Linear Algebra

Vector Spaces and Subspaces

The Nullspace of A: Solving $Ax = 0$

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Nullspace of A in $A\mathbf{x} = \mathbf{0}$

- Solving $A\mathbf{x} = \mathbf{0}$, where A is mxn matrix
  - If A is invertible, then $\mathbf{x} = \mathbf{0}$ is the unique solution.
  - If A is not invertible, then there are many nonzero solutions.
- Definition : nullspace of A (notation $\boldsymbol{N}(A)$)
  - $\boldsymbol{N}(A)$: set of all solutions to $A\mathbf{x} = \mathbf{0}$
  - $\boldsymbol{N}(A)$: vector space
  - $\boldsymbol{N}(A)$: subspace of $\mathbf{R}^n$ (cf. $\boldsymbol{C}(A)$: subspace of $\mathbf{R}^m$)

    Suppose $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{N}(A)$.

    $\Rightarrow A\boldsymbol{x} = \mathbf{0},\ A\boldsymbol{y} = \mathbf{0}$

    $\Rightarrow A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0},\ A(c\boldsymbol{x}) = cA\boldsymbol{x} = \mathbf{0}$

    $\Rightarrow \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{N}(A),\quad c\boldsymbol{x} \in \boldsymbol{N}(A)$

    $\Rightarrow \boldsymbol{N}(A)$: subspace

Nullspace of A – Example 1

- Example : $x + 2y + 3z = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

  – A = [1 2 3] : 1x3 matrix.
  – (0, 0, 0) $\in$ { solutions of $A\mathbf{x} = \mathbf{0}$ } : trivial solution.
    - Are there other solutions? Yes. Can you answer for example?
  – Solutions of $A\mathbf{x} = \mathbf{0}$, $N(A)$, is a subspace of $\mathbf{R}^3$. Why?

  – Solutions of $A\mathbf{x} = [6]$, is not a subspace of $\mathbf{R}^3$. Why?

Nullspace of A – Example 2

- Example : nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ in $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow[\text{Elimination}]{} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 0$$

  - Choose any one special solution : $\boldsymbol{s} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
    - For convenience, choose components as 1 or 0.
  - Span of { $\boldsymbol{s}$ } = { $c\boldsymbol{s}$ } : linear combinations of $\boldsymbol{s}$

$$c\boldsymbol{s} = c\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2c \\ c \end{bmatrix} \Rightarrow (-2c) + 2(c) = 0 \Rightarrow c\boldsymbol{s} : \text{solution}$$

$\Rightarrow$ Span of { $\boldsymbol{s}$ }: nullspace of $A = \boldsymbol{N}(A)$. Why?

※ The nullspace consists of all combinations of the special solutions. (span of the special solutions)

$N(A)$: Span of Special Solutions – Ex

- Example : $x + 2y + 3z = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

  2 special solutions : $\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ Chosen specially

  $\Rightarrow N(A)$: span of $\{\boldsymbol{s}_1, \ \boldsymbol{s}_2\}$

- Example : $\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 8x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : A\mathbf{x} = \mathbf{0}$

  $$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow[\text{Elimination}]{} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \Rightarrow N(A) = \{\,\mathbf{0}\,\}$$

  $$\boxed{\begin{cases} x_1 + 2x_2 = 0 \\ \qquad 2x_2 = 0 \end{cases}}$$

  $$B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \xrightarrow[\text{Elimination}]{} \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow N(B) = N(A) = \{\,\mathbf{0}\,\}$$

  Extra rows/equations

Free/Pivot Variable

- Variables can be divided into two groups
  - Free variable : column without pivot
  - Pivot variable : column with pivot

$$C=\begin{bmatrix}1&2&2&4\\3&8&6&16\end{bmatrix}\Rightarrow U=\Bigg[\underbrace{\boxed{\begin{matrix}1&2\\0&2\end{matrix}}}_{\text{Pivot columns}}\ \underbrace{\boxed{\begin{matrix}2&4\\0&4\end{matrix}}}_{\text{Free columns}}\Bigg]\Rightarrow U\boldsymbol{x}=\begin{bmatrix}1&2&2&4\\0&2&0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Pivot variables : x_1, x_2 Free variables : x_3, x_4

$$\text{Choose 2 special solutions}\begin{cases}\boldsymbol{s}_1 : x_3=1, x_4=0 \;\Rightarrow\; x_1=-2, x_2=0\\ \boldsymbol{s}_2 : x_3=0, x_4=1 \;\Rightarrow\; x_1=0, x_2=-2\end{cases}$$

The pivot variables x_1, x_2 are determined by $U\boldsymbol{x}=\boldsymbol{0}$

$$\boldsymbol{s}_1=\begin{bmatrix}-2\\0\\\boxed{\begin{matrix}1\\0\end{matrix}}\end{bmatrix},\quad \boldsymbol{s}_2=\begin{bmatrix}0\\-2\\\boxed{\begin{matrix}0\\1\end{matrix}}\end{bmatrix}$$

$N(A)$: span of $\{\boldsymbol{s}_1, \boldsymbol{s}_2\}$ Why?

Reduced Row Echelon Form

- Reduced Row Echelon Form (rref)
  1. Produce zeros below the pivots, by eliminating downward.
  2. Produce zeros above the pivots, by eliminating upward.
  3. produce ones in the pivots, by dividing the whole row by its pivot
- Example

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = R$$

$$\boldsymbol{s}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Finding special solutions is much easier.

Solving $A\boldsymbol{x} = \mathbf{0}$ by Elimination

- Solving $A\boldsymbol{x} = \mathbf{0}$ by elimination

  where A : mxn matrix (rectangular)

  1. Forward elimination from A to a U or R = rref(A)
  2. Back substitution in $U\boldsymbol{x} = \mathbf{0}$ or $R\boldsymbol{x} = \mathbf{0}$ to find $\boldsymbol{x}$

  ※ When $m < n$ and a column doesn't have no pivot, go on to the next column.

- Note :
  - $\boldsymbol{N}(A) = \boldsymbol{N}(U) = \boldsymbol{N}(R) = \boldsymbol{N}(\mathrm{rref}(A))$
  - If $\boldsymbol{N}(A) = \mathbf{Z} = \{ \mathbf{0} \}$,
    - Columns of A are independent.
    - All columns have pivots.
    - No columns are free.

Column without Pivot – Example 1

- Example : 3x4 matrix with 2 pivots

$$A=\begin{bmatrix}1&1&2&3\\2&2&8&10\\3&3&10&13\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&4&4\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&0&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&2&3\\0&0&1&1\\0&0&0&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

$$\Rightarrow\begin{cases}x_1+x_2\quad\quad+x_4=0\\ \quad\quad\quad x_3+x_4=0\end{cases}$$

pivot variable : x_1, x_3

free variable : x_2, x_4

$$\text{Choose special solutions}\begin{cases}\boldsymbol{s}_1:\ x_2=1, x_4=0\quad\text{then}\quad x_1=-1, x_3=0\\ \boldsymbol{s}_2:\ x_2=0, x_4=1\quad\text{then}\quad x_1=-1, x_3=-1\end{cases}$$

$$\Rightarrow\ \boldsymbol{s}_1=\begin{bmatrix}-1&1&0&0\end{bmatrix}^{\mathrm{T}},\quad \boldsymbol{s}_2=\begin{bmatrix}-1&0&-1&1\end{bmatrix}^{\mathrm{T}}$$

$\boldsymbol{N}(A)$: subspace spanned by $\{\boldsymbol{s}_1,\boldsymbol{s}_2\}$

Complete solution :

Since x_2, x_4 are free variables,

$$\boldsymbol{x}=x_2\underbrace{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}}_{\text{special}}+x_4\underbrace{\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}}_{\text{specia}}=\underbrace{\begin{bmatrix}-x_2-x_4\\x_2\\-x_4\\x_4\end{bmatrix}}_{\text{complete}}$$

Linear combination of special solutions

Column without Pivot – Example 2

- Example : nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot variable : x_1, x_3
free variable : x_2

Choose special solution $\boldsymbol{s} : x_2 = 1$ then $x_1 = -5, x_3 = 0$

$$\Rightarrow \boldsymbol{s} = \begin{bmatrix} -5 & 1 & 0 \end{bmatrix}^T$$

$\boldsymbol{N}(U)$: subspace spanned by $\{\boldsymbol{s}\}$

Complete solution : $\boldsymbol{N}(U)$

Since x_2 is a free variable,

$$\boldsymbol{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

Linear combinations of special solutions

Line in $\mathbf{R}^3$

Forward Elimination for Echelon Matrices

- Forward elimination for Echelon Matrices goes from A to U
  - By row operations including row exchanges
  - Go on to the next column when no pivot is available in the current column
- Form of echelon matrix (4x7)

$$U = \begin{bmatrix} p_1 & x & x & x & x & x & x \\ 0 & p_2 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p_3 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} &\text{pivot variables}: x_1, x_2, x_6 \\ &\text{free variables}: x_3, x_4, x_5, x_7 \end{aligned}$$

Column Space and Nullspace of Echelon Matrix

$$U = \begin{bmatrix} p_1 & x & x & x & x & x & x \\ 0 & p_2 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p_3 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \text{pivot variables}: x_1, x_2, x_6 \\ \text{free variables}: x_3, x_4, x_5, x_7 \end{array}$$

- Column space, $\boldsymbol{C}(U)$: subspace of $\mathbf{R}^4$
  - Columns have 4 components.
  - 4th component of every column is 0.

  $\Rightarrow$ $\boldsymbol{C}(U) = \{ [b_1 \;\; b_2 \;\; b_3 \;\; 0]^{\mathrm{T}} \mid b_1, b_2, b_3 \in \mathbf{R} \}$
- Nullspace, $\boldsymbol{N}(U)$: subspace of $\mathbf{R}^7$
  - Free variables : x_3, x_4, x_5, x_7
  - Choose 4 set of special values of free variables: set one free variable to 1 and set the others 0
  - For each set of special values of free variables, solve $U\boldsymbol{x} = 0$ for the pivot variables x_1, x_2, x_6 $\Rightarrow$ we get 4 special solutions, $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$, $\boldsymbol{s}_4$.
  - $\boldsymbol{N}(U)$: subspace spanned by $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4\}$

Remarks on $A\boldsymbol{x} = \mathbf{0}$

- When A: mxn matrix, $m < n$
  - There is at least $(n-m)$ free variables.
    - There are at least $(n-m)$ special solutions, $\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_{n-m}$
    - Special solutions are not $\mathbf{0}$, i.e. nontrivial solutions.
  - $\boldsymbol{N}(A)$: subspace spanned by $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_{n-m}\}$
    - Dimension of $\boldsymbol{N}(A)$: the number of free variables $\geq n-m$
- When $m < n$, dimension of $\boldsymbol{N}(A)$ in $\mathbf{R}^n$?
  - If there is no free variable, dimension of $\boldsymbol{N}(A)$ = ?
  - If there is 1 free variable, dimension of $\boldsymbol{N}(A)$ = ?
  - If there are 2 free variables, dimension of $\boldsymbol{N}(A)$ = ?

Reduced Row Echelon Matrix R

- Getting Reduced Row Echelon Form (rref)
  1. Produce zeros below the pivots, by eliminating downward.
     - Row exchanges may be required.
     - We get echelon matrix U.
  2. Produce zeros above the pivots, by eliminating upward.
  3. produce ones in the pivots, by dividing the whole row by its pivot

$$R = \text{rref}(A) = \begin{bmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l} \text{pivot variables}: x_1, x_2, x_6 \\ \text{free variables}: x_3, x_4, x_5, x_7 \end{array}$$

$R = \text{rref}(A)$ – Example

- Example : 3x4

$$A \Rightarrow U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Pivot variables : x_1, x_3
Free variables : x_2, x_4

Choose 2 special solutions $\begin{cases} \boldsymbol{s}_1 : x_2 = 1, x_4 = 0 \Rightarrow x_1 = -1, x_3 = 0 \\ \boldsymbol{s}_2 : x_2 = 0, x_4 = 1 \Rightarrow x_1 = -1, x_3 = -1 \end{cases}$

$\boldsymbol{N}(A) = \boldsymbol{N}(U) = \boldsymbol{N}(R)$: all linear combinations of the special solutions

Complete solution :

Since x_2, x_4 are free variables,

$$\boldsymbol{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Invertible Matrix A and rref(A)

- If A is invertible,
  - A is nxn square matrix.
  - rref(A) is the identity matrix, i.e. $R = I$
  - $\boldsymbol{N}(A) = \mathbf{Z}$

Question?