Linear Algebra

Solving Linear Equation

충남대학교
컴퓨터공학과
이만호

Review 1

- Vector
- Vector operations : addition, scalar multiplication
- Matrix form
- Length, dot product
- Cosine formula (angle formula)

$$\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \cos\theta$$

- Schwarz inequality

$$|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- Triangle inequality

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

Review 2

- Let $\boldsymbol{v}$ = (1,1,0) and $\boldsymbol{w}$ = (0,1,1)

  The linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill a plane.
  - Describe that plane.
  - Find a vector that is not a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$
  - The linear combination $c\mathbf{v} + d\mathbf{w}$ fill a plane in ???

- The vectors in that plane allow any c and d where $c,d \in$ R

$$c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix}$$

- Questions
  - Vector (1,2,3) is not in the plane. Why?
  - Find a vector $\boldsymbol{u}$ in the plane with $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$.
  - Find a vector $\boldsymbol{u}$ not in the plane with $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$.

Vectors and Linear Equation

System of Linear Equations

- How to solve a system of linear equations?

  Example: $\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$

  2 unknowns, 2 equations

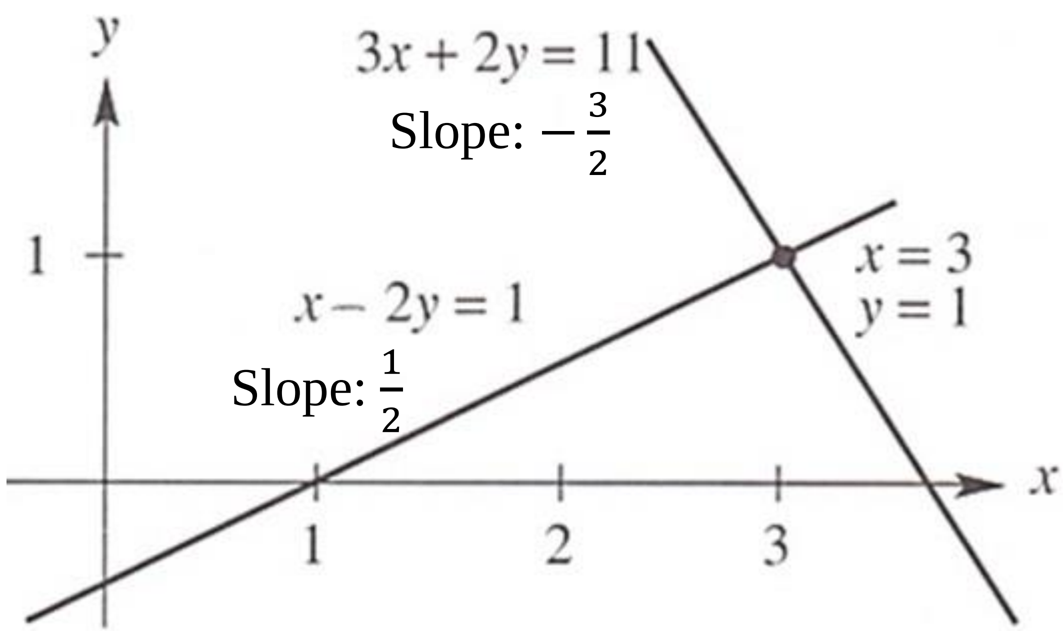
- Solving methods
  - Row picture
  - Column picture
  - Matrix equation

Row Picture

- A system of linear equations

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

- Draw the graph of each row : two lines meet at a single point.



Row picture

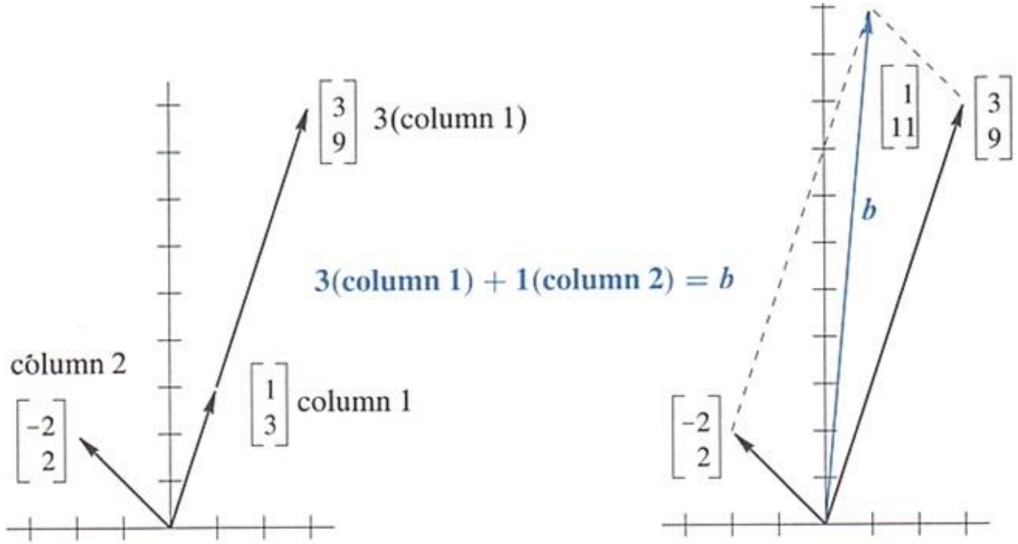
Column Picture

- A system of linear equations

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \Rightarrow \quad x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

- Recognize the linear system as a **vector equation**.
  - Find the linear combination of vectors (1,3) and (–2,2) that equals to the vector (1,11).

Do you prefer which method, row picture or column picture?



Matrix Form of the Equations

- A system of linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \quad \Rightarrow \quad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

- Matrix form: $A\boldsymbol{x} = \boldsymbol{b}$

$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$: Coefficient matrix, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

$\boldsymbol{x} = A^{-1}\boldsymbol{b}$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

Used for row picture

Dot product with rows

Used for column picture

Combination of columns

3 Equations in 3 Unknowns

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

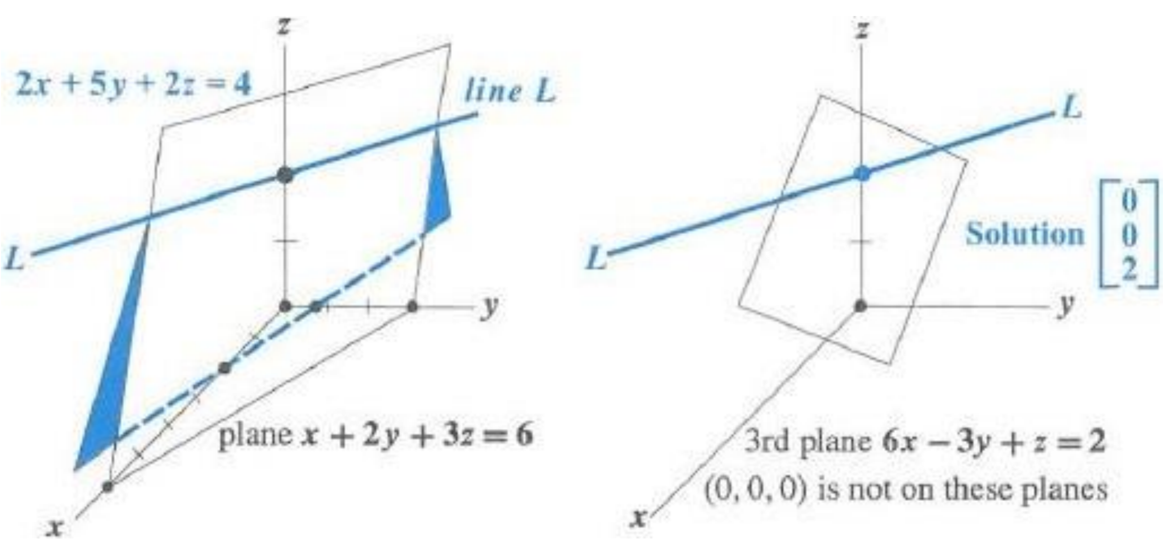
- Solving methods
  - Row picture
  - Column picture
  - Matrix equation

Row Picture of 3 Equations

- System of linear equations

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

The row picture shows three planes meet at a single point (0,0,2)

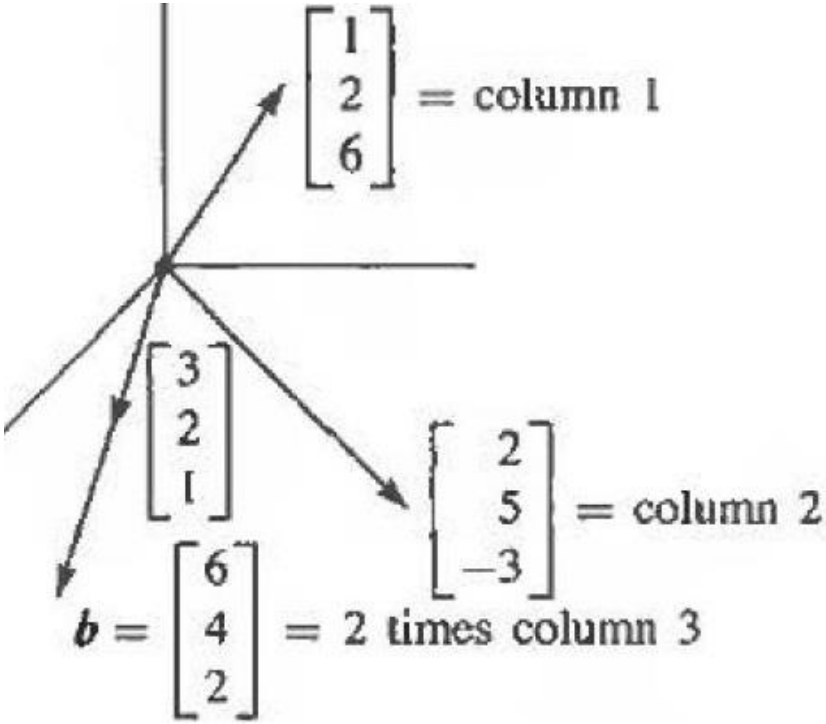


- Each equation produces a plane in 3-dim. space.
- The result of two equations in three unknowns is a line.

Column Picture of 3 Equations

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned} \implies x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$



The **column picture** combines three columns to produce the vector $\begin{bmatrix}6\\4\\2\end{bmatrix}$

Matrix Form of the Equations

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$A \quad \boldsymbol{x} \quad = \quad \boldsymbol{b}$$

- Multiplication **by row**: dot product

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (row1) \cdot \boldsymbol{x} \\ (row2) \cdot \boldsymbol{x} \\ (row3) \cdot \boldsymbol{x} \end{bmatrix}$$

- Multiplication **by columns**: linear combination of column vectors

$$A\boldsymbol{x} = x(column1) + y(column2) + z(column3)$$

Matrix Form of the Equations – Example

- Example 1 :

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \quad I\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

  - Row person : by dot product
  - Column person : by linear combination

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \text{Identity matrix}$$

$$I\boldsymbol{x} = \boldsymbol{x}$$

Matrix Notation

- Let A be mxn matrix

  a_{ij} : an entry in i^{th} row and j^{th} column

  $$1 \leq i \leq m,\ 1 \leq j \leq n$$

  $A(i,j) = a_{ij}$

- Identity Matrix I (nxn matrix)

  $$I(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

  - **Remark :** $I\boldsymbol{x} = \boldsymbol{x}$

Rules for Matrix operations

Matrix Addition and Scalar Multiplication

$$\text{Let } A=\left(a_{ij}\right)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}, \quad B=\left(b_{ij}\right)=\begin{bmatrix}2 & 1\\ 4 & 5\\ 3 & 7\end{bmatrix}$$

$$\text{Addition}: A+B=\left(a_{ij}+b_{ij}\right)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}+\begin{bmatrix}2 & 1\\ 4 & 5\\ 3 & 7\end{bmatrix}=\begin{bmatrix}3 & 3\\ 7 & 9\\ 8 & 13\end{bmatrix}$$

$$\text{Scalar multiplication}: kA=k\left(a_{ij}\right)=\left(ka_{ij}\right)=k\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}=\begin{bmatrix}k & 2k\\ 3k & 4k\\ 5k & 6k\end{bmatrix}$$

Matrix Multiplication

- Matrix multiplication : AB
  - To multiply A and B, # of columns of A = # of rows of B
  - A : mxn matrix, B: nxp matrix → AB : mxp matrix

$$\text{Let } A = \left(a_{ij}\right)_{mxn}, \quad B = \left(b_{ij}\right)_{nxp}$$

$$AB = \left(c_{ij}\right): (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B), \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

  - Example

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Laws for Matrix Operations

- Commutative law:
  $A+B = B+A$ but $AB \neq BA$
- Distributive law:
  $c(A+B) = cA+cB$
  $C(A+B) = CA+CB$
  $(A+B)C = AC+BC$
- Associative law:
  $(A+B)+C = A+(B+C)$
  $A(BC) = (AB)C$
- $A^p = AA\ldots A$ (p factors), $A^pA^q = A^{p+q}$, $(A^p)^q = A^{pq}$
  $A^0 = I$: identity matrix (square matrix)
  A^{-1}: inverse matrix of A(square matrix)

Basic Matrix Operations – Identity

$$\text{Let a matrix and a vector are given}: A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\text{Identity}: I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = A$$

$$I\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Basic Matrix Operations

Interchange rows

Let a matrix and a vector are given : $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Interchange row1 and row2

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 9 & -3 \\ 2 & 4 & -2 \\ -2 & -3 & 7 \end{bmatrix}$$

$$P_{12}\boldsymbol{b} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 10 \end{bmatrix}$$

Interchange row1 and row3

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{13}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 7 \\ 4 & 9 & -3 \\ 2 & 4 & -2 \end{bmatrix}$$

$$P_{13}\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 8 \end{bmatrix}$$

Interchange row2 and row3

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ -2 & -3 & 7 \\ 4 & 9 & -3 \end{bmatrix}$$

$$P_{23}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 8 \end{bmatrix}$$

Basic Matrix Operations

Multiplying Row by a Scalar value

$$\text{Let a matrix and a vector are given}: A=\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}2\\ 8\\ 10\end{bmatrix}$$

Multiply row1 by 1/2

$$M_1=\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad M_1A=\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix}=\begin{bmatrix}1 & 2 & -1\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix}$$

$$M_1\boldsymbol{b}=\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 8\\ 10\end{bmatrix}=\begin{bmatrix}1\\ 8\\ 10\end{bmatrix}$$

Multiply row2 by 2

$$M_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad M_2A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix}=\begin{bmatrix}2 & 4 & -2\\ 8 & 18 & -6\\ -2 & -3 & 7\end{bmatrix}$$

$$M_2\boldsymbol{b}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 8\\ 10\end{bmatrix}=\begin{bmatrix}2\\ 16\\ 10\end{bmatrix}$$

Multiply row3 by −1

$$M_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix} \qquad M_3A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ -2 & -3 & 7\end{bmatrix}=\begin{bmatrix}2 & 4 & -2\\ 4 & 9 & -3\\ 2 & 3 & -7\end{bmatrix}$$

$$M_3\boldsymbol{b}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}2\\ 8\\ 10\end{bmatrix}=\begin{bmatrix}2\\ 8\\ -10\end{bmatrix}$$

Basic Matrix Operations

Adding a Row to Other Row

Let a matrix and a vector are given : $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Add row1 to row2

$$R_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 6 & 13 & -5 \\ -2 & -3 & 7 \end{bmatrix}$$

$$R_{12}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$$

Add row1 to row3

$$R_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad R_{13}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

$$R_{13}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 12 \end{bmatrix}$$

Add row3 to row2

$$R_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 2 & 6 & 4 \\ -2 & -3 & 7 \end{bmatrix}$$

$$R_{32}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \\ 10 \end{bmatrix}$$

Basic Matrix Operations

Sequence of Basic Matrix Operations

Let a matrix and a vector are given : $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Multiply row1 by 2, and subtract it from row2

- row1 and row3 are not changed.

- row2 = row2 − 2 * row1: $\boxed{E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

Multiply row1 by −1, and subtract it from row3

- row1 and row2 are not changed.

- row3 = row3 − (−1) * row1: $\boxed{E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$

$$E_{31}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

Elimination matrix

Block matrices & Block multiplication

- Matrices can be cut into blocks which are smaller matrices.
- When matrices split into blocks, it is often simpler to see how they act.
- Example : 4x6 matrix is broken into six 2x2 blocks.

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}$$

  – If A and B are same size and the block sizes match, then $A+B$ can be got by block by block addition.
- Example : Augmented matrix $[A \ \ \boldsymbol{b}]$ has two blocks of different sizes.
  – After elimination, $[EA \ \ E\boldsymbol{b}]$

Block Multiplication

- Block multiplication:
  - If the <u>cuts between columns of A</u> match the <u>cuts between rows of B</u>, then block multiplication of AB is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}$$

- Example : columns times rows

$$\left[\begin{array}{c|c} 1 & 4 \\ 1 & 5 \end{array}\right]\left[\begin{array}{cc} 3 & 2 \\ \hline 1 & 0 \end{array}\right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Block Multiplication – Example

- Example : Elimination by blocks
  - $E=\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3 & 1 & 0 \\ -4 & 0 & 1 \end{array}\right] \quad A=\left[\begin{array}{c|cc} 1 & any_{12} & any_{13} \\ \hline 3 & any_{22} & any_{23} \\ 4 & any_{32} & any_{33} \end{array}\right] \quad EA=\left[\begin{array}{c|cc} 1 & any_{12} & any_{13} \\ \hline 0 & any'_{22} & any'_{23} \\ 0 & any'_{32} & any'_{33} \end{array}\right]$
  - $E=\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3/2 & 1 & 0 \\ -4/2 & 0 & 1 \end{array}\right] \quad A=\left[\begin{array}{c|cc} 2 & any_{12} & any_{13} \\ \hline 3 & any_{22} & any_{23} \\ 4 & any_{32} & any_{33} \end{array}\right] \quad EA=\left[\begin{array}{c|cc} 2 & any_{12} & any_{13} \\ \hline 0 & any'_{22} & any'_{23} \\ 0 & any'_{32} & any'_{33} \end{array}\right]$
- Block Elimination

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=\left[\begin{array}{c|c} A & B \\ \hline 0 & D-CA^{-1}B \end{array}\right]$$

$S=D-CA^{-1}B:$ Schur complement

The idea of Elimination

Solving 2 Equations in 2 Unknowns

- System of linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 \\ \quad\;\; 8y = 8 \end{cases} : \text{upper triangular matrix}$$

  - The order of 2 equations is not important.

- Solving methods
  1. Eliminate variables
     1) Eliminate one variable from 2 equations producing 1 equations.
  2. Substitute variables
     1) Get a solution of one variable from the equation got in the step 1-1).
     2) Substitute the solutions of 1 variables into the one of 2 original equations.

Solving 3 Equations in 3 Unknowns

- System of linear equations

$$\begin{cases} x+2y+3z=6 \\ 2x+5y+2z=4 \\ 6x-3y+z=2 \end{cases} \Rightarrow \begin{cases} -17x+11y=0 \\ -10x+11y=0 \\ 6x-3y+z=2 \end{cases} \Rightarrow \begin{cases} -17x+11y=0 \\ 7x=0 \\ 6x-3y+z=2 \end{cases}$$

  – The order of equations and variables is not important.

$$\begin{cases} 6x-3y+z=2 \\ x+2y+3z=6 \\ 2x+5y+2z=4 \end{cases} \Rightarrow \begin{cases} 6x-3y+z=2 \\ -17x+11y=0 \\ -10x+11y=0 \end{cases} \Rightarrow \begin{cases} 6x-3y+z=2 \\ -17x+11y=0 \\ 7x=0 \end{cases}$$

$$\begin{cases} z-3y+6x=2 \\ 3z+2y+x=6 \\ 2z+5y+2x=4 \end{cases} \Rightarrow \begin{cases} z-3y+6x=2 \\ 11y-17x=0 \\ 11y-10x=0 \end{cases} \Rightarrow \begin{cases} z-3y+6x=2 \\ 11y-17x=0 \\ 7x=0 \end{cases}$$

upper triangular matrix

The idea of Elimination

- **Elimination** :

$$\begin{cases} x-2y=1 & (1) \\ 3x+2y=11 & (2) \end{cases} \rightarrow (2)-(1)\cdot 3 \rightarrow \begin{cases} x-2y=1 & (1') \\ 8y=8 & (2') \end{cases}$$

$$\Rightarrow \quad y=1, \quad x=3$$

- Matrix format

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

  – Upper triangular system/matrix
  – Back substitution

Pivot and Multiplier

- **Definition :**
  - Pivot = first nonzero in the row that does the elimination
  - Multiplier = (entry to eliminate) / (Pivot) : divide
- Solving linear system of *n* equations
  - Need *n* pivots
  - After eliminations, we get the upper triangular matrix.
    - The pivots are on the diagonal of the triangular matrix.

$$\begin{cases} z-3y+6x=2 \\ 3z+2y+\ \ x=6 \\ 2z+5y+2x=4 \end{cases} \Rightarrow \begin{cases} z-3y+\ 6x\ =2 \\ \quad 11y-17x\ =0 \\ \quad 11y-10x\ =0 \end{cases} \Rightarrow \begin{cases} z-3y+\ 6x=2 \\ \quad 11y-17x\ =0 \\ \quad\quad\quad 7x\ =0 \end{cases}$$

Pivot and Multiplier – Example (2 vars)

– Pivot? Multiplier?

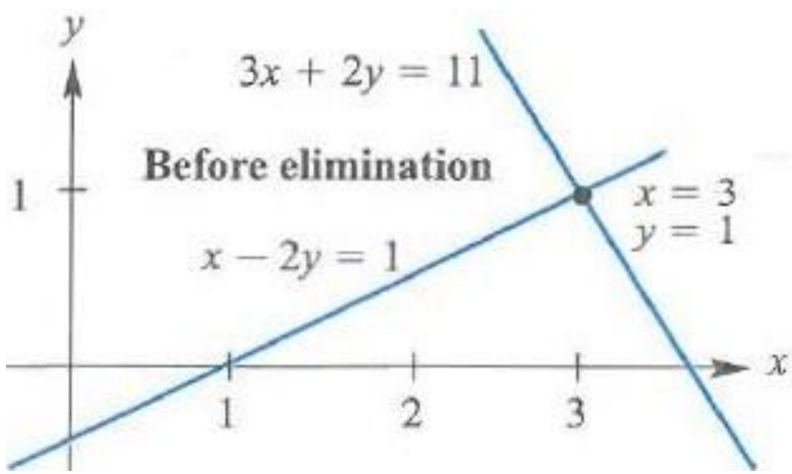
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ \quad\ 8y=8 \end{cases}$$

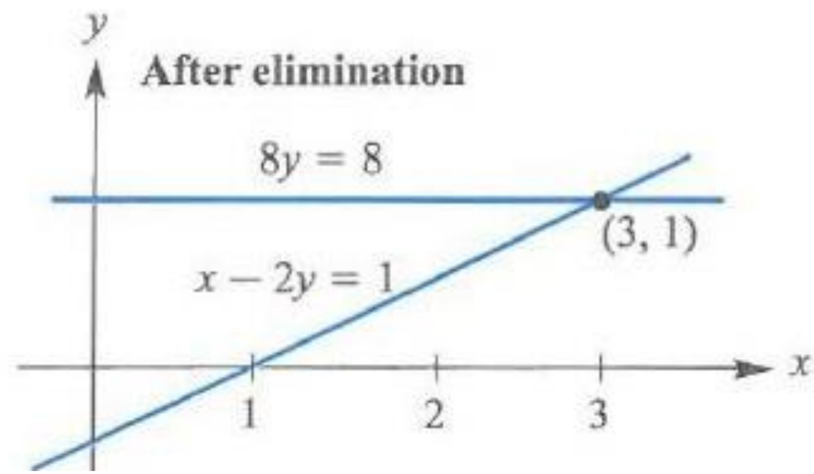
1^{st} pivot is 1, multiplier is $3/1$, 2^{nd} pivot is 8

$$\begin{cases} 5x-10y=5 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} 5x-10y=5 \\ \quad\quad\ 8y=8 \end{cases}$$

1^{st} pivot is 5, multiplier is $3/5$, 2^{nd} pivot is 8

- After elimination, pivots are on the diagonal of the triangular system





Pivot and Multiplier – Example (3 vars)

$$\begin{cases} 6x - 3y + z = 2 \\ x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \end{cases} \Rightarrow \begin{cases} 6x - 3y + z = 2 & 1^{\text{st}} \text{ pivot} = 6 \\ \frac{5}{2}y + \frac{17}{6}z = \frac{17}{3} & \text{multiplier} = \frac{1}{6} \\ 6y + \frac{5}{3}z = \frac{10}{3} & \text{multiplier} = \frac{2}{6} \end{cases}$$

$$\Rightarrow \begin{cases} 6x - 3y + z = 2 & \\ \frac{5}{2}y + \frac{17}{6}z = \frac{17}{3} & 2^{\text{nd}} \text{ pivot} = \frac{5}{2} \\ -\frac{77}{15}z = -\frac{154}{15} & \text{multiplier} = \frac{6}{5/2} = \frac{12}{5} \end{cases}$$

Breakdown of Elimination

- Success of eliminations :
  - Produces the *full set of Pivots* and get the solution.
- Failure with no solution
  - A is singular.

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 \\ 0y = 8 \end{cases}$$

- Failure with infinitely many solutions
  - A is singular.

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 3 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 \\ 0y = 0 \end{cases}$$

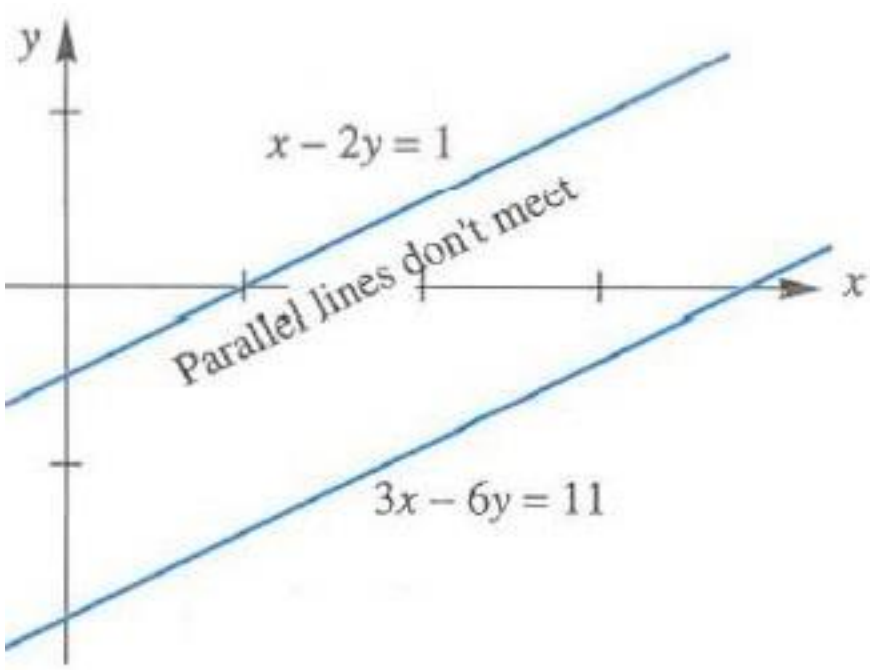
- Temporary failure (zero in pivot), A row exchange produces full set of pivots.
  - A is not singular.

$$\begin{cases} 0x + 2y = 4 \\ 3x - 2y = 5 \end{cases} \Rightarrow \begin{cases} 3x - 2y = 5 \\ 2y = 4 \end{cases}$$

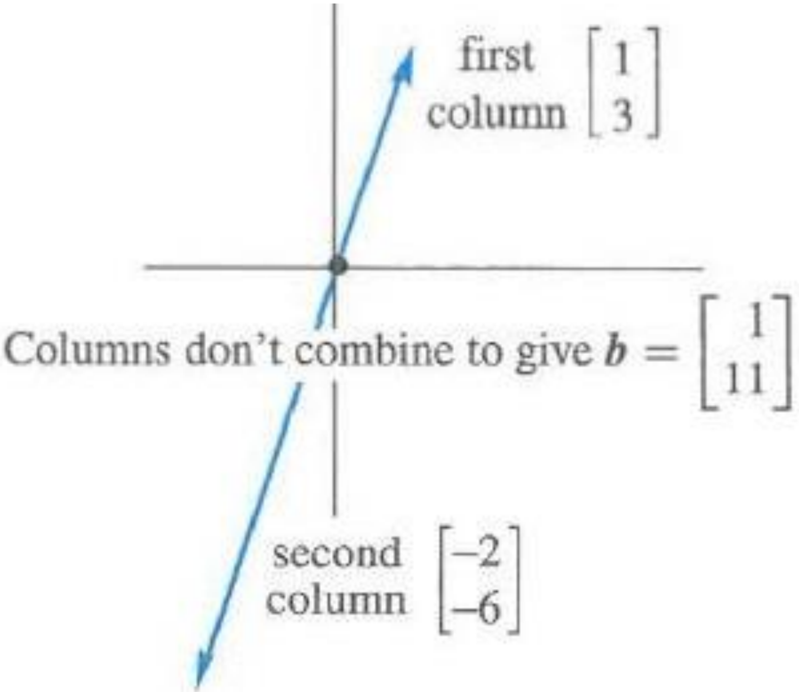
Graph of a Linear System with No Solution

$$\begin{cases} x-2y=1 \\ 3x-6y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ \quad\;\; 0y=8 \end{cases}$$

- One pivot
- $0y = 8$: no solution



Row picture

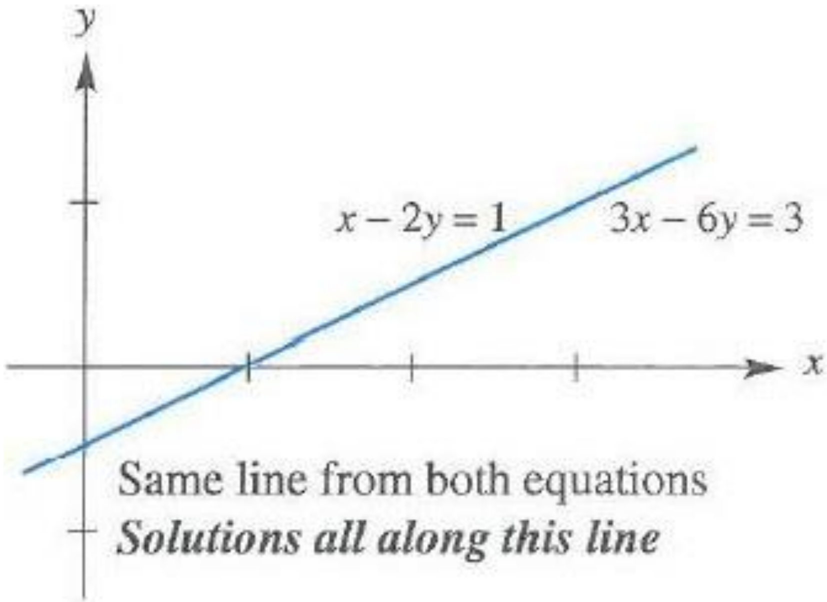


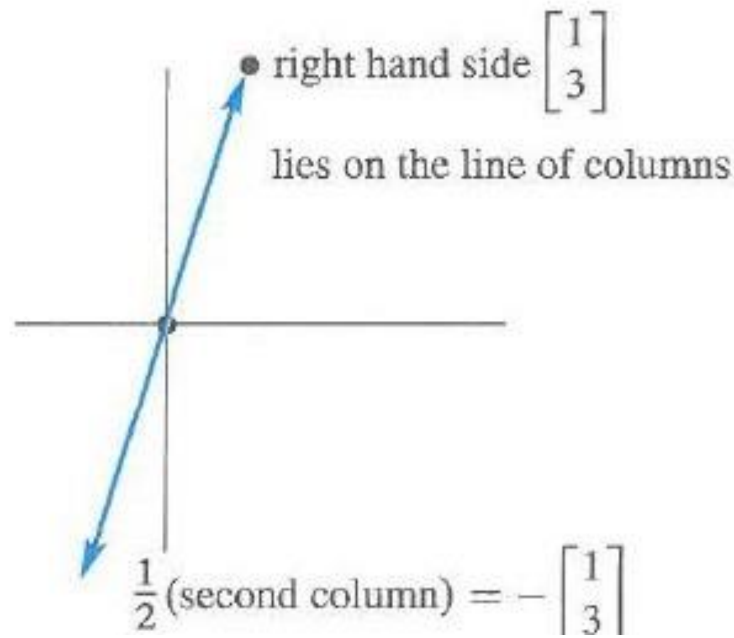
Column picture

Graph of a Linear System with Infinitely Many Solutions

$$\begin{cases} x-2y=1 \\ 3x-6y=3 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ 0y=0 \end{cases}$$

- One pivot
- y : free variable





Row picture

Column picture

Elimination of 3 Equations in 3 unknowns

$$\text{Example}: \begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$\begin{cases} 2x+4y-2z=2 & ① & \text{pivot}: p_1 = 2 \\ 4x+9y-3z=8 & ② & \text{multiplier}: l_{21} = a_{21} / p_1 = 4/2,\ ②-①*(4/2) \\ -2x-3y+7z=10 & ③ & \text{multiplier}: l_{31} = a_{31} / p_1 = -2/2,\ ③-①*(-2/2) \end{cases}$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① & \\ \quad y+\ \ z=4 & ②' & \text{pivot}: p_2 = 1 \\ \quad y+5z=12 & ③' & \text{multiplier}: l_{32} = a_{32} / p_2 = 1/1,\ ③'-②'*1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① & \\ \quad y+\ \ z=4 & ②' & \\ \quad 4z=8 & ③'' & \text{pivot} = 4 \end{cases}$$

Back substitution

From ③$''$, $z=2$

From ②$'$ and $z=2,\ y=2$

From ① and $z=2,\ y=2,\ x=-1$

Forward Elimination

- Forward Elimination

  Step-1: in the 1st column, make <u>the coefficients below the diagonal</u> zero.

  Step-2: in the 2nd column, make <u>the coefficients below the diagonal</u> zero.

  ……

  Step-(n-1): in the (n-1)th column, make <u>the coefficients below the diagonal</u> zero.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} + a_{3n}x_n = b_3 \\ \cdots\cdots \\ a_{n-1,1}x_1 + a_{n-1,2}x_2 + a_{n-1,3}x_3 + \cdots + a_{n-1,,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} + a_{nn}x_n = b_n \end{cases} \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3,n-1}x_{n-1} + a'_{3n}x_n = b'_3 \\ \cdots\cdots \\ a'_{n-1,2}x_2 + a'_{n-1,3}x_3 + \cdots + a'_{n-1,,n-1}x_{n-1} + a'_{n-1,n}x_n = b'_{n-1} \\ a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{n,n-1}x_{n-1} + a'_{nn}x_n = b'_n \end{cases}$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \cdots + a''_{3,n-1}x_{n-1} + a''_{3n}x_n = b''_3 \\ \cdots\cdots \\ a''_{n-1,3}x_3 + \cdots + a''_{n-1,,n-1}x_{n-1} + a''_{n-1,n}x_n = b''_{n-1} \\ a''_{n3}x_3 + \cdots + a''_{n,n-1}x_{n-1} + a''_{nn}x_n = b''_n \end{cases} \Rightarrow \cdots\cdots \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \cdots + a''_{3,n-1}x_{n-1} + a''_{3n}x_n = b''_3 \\ \cdots\cdots \\ a^{(n-2)}_{n-1,n-1}x_{n-1} + a^{(n-2)}_{n-1,n}x_{n-1}x_n = b^{(n-2)}_{n-1} \\ a^{(n-1)}_{nn}x_n = b^{(n-1)}_n \end{cases}$$

Elimination Using Matrices

Matrix Form of Linear System

- Consider a linear system and its Matrix form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \sum_{j=1}^{n} a_{1j}x_j = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \sum_{j=1}^{n} a_{2j}x_j = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \sum_{j=1}^{n} a_{nj}x_j = b_n \end{cases} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} : A\boldsymbol{x} = \boldsymbol{b}$$

  – Multiplication **by row**: dot product

$$A\boldsymbol{x} = \begin{bmatrix} (a_{11}, a_{12}, \cdots, a_{1n}) \cdot (x_1, x_2, \cdots, x_n) \\ (a_{21}, a_{22}, \cdots, a_{2n}) \cdot (x_1, x_2, \cdots, x_n) \\ \vdots \\ (a_{n1}, a_{n2}, \cdots, a_{nn}) \cdot (x_1, x_2, \cdots, x_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

  – Multiplication **by columns**: combination of column vectors

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

Elimination Matrix – 1

$$\text{Consider}:\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}:\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$\begin{cases} 2x+4y-2z=2 & ① \quad \text{pivot}: p_1=2 \\ 4x+9y-3z=8 & ② \quad \text{multiplier}: l_{21}=a_{21}/p_1=4/2,\ ②-①*(4/2) \\ -2x-3y+7z=10 & ③ \quad \text{multiplier}: l_{31}=a_{31}/p_1=-2/2,\ ③-①*(-2/2) \end{cases}$$

$$\text{Elimination matrix}:\ E_1=\begin{bmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ 2/2 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_1A=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}=A_1$$

$$E_1\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}=\boldsymbol{b}_1$$

$$\Rightarrow\begin{cases} 2x+4y-2z=2 & ① \\ \quad\quad\ \ y+\ \ z=4 & ②' \\ \quad\quad\ \ y+5z=12 & ③' \end{cases}$$

Elimination Matrix – 2

Forward elimination (cont.)

$$\begin{cases} 2x+4y-2z=2 & ① \\ \quad\; y+\;\; z=4 & ②' \quad \text{pivot}: p_2=1 \\ \quad\; y+5z=12 & ③' \quad \text{multiplier}: l_{32}=a_{32}/p_2=1/1,\ ③'-②'*1 \end{cases}$$

$$\text{Elimination matrix}:\ E_2=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$$

$$E_2A_1=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}2&4&-2\\0&1&1\\0&1&5\end{bmatrix}=\begin{bmatrix}2&4&-2\\0&1&1\\0&0&4\end{bmatrix}=A_2:\text{upper triangular matrix}$$

$$E_2\boldsymbol{b}_1=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}2\\4\\12\end{bmatrix}=\begin{bmatrix}2\\4\\8\end{bmatrix}=\boldsymbol{b}_2$$

$$\Rightarrow\begin{cases} 2x+4y-2z=2 & ① \\ \quad\; y+\;\; z=4 & ②' \\ \quad\quad\;\; 4z=8 & ③'' \end{cases}$$

Elimination Matrix – 3

$$\text{Consider}:\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad\Rightarrow\quad A\boldsymbol{x}=\boldsymbol{b}:\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$A\boldsymbol{x}=\boldsymbol{b} \quad\Rightarrow\quad \underbrace{E_1A}_{A_1}\boldsymbol{x}=\underbrace{E_1\boldsymbol{b}}_{\boldsymbol{b}_1} \quad\Rightarrow\quad \underbrace{E_2\underbrace{E_1A}_{A_1}}_{A_2}\boldsymbol{x}=\underbrace{E_2\underbrace{E_1\boldsymbol{b}}_{\boldsymbol{b}_1}}_{\boldsymbol{b}_2} \quad\Rightarrow\quad EA\boldsymbol{x}=E\boldsymbol{b}$$

For matrices A, B, C,
$(AB)C = A(BC)$
$AB \neq BA$

$$\text{where } E_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},\quad E_1=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E=E_2E_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}: \textbf{lower}\text{ triangular matrix}$$

$$EA=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}: \textbf{upper}\text{ triangular matrix}$$

$$E\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{cases} 2x+4y-2z=2 & ① \\ \qquad y+\ z=4 & ②' \\ \qquad\quad 4z=8 & ③'' \end{cases}$$

Elimination Matrix – 4

Consider : $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b} : \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Forward elimination

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad EA\boldsymbol{x}=E\boldsymbol{b}, \quad \text{where } E=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}} \qquad \begin{cases} 2x+4y-2z=2 & ① \\ y+\ \ z=4 & ②' \\ 4z=8 & ③'' \end{cases}$$

This matrix and vector can be used to get the solution.

Back substitution

$z=8/4=2$

$y=4-z=4-2=2$

$x=(2-4y+2z)/2=(2-4\cdot 2+2\cdot 2)/2=-1$

※ Multiply elimination matrix with A and $\boldsymbol{b}$.

※ EA and $E\boldsymbol{b}$ are enough to get solution.

Augmented Matrix

Consider : $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b} : \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad EA\boldsymbol{x}=E\boldsymbol{b}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Augmented matrix of $A\boldsymbol{x}=\boldsymbol{b}$: $\boxed{\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}}$: rectangular matrix

$$E\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Back substitution

$z=8/4=2$

$y=4-z=4-2=2$

$x=(2-4y+2z)/2=(2-4\cdot 2+2\cdot 2)/2=-1$

Inverse Matrices