Linear Algebra

Vector Spaces and Subspaces

Spaces of Vectors

충남대학교
컴퓨터공학과
이만호

Vector Space

- Definition : $\{\mathbf{X}, +, \cdot\}$ is called a vector space

  where $\mathbf{X}$: set,

  $+$: a vector addition,

  $\cdot$: scalar multiplication (usually omitted)

  and the operators $+, \cdot$ satisfy the following eight rules.

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ $(\boldsymbol{x}, \boldsymbol{y} \in \mathbf{X})$
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (x + \boldsymbol{y}) + \boldsymbol{z}$ $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{X})$
3. There is a unique zero vector such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$ (identity for +)
4. For each $\boldsymbol{x}$, there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$ (inverse for +)
5. $1\cdot\boldsymbol{x} = \boldsymbol{x}$ (identity for $\cdot$)
6. $(c_1 \cdot c_2)\cdot\boldsymbol{x} = c_1\cdot(c_2\cdot\boldsymbol{x})$ // $(c_1 c_2)\,\boldsymbol{x} = c_1(c_2\boldsymbol{x})$
7. $c\cdot(\boldsymbol{x} + \boldsymbol{y}) = c\cdot\boldsymbol{x} + c\cdot\boldsymbol{y}$ // $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$
8. $(c_1 + c_2)\cdot\boldsymbol{x} = c_1\cdot\boldsymbol{x} + c_2\cdot\boldsymbol{x}$ // $(c_1 + c_2)\boldsymbol{x} = c_1\boldsymbol{x} + c_2\boldsymbol{x}$

※ Usually $\mathbf{X}$ is called a vector space.

Vector Space – Example

- $\mathbf{R}^2$ space with usual + and ·
- $\mathbf{R}^n$ consists of all column vectors $\boldsymbol{v}$ with n real components
- $\mathbf{C}^n$ consists of n complex components
- $\mathbf{M}_{2x2}$ consists all real 2 by 2 matrix
- $\mathbf{M}_{mxn}$ consists all real m by n matrix
- **F** all real functions $\boldsymbol{f(x)}$
- $\mathbf{P}_n$ containing all polynomials of degree n
- **Z** consists only of a zero vector

※ Vector space must contain **0**(zero vector)

Subspaces

- Definition : A subspace $\mathbf{Y}$ of a vector space $\mathbf{X}$ is a subset that satisfies two requirements

  (1) for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{Y}$, $\boldsymbol{v} + \boldsymbol{w} \in \mathbf{Y}$ (closed under +)

  (2) for any $\boldsymbol{v} \in \mathbf{Y}$, $c\boldsymbol{v} \in \mathbf{Y}$ for any scalar c (closed under $\cdot$)

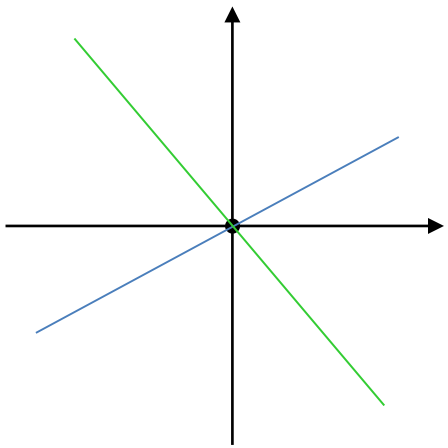
- **Note**

  $\boxed{\text{for any } \boldsymbol{v}, \boldsymbol{w} \in \mathrm{Y},\ c\boldsymbol{v} + d\boldsymbol{w} \in \mathrm{Y}}$

  1. $\mathbf{Y}$: subspace of $\mathbf{X}$ $\Rightarrow$ $\mathbf{0} \in \mathbf{Y}$
  2. A subspace is a subset which is closed under vector addition and scalar multiplication.
  3. A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$
  4. The smallest subspace of $\mathbf{X}$ is $\{\mathbf{0}\}$, not empty set.
  5. The largest subspace of $\mathbf{X}$ is $\mathbf{X}$ itself.

Subspaces – Example 1

- Possible subspaces of $\mathbf{R}^2$
  - The single vector (0,0)
  - Any line through (0,0)
  - The whole space
- Possible subspaces of $\mathbf{R}^3$
  - The single vector (0,0,0)
  - Any line through (0,0,0)
  - Any plane through (0,0,0)
  - The whole space
- Possible subspaces of $\mathbf{M}_{2x2}$
  - Single zero matrix
  - All diagonal matrices
  - All upper triangular matrices
  - All lower triangular matrices



$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}$$

Subspace in $\mathbf{R}^2$ – Example 2

- Example : Vectors (x, y) whose components are <u>positive or zero</u>. (quadrant-1)

$$\begin{array}{l} \text{Let } \boldsymbol{v} = (2,3) \in (\text{quadrant-1}) \\ (-1)\cdot\boldsymbol{v} = (-2,-3) \notin (\text{quadrant-1}) \\ \text{So (quadrant-1) is not a subspace.} \end{array}$$

- Example : Vectors (x, y) whose components are both same sign. (quadrant-1&3)

$$\begin{array}{l} \text{Let } \boldsymbol{v} = (2,3) \in (\text{quadrant-1\&3}), \\ \qquad \boldsymbol{w} = (-3,-2) \in (\text{quadrant-1\&3}) \\ \boldsymbol{v}+\boldsymbol{w} = (-1,1) \notin (\text{quadrant-1\&3}) \\ \text{So (quadrant-1\&3) is not a subspace.} \end{array}$$

※A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

Subspace in $\mathbf{M}_{2x2}$ – Example

- Example :
  - D is a subspaces of M
  - L is a subspaces of M
  - U is a subspaces of M

$$D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : \text{diagonal matrix}$$

$$L = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} : \text{lower triangular matrix}$$

$$U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : \text{upper triangular matrix}$$

Column Space of A

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\Rightarrow \underbrace{x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}}_{\text{Linear combination of columnns of } A} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

- Definition : Column space of A (notation: $\boldsymbol{C}(A)$)

  $\boldsymbol{C}(A)$={all linear combinations of the columns of A}

Column Space of A and $A\boldsymbol{x} = \boldsymbol{b}$

- If $\boldsymbol{b} \in \boldsymbol{C}(A)$, $\boldsymbol{b}$ is a combination of the columns
  - $\Rightarrow$ Solution : the coefficients in that combination
- If $A\boldsymbol{x} = \boldsymbol{b}$ solvable, then $\boldsymbol{b}$ can be represented as a linear combination of the column vectors of A.
- Therefore, $A\boldsymbol{x} = \boldsymbol{b}$ is solvable iff $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$.

Linear System of n Unknowns in $\mathbf{R}^m$

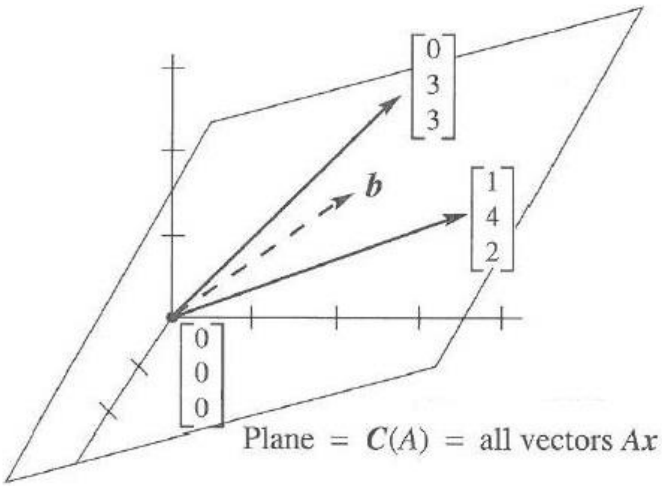
$A\boldsymbol{x} = \boldsymbol{b}$ where A: mxn matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- In A, there are n columns.
- Each column has m components.
- $\boldsymbol{C}(A)$ is a subspace of $\mathbf{R}^m$.

- Example :

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \boldsymbol{b}$$



Plane $=$ $\boldsymbol{C}(A)$ $=$ all vectors $A\boldsymbol{x}$

- $\boldsymbol{C}(A)$ = Plane containing the 2 columns = { $A\boldsymbol{x}$ }
- $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is on the plane $\boldsymbol{C}(A)$.
- $\boldsymbol{b}$ is a combination of the columns.

Span of Vectors

- Let **V**: vector space

  S: set of vectors in **V** (not a space)

  $\mathbf{S} = \{ \boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_N \}$

  $\mathbf{SS} = \{ c_1\boldsymbol{v}_1 + c_1\boldsymbol{v}_2 + \cdots + c_N\boldsymbol{v}_N \}$

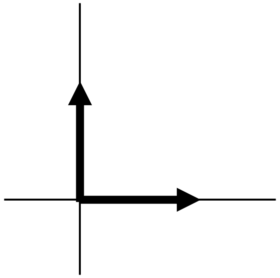
  - **SS** : set of all linear combinations of vectors in **S**
  - **SS** : subspace of **V** spanned by **S** (span of **S**)

- Example : $A\boldsymbol{x} = \boldsymbol{b}$
  - **S** : column vectors of A
  - $\mathbf{SS} = \boldsymbol{C}(A)$: span of column vectors of A

Span of Vectors – Example

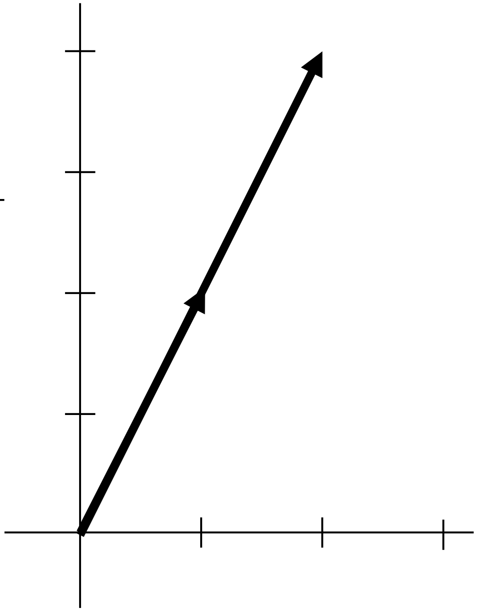
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C}(I) = \mathbf{R}^2$$



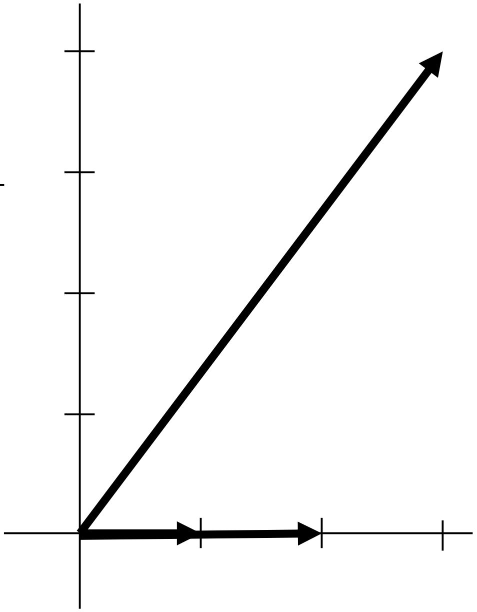
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\boldsymbol{C}(A) = \{(c, 2c)\} : \text{line}$$



$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\boldsymbol{C}(B) = \mathbf{R}^2$$



Question?