : 1 \leq i \leq n\}$ be a basis, and $\mathfrak{C} = \{w_j : 1 \leq j \leq m\}$ span V . Then $m \geq n$.

Proof. If there is a vector v of \mathfrak{B} which is not in \mathfrak{C} , without loss of generality, rename it $v_1 \neq w_i$. Then $\mathfrak{C} \cup \{v_1\}$ is linearly dependent: since \mathfrak{C} spans V , there is a linear combination of \mathfrak{B} represents $-v_1$ whence $\mathfrak{C} \cup \{v_1\}$ is linearly dependent. Hence there is at least one vector w_t which is represented by a linear combination of others. Then let $\mathfrak{C}_1 = \mathfrak{C} \cup \{v_1\} - \{w_t\}$.

Repeat this process. It is possible up to n -th stage since \mathfrak{B} is linearly independent: if there is a linear dependence, it must contain a w -vector. Hence we obtain

$$\mathfrak{C}_n = \{v_1, \dots, v_n, w_{r_1}, \dots, w_{r_{m-n}}\}$$

and it is linearly dependent. □

Theorem 1.2.1 (uniqueness of dimension). Let \mathfrak{B} and \mathfrak{C} be two bases of finite dimensional vector space V . Then $|\mathfrak{B}| = |\mathfrak{C}|$.

Proof. $|\mathfrak{B}| \geq |\mathfrak{C}|$ and $|\mathfrak{C}| \geq |\mathfrak{B}|$ by **Lemma 1.2.1**. □

Well... How about the existence? The existence of the dimension needs the existence of the basis of V .

Theorem 1.2.2 (existence of basis). Every vector space has a basis, if **AC** (Axiom of Choice) assumed. In addition, it is equivalent to **AC**.

Proof. $\exists \mathfrak{B} \iff \mathbf{ZL} \iff \mathbf{AC}$. See a set theory textbook. □

[†]It is the best way for defining finite dimensional vector space since the dimension is *not* well-defined yet.

1.2.4 Basis extension

We can *extend* a basis of smaller space to a larger space.

Theorem 1.2.3 (basis extension). *Let $W \leq V$ be two vector spaces and \mathfrak{C} be a basis for W . Then there is a basis \mathfrak{B} of V which contains \mathfrak{C} .*

Proof. Induction on $n-m$, where $n = \dim V$ and $m = \dim W$. If $n - m = 0$, just let $\mathfrak{B} = \mathfrak{C}$. (Why?) Now, assuming there is a vector in $V - W$, take a vector v in $V - W$. Then v is linearly independent with \mathfrak{B} (that is, $\tilde{\mathfrak{B}} = \mathfrak{B} \cup \{v\}$ is linearly independent) and hence $\tilde{W} = \langle \tilde{\mathfrak{B}} \rangle$ is a vector space which $\tilde{W} \leq V$. Since $n - m$ decreases, the induction proceeds. \square

1.2.5 Sum and direct sum

Definition 1.2.8. For a set $\{S_i\}_{i \in I}$ of sets with a common addition, we define the **sum** of $\{S_i\}$ as follows:

$$\begin{aligned} \sum_{i \in I} S_i &= \left\{ \sum_{\text{finite}} s_i : s_i \in S_i \right\} \\ &= \left\{ \sum_{i \in I} s_i : s_i \in S_i \text{ and all } s_i = 0 \text{ but for finitely many } i \right\}. \end{aligned}$$

Definition 1.2.9. For a set $\{W_i\}_{i \in I}$ of *subspaces* of V which is mutually disjoint:

$$W_i \cap W_j = \{0\}, \quad \text{for } i \neq j,$$

we define the **direct sum** of $\{W_i\}$ just the sum of them:

$$\bigoplus_{i \in I} W_i = \sum_{i \in I} W_i.$$

If the set is not mutually disjoint, even if it is mutually disjoint itself, we *make it be* mutually disjoint: isolate the vectors with giving different coordinates for each vector space. For example, if $V \oplus W \neq \{0\}$,

$$V \oplus W := \{(v, w) : v \in V, w \in W\}.$$

Trivially the direct sum of some vector spaces is a vector space.

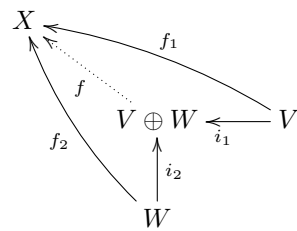
Example 1.2.5.

- $\mathbb{R} \oplus \mathbb{R} \approx \mathbb{R}^2$,
- $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \approx \mathbb{R}^\infty$.

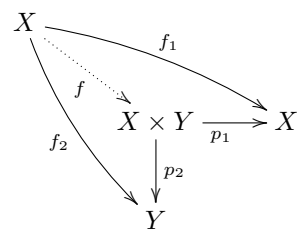
Additional.

- For finite spaces, the direct sum of them is *the same* (isomorphic) with the cartesian product.

- The direct sum can be represented by a commutative diagram:



where the cartesian product is represented by:



1.3 Linear Transformation

Definition 1.3.1. Let V and W be two F -vector spaces, then a map $f : V \rightarrow W$ is **linear** if

$$f(cv + dw) = cf(v) + df(w)$$

for every $c, d \in F$ and $v, w \in V$.

In another viewpoint, a linear map is a **vector space homomorphism** since it *preserves* the operations of vector spaces.

Example 1.3.1.

- A map $f : F \rightarrow F, \quad x \mapsto ax$ is a linear map from and to F . It is why maps of this kind are called *linear*.
- Producting a matrix is a linear map. For a matrix $A \in \mathfrak{M}_{m,n}(F)$,

$$L_A : F^m \rightarrow F^n, \quad X \mapsto AX$$

is a linear map from F^m to F^n . We will show that every linear map (from and to finite dimensional vector spaces) can be represented in this way, i.e., *matrices and linear transformations are the same things*.

- Another familiar linear maps are differentiation and integration. Let \mathcal{C}^n be the space of function from and to \mathbb{R} which is n -th differentiable and has continuous n -th derivative. Then for $n \in \mathbb{N}$, the **differentiation operator**

$$D : \mathcal{C}^n \rightarrow \mathcal{C}^{n-1}, \quad f \mapsto f'$$

is a linear transformation since $(cf + dg)' = cf' + dg'$. Similarly, the **integration operator**

$$J : \mathcal{C}^{n \geq 0} \rightarrow \mathcal{C}^{n+1}, \quad f \mapsto \int_0^x f \, dx$$

is linear. (Here, F is not a field but the antiderivative of f .) Similarly, partial differentiation operators are linear.

- Transpose and trace are linear.

Theorem 1.3.1. *If f is linear, values of f at the basis elements determine f . Equivalently, there is an one-to-one correspondence between f and $f(\mathbf{e}_i)$'s.*

Proof.

$$\begin{array}{ccc} & v = \mathbf{e}_i & \\ f(v) & \xrightarrow{\quad} & f(\mathbf{e}_i) \\ & v = \sum_i a_i \mathbf{e}_i & \end{array}$$

□

Proposition 1.3.1. *Let denote the space of all linear transformations from V to W as $\mathfrak{L}(V, W)$. Then $\mathfrak{L}(V, W)$ is a vector space dimension of mn .*

Definition 1.3.2. For an F -vector space V , a linear map $f : V \rightarrow F$ is called a **linear functional**. And it forms a vector space, which is called the **dual space** of V .

$$V^* = \{f : V \rightarrow F \mid f \text{ linear}\}.$$

Since V^* is an F -vector space, we can define the **double dual** V^{**} of V as follows:

$$V^{**} = (V^*)^* = \{\alpha : V^* \rightarrow F \mid \alpha \text{ linear}\}.$$

An example of elements of V^{**} is **evaluation**:

$$\alpha_a(f) = f(a), \quad f \in V^*.$$

Definition 1.3.3. For a linear map $f : V \rightarrow W$, the **kernel** or the **null space** of f is

$$\ker f = f^{-1}(0) = \{v \in V : f(v) = 0\},$$

where 0 is the zero vector of W . The **image** of f is just $\text{im } f = f(V)$.

Proposition 1.3.2. *The kernel and the image of a linear map form subspaces of V , respectively.*

Theorem 1.3.2 (dimension theorem). *For a linear map $f : V \rightarrow W$,*

$$\dim \ker f + \dim \text{im } f = \dim V.$$

Proof. Proof by basis extension. Let $\mathfrak{B} = \{v_i : 1 \leq i \leq n\}$ be a basis of $\ker f$. Then there is a basis $\mathfrak{C} = \mathfrak{B} \cup \{w_j : 1 \leq j \leq m\}$ of V which contains \mathfrak{B} , and we will show that $f(\mathfrak{C} - \mathfrak{B})$ is a basis of $\text{im } f$.

For an arbitrary vector $v = \sum_i a_i v_i + \sum_j b_j w_j$ of V ,

$$f(v) = f\left(\sum_i a_i v_i + \sum_j b_j w_j\right) = \sum_j b_j f(w_j)$$

since v 's are in the kernel. Since v 's and w 's are linearly independent, so are $f(w)$'s. Hence $f(w)$'s form a basis of $\text{im } f$. \square

We call $\dim \ker f$ the **nullity** of f and denote it as $\text{null } f$.

Definition 1.3.4.

- A **monomorphism** is an injective homomorphism.
- An **epimorphism** is an surjective homomorphism.
- An **isomorphism** is an bijective homomorphism.
- An **automorphism** is an bijective homomorphism from and to itself.



Figure 1.1: A pigeon.

Theorem 1.3.3 (vector space version of pigeonhole principle). *Let $f : V \rightarrow W$ be linear, and suppose $\dim V = \dim W = n < \infty$. Then the followings hold:*

- *if f is a monomorphism, then it is an isomorphism;*
- *if f is an epimorphism, then it is an isomorphism.*

Proof. Let f be a monomorphism; suppose that f is not surjective. Then there is a vector $w \in W$ such that $\forall v \in V, w \neq f(v)$. Since $f(0) = 0$, other vectors in V are not mapped to 0 and hence $\ker f = \{0\}$. And we get $\dim \operatorname{im} f = n$ from $\dim \ker f = 0$. Hence $\operatorname{im} f = W$.

Changing im and \ker proves the rest part of the theorem. \square

Now we can prove the question in **Section 1.2**.

Theorem 1.3.4. *For two square matrices A and $B \in \mathfrak{M}_{n,n}(F)$, if $AB = I$, then $A = B^{-1}$.*

Proof. $AB = I$ implies that B is left-invertible, which is equivalent to that L_B is a monomorphism. Since $L_B : F^n \rightarrow F^n$, L_B is an isomorphism whence B is invertible:

$$B^{-1} = \begin{pmatrix} \left| \begin{array}{c} L_B^{-1} \mathbf{e}_1 \\ \vdots \end{array} \right| & \cdots & \left| \begin{array}{c} L_B^{-1} \mathbf{e}_n \\ \vdots \end{array} \right| \end{pmatrix}.$$

Multiplying B^{-1} right in the both sides of $AB = I$, we obtain $A = B^{-1}$. Similarly, A is invertible and $A^{-1} = B$. \square

1.3.1 Rank

Definition 1.3.5. For a matrix $A \in \mathfrak{M}_{m,n}(F)$, the **row space** is a space which is generated by the row vectors of A . Similarly, the **column space** is a space which is generated by the column vectors of A . Then the **row(column) rank** is the dimension of the row(column) space.

Example 1.3.2. We can know the row rank and the column rank in the (R, C)-REF of the matrix; one can do elementary row operations:

$$\begin{aligned}
 A &= \begin{pmatrix} 1 & 0 & 1 \\ 5 & 6 & 7 \\ 0 & 3 & 1 \end{pmatrix} \\
 &\sim_r \begin{pmatrix} 1 & 0 & 1 \\ 0 & 6 & 2 \\ 0 & 3 & 1 \end{pmatrix} \\
 &\sim_r \begin{pmatrix} 1 & 0 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
 &\sim_r \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix};
 \end{aligned}$$

hence the row space is $\{(a, b, a + b/3)^\top : a, b \in R\}$ whence the row rank is 2; while the column space is $\{(a + c, b + c/3, 0)^\top = (\tilde{a}, \tilde{b}, 0)^\top : \tilde{a}, \tilde{b} \in R\}$ whence the column rank is also 2. Otherwise one can do elementary column operations:

$$\begin{aligned}
 A &= \begin{pmatrix} 1 & 0 & 1 \\ 5 & 6 & 7 \\ 0 & 3 & 1 \end{pmatrix} \\
 &\sim_c \begin{pmatrix} 1 & 0 & 0 \\ 5 & 6 & 2 \\ 0 & 3 & 1 \end{pmatrix} \\
 &\sim_c \begin{pmatrix} 1 & 0 & 0 \\ 5 & 6 & 0 \\ 0 & 3 & 0 \end{pmatrix} \\
 &\sim_c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{5}{2} & \frac{1}{2} & 0 \end{pmatrix};
 \end{aligned}$$

which makes the same result.

Are the row rank and the column rank the same? The answer is...

Lemma 1.3.1. For a matrix $A \in \mathfrak{M}_{m,n}(F)$ over an ordered field F ,

$$\text{col rk } A = \text{col rk } A^\top.$$

Proof. It suffices to show that $\text{col rk } A \leq \text{col rk } A^\top$, since it implies

$$\text{col rk } A^\top \leq \text{col rk } (A^\top)^\top = \text{col rk } A$$

which completes the proof of lemma.

We will show that $Av = 0$ if and only if $(A^T A)v = 0$ whence

$$\text{col rk } A = \text{col rk}(A^T A) \leq \text{col rk } A^T$$

; the last inequality follows because each column of $A^T A$ is a linear combination of the columns of A^T . First, $Av = 0 \implies A^T Av = 0$ trivially. Conversely,

$$A^T Av = 0 \implies v^T A^T Av = 0 \implies (Av)^T Av = 0 \implies Av = 0,$$

by positive-definiteness of the dot product. (More generalized version of proof uses the orthogonal complement, or even **Erdős-Kaplansky Theorem**(?). See <http://math.stackexchange.com/questions/2315/is-the-rank-of-a-matrix-the-same-of-its-transpose-if-yes-how-can-i-prove-it>) \square

Theorem 1.3.5 (rank theorem). *The row rank and the column rank are the same, and we call it the **rank** of the given matrix.*

Proof 1. Count the number of *leading 1* in RREF. \square

Proof 2 assuming that F is an ordered field. It is trivial that the row rank of A equals the column rank of A^T . Since the column rank of A^T is the same with of A , the proof completed. \square

Theorem 1.3.6 (rank-nullity theorem). *Let $A \in \mathfrak{M}_{m,n}(F)$ be a matrix, then*

$$\text{rank } A + \dim \ker L_A = m.$$

*We call $\dim \ker L_A$ the **nullity** of A , and denote $\text{null } A$. Hence*

$$\text{rank } A + \text{null } A = m = \dim \text{dom } L_A.$$

Proof. We know that Ae_i is i -th column of A . Hence the column space of A is just the image of $L_A : F^m \rightarrow F^n$, hence $\text{col rk } A = \dim \text{im } L_A$. By the dimension theorem(**Theorem 1.3.2**),

$$\dim \ker L_A + \text{col rk } A = \dim F^m = m.$$

\square

Definition 1.3.6. Given an m by n matrix A of rank r , a **rank decomposition** of A is a representation by a product $A = PQ$ of two matrices $P \in \mathfrak{M}_{m,r}(F)$ and $Q \in \mathfrak{M}_{r,n}(F)$.

Theorem 1.3.7. *A rank decomposition of a matrix exists, but not uniquely.*

Proof. \square

1.3.2 Matrix representation and similarity

Definition 1.3.7. For a finite dimensional vector space V and a basis $\mathfrak{B} = \{v_i\}_{i \in I}$ of V , every vector v in V can be represented as a linear combination of \mathfrak{B} *uniquely*, namely

$$v = \sum a_i v_i;$$

and we call the row vector

$$[v]_{\mathfrak{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

the **coordinate** vector.

Theorem 1.3.8. For a F -vector space homomorphism $f : V \rightarrow W$ and the bases \mathfrak{B} and \mathfrak{C} of V and W , respectively, there is a unique matrix $[f]_{\mathfrak{C}}^{\mathfrak{B}} \in \mathfrak{M}_{\dim W, \dim V}(F)$ such that

$$[f]_{\mathfrak{C}}^{\mathfrak{B}} [v]_{\mathfrak{B}} = [fv]_{\mathfrak{C}}.$$

Proof. Let $n = \dim V$, $m = \dim W$, $\mathfrak{B} = \{v_i\}_{i=1}^n$ and $\mathfrak{C} = \{w_j\}_{j=1}^m$, then

$$\begin{aligned} [f(v)]_{\mathfrak{C}} &= \left[f \left(\sum_i a_i v_i \right) \right]_{\mathfrak{C}} \\ &= \left[\sum_i a_i f(v_i) \right]_{\mathfrak{C}} \\ &= \left[\sum_i a_i \sum_j b_{ij} w_j \right]_{\mathfrak{C}} \\ &= \left[\sum_j \left(\sum_i a_i b_{ij} \right) w_j \right]_{\mathfrak{C}} \\ &= \begin{pmatrix} \sum_i a_i b_{i1} \\ \vdots \\ \sum_i a_i b_{im} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1m} & \cdots & b_{nm} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= [f]_{\mathfrak{C}}^{\mathfrak{B}} [v]_{\mathfrak{B}}, \end{aligned}$$

where $[f(v_i)]_{\mathfrak{C}} = [b_{i1} \ \cdots \ b_{im}]^T$ whence

$$[f]_{\mathfrak{C}}^{\mathfrak{B}} = \begin{pmatrix} [f(v_1)]_{\mathfrak{C}} & \cdots & [f(v_n)]_{\mathfrak{C}} \end{pmatrix}.$$

Uniqueness follows from the uniqueness of the coordinate representation. \square

Proposition 1.3.3 (composition and product). *Let $V \xrightarrow{f} W \xrightarrow{g} U$ be two homomorphisms. Then*

$$[g \circ f]_{\mathfrak{D}}^{\mathfrak{B}} = [g]_{\mathfrak{D}}^{\mathfrak{C}} [f]_{\mathfrak{C}}^{\mathfrak{B}}.$$

Proof. Easy. □

Therefore, if the bases are fixed, there is a *one-to-one correspondence* between the space of matrices and the space of linear transformations; where the operations are preserved under the correspondence, as follows:

$$f + g \quad \longleftrightarrow \quad [f] + [g]$$

and

$$f \circ g \quad \longleftrightarrow \quad [f][g].$$

We just say that,

“the matrices are the same thing as the linear transformations.”

1.3.3 Basis transition

Theorem 1.3.9. *Let \mathfrak{B} be a basis of F^n and let A is an n by n square matrix. Then $A\mathfrak{B} = \{Av_i : v_i \in \mathfrak{B}\}$ is a basis of F^n if and only if A is invertible.*

Proof. Denote $\mathfrak{B} = \{v_i : 1 \leq i \leq n\}$ like a column vector, although F^n is not a field, namely,

$$\mathfrak{B} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

And define $A\mathfrak{B}$ as componentwise product:

$$A\mathfrak{B} = \begin{pmatrix} Av_1 \\ Av_2 \\ \vdots \\ Av_n \end{pmatrix}.$$

(\Rightarrow) If Y is a basis, then $\exists C \in \mathfrak{M}_{n,n}(F)$, $C(A\mathfrak{B}) = \mathfrak{B}$ since $A\mathfrak{B}$ spans F^n . $(CA)\mathfrak{B} = \mathfrak{B}$ implies that $(CA)v_i = v_i$ for every basis element $v_i \in \mathfrak{B}$ whence $CA = I$ and it implies that A is invertible.

(\Leftarrow) Let $v = \sum_i a_i v_i$ and

$$R = (a_1 \quad \cdots \quad a_n)$$

be the coefficient matrix. Then

$$v = \sum a_i v_i = R\mathfrak{B} = (RA^{-1})(A\mathfrak{B}) = \tilde{R}(A\mathfrak{B})$$

whence $A\mathfrak{B}$ is a basis of F^n . □

Theorem 1.3.10. *If \mathfrak{B} and $\tilde{\mathfrak{B}}$ are bases of V and \mathfrak{C} and $\tilde{\mathfrak{C}}$ are bases of W , then for linear $f : V \rightarrow W$,*

$$[\text{id}_W]_{\tilde{\mathfrak{C}}}^{\mathfrak{C}} [f]_{\tilde{\mathfrak{C}}}^{\mathfrak{C}} [\text{id}_V]_{\tilde{\mathfrak{B}}}^{\mathfrak{B}} = [f]_{\tilde{\mathfrak{C}}}^{\tilde{\mathfrak{B}}}.$$

Furthermore, $[\text{id}]_{\bullet}^{\bullet}$ are all invertible.

Proof. For the first assertion, just use **Proposition 1.3.3**. For the second one, since

$$[\text{id}]_{\mathfrak{B}}^{\tilde{\mathfrak{B}}} [\text{id}]_{\tilde{\mathfrak{B}}}^{\mathfrak{B}} = [\text{id}]_{\mathfrak{B}}^{\mathfrak{B}} = I_{\dim \bullet},$$

they are invertible by **Theorem 1.3.4**. □

Proposition 1.3.4. *For two bases \mathfrak{B} and $\tilde{\mathfrak{B}}$ of V , the **transition matrix** $[\text{id}_V]_{\tilde{\mathfrak{B}}}^{\mathfrak{B}}$ is invertible. Conversely, for a basis \mathfrak{B} of V and an invertible square matrix U whose the number of row is the dimension of the space V , there is a basis $\tilde{\mathfrak{B}}$ of V such that*

$$U = [\text{id}_V]_{\tilde{\mathfrak{B}}}^{\mathfrak{B}}.$$

Proof. The first assertion was proved in **Theorem 1.3.10**. For the second, let

$$U = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

and $\mathfrak{B} = \{v_i\}$. We want another basis $\tilde{\mathfrak{B}} = \{w_i\}$ which satisfies

$$U = [\text{id}_V]_{\tilde{\mathfrak{B}}}^{\mathfrak{B}} = \left(\begin{array}{c|ccc|c} & & & & \\ \hline & [v_1]_{\tilde{\mathfrak{B}}} & \cdots & [v_n]_{\tilde{\mathfrak{B}}} & \\ \hline & & & & \end{array} \right),$$

that is,

$$v_i = \sum_j a_{ij} w_j, \quad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = U \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Hence we have

$$\tilde{\mathfrak{B}} = U^{-1} \mathfrak{B}. \quad \square$$

1.3.4 Similarity

Definition 1.3.8 (similarity). For two square matrices A and \tilde{A} , we say those are **similar** if there is an invertible matrix U such that

$$\tilde{A} = U^{-1} A U,$$

and denote $A \sim \tilde{A}$.

Proposition 1.3.5. *Similarity relation is an **equivalence relation**, that is, satisfies the following three properties:*

- (Reflexivity) $A \sim A$,
- (Symmetricity) $A \sim B \implies B \sim A$,
- (Transitivity) $A \sim B \sim C \implies A \sim C$.

Example 1.3.3 (similarity). For two square matrices A and \tilde{A} , we say those are **similar** if there is an invertible matrix U such that

$$\tilde{A} = U^{-1}AU,$$

and denote $A \sim \tilde{A}$.

Proposition 1.3.6 (similarity). *For a linear operator $f : V \rightarrow V$ and the bases $\mathfrak{B}, \tilde{\mathfrak{B}}$ of V , two matrix representations of f are similar, that is,*

$$[f]_{\mathfrak{B}}^{\mathfrak{B}} \sim [f]_{\tilde{\mathfrak{B}}}^{\tilde{\mathfrak{B}}}.$$

Proof.

$$[\text{id}_V]_{\mathfrak{B}}^{\mathfrak{B}} [f]_{\mathfrak{B}}^{\mathfrak{B}} [\text{id}_V]_{\tilde{\mathfrak{B}}}^{\tilde{\mathfrak{B}}} = [f]_{\tilde{\mathfrak{B}}}^{\tilde{\mathfrak{B}}}.$$

□

Chapter 2

Matrix Group

2.1 Linear Groups

Definition 2.1.1 (normal subgroup). A **normal subgroup** N of a given group G is a subgroup which left and right cosets gN and Ng are the same: $gN = Ng$, i.e.,

$$gNg^{-1} = N$$

for every $g \in G$. We denote it as $N \triangleleft G$.

Proposition 2.1.1 (why normal?). *The quotient group G/N (read $G \bmod N$) is well-defined if and only if N is normal of G .*

Proof. Whatever we take, the equivalence class must be the same. If N is normal and letting $x \sim \tilde{x}$, i.e., $x^{-1}\tilde{x} \in N$,

$$\tilde{x}N \subseteq (xN)N = xN$$

and *vice versa*. If $\bar{x} = \bar{\tilde{x}}$ if $x \sim \tilde{x}$ and $\bar{x}\bar{y} = \bar{\tilde{x}}\bar{y}$, we have

$$gN = (1g)N = NgN = hNgN = (hg)N$$

for every $h \in N$, hence $N = g^{-1}hgN$ so that N is normal: letting $n = g^{-1}hg\tilde{n}$, we have $g^{-1}hg = n\tilde{n}^{-1} \in N$ for every $h \in N$ whence $g^{-1}Ng \subseteq N$. \square

Definition 2.1.2 (general linear group and special linear group). The **general linear group** of a given vector space V is a (multiplicative) group of automorphism on V ; that is, $\text{GL}(V) = \mathfrak{L}(V, V)^\times$. And the **special linear group** of V is the subgroup of $\text{GL}(V)$ which is consisted by linear transformations whose determinant are all 1.

We denote $\text{GL}(n, F) = \text{GL}(F^n)$, and $\text{SL}(n, F) = \text{SL}(F^n)$.

Theorem 2.1.1 (first isomorphism theorem). *For any homomorphism $\varphi : G \rightarrow H$ for two groups G and H ,*

$$G/\ker \varphi \approx \text{im } \varphi.$$

Proof.

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \text{im } \varphi \\ & \searrow & \nearrow \approx \\ & G/\ker \varphi & \end{array}$$

□

Proposition 2.1.2 (GL and SL).

$$\text{GL}(V)/\text{SL}(V) \approx F^\times.$$

Definition 2.1.3 (center). The **center** $Z(G)$ of a group G is the subgroup of elements which satisfy the ‘commutative law’, i.e.,

$$Z(G) = \{z \in G : zg = gz \text{ for every } g \in G\}.$$

Z for *zentrum*, which means ‘center’ in German.

Proposition 2.1.3 (normality of the center).

$$Z(G) \triangleleft G.$$

Proof. Trivially,

$$gZ(G) = \{gz : z \in Z(G)\} = \{zg : z \in Z(G)\} = Z(G)g.$$

□

Example 2.1.1. What are the centers of (a) $\text{GL}(n, F)$ and (b) $\text{SL}(n, F)$?

Answer: (a) $0 \neq cI$ ’s, (b) αI ’s where $\alpha^n = 1$.

Proof. (a) is just all. Let $AZ = ZA$ for all invertible A . Then, especially, for all *elementary matrices*, $EZ = ZE$. Note that multiplying E left is the same with elementary *row* operating, while multiplying right is for elementary *column* operating. (Especially, for E_{i+cj} ’s.) Hence we obtain that Z is diagonal. Instead a more detailed explanation, we see an example:

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & *_1 \\ *_2 & b \end{pmatrix} = \begin{pmatrix} a + 3*_2 & *_1 + 3b \\ *_2 & b \end{pmatrix},$$

$$\begin{pmatrix} a & *_1 \\ *_2 & b \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & *_1 + 3a \\ *_2 & b + 3*_2 \end{pmatrix},$$

hence $*_1$ and $*_2$ are zero. Similar details says that Z must be diagonal, for bigger matrices.

Now, the proof is done: since $E_{i \leftrightarrow j}$ is an elementary matrix, $Z_{ii} = Z_{jj}$, for every i and j pair. Therefore Z is a ‘nonzero’ (since Z is invertible!) multiple of I .

Not so surprisingly, the proof works on any *ring* with 1; if we modify ‘nonzero’ to ‘invertible’, that is, $c \in R^\times$. □

Additional (divide by center?). Dividing by center means to ignore the difference due to the elements of Z . Since

$$Z(G) = \{z \in G : z = gzg^{-1} \text{ for every } g \in G\},$$

we have some ‘morphisms’ $\varphi_g : a \mapsto b = gzg^{-1}$ and *their group*

$$\text{Inn}(G) = \{\varphi_g : g \in G\}.$$

We call this group the **inner automorphism group** of G .

We want to show that $G/Z(G) \approx \text{Inn}(G)$. The idea is easy: use the homomorphism φ_\bullet above:

$$\varphi_\bullet : G \rightarrow \text{Inn}(G).$$

The kernel of this homomorphism is just the center of G , since the (multiplicative) identity of $\text{Inn}(G)$ is the identity function id_G and, from

$$\varphi_z = z \bullet z^{-1} = \text{id}_G, \quad \forall g \in G,$$

i.e.,

$$\varphi_z(g) = zgz^{-1} = g, \quad \forall g \in G,$$

we get $zg = gz$ whence $z \in Z(G)$. Therefore $\ker(\varphi_\bullet) = Z(G)$, and by the first isomorphism theorem, we obtain

$$G/Z(G) \approx \text{Inn}(G).$$

Definition 2.1.4 (PGL and PSL). The **projective general linear group** is defined by

$$\text{PGL}(V) = \text{GL}(V)/Z(\text{GL}(V)).$$

The **projective special linear group** is defined by

$$\text{PSL}(V) = \text{SL}(V)/Z(\text{SL}(V)).$$

Projective geometry is difficult...

2.2 Orthogonal group

Definition 2.2.1 (orthogonal transformation). For an *inner product space* $(V, \langle \bullet, \bullet \rangle)$ (or even just a quadratic space with non-degenerate symmetric bilinear form), an **orthogonal transformation** of V is an invertible linear transformation which preserves the given inner product, that is, such $A \in \text{GL}(V)$:

$$\langle v, w \rangle = \langle Av, Aw \rangle.$$

The group of such transformations is called the **orthogonal group** $O(V)$ of V , and also denote $O(n, F) = O(F^n)$ and $O(n) = O(n, \mathbb{R})$. F^n is considered with *dot product*.

Similarly, $\text{SO}(V) = \{T \in O(V) : \det T = 1\}$, and analogous definitions for $\text{SO}(n, F)$ and $\text{SO}(n)$. Obviously, it is called the **special orthogonal group** of V .

Definition 2.2.2 (unitary group). If we give a *hermitian form* $(V, \langle \bullet, \bullet \rangle)$ rather than an inner product, where the given field is ‘trivially’ the field \mathbb{C} of complex number, we define analogously **unitary group** $U(n)$ as we defined the orthogonal group:

$$\langle v, w \rangle = \langle Av, Aw \rangle, \quad A \in GL(n, \mathbb{C}).$$

We *already know* what is $SU(n)$ and how to call it :D.

Proposition 2.2.1.

- $SO(V) \triangleleft O(V) \triangleleft GL(V)$;
- $SO(V) \triangleleft SL(V) \triangleleft GL(V)$;
- $O(n, F) = \{A \in \mathfrak{M}_{n,n}(F)^\times : A^{-1} = A^\top\}$, if the inner product is a standard one, so-called dot product. (Canonically isomorphic!)

Also the followings hold: for $O(n, F)$, every element is a matrix with pairwise orthonormal columns (or rows).

Good, well, why it is called ‘orthogonal’? It is because these preserves the ‘angle’ of two vectors, especially the *orthogonality*. Then, *what* is orthogonal? Which matrices are orthogonal?

Proposition 2.2.2.

In \mathbb{R}^2 , $O(2)$ consists of rotations and reflections. And the group of rotations is just $SO(2)$.

Proof. From

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad AA^\top = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = I,$$

we obtain $a^2 + b^2 = 1 = c^2 + d^2$ and $ac + bd = 0$. Solutions for the first equality are just sines and cosines, namely:

$$a = \cos x, \quad b = \sin x, \quad c = \cos y, \quad d = \sin y.$$

(Orders of sine and cosine do not have to consider; since there is an inversion $\theta \mapsto \frac{\pi}{2} - \theta$.) Evaluating to another equality,

$$\cos x \cos y + \sin x \sin y = \cos(y - x) = 0, \quad y - x = \frac{2k-1}{2}\pi.$$

Hence $y = x + \frac{2k-1}{2}\pi$. Substituting it, we get

$$c = -\sin x \sin\left(\frac{2k-1}{2}\pi\right) = \mp \sin x, \quad d = \cos x \sin\left(\frac{2k-1}{2}\pi\right) = \pm \cos x.$$

Due to a *custom* in math and other sciences, we use $\theta = -x$ and finally get

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \pm \sin \theta & \pm \cos \theta \end{pmatrix}.$$

If the signa of second row are pluses,

$$A_+ = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = R_\theta,$$

where R_θ is the **rotation matrix** of angle θ . Since $\det R_\theta = 1$, $R_\theta \in \text{SO}(2)$.

If the signa of second row are minuses,

$$A_- = \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_\theta = S_{-\theta/2},$$

where S_φ is a **reflection matrix** w.r.t. a line $\theta = \varphi$ in polar coordinate system. (Draw it \sim .) Note that $\det S_\varphi = -1$. \square

How about 3-dimensional space? We consider a rotation on a line, the *axis*. For example, there are ‘basic’ three rotations:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix},$$

$$R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix},$$

$$R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Surprisingly, they are almost *all*, i.e., the following holds. (Details are omitted.)

Theorem 2.2.1 (decomposition of rotation). *For every ‘rotation’ $R \in \text{SO}(3)$,*

$$R = R_z(\alpha) R_y(\beta) R_x(\gamma)$$

where Tait-Bryan angles of R are α, β, γ , about axes z, y, x respectively.

Following our knowledge, there efnition for *arbitrary rotation* is quite obvious, and only acceptable:

Definition 2.2.3 (rotation). **Rotation** is an element of SO .

We must figure out the following definitions.

Definition 2.2.4 (PGO, PSO, PGU, PSU). Projective (general) orthogonal group $\text{PGO}(V)$ and projective special orthogonal group $\text{PSO}(V)$. Similarly for U ’s...

Example 2.2.1. Calculate them! What is $Z(\text{O}(V))$ and $Z(\text{SO}(V))$?

Proof. Same with **Example 1.1**. A difference is that $\det Z = \pm 1$ in $O(V)$. Another one is for SO : for odd-dimensional V , $Z(SO(V)) = \{I\}$ (a trivial group) since $\det \pm I = \pm 1$; while $Z(SO(V)) = \{\pm I\}$ for even-dimensional V since $\det \pm I = 1$. \square

Corollary 2.2.1. $PSO \approx SO$ for odd-dimensional vector space V .

Proposition 2.2.3.

$$PSU(2) \approx SO(3), \quad SU(2) \xrightarrow{\text{double}} SO(3),$$

where $\xrightarrow{\text{double}}$ means that there is a double covering.

“The shortest path between two truths in the real domain passes through the complex domain.” —Jacques Hadamard.

Proof. $Z(SU(2)) = \{\pm I\}$? Trivial. Then it suffices to show that $PSU(2) \approx SO(3)$. A transformation A of $PSU(2)$ satisfies (U) $AA^\dagger = I$ by the definition. Use same method as **Proposition 1.5.:** let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$AA^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} a\bar{a} + b\bar{b} & a\bar{c} + b\bar{d} \\ \bar{a}c + \bar{b}d & \bar{c}c + \bar{d}d \end{pmatrix} = I$$

whence

$$|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2, \quad a\bar{c} + b\bar{d} = 0.$$

The first equality gives us

$$a = e^{i\varphi_1} \cos x, \quad b = e^{i\varphi_2} \sin x, \quad c = e^{i\varphi_3} \cos y, \quad d = e^{i\varphi_4} \sin y,$$

and the second equality gives

$$\cos x \cos y + e^{i(-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4)} \sin x \sin y = 0,$$

since $\overline{e^{i\theta}} = e^{-i\theta}$ for real θ . If $-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4 \neq 0$, the equality must not hold unless $b = d = 0$, which leads to a contradiction. Also, since (S) $\det A = 1$,

$$ad - bc = e^{i(\varphi_1 + \varphi_4)} (\cos x \sin y - \sin x \cos y) = e^{i(\varphi_1 + \varphi_4)} \sin(y - x) = 1$$

whence $\varphi_1 + \varphi_4 = \varphi_2 + \varphi_3 = k\pi$ and $y - x = \frac{2k-1}{2}\pi$. Therefore

$$A = \begin{pmatrix} e^{i\varphi_1} \cos x & e^{i\varphi_2} \sin x \\ \mp e^{-i\varphi_2} \sin x & \pm e^{-i\varphi_1} \cos x \end{pmatrix}.$$

Finally, (P) ignore one signum of them, then we have

$$A = \begin{pmatrix} e^{i\varphi_1} \cos \theta & -e^{i\varphi_2} \sin \theta \\ e^{-i\varphi_2} \sin \theta & e^{-i\varphi_1} \cos \theta \end{pmatrix}.$$

Hence, for example, there is an ‘isomorphism’

$$\begin{pmatrix} e^{i\varphi_1} \cos \theta & -e^{i\varphi_2} \sin \theta \\ e^{-i\varphi_2} \sin \theta & e^{-i\varphi_1} \cos \theta \end{pmatrix} \leftrightarrow (\theta, \varphi_1, \varphi_2) \leftrightarrow R_z(\theta) R_y(\varphi_1) R_x(\varphi_2),$$

since $R_\bullet(\alpha + \beta) = R_\bullet(\alpha) R_\bullet(\beta)$. Therefore $\text{PSU}(2) \approx \text{SO}(3)$. \square

2.3 $SO(1,1)$

Definition 2.3.1 (indefinite orthogonal group). Consider the Euclidean space only, i.e., $F = \mathbb{R}$. The **indefinite orthogonal group** $O(p, q)$ is something like O , but the inner product is not provided while the following bilinear form is given:

$$\langle v, w \rangle = v^T \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q) w,$$

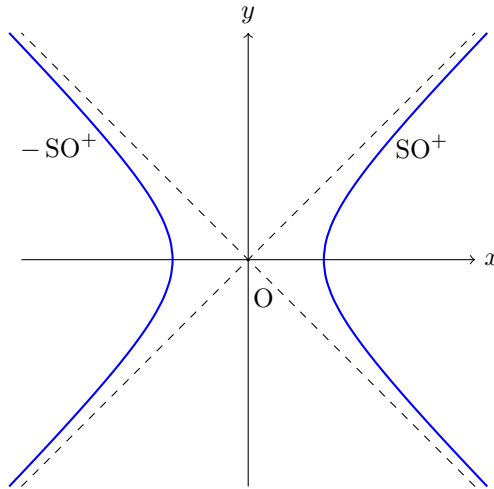
for $(p+q)$ -dimensional vectors v and w . For instance, $O(n) = O(n, 0) = O(0, n)$. And $SO(p, q)$ is ...

We are interested in $O(1,1)$ and $O(1,3)$ in particular.

Proposition 2.3.1. $SO(1,1)$ can be represented by a hyperbolae $x^2 - y^2 = 1$, hence 2 connected curves. SO^+ is the ‘connected’ component of this group which contains the identity I ,

$$SO^+ = \left\{ \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

In fact, we call the connected component of a given ‘topological’ group that contains the identity element the **identity component** of given group.



Proof. Completely same process. Note that if $A \in \text{SO}(1, 1)$,

$$v^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w = \langle v, w \rangle = \langle Av, Aw \rangle = v^\top A^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Aw,$$

hence

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A.$$

Then we have

$$\text{SO}(1, 1) = \left\{ \pm \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

We *can(?)* represent it as a parametrized hyperbola:

$$\pm \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \longleftrightarrow \pm \begin{pmatrix} \cosh \theta \\ \sinh \theta \end{pmatrix},$$

then SO^+ and $-\text{SO}^+$ are connected components of $\text{SO}(1, 1)$. □

What does ‘connected’ means? Detail definition is in *topology*: it cannot separated by some open sets.

We will stop our work here about groups for the time being. If we learn *topology* or *Lie group theory*, it will continue...

Chapter 3

Similarity

3.1 Eigen-*something*

The prefix *eigen-* is adopted from the German word *eigen* for “own-” or “unique to”, “peculiar to”. We will study about some *unique* something, up to similarity.

Definition 3.1.1 (eigenvalue, eigenvector, eigenspace). For a linear *operator* $T \in \mathcal{L}(V, V)$, if

$$Tv = \lambda v$$

for a scalar $\lambda \in F$ and a *nonzero* vector $v \in V$, we call λ an **eigenvalue** and v an **eigenvector** of T . The **eigenspace** of λ is a subspace

$$E_\lambda = \{v \in V : Tv = \lambda v\} = \ker(T - \lambda I)$$

of all vectors whose eigenvalue is λ .

Example 3.1.1. Check whether E_λ is a subspace of V .

Proposition 3.1.1. If $f(t) \in F[t]$ and $v \in E_\lambda$, then $f(T)v = f(\lambda)v$.

Theorem 3.1.1. *TFAE (the followings are equivalent):*

- (a) λ is an eigenvalue of T .
- (b) $T - \lambda I$ is singular, i.e., non-invertible.
- (c) $\det(T - \lambda I) = 0$.

Proof. We already (and MUST) know that (b) and (c) are equivalent. If $T - \lambda I$ is invertible,

$$Tv = \lambda v \iff (T - \lambda I)v = 0 \iff v = 0$$

and hence λ is not an eigenvalue. And if λ is an eigenvalue, $T - \lambda I$ is not bijective and hence singular. \square

Thus, determinant of $\lambda I - T$ is important to decide whether or not λ is an eigenvalue of T . Hence we define a *polynomial*:

Definition 3.1.2 (characteristic polynomial). The **characteristic polynomial** $\phi_T(t)$ is a polynomial defined by

$$\phi_T(t) = \det(tI - T).$$

We will write $\chi\phi$ instead of the term ‘characteristic polynomial’ since it is so long. XD

Here, t behaves like a *scalar* since it used instead of a scalar λ .

Note that λ is an eigenvalue iff $\phi_T(\lambda) = 0$.

Example 3.1.2. Check if $\chi\phi$ is really a polynomial.

Example 3.1.3. Calculate the $\chi\phi$ of a matrix

$$A = \begin{pmatrix} 3 & 3 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

Find its eigenvalues and eigenspaces, and calculate $\phi_A(A)$. Note that

$$f(T) = \sum a_n T^n$$

for a polynomial $f(t) = \sum a_n t^n \in F[t]$ and a linear operator T .

3.2 Diagonalizability

Why we consider it? A big *raison d’etre* of eigen-something is *diagonalization* of a linear operator. First, from its name, we can define as follows:

Definition 3.2.1 (diagonalization). A **diagonalization** of a linear operator $T \in \mathfrak{L}(V, V)$ is a representation T as a similar operator of a diagonal operator $D = \text{diag}(d_1, \dots, d_n)$. If there is a diagonalization of T , that is $T \sim D$ for a diagonal operator D , then we call T is **diagonalizable**.

Example 3.2.1. Determine whether the following matrices are diagonalizable, where $F = \mathbb{Q}$:

$$A = \begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If a matrix is not diagonalizable in given field, consider $F = \mathbb{R}$ and $F = \mathbb{C}$.

If T is diagonalizable, then $[T]_{\mathfrak{B}}^{\mathfrak{B}} = U^{-1}DU$ for a diagonal matrix D , supposing the basis \mathfrak{B} is given for the vector space V ; and there is another basis

\mathfrak{C} for V such that $U = [\text{id}_V]_{\mathfrak{B}}^{\mathfrak{C}}$. Evaluating this, we obtain $D = [T]_{\mathfrak{C}}^{\mathfrak{C}}$. Since it is diagonal, we have

$$[D]^i = D\mathbf{e}_i = [T]_{\mathfrak{C}}^{\mathfrak{C}}[w_i]_{\mathfrak{C}} = [Tw_i]_{\mathfrak{C}}$$

and

$$D\mathbf{e}_i = d_i\mathbf{e}_i = [d_iw_i]_{\mathfrak{C}},$$

where $D = \text{diag}(d_1, \dots, d_n)$ and $\mathfrak{C} = \{w_1, \dots, w_n\}$. Hence we have $Tw_i = d_iw_i$, i.e., new basis must consist of eigenvectors, and the diagonal matrix contains corresponding eigenvalues. It is equivalent to the original definition. Hence we can re-define diagonalizability of a linear operator without matrices:

Definition 3.2.2 (redefine of diagonalizability). A linear operator $T \in \mathfrak{L}(V, V)$ is diagonalizable if there is a basis for V whose elements are all eigenvectors of V .

Since eigenvectors span V , there are n linearly independent eigenvectors.

Proposition 3.2.1. *If the eigenvalues of T are mutually different, T is diagonalizable.*

Proof. If λ 's are different, eigenvectors are linearly independent. \square

Proposition 3.2.2. *If H is Hermitian, that is $H = H^\dagger$, then H can be diagonalized by a unitary operator U , i.e., $U^{-1} = U^\dagger$.*

Proof. Exercise. \square

Theorem 3.2.1. *Let $T \in \mathfrak{L}(V, V)$ and λ_i 's are eigenvalues of T . Then TFAE:*

- (a) T is diagonalizable,
- (b) $\phi_T(t) = \prod (t - \lambda_i)^{e_i}$, $e_i = \dim E_{\lambda_i}$,
- (c) $V = \bigoplus E_{\lambda_i}$,
- (d) $\dim V = \sum \dim E_{\lambda_i}$.

Proof. (a) \Rightarrow (b) $\chi\phi$ is invariant under similarity, since

$$tI - T = U^{-1}(tI - D)U,$$

for example. Hence

$$\phi_T(t) = \phi_D(t) = \prod (t - \lambda_i)^{e_i}.$$

A term due to a basis element appears once in the characteristic polynomial, hence the exponent of $t - \lambda_i$ is the (maximum) number of independent vectors in E_{λ_i} , i.e., dimension.

(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a) $\Leftarrow \Leftarrow$. For (b) to (c), use dimension argument. Note that $\sum e_k = n$. \square

3.3 Cayley-Hamilton Theorem and Minimal Polynomial

From **Example 3.1.3**, we can know $\phi_A(A)$ for some matrices. Is it a general result? The answer is YES, and it is called *Cayley-Hamilton theorem*!

Theorem 3.3.1 (Cayley-Hamilton).

$$\phi_T(T) = 0.$$

For $n = 2$, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\phi_A(t) = (t - a)(t - d) - bc = t^2 - (a + d)t + (ad - bc)$$

and we get a *familiar(?)* form:

$$T^2 - (a + d)T + (ad - bc) = 0.$$

Proof(?). Evaluating $t = T$,

$$\phi_T(T) = \det(TI - T) = \det(T - T) = 0.$$

(NOT A PROOF.)

⚡

First, t behaves as a scalar. And also 0 is a zero matrix, rather than a scalar 0, in a formula $\phi_T(T) = 0$.

Then how to prove it? We will consider $t^n \phi_T(t^{-1})$.

Proof. Let $\phi_T(t) = \sum_{i=0}^n c_i t^i$, then

$$t^n \phi_T(t^{-1}) = \sum_{i=0}^n c_i t^{n-i} = t^n \det(t^{-1}I - T) = \det(I - tT).$$

From

$$\det(A)I = A \cdot \text{adj } A,$$

we get

$$\det(I - tT)I = (I - tT) \text{adj}(I - tT).$$

In order to ‘remove’ $I - tT$ in the RHS, multiplying $\sum_{i=0}^m (tT)^i$ left,

$$\begin{aligned} \left(\sum_{i=0}^m (tT)^i \right) \left(\sum_{i=0}^n c_i t^{n-i} \right) &= \left(\sum_{i=0}^m (tT)^i \right) \det(I - tT)I \\ &= \left(\sum_{i=0}^m (tT)^i \right) (I - tT) \text{adj}(I - tT) \\ &= (I - (tT)^{m+1}) \text{adj}(I - tT). \end{aligned}$$

By definition of classical adjoint, every entry of this matrix is a polynomial of degree less than n . Hence RHS have terms of degree less than n or greater than or equal to m ; for big m , the terms of degree $d \in [n, m)$ in LHS must be vanished. Hence, with m big enough, we obtain that the coefficient of the term of degree n is zero. Now, observing the coefficient of the term of degree n , we get

$$\sum_{i=0}^n c_i T^i = 0.$$

Hence $\phi_T(T) = 0$. □

Definition 3.3.1 (annihilating ideal).

$$\mathcal{I}_T = \{p(t) \in F[t] : p(T) = 0\}.$$

A polynomial in \mathcal{I}_T is called an **annihilating polynomial**.

Since $\phi_T(t) \in \mathcal{I}_T$, by Cayley-Hamilton theorem, $\mathcal{I}_T \neq \emptyset$.

Theorem 3.3.2 (minimal polynomial). *There is a monic annihilating polynomial which has the smallest degree. ‘Monic’ means that the coefficient of the highest order term is 1. We call this polynomial the **minimal polynomial** $m_T(t)$. And also,*

$$m_T(t) | p(t), \quad p(t) \in \mathcal{I}_T;$$

especially, $m_T(t) | \phi_T(t)$.

Proof. Since $\deg \mathcal{I}_T$ is a subset of \mathbb{N} , there is the minimal degree d . If there is two different monic annihilating polynomial of degree d , denoting m_1 and m_2 , we have $m_1 - m_2 \in \mathcal{I}_T$ which leads to a contradiction.

If $m_T(t) \nmid p(t)$ for every $p(t) \in \mathcal{I}_T$, by division algorithm, we get that the remainder $r(t) = p(t) \bmod m_T(t)$ is also an annihilating polynomial which has the degree less than of $m_T(t)$, a contradiction. □

Proposition 3.3.1. \mathcal{I}_T is really an ideal. (Of a ring $F[t]$.)

Proof. Exercise. □

Example 3.3.1. Find the $\chi\phi$ and $m\phi$ of a matrix

$$A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}.$$

Answer.

$$\phi_A(t) = (t-1)(t-2)^2, \quad m_A(t) = (t-1)(t-2).$$

□

3.4 Invariant and Triangularizability

Definition 3.4.1 (invariant subspace). For $T \in \mathfrak{L}(V, V)$ and $W \leq V$, W is called invariant under T if $TW \leq W$.

Example 3.4.1. • $F[t]$ is invariant under $D = \frac{d}{dt}$.

- Every space is invariant under a projection.
- Suppose there are two linear operator T and S on V , which commute, i.e., $TS = ST$. Let $W = \text{im } S$ and $N = \ker S$, then W and N are invariant under T , since $TW = TSV = STV \leq SV = W$ and $Sn = 0 \implies STn = TS n = 0$.

Let $W \leq V$ be invariant under T , and $\mathfrak{C} = \{w_i\}_{i=1}^m$ be a basis of W . Then,

$$[T]_{\mathfrak{C}}^{\mathfrak{C}} = \begin{pmatrix} [T \upharpoonright_W]_{\mathfrak{C}}^{\mathfrak{C}} & * \\ \mathbf{0} & * \end{pmatrix},$$

since, letting $\mathfrak{B} = \{w_i, v_j\}_{i=1, j=1}^{m, n-m} \supseteq \mathfrak{C}$ be a basis of V ,

$$Tw_i = \sum a_i w_i + \sum 0 v_j.$$

Theorem 3.4.1. Let $W \leq V$ be invariant under T , then

$$\phi_{T \upharpoonright_W} | \phi_T \quad \text{and} \quad m_{T \upharpoonright_W} | m_T.$$

Proof. Let W has a basis \mathfrak{C} and \mathfrak{B} is a basis of V which is extended from \mathfrak{C} . Then we have

$$[T]_{\mathfrak{B}}^{\mathfrak{B}} = \begin{pmatrix} [T \upharpoonright_W]_{\mathfrak{C}}^{\mathfrak{C}} & * \\ \mathbf{0} & * \end{pmatrix}.$$

For $\chi\phi$,

$$0 = \phi_T ([T]_{\mathfrak{B}}^{\mathfrak{B}}) = \begin{pmatrix} \phi_T ([T \upharpoonright_W]_{\mathfrak{C}}^{\mathfrak{C}}) & * \\ \mathbf{0} & * \end{pmatrix}.$$

And by above, we have

$$\phi_T ([T]_{\mathfrak{B}}^{\mathfrak{B}}) = 0 \implies \phi_T ([T \upharpoonright_W]_{\mathfrak{C}}^{\mathfrak{C}}) = 0,$$

which completes the remained part of proof. \square

Example 3.4.2. Consider a diagonalizable transformation T , and let W_i 's be its eigenspaces, then it suits perfectly to above theorem, and it makes the 'sufficient-necessary condition' of diagonalizability clear. But if T is not diagonalizable, it cannot be adopted since we do not know the other components of given block matrix.

We define the following as a generalization of 'annihilator ideal':

Definition 3.4.2 (conductor (ideal)). Let W be an *invariant* subspace for T and let v be a vector in V . The **T-conductor of v into W** is the set $S_T(v; W)$ which consists of all polynomials $g \in F[t]$ such that $g(T)v \in W$.

If $W = 0$, we denote it as $\mathcal{I}_T(v) = S_T(v; 0)$ and call the **T-annihilator of v** . And $\mathcal{I}_T = \bigcap_v \mathcal{I}_T(v)$ is the T -annihilator of V , which annihilates all the vectors of V .

Proposition 3.4.1. *Conductor is an ideal in $F[t]$.*

Definition 3.4.3 (conductor (vector)). The monic generator of the ideal $S(v; W)$ is also called the **conductor** of v into W .

Analogous proofs of one for uniqueness of minimal polynomial prove also for the conductors, trivially. And, since \mathcal{I}_T is the *strongest* polynomials, the T -conductors divide the minimal polynomial for T .

3.5 Minimal Polynomials and Triangular-/Diagonal-izability

Lemma 3.5.1. *Suppose*

$$m_T(t) = \prod (t - c_i)^{r_i}, \quad c_i \in F,$$

and let $W \subsetneq V$ be invariant under T . Then there exists a vector $v \notin W$ such that

$$\exists \lambda: \text{eigenvalue of } T : \quad (T - \lambda I)v \in W,$$

that is, a linear polynomial is a T -conductor for some v .

Proof. Let $w \in V \setminus W$, and g be the T -conductor of w into W . ($g(T)w = 0$.) Then $g|m_T$, and since $w \notin W$, g cannot be a constant. ($g(T)w = kw \in W \implies k = 0 = g$ which is contradict to the fact that g is a generator of a nontrivial ideal.) Therefore

$$g(t) = \prod (t - c_i)^{e_i}; \quad \sum e_i > 0.$$

Choose j so that $e_j > 0$, then $g = (t - c_j)h$ for some h . Since $v = h(T)w \notin W$ (g is minimal in the sense of degree) and $g(T)w = (T - c_j I)v \in W$, we just found v ! Obviously, c_j is an eigenvalue. \square

We conclude(?) with the following necessary-sufficient condition of diagonalizability and trigonalizability(trivial meaning), in the sense of minimal polynomial:

Theorem 3.5.1. *T is triangularizable iff $m_T = \prod (t - \lambda_i)^{e_i}$, where λ_i 's are distinct.*

Proof. (\Leftarrow) Let $W = 0$, then above lemma says $\exists v \exists \lambda (T - \lambda I)v = 0$. Hence it forms an eigenspace, and there is a basis \mathfrak{B} of V extending $\{v\}$; therefore we have

$$[T]_{\mathfrak{B}}^{\mathfrak{B}} = \begin{pmatrix} \lambda & ** \\ \mathbf{0} & * \end{pmatrix}.$$

By an induction on the dimension of square matrix (* for above), we obtain a triangularization of T :

$$[T]_{\mathfrak{B}}^{\mathfrak{B}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

(\Rightarrow) Calculate it \sim . □

Corollary 3.5.1. *Every linear operator is triangularizable if the given field is algebraically closed.*

Another proof of Corollary: using induction. There is an eigenvalue and an eigenvector since the field is algebraically closed. Hence let λ and v the chosen ones, extend v to a basis \mathfrak{B} of V , and denote $\mathfrak{C} = \mathfrak{B} - \{v\}$ and $W = \langle \mathfrak{C} \rangle$. Then:

$$[T]_{\mathfrak{B}}^{\mathfrak{B}} = \begin{pmatrix} \lambda & ** \\ \mathbf{0} & * \end{pmatrix}.$$

And we know that

$$* = [\pi \circ T]_{\mathfrak{C}}^{\mathfrak{C}}$$

where π ‘removes’ v -component:

$$\pi : V \rightarrow W; \quad av + \sum_{w_i \in \mathfrak{C}} b_i w_i \mapsto \sum_{w_i \in \mathfrak{C}} b_i w_i.$$

$$\begin{array}{ccc} W & \xrightarrow{T} & TW \\ & \searrow \pi \circ T & \downarrow \pi \\ & & W \end{array}$$

Since $\pi \circ T$ is linear, an induction completes the proof. □

Theorem 3.5.2. *T is diagonalizable iff $m_T = \prod (t - \lambda_i)$, where λ_i ’s are distinct.*

Proof. (\Rightarrow) Trivial. Think as a linear transformation each of $T - \lambda_i I$ ’s.

(\Leftarrow) Let $W = \bigoplus E_{\lambda_i}$ be the space spanned by all of the characteristic vectors of T , and suppose $W \neq V$. By **Lemma 3.5.1**, there is a vector $v \notin W$ and an eigenvalue λ_j such that $w = (T - \lambda_j I)v \in W$. Since $w \in W$, it is represented by a linear combination of eigenvectors uniquely:

$$w = \sum_{w_i \in E_{\lambda_i}} w_i,$$

noting that $Tw = \sum \lambda_i w_i$.

Let $m_T = (t - \lambda c_j)g$ for some polynomial g , and

$$g(t) - g(c_j) = (t - c_j)h(t)$$

for some polynomial h . Then we have

$$g(T)v - g(c_j)v = h(T)(T - c_jI)v = h(T)w \in W$$

and $g(T)v \in W$ whence $g(c_j)v \in W$. Since $v \notin W$, $g(c_j) = 0$. It contradicts the assumption that m_T has distinct roots. \square

3.6 Simultaneous Triangular-/Diagonal-ization

We want to find a basis which triangularizes all of the transformations in a family \mathcal{F} *simultaneously*.

The subspace W is **invariant under \mathcal{F}** if W is invariant under each operators.

Since all diagonal matrices commute, if T and S diagonalized simultaneously, then

$$(U^{-1}TU)(U^{-1}SU) = (U^{-1}SU)(U^{-1}TU)$$

and hence $TS = ST$. Therefore we consider only a family whose elements commute mutually, for simultaneous diagonalization.

For simultaneous triangularization, one does not have to satisfy the commuting condition; however it is a *sufficient* condition for simultaneous triangularization, as we will see.

Lemma 3.6.1. *Let \mathcal{F} be a commuting family of triangularizable linear operators on V . Let W be a proper subspace of V which is invariant under \mathcal{F} , then there is a vector $v \in V \setminus W$ such that*

$$\forall T \in \mathcal{F}, \quad Tv \in \langle v \rangle \oplus W.$$

Proof. It is too taxing to deal with infinitely many operators; hence we use a basis: let $\{T_1, \dots, T_r\}$ be ‘a’(need not to be unique) maximal linearly independent subset of \mathcal{F} ; i.e. a basis for $\langle \mathcal{F} \rangle \leq \mathfrak{L}(V, V)$. ($\mathfrak{L}(V, V)$ is a f.d.v.s.) Then it is sufficient to check for these basis elements only.

By **Lemma 3.5.1**, for a single operator, we can find a vector $v_1 \in V \setminus W$ and a scalar λ_1 such that $(T_1 - \lambda_1 I)v_1 \in W$. Since W is invariant under T_1 ,

$$V_1 = \{v \in V : (T_1 - \lambda_1 I)v \in W\} \supsetneq W.$$

And V_1 is invariant under \mathcal{F} .

Now, in order to use induction, consider V_1 instead of V . Let W be a proper subspace of V_1 , and $U_2 = T_2 \upharpoonright_W$ instead of T_1 of above procedure. Since $m_{U_2} \mid m_{T_2}$, we may apply **Lemma 3.5.1** to new W and U_2 and consider as of T_2 . We obtain a vector $v_2 \in V_1 \setminus W$ and a scalar λ_2 such that $(T_2 - \lambda_2 I)v_2 \in W$.

Note that, since $v_2 \in V_1$, both of $(T_1 - \lambda_1 I)v_2$ and $(T_2 - \lambda_1 I)v_2$ belong to W . And let

$$V_2 = \{v \in V_1 : (T_2 - \lambda_2 I)v \in W\},$$

then V_2 is invariant under \mathcal{F} .

Continue this process by an induction, then we can find $v = v_r$ as the desired vector. \square

Theorem 3.6.1. *Let \mathcal{F} be a commuting family of triangularizable linear operators on V . Then it can be triangularized simultaneously.*

Proof. Induction. Now it is easy. (Same with the proof of **Theorem 3.5**.) \square

Now, finish with diagonalization.

Theorem 3.6.2. *Let \mathcal{F} be a commuting family of diagonalizable linear operators on V . Then it can be diagonalized simultaneously.*

Proof. Almost same process, at this point, however, it is easier to proceed by induction on $\dim V$.

If $\dim V = 1$, automatically proved. Let $\dim V = n$ and choose any $cI \neq T \in \mathcal{F}$. Let λ_i 's be the distinct eigenvalues of T and let $W_i = E_{\lambda_i} = \ker(T - c_i I)$. W_i is invariant under every operator which commutes with T ; and each operator in

$$\mathcal{F}_i = \{T \upharpoonright_{W_i} : T \in \mathcal{F}\}$$

is diagonalizable since its minimal polynomial divides the minimal polynomial for the corresponding operator in \mathcal{F} . Operators in \mathcal{F}_i can be diagonalized simultaneously since $\dim W_i < \dim V$ by a basis \mathfrak{B}_i . Then $\mathfrak{B} = (\mathfrak{B}_i)$ is a desired basis. \square