Linear Algebra

Vector Spaces and Subspaces

The Complete Solution to $A\boldsymbol{x} = \boldsymbol{b}$

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Solution to $A\boldsymbol{x} = \boldsymbol{b} \neq \boldsymbol{0}$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad R\boldsymbol{x} = \boldsymbol{b}'$$

$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$: Augmented matrix $\qquad \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} \Rightarrow \begin{bmatrix} R & \boldsymbol{b}' \end{bmatrix}$

Example:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right] = \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} R & \boldsymbol{b}' \end{bmatrix}$$

One Particular Solution

Solving one particular solution of $A\boldsymbol{x} = \boldsymbol{b}$ (i.e. $R\boldsymbol{x} = \boldsymbol{b}'$)

$$R\boldsymbol{x} = \boldsymbol{b}' : \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \qquad \begin{cases} \text{pivot variables} : x_1, x_3 \\ \text{free variables} : x_2, x_4 \end{cases}$$

For one $\boxed{\text{particular}}$ solution $\boldsymbol{x}_p$ of $R\boldsymbol{x} = \boldsymbol{b}'$,

choose free variables to be $x_2 = x_4 = 0 \Rightarrow x_1 = 1, x_3 = 6$ (These values are in $\boldsymbol{b}'$.)

$$\Rightarrow \quad \boldsymbol{x}_p = \begin{bmatrix} 1 & 0 & 6 & 0 \end{bmatrix}^{\mathrm{T}}$$

Note : $\boldsymbol{x}_p$ is the solution of $A\boldsymbol{x} = \boldsymbol{b}$ (i.e. $R\boldsymbol{x} = \boldsymbol{b}'$)

$$\{x_1 = 1, x_3 = 6\} \text{ is the solution of } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

pivot rows and pivot columns of R

$A\boldsymbol{x}_p = \boldsymbol{b}$ (i.e. $R\boldsymbol{x}_p = \boldsymbol{b}'$)

Complete Solution

Solving complete solution of $A\boldsymbol{x} = \boldsymbol{b}$ (i.e. $R\boldsymbol{x} = \boldsymbol{b}'$)

$$R\boldsymbol{x} = \boldsymbol{b}' : \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \qquad \begin{cases} \text{pivot variables} : x_1, x_3 \\ \text{free variables} : x_2, x_4 \end{cases}$$

One <u>particular</u> solution of $R\boldsymbol{x} = \boldsymbol{b}'$: $\boldsymbol{x}_p = \begin{bmatrix} 1 & 0 & 6 & 0 \end{bmatrix}^{\mathrm{T}}$

Solve <u>special</u> solutions of $R\boldsymbol{x} = \boldsymbol{0}$

Choose special solutions : $\begin{cases} \boldsymbol{s}_1 : x_2 = 1, x_4 = 0 \;\Rightarrow\; x_1 = -3, x_3 = 0 \\ \boldsymbol{s}_2 : x_2 = 0, x_4 = 1 \;\Rightarrow\; x_1 = -2, x_3 = -4 \end{cases}$

$$\Rightarrow \quad \boldsymbol{s}_1 = \begin{bmatrix} -3 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \boldsymbol{s}_2 = \begin{bmatrix} -2 & 0 & -4 & 1 \end{bmatrix}^{\mathrm{T}}$$

Note : Special solutions are the solutions of $A\boldsymbol{x} = \boldsymbol{0}$ (i.e. $R\boldsymbol{x} = \boldsymbol{0}$)

$\boldsymbol{N}(A) = \boldsymbol{N}(R) = \{\boldsymbol{x}_n \mid A\boldsymbol{x}_n = \boldsymbol{0} \text{ (i.e. } R\boldsymbol{x}_n = \boldsymbol{0})\}$: linear combination of $\boldsymbol{s}_1, \boldsymbol{s}_2$

$\therefore$ <u>Complete</u> solutions of $A\boldsymbol{x} = \boldsymbol{b}$ (i.e. $R\boldsymbol{x} = \boldsymbol{b}'$)

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Complete Solution of Square Invertible Matrix

- Square invertible matrix A
  - A^{-1} exists,
  - $m = n = r$
- Complete solution of $A\boldsymbol{x} = \boldsymbol{b}$
  - One particular solution: $\boldsymbol{x}_p = A^{-1}\boldsymbol{b}$
  - No special solution or free variable $\Rightarrow$ $\boldsymbol{x}_n = \mathbf{0}$

  $\Rightarrow$ Complete solution: $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = A^{-1}\boldsymbol{b} + \mathbf{0}$

Complete Solution of Square Matrix
– Example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} R & \boldsymbol{b}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_1 + b_2 + b_3 \end{bmatrix} \quad \text{No free variable}$$

For $A\boldsymbol{x} = \boldsymbol{b}$ to be solvable, $b_1 + b_2 + b_3 = 0$

One particular solution : $\boldsymbol{x}_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$

No free variable $\Rightarrow$ no special solution $\Rightarrow \boldsymbol{x}_n = \mathbf{0} \Rightarrow N(A) = \{\mathbf{0}\}$

$$\therefore \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \mathbf{0} = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$$

Remark : A has full column rank.

A has Full Column Rank

- A has Full Column Rank
  - rank(A) = r = n
  - All columns of A are pivot columns.
  - There are no free columns/variables or special solutions.
  - $\boldsymbol{N}(A)$ contains only the zero vector.
  - $A\boldsymbol{x} = \boldsymbol{b}$ may not have a solution. (overdetermined)
  - If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has only one solution.
  - A is tall and thin. ($m \geq n$) $R = \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} n\text{x}n \text{ identity matrix} \\ (m-n) \text{ rows of } 0 \end{bmatrix}$

A has Full Row Rank

- A has Full Row Rank
  - rank(A) = r = m
  - All rows of A have pivots. R has no zero rows.
  - $A\boldsymbol{x}$ = $\boldsymbol{b}$ has a solution for every right side $\boldsymbol{b}$.
  - There are $n - r = n - m$ special solutions in the nullspace of A.
  - $\boldsymbol{C}(A)$ is the whole space $\mathbf{R}^m$.
  - $A\boldsymbol{x}$ = $\boldsymbol{b}$ has one or infinitely many solutions. (underdetermined)
  - A is short and wide. ($m \leq n$)
  - $R = [I \quad F] = [mxm \text{ identity matrix} \quad (n-m)\ columns]$

A has Full Row Rank – Example

$$\begin{cases} x+y+z=3 \\ x+2y-z=4 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

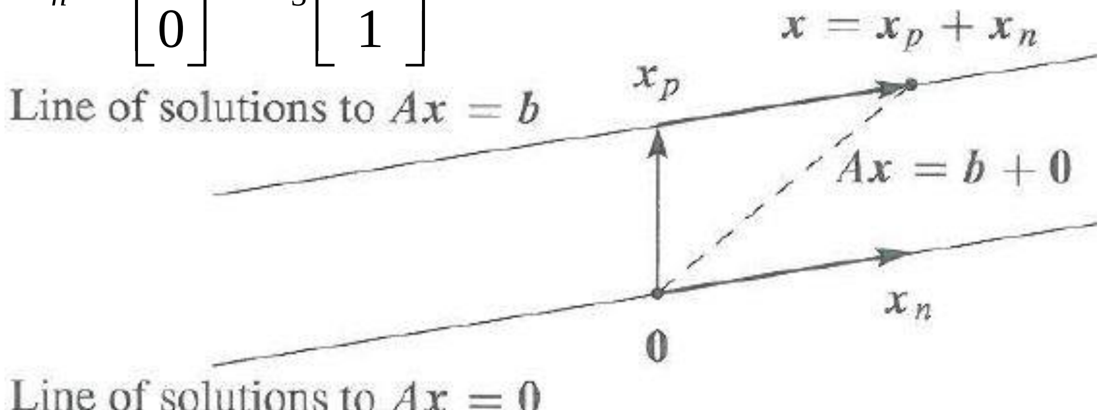
$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} R & \boldsymbol{b}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix} \quad \begin{cases} \text{pivot variables}: x_1, x_2 \\ \text{free variable}: x_3 \end{cases}$$

One particular solution : set $x_3 = 0 \Rightarrow x_1 = 2, x_2 = 1 \Rightarrow \boldsymbol{x}_p = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$

Special solution $\boldsymbol{s}$: set $x_3 = 1 \Rightarrow x_1 = -3, x_2 = 2 \Rightarrow \boldsymbol{s} = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^{\mathrm{T}}$

$N(A)$: span of $\{\boldsymbol{s}\} = \{ \boldsymbol{x}_n \mid \boldsymbol{x}_n = c\boldsymbol{s} \}$: linear combination of $\boldsymbol{s}$

$$\therefore \text{ Complete solution}: \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$



Four Cases of Linear Systems

- The four possibilities for linear equations depend on the rank r:

Solving $A\boldsymbol{x} = \boldsymbol{b}$

	A	R	# of solutions	
$r = m = n$	Square and invertible	$[I]$	1	Full row rank Full column rank
$r = m < n$	Short and wide	$[I \quad F]$	∞	Full row rank
$r = n < m$	Tall and thin	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	0 or 1	Full column rank
$r<m$, $r<n$	Not full rank	$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$	0 or ∞	

Question?