Linear Algebra

Vector Spaces and Subspaces

Independence, Basis and Dimension

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Independence, Basis and Dimension

- Important issues:

  Let A be an mxn matrix,

  what is the true size (dimension) of subspaces, $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$?
- Dimension of $\boldsymbol{C}(A)$:
  - The number of independent columns
  - rank(A) = r
- Definition:

  Basis: independent vectors that span the space.
- Every vector in the space is a unique combination of the basis vectors.

Essential Ideas

- Four essential ideas:
  - Independent vectors (no extra vectors)
  - Spanning a space (enough vectors to produce the rest)
  - Basis for a space (not too many or too few)
  - Dimension of a space (the number of vectors in a basis)

Linear Independence

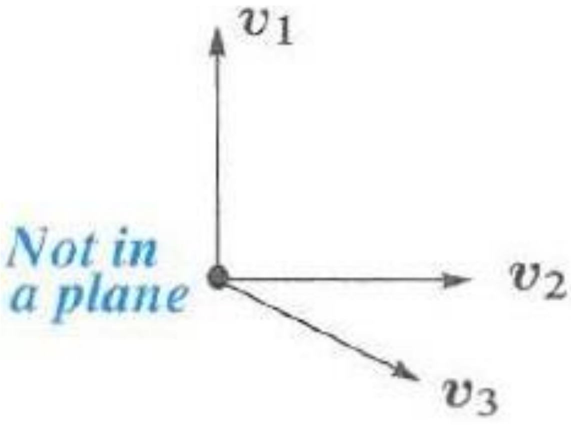
- Linear independence:
  - The columns of A are linearly independent when the only solutions to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.
  - No other combination of the columns gives zero vector. i.e. $\boldsymbol{N}(A)$ contains only zero vectors.
- Linear independence: other definition
  - The sequence of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ is *linearly independent* if the only combination that gives the zero vector is

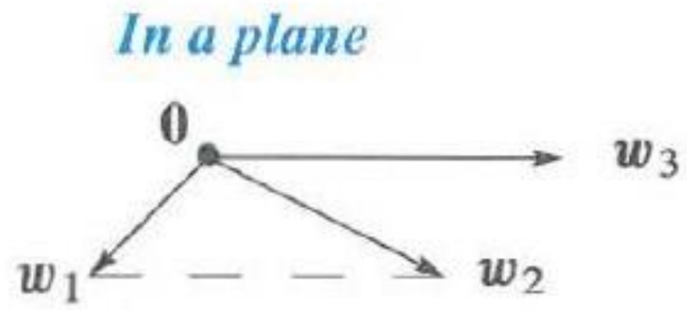
    $$0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_n$$

    (i.e. the only solution of $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$)

Linear Independence in $\mathbf{R}^3$

- Linear Independence with 3 vectors in $\mathbf{R}^3$
  - If three vectors are *not* in the same plane, they are independent. No combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ gives zero except $0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + 0\boldsymbol{v}_3$.
  - If three vectors $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ are *in the same plane*, they are dependent.





Remark on Linear Independence

- Columns of A are independent exactly when rank(A) = n.
  - There are n pivots and no free variables
  - $\boldsymbol{N}(A) = \{\ \mathbf{0}\ \}$
- Any set of n vectors in $\mathbf{R}^m$ must be linearly dependent if $n > m$

Linear Independence – Example

- In $\mathbf{R}^2$
  - (1, 0) & (0, 1) : independent
  - (1, 0) & (1, 0.00001) : independent
  - (1, 1) & (−1, −1) : dependent
  - (1, 1) & (0, 0) : dependent
  - Any three vectors are dependent!
- In $A\boldsymbol{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank}(A) = \text{rank}(R) = 2 \neq 3$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\boldsymbol{x} = \mathbf{0}$ has a nonzero solution $(-3, 1, 1) \in \boldsymbol{N}(A)$

$\therefore$ Columns of A are not linearly independent.

Span

- Column space
  - Is spanned by the columns.
  - Consists of all linear combinations of the columns. — $A\boldsymbol{x}$
- Definition

  A set of vectors *spans* a space if their linear combinations fill the space.
- Example

  Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix},\ \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \boldsymbol{w}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
  - $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ span the full space $\mathbf{R}^2$.
  - $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ also span the full space $\mathbf{R}^2$. Relation among $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$?
  - $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ only span a line in $\mathbf{R}^2$.

Row Space

- Definition

  The row space of matrix is the subspace of $\mathbf{R}^n$ spanned by the rows. (A: mxn)
  - The row space of A is $\boldsymbol{C}(A^{\mathrm{T}})$
- Example

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}, \qquad A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}$$

  - $\boldsymbol{C}(A)$: plane in $\mathbf{R}^3$ spanned by 2 columns of A.
  - $\boldsymbol{C}(A^{\mathrm{T}})$: all of $\mathbf{R}^2$ spanned by 3 rows of A (3 columns of A^{T}).
  - The rows are in $\mathbf{R}^n$ spanning the row space.
  - The columns are in $\mathbf{R}^m$ spanning the column space.

Basis for a Vector Space

- Definition : A basis for a vector space is a sequence of vectors with two properties
  - The basis vectors are linearly independent.
  - The basis vectors span the space.
- Notes
  - There is one and only one way to write $\boldsymbol{v}$ as a combination of the basis vectors.
    - Every vector $\boldsymbol{v}$ in the space is a linear combination of the basis vectors, and the combination is unique because the basis vectors are independent.
  - 2 vectors can't span all of $\mathbf{R}^3$, even if they are independent.
  - 4 vectors can't be independent, even if they span $\mathbf{R}^3$.

Standard Basis for $\mathbf{R}^n$

- Standard basis for $\mathbf{R}^2$:

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Standard basis for $\mathbf{R}^3$:

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Standard basis for $\mathbf{R}^n$:

Column vectors of the nxn identity matrix $\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$

Basis is not Unique

- The columns of *every invertible nxn matrix* give a basis for $\mathbf{R}^n$:
- In $\mathbf{R}^3$, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$(standard basis) is a basis

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis.}$$

$$\text{Consider } A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible, full column rank, full row rank.

$\therefore$ Column vectors of A is also a basis.

$$\boxed{B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} : ?}$$

Properties of Basis

- Columns of A is a basis: (A: nxn)
  - $N(A)$ contains **only the zero vector**.
  - The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$.
  - The columns are independent.
  - The column space is the whole space $\mathbf{R}^n$.
  - $A\boldsymbol{x} = \boldsymbol{b}$ can always be solved by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

$\Rightarrow$ in one sentence

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are basis for $\mathbf{R}^n$ exactly when they are the columns of an nxn invertible matrix. Thus $\mathbf{R}^n$ has infinitely many different bases.

Pivot Columns are a Basis

- When the columns are dependent, keep only the pivot columns. They are independent and span the column space.

The pivot columns of A are a basis for $\boldsymbol{C}(A)$.

The pivot rows of A are basis for its row space.

So are the pivot rows of its echelon form R.

Pivot Columns are a Basis – Example

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

- The pivot column (2, 3) of A is a basis for $\boldsymbol{C}(A)$. The second column (4, 6) of A would be a different basis.
- The pivot column (1, 0) of R is a basis for $\boldsymbol{C}(R)$.
- $\boldsymbol{C}(A)$ and $\boldsymbol{C}(R)$ are different. Their bases are different.
- Dimensions of $\boldsymbol{C}(A)$ and $\boldsymbol{C}(R)$ are the same.
- The row space of A is the *same* as the row space of R.
- There are infinitely many bases to choose from.
  - One natural choice is to pick the nonzero rows of R (rows with a pivot).

Basis for $\boldsymbol{C}(A)$: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Basis for the row space: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Pivot Columns are a Basis – Example

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Pivot columns: column 1 and 3
  - (1, 0, 0), (0, 1, 0) are basis for $\boldsymbol{C}(R)$. Subspace of $\mathbf{R}^3$.
  - Column 1 and 4 are also a basis for $\boldsymbol{C}(R)$. Subspace of $\mathbf{R}^3$.
  - Column 2 and 3 are also a basis for $\boldsymbol{C}(R)$. Subspace of $\mathbf{R}^3$.
  - Column 2 and 4 are also a basis for $\boldsymbol{C}(R)$. Subspace of $\mathbf{R}^3$.
  - Column 1 and 2 are not a basis for $\boldsymbol{C}(R)$.
- Pivot rows: row 1 and 2
  - (1,2,0,3), (0,0,1,4) are basis for row space. Subspace of $\mathbf{R}^4$.

Dimension of a Vector Space

- There are many choices for the basis vectors, but the *number* of basis vectors doesn't change.
- Theorem

  If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_n$ are both bases for the same vector space, then $m = n$.

  – The number of basis vectors doesn't change.
- Definition

  Dimension of a space is the number of vectors in every basis.

  – Dimension of $\boldsymbol{C}(A) = \text{rank}(A)$

Independence, Basis, Dimension

Extended Concept of Basis

- The words independence, basis, dimension are not restricted to column vectors.
- Examples: see next slide
  - Matrix spaces
  - Function spaces

Matrix Spaces

- **M**: set of all 2x2 matrices.
  - Dimension is 4.
  - One basis: $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
  - Every 2x2 matrix is the linear combination of A_1, A_2, A_3, A_4.

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

  - A_1, A_4 are a basis for a subspace — the diagonal matrices.
  - A_1, A_2, A_4 are a basis for a subspace — the upper triangular matrices. Its dimension is 3.
  - What is a basis for the symmetric matrices?
- **M**: set of all *n*x*n* matrices. ???

Function Spaces

- $y'' = 0$
  - Solution: $y = cx + d$
  - One basis: x, 1
- $y'' = y$
  - Solution: $y = ce^{x} + de^{-x}$
  - One basis: e^{x}, e^{-x}
- $y'' = -y$
  - Solution: $y = c \sin x + d \cos x$
  - One basis: $\sin x$, $\cos x$

Z (zero vector)

- In $\mathbf{R}^2$
  - $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Dimension of $\boldsymbol{C}(A) =$?
    One basis: ?
  - $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Dimension of $\boldsymbol{C}(A) =$?
    One basis: ?
  - $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Dimension of $\boldsymbol{C}(A) =$?
    One basis: ?
  - $\mathbf{Z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Dimension of $\boldsymbol{Z} =$?
    One basis: ?
- Basis for **Z**: ?
- **Z** can/never? be allowed as a basis.

Question?