Optimization Assignment 1

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1 Q6. (Graduate Student)

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be defined as $f(x) = \|Ax - b\| + \lambda\|Cx\| + \gamma\|Eb\|$ where $A, C, E \in \mathbb{R}^{4\times 4}$, $x, b \in \mathbb{R}^4$ and $\lambda, \gamma \in \mathbb{R}$.

Let D denote the differential operator.

$$
\begin{aligned}
Df(x) &= D\left(\|Ax - b\| + \lambda\|Cx\| + \gamma\|Eb\|\right) \\
&= D\left(\|Ax - b\|\right) + \lambda D\left(\|Cx\|\right) + \gamma D\left(\|Eb\|\right) && \text{[using linearity of D]} \\
&= D\left(\|Ax - b\|\right) + \lambda D\left(\|Cx\|\right) + 0 \\
&= D\left(\|Ax - b\|\right) + \lambda D\left(\|Cx\|\right) \\
&= \frac{(Ax-b)^T A}{\|Ax - b\|} + \lambda \frac{(Cx)^T C}{\|Cx\|} && \text{[using derivations from next sections]}
\end{aligned}
$$

1.1 $D\left(\|Cx\|\right)$

$$
\begin{aligned}
D\left(\|Cx\|\right) &= D\left(\sqrt{(Cx)^T(Cx)}\right) \\
&= D\left(\left((Cx)^T(Cx)\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left((Cx)^T(Cx)\right)^{\frac{-1}{2}} D\left((Cx)^T(Cx)\right) && \text{[using Chain Rule]} \\
&= \frac{1}{2\sqrt{(Cx)^T(Cx)}} D\left((Cx)^T(Cx)\right) \\
&= \frac{1}{2\|Cx\|} D\left((Cx)^T(Cx)\right) \\
&= \frac{\left((Cx)^T D(Cx)\right) + \left((Cx)^T D(Cx)\right)}{2\|Cx\|} && \text{[using Product Rule]} \\
&= \frac{2\left((Cx)^T D(Cx)\right)}{2\|Cx\|} \\
&= \frac{(Cx)^T C}{\|Cx\|}
\end{aligned}
$$

1.2 $D\left(\|Ax - b\|\right)$

$$
\begin{aligned}
D\left(\|Ax - b\|\right) &= D\left(\sqrt{(Ax-b)^T(Ax-b)}\right) \\
&= D\left(\left((Ax-b)^T(Ax-b)\right)^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\left((Ax-b)^T(Ax-b)\right)^{\frac{-1}{2}} D\left((Ax-b)^T(Ax-b)\right) \quad \text{[using Chain Rule]} \\
&= \frac{1}{2\sqrt{(Ax-b)^T(Ax-b)}} D\left((Ax-b)^T(Ax-b)\right) \\
&= \frac{1}{2\|Ax-b\|} D\left((Ax-b)^T(Ax-b)\right) \\
&= \frac{\left((Ax-b)^T D(Ax-b)\right) + \left((Ax-b)^T D(Ax-b)\right)}{2\|Ax-b\|} \quad \text{[using Product Rule]} \\
&= \frac{2(Ax-b)^T D(Ax-b)}{2\|Ax-b\|} \\
&= \frac{2(Ax-b)^T A}{2\|Ax-b\|} \\
&= \frac{(Ax-b)^T A}{\|Ax-b\|}
\end{aligned}
$$