Linear Algebra

Vector Spaces and Subspaces

Dimensions of the Four Subspaces

충남대학교
컴퓨터공학과
이만호

Four Fundamental Subspaces

- Rank of a matrix is the number of pivots.
- Dimension of a subspace is the number of vectors in a basis.
- The rank of A reveals the dimensions of all four fundamental subspaces.
- Four fundamental subspaces of A (mxn matrix)

  1. The *row space* is $\boldsymbol{C}(A^T)$, a subspace of $\mathbf{R}^n$.

  2. The *column space* is $\boldsymbol{C}(A)$, a subspace of $\mathbf{R}^m$.

  3. The *nullspace* is $\boldsymbol{N}(A)$, a subspace of $\mathbf{R}^n$.

  4. The *left nullspace* is $\boldsymbol{N}(A^T)$, a subspace of $\mathbf{R}^m$.

Left Nullspace

- Solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, A^{T}: nxm matrix, is in $\mathbf{R}^m$

  $\Rightarrow$ get nullspace of A^{T}

  $\Rightarrow$ $(A^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$

  $\Rightarrow$ $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$ (vectors $\boldsymbol{y}$ are in the left side of A)

The Four Subspaces for rref R – 2

- Example : 3x5 R matrix

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} m=3, n=5, r=2 \\ \text{pivot rows : row1, row2} \\ \text{pivot columns : col1, col4} \end{array}$$

1. Column space of R: $\boldsymbol{C}(R)$
   - Dimension: $r = 2$
     - The number of independent columns = the number of independent rows
   - One basis: col1, col4 (2 pivot columns)
   - Pivot columns col1, col4 span $\boldsymbol{C}(R)$
   - Pivot columns col1, col4 are independent.
   - Every free column is a combination of the pivot columns

> The dimension of the column space is r.
> The pivot columns of R form a basis.

The Four Subspaces for rref R – 1

2. Row space of R: $\boldsymbol{C}(R^{\mathrm{T}})$:
   - Dimension: $r = 2$
     - The number of independent columns = the number of independent rows
   - One basis: row1, row2 (2 pivot rows, nonzero rows)
   - Pivot rows row1 & row2 span $\boldsymbol{C}(R^{\mathrm{T}})$
   - Pivot rows row1 & row2 are independent.

The dimension of the row space is r.
The pivot rows of R form a basis.

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{bmatrix}$$

The Four Subspaces for rref R – 3

3. $\boldsymbol{N}(R)$:

- Dimension: $n - r = 5 - 2 = 3$ (same as the dimension of $\boldsymbol{N}(A)$)
- The basis of $\boldsymbol{N}(R)$ is same as the basis of $\boldsymbol{N}(A)$.
  - The special solutions of R are same as the special solution of A.
- There are $n - r$ free variables.
- They yield $n - r$ special solutions to $R\boldsymbol{x} = \mathbf{0}$.

  $\boldsymbol{s}_1 = (-3, 1, 0, 0, 0)$, $\boldsymbol{s}_2 = (-5, 0, 1, 0, 0)$, $\boldsymbol{s}_3 = (-7, 0, 0, -2, 1)$,
- 3 special solutions are independent.

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The dimension of the nullspace is $n - r$.
The special solutions form a basis.

The Four Subspaces for rref R – 4

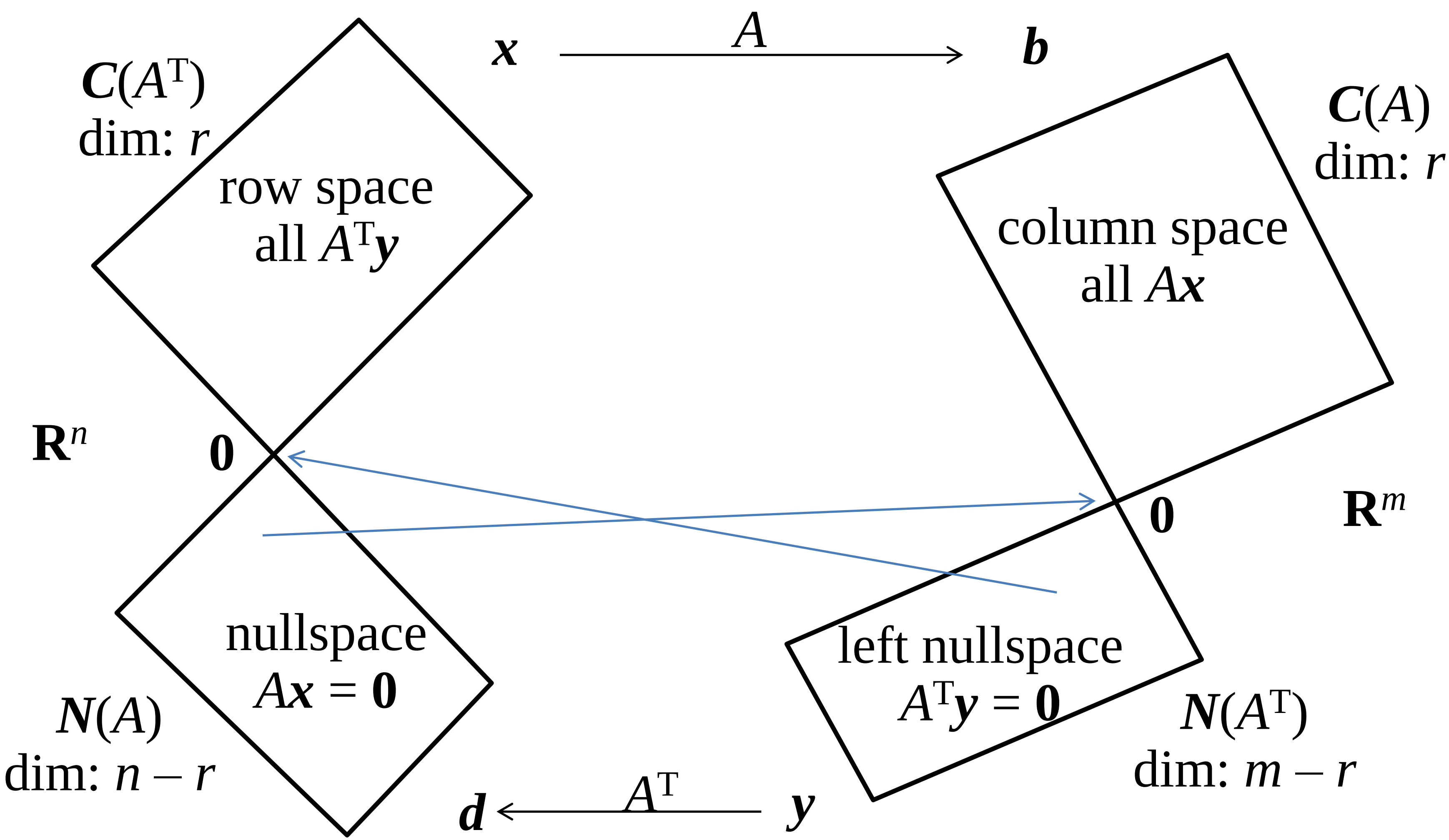
4. Left nullspace of R, $\boldsymbol{N}(R^{\mathrm{T}})$:
   - Dimension: $m - r = 3 - 2 = 1$
   - $R^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has a solution $\boldsymbol{y} = (0, 0, y_3)$

$$R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 2 & 0 \end{bmatrix}$$

In R^n, the dimension of the row space is r.
the dimension of nullspace is $n - r$.
In R^m, the dimension of the column space is r.
the dimension of left nullspace is $m - r$.

Dimensions of the Four Fundamental Subspaces for A(for R)



The Four Subspaces for A – 0

- The dimensions of subspaces for A are the same as for R.

  row space, column space, nullspace, left nullspace
- Consider

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Notice}: \ \boldsymbol{C}(A) \neq \boldsymbol{C}(R)$$

The Four Subspaces for A – 1

- Row space for A: $\boldsymbol{C}(A^{\mathrm{T}})$
  - A has the same row space as R.
  - The dimension of A is same as the dimension of R. r
  - Same dimension and same basis.
  - Every row of $A(R)$ is a combination of the rows of $R(A)$, respectively. — Elimination changes rows, but not row spaces.
  - One basis: the first r rows of R

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}} \\ \begin{matrix} 1 & 3 & 5 & 1 & 9 \end{matrix} \end{bmatrix} \Rightarrow R = \begin{bmatrix} \boxed{\begin{matrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \end{bmatrix}$$

The Four Subspaces for A – 2

- Column space for A: $\boldsymbol{C}(A)$
  - Dimension: r
  - The number of independent columns equals the number of independent rows.
  - The r pivot columns (of A and R) are independent.
  - One basis: the r pivot columns of A.

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\end{matrix}} & \begin{matrix}3\\0\\3\end{matrix} & \begin{matrix}5\\0\\5\end{matrix} & \boxed{\begin{matrix}0\\1\\1\end{matrix}} & \begin{matrix}7\\2\\9\end{matrix} \end{bmatrix} \Rightarrow R = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}3\\0\\0\end{matrix} & \begin{matrix}5\\0\\0\end{matrix} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}7\\2\\0\end{matrix} \end{bmatrix}$$

The Four Subspaces for A – 3

- Nullspace for A: $\boldsymbol{N}(A)$
  - A has the same nullspace as R.
  - Dimension: $n - r$ (same as R)
  - Basis: same as R

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Four Subspaces for A – 4

- Left nullspace for A: $\boldsymbol{N}(A^{\mathrm{T}})$
  - Dimension: $m - r$

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fundamental Theorem of Linear Algebra, Part 1

- The column space and row space both have dimension r.
- The nullspaces have dimensions $n - r$ and $m - r$.

The Four Subspaces – Example

- $A = [1 \;\; 2 \;\; 3]$, $m = 1$, $n = 3$, $r = 1$
  - Row space: $\boldsymbol{C}(A^{\mathrm{T}})$

    $\boldsymbol{C}(A^{\mathrm{T}})$: line through (1, 2, 3) in $\mathbf{R}^3$: dim. =1
  - Nullspace: $\boldsymbol{N}(A)$

    $\boldsymbol{N}(A)$: plane $Ax = x_1 + 2x_2 + 3x_3 = 0$: dim. = 2 (= 3 – 1)
  - Column space: $\boldsymbol{C}(A)$

    $\boldsymbol{C}(A)$: all of real numbers in $\mathbf{R}^1$: dim. = 1
  - Left nullspace: $\boldsymbol{N}(A^{\mathrm{T}})$

    $\boldsymbol{N}(A^{\mathrm{T}})$: $\mathbf{Z}$ (zero space with dimension 0) : dim. = 0

The Four Subspaces – Example

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $\quad m = 2, n = 3, r = 1$
  - Row space

    $\boldsymbol{C}(A^{\mathrm{T}})$: line through (1, 2, 3) in $\mathbf{R}^3$: dim. = 1
  - Nullspace

    $\boldsymbol{N}(A)$: plane $A\boldsymbol{x} = x_1 + 2x_2 + 3x_3 = 0$: dim. = 2 (= 3 – 1)
  - Column space

    $\boldsymbol{C}(A)$: multiples of the first column (1, 2) in $\mathbf{R}^1$: dim. = 1
  - Left nullspace

    $\boldsymbol{N}(A^{\mathrm{T}})$: $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has the solution $\mathbf{y} = (–2, 1)$: dim. =1
  - $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ are perpendicular lines in $\mathbf{R}^2$

Matrices of Rank One

- Rank one matrices are special.
- Let A be mxn matrix with rank r = 1
  - Dimension of row space = dimension of column space = r = 1
  - Basis of row space : first row
    - Every row is a multiple of the first row.
  - Basis of column space : first column
    - Every column is a multiple of the first column.
  - Therefore every rank one matrix has the special form

    $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$: (column vector) * (row vector)
- Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 4 & 8 & 12 \end{bmatrix}, \ \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, A = \boldsymbol{u}\boldsymbol{v}^T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Question?