Linear Algebra

Solving Linear Equations

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Review 1

- Vector
- Vector operations : addition, scalar multiplication
- Matrix form
- Length, dot product
- Cosine formula (angle formula)

$$\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \cos\theta$$

- Schwarz inequality

$$|\boldsymbol{u}\cdot\boldsymbol{v}| \le \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- Triangle inequality

$$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$$

Review 2

- Let $\boldsymbol{v}$ = (1,1,0) and $\boldsymbol{w}$ = (0,1,1)

  The linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill a plane.
  - Describe that plane.
  - Find a vector that is not a combination of $\boldsymbol{v}$ and $\boldsymbol{w}$
  - The linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ fill a plane in ???

- The vectors in that plane allow any c and d where $c,d \in$ R

$$c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix}$$

- Questions
  - Vector (1,2,3) is not in the plane. Why?
  - Find a vector $\boldsymbol{u}$ in the plane with $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$.
  - Find a vector $\boldsymbol{u}$ not in the plane with $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$.

Vectors and Linear Equation

System of Linear Equations

- How to solve a system of linear equations?

  Example: $\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases}$

  2 unknowns, 2 equations

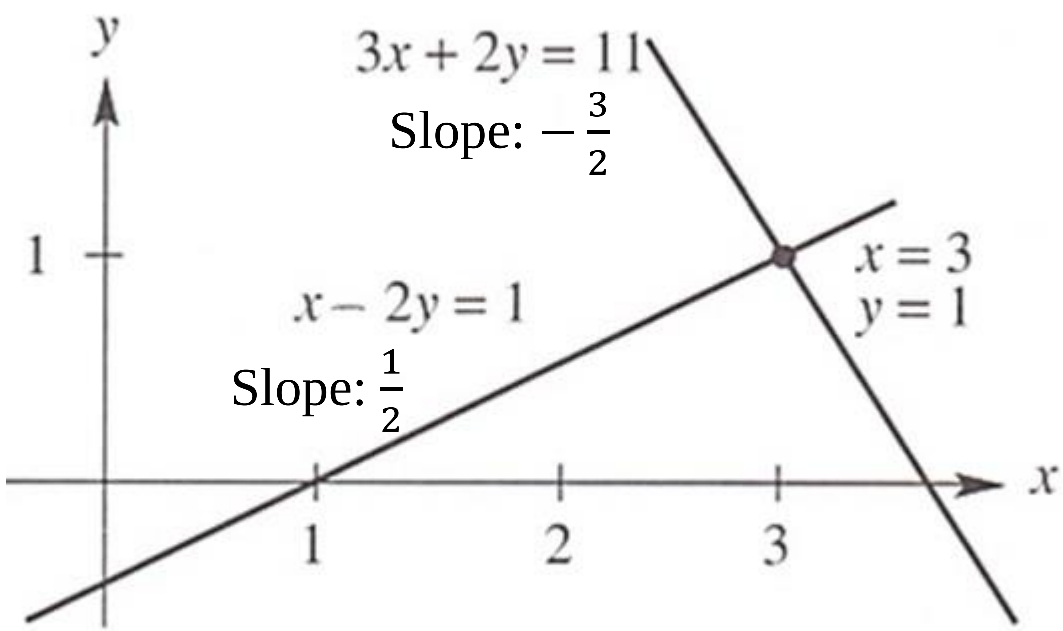
- Solving methods
  - Row picture
  - Column picture
  - Matrix equation

Row Picture

- A system of linear equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

- Draw the graph of each row : two lines meet at a single point.



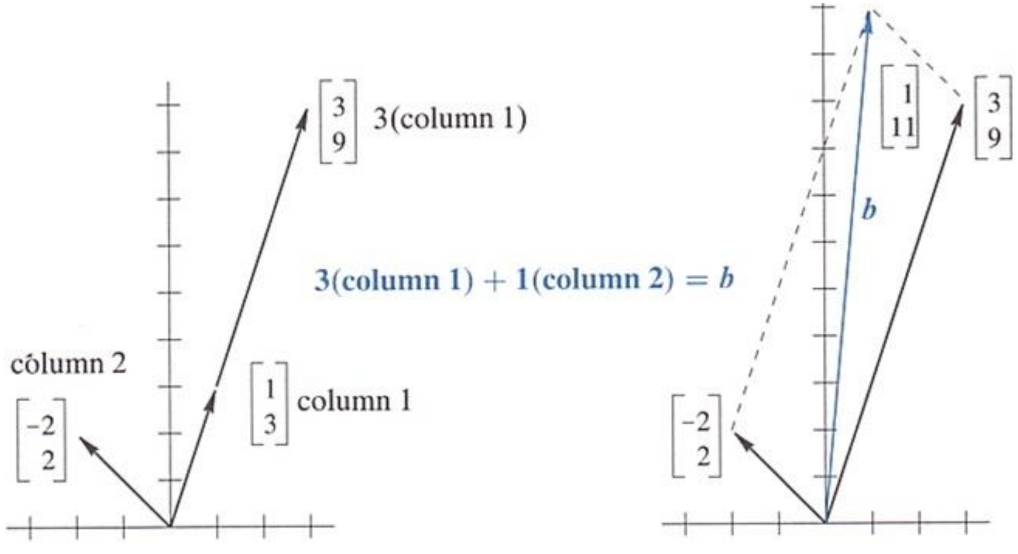
Row picture

Column Picture

- A system of linear equations

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \quad \Rightarrow \quad x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

- Recognize the linear system as a **vector equation**.
  – Find the linear combination of vectors (1,3) and (–2,2) that equals to the vector (1,11).



Do you prefer which method, row picture or column picture?

Matrix Form of the Equations

- A system of linear equations

$$\begin{cases} x - 2y = 1 \\ 3x + 2y = 11 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

- Matrix form: $A\boldsymbol{x} = \boldsymbol{b}$

$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$: Coefficient matrix, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

$\boldsymbol{x} = A^{-1}\boldsymbol{b}$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

Used for row picture

Dot product with rows

Used for column picture

Combination of columns

3 Equations in 3 Unknowns

- System of linear equations

$$
\begin{aligned}
x + 2y + 3z &= 6 \\
2x + 5y + 2z &= 4 \\
6x - 3y + z &= 2
\end{aligned}
$$

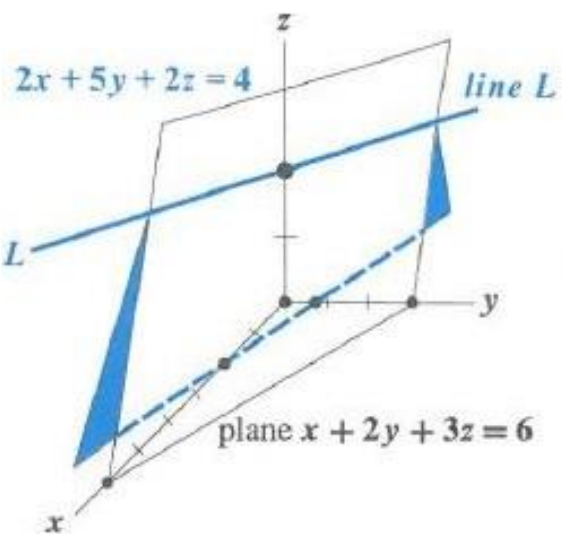
- Solving methods
  - Row picture
  - Column picture
  - Matrix equation

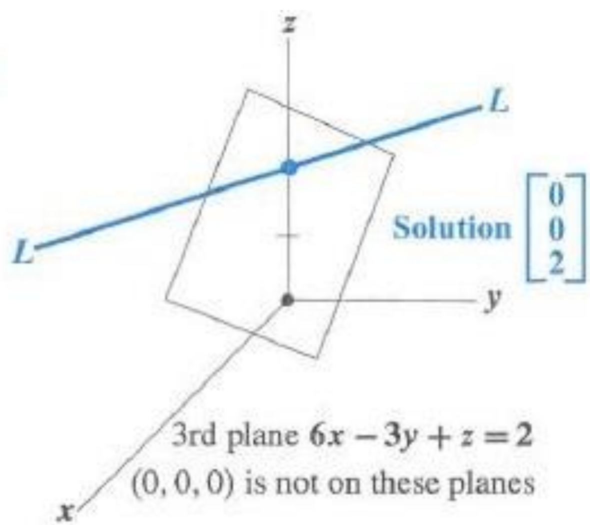
Row Picture of 3 Equations

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

The row picture shows three planes meet at a single point (0,0,2)



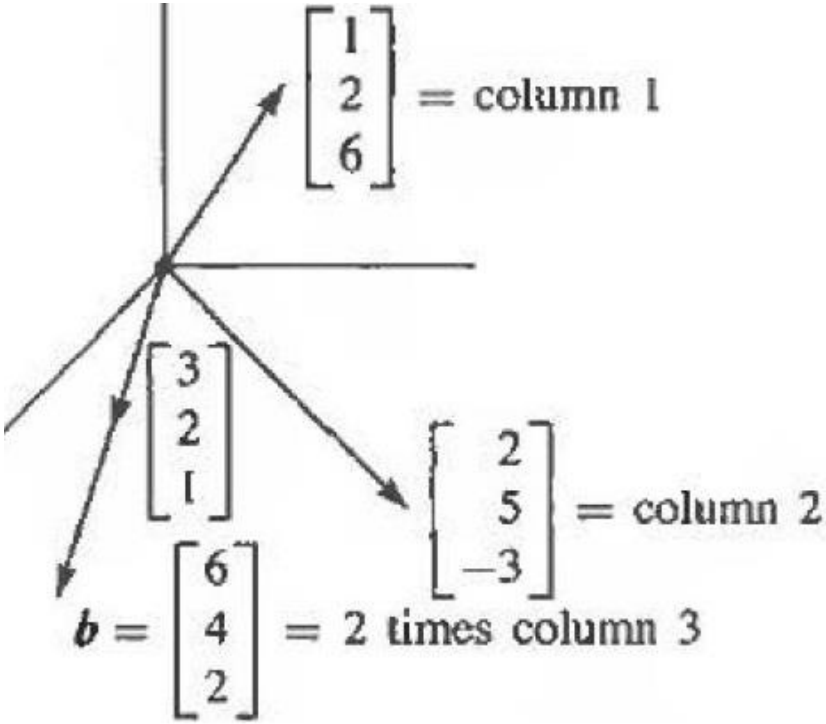


- Each equation produces a plane in 3-dim. space.
- The result of two equations in three unknowns is a line.

Column Picture of 3 Equations

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned} \quad \Longrightarrow \quad x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$



The **column picture** combines three columns to produce the vector $\begin{bmatrix}6\\4\\2\end{bmatrix}$

Matrix Form of the Equations

- System of linear equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

$$A \quad \boldsymbol{x} \quad = \quad \boldsymbol{b}$$

- Multiplication **by row**: dot product

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (row1) \cdot \boldsymbol{x} \\ (row2) \cdot \boldsymbol{x} \\ (row3) \cdot \boldsymbol{x} \end{bmatrix}$$

- Multiplication **by columns**: linear combination of column vectors

$$A\boldsymbol{x} = x(column1) + y(column2) + z(column3)$$

Matrix Form of the Equations – Example

- Example:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \quad I\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

  - Row person : by dot product
  - Column person : by linear combination

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \text{Identity matrix}$$

$$I\boldsymbol{x} = \boldsymbol{x}$$

Matrix Notation

- Let A be mxn matrix

  a_{ij} : an entry in i^{th} row and j^{th} column

  $$1 \le i \le m,\ 1 \le j \le n$$

  $A(i,j) = a_{ij}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Identity Matrix I (nxn matrix)

  $$I(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

  – **Remark :** $I\boldsymbol{x} = \boldsymbol{x}$

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Rules for Matrix Operations

Matrix Addition and Scalar Multiplication

$$\text{Let } A=\left(a_{ij}\right)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix},\quad B=\left(b_{ij}\right)=\begin{bmatrix}2 & 1\\ 4 & 5\\ 3 & 7\end{bmatrix}$$

$$\text{Addition}: A+B=\left(a_{ij}+b_{ij}\right)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}+\begin{bmatrix}2 & 1\\ 4 & 5\\ 3 & 7\end{bmatrix}=\begin{bmatrix}3 & 3\\ 7 & 9\\ 8 & 13\end{bmatrix}$$

$$\text{Scalar multiplication}: kA=k\left(a_{ij}\right)=\left(ka_{ij}\right)=k\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}=\begin{bmatrix}k & 2k\\ 3k & 4k\\ 5k & 6k\end{bmatrix}$$

Matrix Multiplication

- Matrix multiplication : AB
  - To multiply A and B, # of columns of A = # of rows of B
  - A : mxn matrix, B: nxp matrix → AB : mxp matrix

Let $A = (a_{ij})_{mxn}$, $B = (b_{ij})_{nxp}$

$$AB = (c_{ij}) : (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B), \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Example

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & c_{ij} & \\ & & \end{bmatrix} = C$$

Laws for Matrix Operations

- Commutative law:

  $A+B = B+A$ but $AB \neq BA$
- Distributive law:

  $c(A+B) = cA+cB$

  $C(A+B) = CA+CB$

  $(A+B)C = AC+BC$
- Associative law:

  $(A+B)+C = A+(B+C)$

  $A(BC) = (AB)C$
- $A^p = AA\ldots A$ (p factors), $A^pA^q = A^{p+q}$, $(A^p)^q = A^{pq}$

  $A^0 = I$: identity matrix (square matrix)

  A^{-1}: inverse matrix of A(square matrix)

Basic Matrix Operations – Identity

$$\text{Let a matrix and a vector are given}: A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\text{Identity}: I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = A$$

$$I\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

Basic Matrix Operations

Interchange rows

Let a matrix and a vector are given : $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Interchange row1 and row2

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 9 & -3 \\ 2 & 4 & -2 \\ -2 & -3 & 7 \end{bmatrix}$$

$$P_{12}\boldsymbol{b} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 10 \end{bmatrix}$$

Interchange row1 and row3

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{13}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 7 \\ 4 & 9 & -3 \\ 2 & 4 & -2 \end{bmatrix}$$

$$P_{13}\boldsymbol{b} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix}$$

Interchange row2 and row3

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{23}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ -2 & -3 & 7 \\ 4 & 9 & -3 \end{bmatrix}$$

$$P_{23}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 8 \end{bmatrix}$$

Basic Matrix Operations

Multiplying Row by a Scalar value

Let a matrix and a vector are given : $A=\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Multiply row1 by 1/2

$$M_1=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_1A=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 1 & 2 & -1 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

$$M_1\boldsymbol{b}=\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 1 \\ 8 \\ 10 \end{bmatrix}$$

Multiply row2 by 2

$$M_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 8 & 18 & -6 \\ -2 & -3 & 7 \end{bmatrix}$$

$$M_2\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 16 \\ 10 \end{bmatrix}$$

Multiply row3 by -1

$$M_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad M_3A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 2 & 3 & -7 \end{bmatrix}$$

$$M_3\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ -10 \end{bmatrix}$$

Basic Matrix Operations

Adding a Row to Other Row

Let a matrix and a vector are given : $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Add row1 to row2

$$R_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 6 & 13 & -5 \\ -2 & -3 & 7 \end{bmatrix}$$

$$R_{12}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$$

Add row1 to row3

$$R_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad R_{13}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

$$R_{13}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 12 \end{bmatrix}$$

Add row3 to row2

$$R_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 2 & 6 & 4 \\ -2 & -3 & 7 \end{bmatrix}$$

$$R_{32}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \\ 10 \end{bmatrix}$$

Basic Matrix Operations

Sequence of Basic Matrix Operations

Let a matrix and a vector are given : $A=\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$, $\boldsymbol{b}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Multiply row1 by 2, and subtract it from row2

- row1 and row3 are not changed.

- row2 = row2 − 2 * row1: $\boxed{E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

$$E_{21}A=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

Elimination matrix

Multiply row1 by −1, and subtract it from row3

- row1 and row2 are not changed.

- row3 = row3 − (−1) * row1: $\boxed{E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$

$$E_{31}A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 0 & 1 & 5 \end{bmatrix}$$

Block matrices & Block multiplication

- Matrices can be cut into blocks which are smaller matrices.
- When matrices split into blocks, it is often simpler to see how they act.
- Example : 4x6 matrix is broken into six 2x2 blocks.

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}$$

  – If A and B are same size and the block sizes match, then $A+B$ can be got by block by block addition.
- Example : Augmented matrix $[A \;\; \boldsymbol{b}]$ has two blocks of different sizes.
  – After elimination, $[EA \;\; E\boldsymbol{b}]$

Block Multiplication

- Block multiplication:
  - If the <u>cuts between columns of *A*</u> match the <u>cuts between rows of *B*</u>, then block multiplication of *AB* is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}$$

- Example : columns times rows

$$\left[\begin{array}{c|c} 1 & 4 \\ 1 & 5 \end{array}\right] \left[\begin{array}{cc} 3 & 2 \\ \hline 1 & 0 \end{array}\right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Block Multiplication – Example

- Example : Elimination by blocks
  - $E=\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3 & 1 & 0 \\ -4 & 0 & 1 \end{array}\right] \quad A=\left[\begin{array}{c|cc} 1 & any_{12} & any_{13} \\ \hline 3 & any_{22} & any_{23} \\ 4 & any_{32} & any_{33} \end{array}\right] \quad EA=\left[\begin{array}{c|cc} 1 & any_{12} & any_{13} \\ \hline 0 & any'_{22} & any'_{23} \\ 0 & any'_{32} & any'_{33} \end{array}\right]$
  - $E=\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3/2 & 1 & 0 \\ -4/2 & 0 & 1 \end{array}\right] \quad A=\left[\begin{array}{c|cc} 2 & any_{12} & any_{13} \\ \hline 3 & any_{22} & any_{23} \\ 4 & any_{32} & any_{33} \end{array}\right] \quad EA=\left[\begin{array}{c|cc} 2 & any_{12} & any_{13} \\ \hline 0 & any'_{22} & any'_{23} \\ 0 & any'_{32} & any'_{33} \end{array}\right]$
- Block Elimination

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=\left[\begin{array}{c|c} A & B \\ \hline 0 & D-CA^{-1}B \end{array}\right]$$

$S=D-CA^{-1}B:$ Schur complement

The idea of Elimination

Solving 2 Equations in 2 Unknowns

- System of linear equations

$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ \quad\;\; 8y=8 \end{cases} : \text{upper triangular matrix}$$

  – The order of 2 equations is not important.

- Solving methods
  1. Eliminate variables
     1) Eliminate one variable from 2 equations producing 1 equations.
  2. Substitute variables
     1) Get a solution of one variable from the equation got in the step 1-1).
     2) Substitute the solutions of 1 variables into the one of 2 original equations.

Solving 3 Equations in 3 Unknowns

- System of linear equations

$$\begin{cases} x+2y+3z=6 \\ 2x+5y+2z=4 \\ 6x-3y+\ \ z=2 \end{cases} \Rightarrow \begin{cases} -17x+11y \qquad =0 \\ -10x+11y \qquad =0 \\ \quad 6x-\ 3y+z=2 \end{cases} \Rightarrow \begin{cases} -17x+11y \qquad =0 \\ \quad 7x \qquad\qquad =0 \\ \quad 6x-\ 3y+z=2 \end{cases}$$

  - The order of equations and variables is not important.

$$\begin{cases} 6x-3y+\ \ z=2 \\ x+2y+3z=6 \\ 2x+5y+2z=4 \end{cases} \Rightarrow \begin{cases} \quad 6x-\ 3y+z=2 \\ -17x+11y \qquad =0 \\ -10x+11y \qquad =0 \end{cases} \Rightarrow \begin{cases} \quad 6x-\ 3y+z=2 \\ -17x+11y \qquad =0 \\ \quad 7x \qquad\qquad =0 \end{cases}$$

$$\begin{cases} z-3y+6x=2 \\ 3z+2y+\ \ x=6 \\ 2z+5y+2x=4 \end{cases} \Rightarrow \begin{cases} z-3y+\ 6x\ =2 \\ \quad 11y-17x\ =0 \\ \quad 11y-10x\ =0 \end{cases} \Rightarrow \begin{cases} z-3y+\ \ 6x=2 \\ \quad 11y-17x\ =0 \\ \qquad\qquad 7x\ =0 \end{cases}$$

upper triangular matrix

The idea of Elimination

- **Elimination** :

$$\begin{cases} x - 2y = 1 & (1) \\ 3x + 2y = 11 & (2) \end{cases} \rightarrow (2) - (1)\cdot 3 \rightarrow \begin{cases} x - 2y = 1 & (1') \\ \quad 8y = 8 & (2') \end{cases}$$

$$\Rightarrow \quad y = 1, \quad x = 3$$

- Matrix format

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

  – Upper triangular system/matrix
  – Back substitution

Pivot and Multiplier

- **Definition :**
  - Pivot = first nonzero in the row that does the elimination
  - Multiplier = (entry to eliminate) / (Pivot) : divide
- Solving linear system of *n* equations
  - Need *n* pivots
  - After eliminations, we get the upper triangular matrix.
    - The pivots are on the diagonal of the triangular matrix.

$$\begin{cases} z-3y+6x=2 \\ 3z+2y+x=6 \\ 2z+5y+2x=4 \end{cases} \Rightarrow \begin{cases} z-3y+6x=2 \\ 11y-17x=0 \\ 11y-10x=0 \end{cases} \Rightarrow \begin{cases} z-3y+6x=2 \\ 11y-17x=0 \\ 7x=0 \end{cases}$$

Pivot and Multiplier – Example (2 vars)

– Pivot? Multiplier?

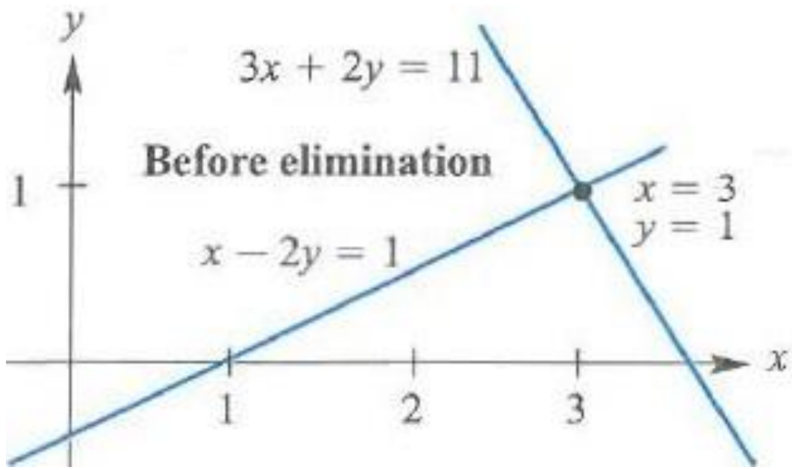
$$\begin{cases} x-2y=1 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ \quad 8y=8 \end{cases}$$

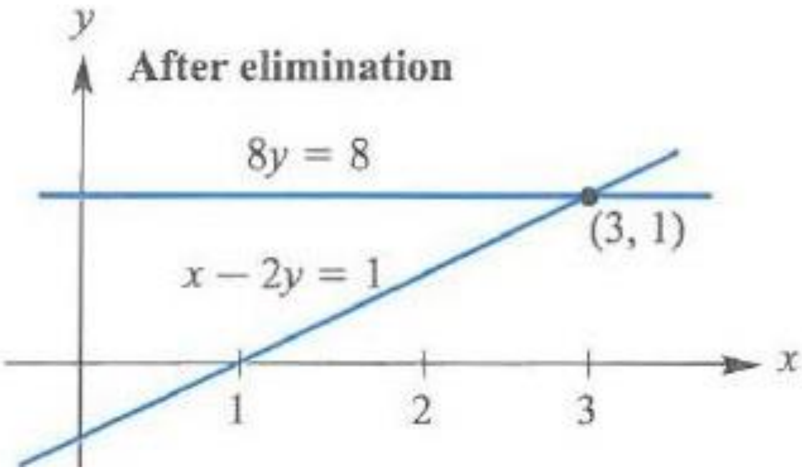
1^{st} pivot is 1, multiplier is $3/1$, 2^{nd} pivot is 8

$$\begin{cases} 5x-10y=5 \\ 3x+2y=11 \end{cases} \Rightarrow \begin{cases} 5x-10y=5 \\ \quad 8y=8 \end{cases}$$

1^{st} pivot is 5, multiplier is $3/5$, 2^{nd} pivot is 8

- After elimination, pivots are on the diagonal of the triangular system





Pivot and Multiplier – Example (3 vars)

$$\begin{cases} 6x - 3y + z = 2 \\ x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \end{cases} \Rightarrow \begin{cases} 6x - 3y + z = 2 & 1^{st} \text{ pivot} = 6 \\ \frac{5}{2}y + \frac{17}{6}z = \frac{17}{3} & \text{multiplier} = \frac{1}{6} \\ 6y + \frac{5}{3}z = \frac{10}{3} & \text{multiplier} = \frac{2}{6} \end{cases}$$

$$\Rightarrow \begin{cases} 6x - 3y + z = 2 & \\ \frac{5}{2}y + \frac{17}{6}z = \frac{17}{3} & 2^{nd} \text{ pivot} = \frac{5}{2} \\ -\frac{77}{15}z = -\frac{154}{15} & \text{multiplier} = \frac{6}{5/2} = \frac{12}{5} \end{cases}$$

Breakdown of Elimination

- Success of eliminations :
  - Produces the *full set of Pivots* and get the solution.
- Failure with no solution
  - *A* is singular.

$$\begin{cases} x-2y=1 \\ 3x-6y=11 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ 0y=8 \end{cases}$$

- Failure with infinitely many solutions
  - *A* is singular.

$$\begin{cases} x-2y=1 \\ 3x-6y=3 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ 0y=0 \end{cases}$$

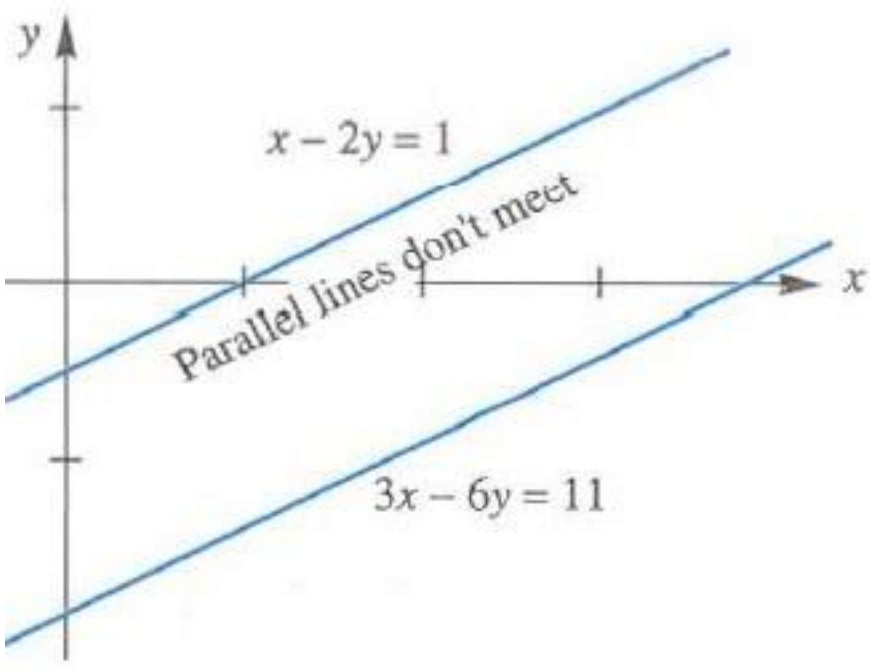
- Temporary failure (zero in pivot)
  - *A* is not singular.
  - A row exchange produces full set of pivots.

$$\begin{cases} 0x+2y=4 \\ 3x-2y=5 \end{cases} \Rightarrow \begin{cases} 3x-2y=5 \\ 2y=4 \end{cases}$$

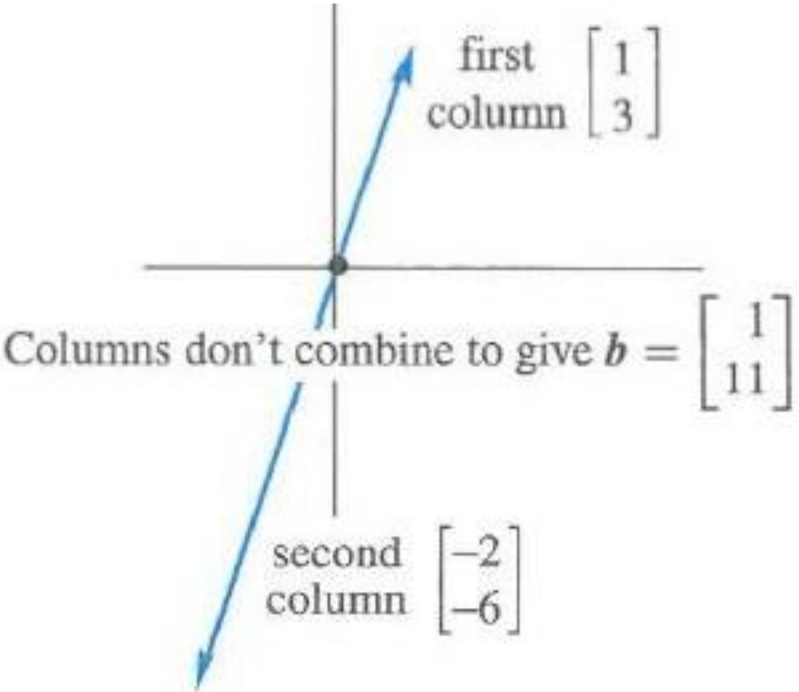
Graph of a Linear System with No Solution

$$\begin{cases} x - 2y = 1 \\ 3x - 6y = 11 \end{cases} \Rightarrow \begin{cases} x - 2y = 1 \\ \quad\quad 0y = 8 \end{cases}$$

- One pivot
- $0y = 8$: no solution



Row picture

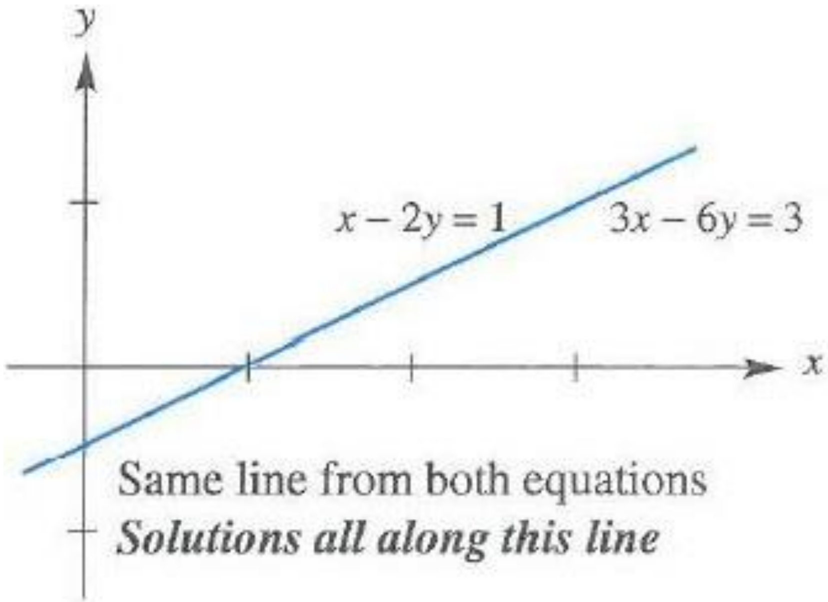


Column picture

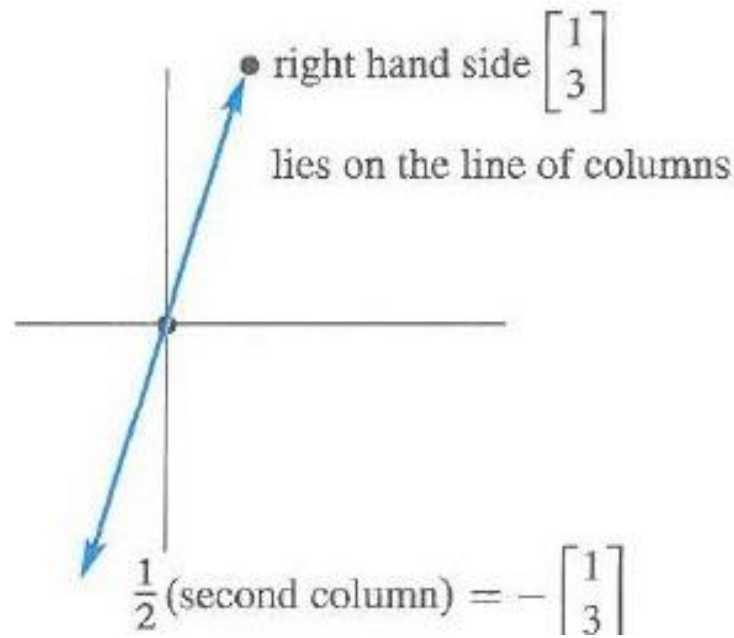
Graph of a Linear System with Infinitely Many Solutions

$$\begin{cases} x-2y=1 \\ 3x-6y=3 \end{cases} \Rightarrow \begin{cases} x-2y=1 \\ \quad 0y=0 \end{cases}$$

- One pivot
- y : free variable



Row picture



Column picture

Elimination of 3 Equations in 3 unknowns

$$\text{Example}: \begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$\begin{cases} 2x+4y-2z=2 & ① & \text{pivot}: p_1 = 2 \\ 4x+9y-3z=8 & ② & \text{multiplier}: l_{21} = a_{21} / p_1 = 4/2,\ ② - ① * (4/2) \\ -2x-3y+7z=10 & ③ & \text{multiplier}: l_{31} = a_{31} / p_1 = -2/2,\ ③ - ① * (-2/2) \end{cases}$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① & \\ \quad\quad y+\ \ z=4 & ②' & \text{pivot}: p_2 = 1 \\ \quad\quad y+5z=12 & ③' & \text{multiplier}: l_{32} = a_{32} / p_2 = 1/1,\ ③' - ②' * 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① & \\ \quad\quad y+\ \ z=4 & ②' & \\ \quad\quad\quad 4z=8 & ③'' & \text{pivot} = 4 \end{cases}$$

Back substitution

From ③'', $z = 2$

From ②' and $z = 2$, $y = 2$

From ① and $z = 2$, $y = 2$, $x = -1$

Forward Elimination

- Forward Elimination

  Step-1: in the 1st column, make <u>the coefficients below the diagonal</u> zero.

  Step-2: in the 2nd column, make <u>the coefficients below the diagonal</u> zero.

  ……

  Step-(n-1): in the (n-1)th column, make <u>the coefficients below the diagonal</u> zero.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} + a_{3n}x_n = b_3 \\ \cdots\cdots \\ a_{n-1,1}x_1 + a_{n-1,2}x_2 + a_{n-1,3}x_3 + \cdots + a_{n-1,,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1} \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} + a_{nn}x_n = b_n \end{cases} \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a'_{32}x_2 + a'_{33}x_3 + \cdots + a'_{3,n-1}x_{n-1} + a'_{3n}x_n = b'_3 \\ \cdots\cdots \\ a'_{n-1,2}x_2 + a'_{n-1,3}x_3 + \cdots + a'_{n-1,,n-1}x_{n-1} + a'_{n-1,n}x_n = b'_{n-1} \\ a'_{n2}x_2 + a'_{n3}x_3 + \cdots + a'_{n,n-1}x_{n-1} + a'_{nn}x_n = b'_n \end{cases}$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \cdots + a''_{3,n-1}x_{n-1} + a''_{3n}x_n = b''_3 \\ \cdots\cdots \\ a''_{n-1,3}x_3 + \cdots + a''_{n-1,,n-1}x_{n-1} + a''_{n-1,n}x_n = b''_{n-1} \\ a''_{n3}x_3 + \cdots + a''_{n,n-1}x_{n-1} + a''_{nn}x_n = b''_n \end{cases} \Rightarrow \cdots\cdots \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1,n-1}x_{n-1} + a_{1n}x_n = b_1 \\ a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2,n-1}x_{n-1} + a'_{2n}x_n = b'_2 \\ a''_{33}x_3 + \cdots + a''_{3,n-1}x_{n-1} + a''_{3n}x_n = b''_3 \\ \cdots\cdots \\ a^{(n-2)}_{n-1,n-1}x_{n-1} + a^{(n-2)}_{n-1,n}x_{n-1}x_n = b^{(n-2)}_{n-1} \\ a^{(n-1)}_{nn}x_n = b^{(n-1)}_n \end{cases}$$

Elimination Using Matrices

Matrix Form of Linear System

- Consider a linear system and its Matrix form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \sum_{j=1}^{n} a_{1j}x_j = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \sum_{j=1}^{n} a_{2j}x_j = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \sum_{j=1}^{n} a_{nj}x_j = b_n \end{cases} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} : A\boldsymbol{x} = \boldsymbol{b}$$

  - Multiplication **<u>by row</u>**: dot product

$$A\boldsymbol{x} = \begin{bmatrix} (a_{11}, a_{12}, \cdots, a_{1n}) \cdot (x_1, x_2, \cdots, x_n) \\ (a_{21}, a_{22}, \cdots, a_{2n}) \cdot (x_1, x_2, \cdots, x_n) \\ \vdots \\ (a_{n1}, a_{n2}, \cdots, a_{nn}) \cdot (x_1, x_2, \cdots, x_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

  - Multiplication **<u>by columns</u>**: combination of column vectors

$$A\boldsymbol{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

Elimination Matrix – 1

$$\text{Consider}:\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}:\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$\begin{cases} 2x+4y-2z=2 & ① & \text{pivot}: p_1=2 \\ 4x+9y-3z=8 & ② & \text{multiplier}: l_{21}=a_{21}/p_1=4/2,\ ②-①*(4/2) \\ -2x-3y+7z=10 & ③ & \text{multiplier}: l_{31}=a_{31}/p_1=-2/2,\ ③-①*(-2/2) \end{cases}$$

$$\text{Elimination matrix}:\ E_1=\begin{bmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ 2/2 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E_1A=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}=A_1$$

$$E_1\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}=\boldsymbol{b}_1$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① \\ \qquad\quad y+\ \ z=4 & ②' \\ \qquad\quad y+5z=12 & ③' \end{cases}$$

Elimination Matrix – 2

Forward elimination (cont.)

$$\begin{cases} 2x+4y-2z=2 & ① \\ \quad\quad y+\quad z=4 & ②' \quad \text{pivot}: p_2=1 \\ \quad\quad y+5z=12 & ③' \quad \text{multiplier}: l_{32}=a_{32}/p_2=1/1,\ ③'-②'*1 \end{cases}$$

$$\text{Elimination matrix}:\ E_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_2A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}=A_2:\text{upper triangular matrix}$$

$$E_2\boldsymbol{b}_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}=\boldsymbol{b}_2$$

$$\Rightarrow \begin{cases} 2x+4y-2z=2 & ① \\ \quad\quad y+\quad z=4 & ②' \\ \quad\quad\quad\quad 4z=8 & ③'' \end{cases}$$

Elimination Matrix – 3

$$\text{Consider}:\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}:\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Forward elimination

$$A\boldsymbol{x}=\boldsymbol{b} \;\Rightarrow\; \underbrace{E_1A}_{A_1}\boldsymbol{x}=\underbrace{E_1\boldsymbol{b}}_{\boldsymbol{b}_1} \;\Rightarrow\; \underbrace{E_2\underbrace{E_1A}_{A_1}}_{A_2}\boldsymbol{x}=\underbrace{E_2\underbrace{E_1\boldsymbol{b}}_{\boldsymbol{b}_1}}_{\boldsymbol{b}_2} \;\Rightarrow\; EA\boldsymbol{x}=E\boldsymbol{b}$$

For matrices $A, B, C,$
$(AB)C = A(BC)$
$AB \neq BA$

$$\text{where } E_2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad E_1=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$E=E_2E_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}: \textbf{lower}\text{ triangular matrix}$$

$$EA=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}: \textbf{upper}\text{ triangular matrix}$$

$$E\boldsymbol{b}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{cases} 2x+4y-2z=2 & ① \\ \quad\;\; y+\;\; z=4 & ②' \\ \qquad\quad 4z=8 & ③'' \end{cases}$$

Elimination Matrix – 4

Consider : $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

Forward elimination

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad EA\boldsymbol{x}=E\boldsymbol{b}, \quad \text{where } E=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\underbrace{\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}} \qquad \begin{cases} 2x+4y-2z=2 & ① \\ \quad\quad\; y+\;\;z=4 & ②' \\ \quad\quad\quad\quad 4z=8 & ③'' \end{cases}$$

This matrix and vector can be used to get the solution.

Back substitution

$z=8/4=2$

$y=4-z=4-2=2$

$x=(2-4y+2z)/2=(2-4\cdot 2+2\cdot 2)/2=-1$

※ Multiply elimination matrix with A and $\boldsymbol{b}$.
※ EA and $E\boldsymbol{b}$ are enough to get solution.

Augmented Matrix

Consider : $\begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad EA\boldsymbol{x}=E\boldsymbol{b}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Augmented matrix of $A\boldsymbol{x}=\boldsymbol{b}$: $\boxed{\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}}$: rectangular matrix

$$E\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

Back substitution

$z = 8/4 = 2$

$y = 4 - z = 4 - 2 = 2$

$x = (2 - 4y + 2z)/2 = (2 - 4\cdot 2 + 2\cdot 2)/2 = -1$

Inverse Matrices

Inverse Matrix

- The matrix A is invertible if there exists a matrix A^{-1} such that $A^{-1}A = I$ and $AA^{-1} = I$
- Notes:
  - Not all matrices have inverses.
  - The inverse exists iff elimination produce n pivots (row exchanges are allowed)
    - Elimination solves $A\boldsymbol{x} = \boldsymbol{b}$ without explicitly using matrix A^{-1}
  - Inverse matrix is unique
    - Suppose B and C are inverses of A.

      $BA = I$ and $AC = I \Rightarrow B(AC) = (BA)C \Rightarrow BI = IC \Rightarrow B = C$

Notes on Inverse Matrix

– If A is invertible,

the one and only one solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

– Suppose there exists a vector $\boldsymbol{x} \neq \mathbf{0}$ such that $A\boldsymbol{x} = \mathbf{0}$,

then A cannot have an inverse.

If A is invertible,

then $A\boldsymbol{x} = \mathbf{0}$ can only have the zero solution $\boldsymbol{x} = A^{-1}\mathbf{0} = \mathbf{0}$

– $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \det(A) = ad - bc$$

A is invertible iff $\det(A) \neq 0$

– Diagonal matrix with $d_i (\neq 0)$ has an inverse $(A^{-1})_{ii} = 1/d_i$.

Inverse of Matrix Multiplication

- Let A, B be nxn matrices and invertible

  Q : AB invertible?

  AB is invertible iff A and B are invertible

  Q : $A+B$ invertible?

  Not invertible. Why?

- Remark:
  - $(AB)^{-1} = B^{-1}A^{-1}$

    $(AB)(B^{-1}A^{-1}) = A(BB^{-1})\, A^{-1} = AIA^{-1} = AA^{-1} = I$
  - $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- How to find A^{-1}?
  - Consider elimination matrix

Inverse of Elimination Matrix

- E subtracts 5 times row 1 from row 2

  E^{-1} : adds 5 times row 1 to row 2

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad EE^{-1} = I$$

- F subtracts 4 times row 2 from row 3

  F^{-1} : adds 4 times row 2 to row 3

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad FF^{-1} = I$$

- FE : sequence of eliminations

  $(FE)^{-1} = E^{-1}F^{-1}$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ \boldsymbol{20} & -4 & 1 \end{bmatrix}$$

row1 feels no effect from other rows
row2 feels an effect from row1 ---- row2′
row3 feels an effect from row2 ---- row3′
row3 feels an effect from row1

$$E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

row2′ feels an effect from row1 ---- back to row2
row3′ feels an effect from row2′ ---- back to row3
row3′ feels no effect from row1

Inverse of Elimination Matrix – Example

$$F(EA): EA = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 11 & 12 & 13 \\ 55 & 22 & 23 \\ 0 & -152 & 33 \end{bmatrix} = \begin{bmatrix} 11 & 12 & 13 \\ 55-5(11) & 22-5(12) & 23-5(13) \\ 0 & -152 & 33 \end{bmatrix} = A'$$

$$FA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 11 & 12 & 13 \\ 55-5(11) & 22-5(12) & 23-5(13) \\ 0 & -152 & 33 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 12 & 13 \\ 55-5(11) & 22-5(12) & 23-5(13) \\ 0-4(55-5(11)) & -152-4(22-5(12)) & 33-4(23-5(13)) \end{bmatrix}$$

$$(FE)A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{} \begin{bmatrix} 11 & 12 & 13 \\ 55 & 22 & 23 \\ 0 & -152 & 33 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix}}_{} \begin{bmatrix} 11 & 12 & 13 \\ 55 & 22 & 23 \\ 0 & -152 & 33 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 12 & 13 \\ 55-5(11) & 22-5(12) & 23-5(13) \\ 0-4(55)+20(11) & -152-4(22)+20(12) & 33-4(23)+20(13) \end{bmatrix} = A''$$

$$E^{-1}F^{-1}A'' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{} \begin{bmatrix} 11 & 12 & 13 \\ 0 & 22-5(12) & 23-5(13) \\ 0 & -152-4(22)+20(12)=0 & 33-4(23)+20(13)=201 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{} \begin{bmatrix} 11 & 12 & 13 \\ 0 & 22-5(12) & 23-5(13) \\ 0 & -152-4(22)+20(12) & 33-4(23)+20(13) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 12 & 13 \\ 0+5(11) & 22-5(12)+5(12) & 23-5(13)+5(13) \\ 0 & -152-4(22)+20(12)+4(22-5(12)) & 33-4(23)+20(13)+4(23-5(13)) \end{bmatrix}$$

Calculating A^{-1}

- Idea
  - Equation $A\boldsymbol{x} = \boldsymbol{b}$ can be solved by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.
  - But, it is not necessary to compute A^{-1}.
  - Elimination goes directly to $\boldsymbol{x}$.
  - It's Gauss-Jordan Elimination Method
- Let A be 3x3 matrix

To solve $AA^{-1} = I$, find each column of $A^{-1} = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3]$.

$$AA^{-1} = A[\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \boldsymbol{x}_3] = [A\boldsymbol{x}_1 \quad A\boldsymbol{x}_2 \quad A\boldsymbol{x}_3] = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3] = I$$

We have to solve three equations

$$A\boldsymbol{x}_1 = \boldsymbol{e}_1 = (1,0,0) \qquad A\boldsymbol{x}_2 = \boldsymbol{e}_2 = (0,1,0) \qquad A\boldsymbol{x}_3 = \boldsymbol{e}_3 = (0,0,1)$$

G-J Elimination : $[A \;\; I] \rightarrow [I \;\; A^{-1}]$

Gauss-Jordan Elimination – 1

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\quad X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix}$

If $AX = I$, then $X = A^{-1}$.

Solving $AX = I$, we get A^{-1}.

$$AX = A\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x}_1 & A\boldsymbol{x}_2 & A\boldsymbol{x}_3 \end{bmatrix} = I = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix}$$

$$\Rightarrow \; A\boldsymbol{x}_1 = \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; A\boldsymbol{x}_2 = \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; A\boldsymbol{x}_3 = \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix} \rightarrow \boldsymbol{x}_1, \; \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 1 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix} \rightarrow \boldsymbol{x}_2, \; \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 1 \end{bmatrix} \rightarrow \boldsymbol{x}_3$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} : \text{Gauss-Jordan Elimination}$$

Gauss-Jordan Elimination – 2

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix} \rightarrow \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} : \text{Gauss - Jordan Elimination}$$

$$\xrightarrow[\text{Eliminate lower diagonal}]{} E \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ 0 & a'_{22} & a'_{23} & 0' & 1' & 0' \\ 0 & 0 & a''_{33} & 0'' & 0'' & 1'' \end{bmatrix}$$

$$\xrightarrow[\text{Eliminate upper diagonal}]{} E' \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ 0 & a'_{22} & a'_{23} & 0' & 1' & 0' \\ 0 & 0 & a''_{33} & 0'' & 0'' & 1'' \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & 1^{(4)} & 0^{(4)} & 0^{(4)} \\ 0 & a'_{22} & 0 & 0''' & 1''' & 0''' \\ 0 & 0 & a''_{33} & 0'' & 0'' & 1'' \end{bmatrix}$$

$$\xrightarrow[\text{Convert into identity}]{} \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a'_{22}} & 0 \\ 0 & 0 & \frac{1}{a''_{33}} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & 1^{(4)} & 0^{(4)} & 0^{(4)} \\ 0 & a'_{22} & 0 & 0''' & 1''' & 0''' \\ 0 & 0 & a''_{33} & 0'' & 0'' & 1'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{1^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} \\ 0 & 1 & 0 & \frac{0'}{a'_{22}} & \frac{1'}{a'_{22}} & \frac{0'}{a'_{22}} \\ 0 & 0 & 1 & \frac{0''}{a''_{33}} & \frac{0''}{a''_{33}} & \frac{1''}{a''_{33}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} \\ \frac{0'}{a'_{22}} & \frac{1'}{a'_{22}} & \frac{0'}{a'_{22}} \\ \frac{0''}{a''_{33}} & \frac{0''}{a''_{33}} & \frac{1''}{a''_{33}} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{1^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} & \frac{0^{(4)}}{a_{11}} \\ \frac{0'}{a'_{22}} & \frac{1'}{a'_{22}} & \frac{0'}{a'_{22}} \\ \frac{0''}{a''_{33}} & \frac{0''}{a''_{33}} & \frac{1''}{a''_{33}} \end{bmatrix} = A^{-1}$$

G-J Elimination – Example

Let $A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$, Solve A^{-1} by G - J Elimination.

$$[A \quad I]=\begin{bmatrix}2 & -1 & 0 & 1 & 0 & 0\\ -1 & 2 & -1 & 0 & 1 & 0\\ 0 & -1 & 2 & 0 & 0 & 1\end{bmatrix}$$

$$(1/2)\text{row}1+\rightarrow\text{row}2: E_{21}=\begin{bmatrix}1 & 0 & 0\\ 1/2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad E_{21}[A \quad I]=\begin{bmatrix}2 & -1 & 0 & 1 & 0 & 0\\ 0 & 3/2 & -1 & 1/2 & 1 & 0\\ 0 & -1 & 2 & 0 & 0 & 1\end{bmatrix}=[A' \quad I']$$

$$(2/3)\text{row}2+\rightarrow\text{row}3: E_{32}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2/3 & 1\end{bmatrix},\quad E_{32}[A' \quad I']=\begin{bmatrix}2 & -1 & 0 & 1 & 0 & 0\\ 0 & 3/2 & -1 & 1/2 & 1 & 0\\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1\end{bmatrix}=[U \quad I'']$$

$$(3/4)\text{row}3+\rightarrow\text{row}2: E_{23}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 3/4\\ 0 & 0 & 1\end{bmatrix},\quad E_{23}[U \quad I'']=\begin{bmatrix}2 & -1 & 0 & 1 & 0 & 0\\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4\\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1\end{bmatrix}=[U' \quad I''']$$

$$(2/3)\text{row}2+\rightarrow\text{row}1: E_{12}=\begin{bmatrix}1 & 2/3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad E_{12}[U' \quad I''']=\begin{bmatrix}2 & 0 & 0 & 3/2 & 1 & 1/2\\ 0 & 3/2 & 0 & 3/4 & 3/2 & 3/4\\ 0 & 0 & 4/3 & 1/3 & 2/3 & 1\end{bmatrix}=[D \quad I^{(4)}]$$

$$\begin{cases}(1/2)\text{row}1\\ (2/3)\text{row}2\\ (3/4)\text{row}3\end{cases}: E_d=\begin{bmatrix}1/2 & 0 & 0\\ 0 & 2/3 & 0\\ 0 & 0 & 3/4\end{bmatrix},\quad E_d[D \quad I^{(4)}]=\begin{bmatrix}1 & 0 & 0 & 3/4 & 1/2 & 1/4\\ 0 & 1 & 0 & 1/2 & 1 & 1/2\\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4\end{bmatrix}=[I \quad X]$$

$$\Rightarrow E_dE_{12}E_{23}E_{32}E_{12}[A \quad I]=[I \quad X] \quad\Rightarrow\quad X=\begin{bmatrix}3/4 & 1/2 & 1/4\\ 1/2 & 1 & 1/2\\ 1/4 & 1/2 & 3/4\end{bmatrix}=A^{-1}=E_dE_{12}E_{23}E_{32}E_{12}$$

Remark on the Previous Example

$$A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$$

Elimination process:

$$A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}, \qquad \text{pivot}:2$$

$$\Rightarrow \quad E_{21}=\begin{bmatrix}1 & 0 & 0\\ 1/2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad E_{21}A=\begin{bmatrix}2 & -1 & 0\\ 0 & 3/2 & -1\\ 0 & -1 & 2\end{bmatrix}=R_1 \qquad \text{pivot}:3/2$$

$$\Rightarrow \quad E_{32}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2/3 & 1\end{bmatrix}, \qquad E_{32}R_1=\begin{bmatrix}2 & -1 & 0\\ 0 & 3/2 & -1\\ 0 & 0 & 4/3\end{bmatrix}=R_2 \qquad \text{pivot}:4/3$$

$$\text{Product of pivot values}=2*\frac{3}{2}*\frac{4}{3}=4=\det(A)$$

$$A^{-1}=\begin{bmatrix}\frac{3}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4}\end{bmatrix}=\frac{1}{\boldsymbol{4}}\begin{bmatrix}3 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 3\end{bmatrix}$$

- A^{-1} involves division by the determinant.
- Invertible matrix cannot have zero determinant.

Inverse and Determinant

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

Applying G - J Elimination

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \qquad \det(A) = 2 * 1 = 2$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7/2 & -3/2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7/2 & -3/2 \\ -2 & 1 \end{bmatrix} = \frac{1}{\boldsymbol{2}} \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

A Property of Triangular Matrix

- A triangular matrix is invertible iff no diagonal entries are zero.
- Example

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$$

Applying G - J Elimination

$$\begin{bmatrix} L & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{bmatrix} = \begin{bmatrix} I & L^{-1} \end{bmatrix}$$

$$E_{32}E_{31}E_{21}L = I \quad \Rightarrow \quad E_{32}E_{31}E_{21} = L^{-1} \quad \Rightarrow \quad E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$$

Elimination = Factorization : $\boldsymbol{A = LU}$

Factorization of Matrix

- A matrix A can be factorized into L(lower triangular matrix) and U(upper triangular matrix).

  $$A = LU$$

  - Factorization can be got from the Gaussian elimination.
- Gaussian Elimination
  - E: elimination matrix
  - $EA = U$: upper triangular matrix with pivots on the diagonal

    $\Rightarrow E^{-1}U = E^{-1}EA = A \Rightarrow A = E^{-1}U \Rightarrow L = E^{-1}$
  - The entries of L are exactly the multipliers l_{ij}

$$EA = U \quad \Rightarrow \quad A = E^{-1}U = LU \quad \Rightarrow \quad L = E^{-1}$$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

Gaussian Elimination and $A = LU$

- Gaussian Elimination
  - E: elimination matrix of A
  - $EA = U$: upper triangular matrix with pivots on the diagonal
    $\Rightarrow E^{-1}U = E^{-1}EA = A \quad \Rightarrow \quad A = E^{-1}U \quad \Rightarrow \quad L = E^{-1}$
  - Pivots are on the diagonal of U.
- For nxn matrix A,

$$E = \left(E_{n(n-1)}\right)\cdots\left(E_{n2}\cdots E_{32}\right)\left(E_{n1}\cdots E_{21}\right)$$

$$L = E^{-1} = \left(E_{21}^{-1}\cdots E_{n1}^{-1}\right)\left(E_{32}^{-1}\cdots E_{n2}^{-1}\right)\cdots\left(E_{n(n-1)}^{-1}\right)$$

- Note on L
  - E_{ij} : lower triangular
  - The main diagonals of E_{ij} and E_{ij}^{-1} contain 1's.
  - The entries of L are exactly the multipliers l_{ij}.

$$EA = U \quad \Rightarrow \quad A = E^{-1}U = LU \quad \Rightarrow \quad L = E^{-1}$$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

Gaussian Elimination and $A = LU$ – Example

Gaussian elimination process

$$A=\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$$

pivot : $p_1 = \boldsymbol{2}$

multiplier : $\boldsymbol{l_{21}} = a_{21} / p_1 = 4/2 = \boldsymbol{2}$

multiplier : $\boldsymbol{l_{31}} = a_{31} / p_1 = -2/2 = \boldsymbol{-1}$

$$E_{21}=\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{21}A=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}=A_1$$

$$E_{31}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_{31}A_1=\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}=A_2$$

pivot : $p_2 = \boldsymbol{1}$

multiplier : $\boldsymbol{l_{32}} = a_{32} / p_2 = 1/1 = \boldsymbol{1}$

$$E_{32}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad E_{32}A_2=\begin{bmatrix} \boldsymbol{2} & 4 & -2 \\ 0 & \boldsymbol{1} & 1 \\ 0 & 0 & \boldsymbol{4} \end{bmatrix}=U$$

- Pivots are on the diagonal of U.

pivot : $p_3 = \boldsymbol{4}$

Factorization

$$E_{32}E_{31}E_{21}A=U \quad \Rightarrow \quad A=\left(E_{32}E_{31}E_{21}\right)^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU \quad \Rightarrow \quad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$$

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boldsymbol{2} & 1 & 0 \\ \boldsymbol{-1} & \boldsymbol{1} & 1 \end{bmatrix}$$

- E_{ij} : lower triangular
- The main diagonals of E_{ij} and E_{ij}^{-1} contain 1's.
- The entries of L are exactly the multipliers l_{ij}.

Detail of Gaussian Elimination

- Pivot rows never change again.
  - 1st row never change. (pivot = a_{11})
    - 2nd ~ nth rows changes.
  - 2nd row never change again. (pivot = a'_{22})
    - 3rd ~ nth rows changes.
  - 3rd row never change again. (pivot = a''_{33})
    - 4th ~ nth rows changes.
  - ……
- When computing the 3rd row,

  (Row3 of U) = (Row3 of A) − l_{31}(Row1 of U) − l_{32}(Row2 of U)

  ⇒ (Row3 of A) = l_{31}(Row1 of U) + l_{32}(Row1 of U) + (Row3 of U)
  - 3rd row of L = $[l_{31} \quad l_{32} \quad 1 \quad 0 \quad \ldots \quad 0]$

$A = LU$ of Matrix with Special Pattern

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E_{21}A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = A_1$$

What pattern is special?
All pivots and multipliers are 1.

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E_{32}A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = A_2$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \ E_{43}A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$$L = E_{21}^{-1}E_{32}^{-1}E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Notes

- When row of A starts with 0, so does that row of L.
- When column of A starts with 0, so does that column of U.

Splitting U into DU'

- $A = LU$
  - Diagonal of $L = 1$
  - Diagonal of U : pivots
- Split U into DU'

$$U = \begin{bmatrix} u_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_2 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_3 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1 & 0 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \cdots & 0 \\ 0 & 0 & u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/u_1 & u_{13}/u_1 & \cdots & u_{1n}/u_1 \\ 0 & 1 & u_{23}/u_2 & \cdots & u_{2n}/up_2 \\ 0 & 0 & 1 & \cdots & u_{3n}/u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = DU'$$

$$\Rightarrow A = LDU'$$

Factorization $A = LDU$ – Example

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LDU$$

Solving $A\boldsymbol{x} = \boldsymbol{b}$ with $A = LU$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LU\boldsymbol{x} = \boldsymbol{b}$$

$$\text{where } L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_2 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_3 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix}$$

Let $U\boldsymbol{x} = \boldsymbol{s}$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LU\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad L\boldsymbol{s} = \boldsymbol{b}, \;\; U\boldsymbol{x} = \boldsymbol{s}$$

$$L\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}$$

Forward substitution to get $\boldsymbol{s}$.

$$U\boldsymbol{x} = \begin{bmatrix} u_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_2 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_3 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \boldsymbol{s}$$

Backward substitution to get $\boldsymbol{x}$.

Solving $A\boldsymbol{x} = \boldsymbol{b}$ with $A = LU$ – Example

$$\text{Consider}: \begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A = LU \quad \text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Let $U\boldsymbol{x} = \boldsymbol{s}$

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad LU\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad L\boldsymbol{s}=\boldsymbol{b}, \quad U\boldsymbol{x}=\boldsymbol{s}$$

$$L\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b}$$

$$s_1 = 2$$
$$s_2 = 8 - 2s_1 = 8 - 2(2) = 4$$
$$s_3 = 10 + s_1 - s_2 = 10 + (2) - (4) = 8$$

$$U\boldsymbol{x} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$$x_1 = (2 - 4x_2 + 2x_3)/2 = (2 - 4(2) + 2(2))/2 = -1$$
$$x_2 = 4 - x_3 = 4 - (2) = 2$$
$$x_3 = 8/4 = 2$$

Solving $A\boldsymbol{x} = \boldsymbol{b}$ with $A = LDU$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LDU\boldsymbol{x} = \boldsymbol{b}$$

$$\text{where } L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}, \quad D = \begin{bmatrix} u_1 & 0 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \cdots & 0 \\ 0 & 0 & u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12}/u_1 & u_{13}/u_1 & \ldots & u_{1n}/u_1 \\ 0 & 1 & u_{23}/u_2 & \ldots & u_{2n}/up_2 \\ 0 & 0 & 1 & \ldots & u_{3n}/u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let $DU\boldsymbol{x} = \boldsymbol{s}, \quad U\boldsymbol{x} = \boldsymbol{t}$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LDU\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad L\boldsymbol{s} = \boldsymbol{b}, \quad D\boldsymbol{t} = \boldsymbol{s}, \quad U\boldsymbol{x} = \boldsymbol{t}$$

$$L\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = \boldsymbol{b}, \qquad D\boldsymbol{t} = \begin{bmatrix} u_1 & 0 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \cdots & 0 \\ 0 & 0 & u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \boldsymbol{s},$$

$$U\boldsymbol{x} = \begin{bmatrix} 1 & u_{12}/u_1 & u_{13}/u_1 & \ldots & u_{1n}/u_1 \\ 0 & 1 & u_{23}/u_2 & \ldots & u_{2n}/up_2 \\ 0 & 0 & 1 & \ldots & u_{3n}/u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_n \end{bmatrix} = \boldsymbol{t}$$

Solving $A\boldsymbol{x} = \boldsymbol{b}$ with $A = LDU$ – Example

$$\text{Consider}: \begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 8 \\ -2x - 3y + 7z = 10 \end{cases} \quad \Rightarrow \quad A\boldsymbol{x} = \boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A = LDU \quad \text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $DU\boldsymbol{x} = \boldsymbol{s}, \quad U\boldsymbol{x} = \boldsymbol{t}$

$$A\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad LDU\boldsymbol{x} = \boldsymbol{b} \quad \Rightarrow \quad L\boldsymbol{s} = \boldsymbol{b}, \quad D\boldsymbol{t} = \boldsymbol{s}, \quad U\boldsymbol{x} = \boldsymbol{t}$$

$$L\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \boldsymbol{b} \qquad \begin{aligned} s_1 &= 2 \\ s_2 &= 8 - 2s_1 = 8 - 2(2) = 4 \\ s_3 &= 10 + s_1 - s_2 = 10 + (2) - (4) = 8 \end{aligned}$$

$$D\boldsymbol{t} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \boldsymbol{s} \qquad \begin{aligned} t_1 &= 2/2 = 1 \\ t_2 &= 4 \\ t_3 &= 8/4 = 2 \end{aligned}$$

$$U\boldsymbol{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{aligned} x_1 &= 1 - 2x_2 + x_3 = 1 - 2(2) + (2) = -1 \\ x_2 &= 4 - x_3 = 4 - (2) = 2 \\ x_3 &= 2 \end{aligned}$$

How Useful Factorization Is?

- We have solved the following linear system by factorization $A = LU$.

$$\text{Consider}: \begin{cases} 2x+4y-2z=2 \\ 4x+9y-3z=8 \\ -2x-3y+7z=10 \end{cases} \quad \Rightarrow \quad A\boldsymbol{x}=\boldsymbol{b}: \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A = LU \quad \text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{Let } U\boldsymbol{x} = \boldsymbol{s}$$

$$A\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad LU\boldsymbol{x}=\boldsymbol{b} \quad \Rightarrow \quad L\boldsymbol{s}=\boldsymbol{b}, \quad U\boldsymbol{x}=\boldsymbol{s}$$

- How to solve the following linear system that is same as the above except $\boldsymbol{b}$? Discuss!!!

$$\begin{cases} 2x+4y-2z=4 \\ 4x+9y-3z=6 \\ -2x-3y+7z=8 \end{cases}$$

Transposes and Permutations

Transpose of Matrix

- Transpose of A (mxn matrix)
  - A^{T} (nxm matrix)
  - The columns of A^{T} are the rows of A
    - $(A^{\mathrm{T}})_{ij} = A_{ji}$
    - $(A^{\mathrm{T}})^{\mathrm{T}} = A$
- Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- Rules of transpose
  - $(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$
  - $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}} \qquad (ABC)^{\mathrm{T}} = C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}$
  - $(A^{-1})^T = (A^T)^{-1}$

$$AA^{-1} = I \;\Rightarrow\; (AA^{-1})^{\mathrm{T}} = I^{\mathrm{T}} \Rightarrow (A^{-1})^{\mathrm{T}}A^{\mathrm{T}} = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$$

Why $(AB)^T = B^T A^T$?

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} : \text{Combination of the columns of } A$$

$$(A\boldsymbol{x})^{\mathrm{T}} = \left(x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \right)^{\mathrm{T}} = \begin{array}{l} x_1 \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \end{bmatrix} \\ + x_2 \begin{bmatrix} a_{12} & a_{22} & \cdots & a_{n2} \end{bmatrix} \\ \cdots \\ + x_n \begin{bmatrix} a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

$$\boldsymbol{x}^{\mathrm{T}} A^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = \begin{array}{l} x_1 \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \end{bmatrix} \\ + x_2 \begin{bmatrix} a_{12} & a_{22} & \cdots & a_{n2} \end{bmatrix} \\ \cdots \\ + x_n \begin{bmatrix} a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \end{array} : \text{Combination of the rows of } A^{\mathrm{T}}$$

$$\Rightarrow (A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}} A^{\mathrm{T}}$$

Let $B = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots \end{bmatrix}$

$$AB = A\begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots \end{bmatrix} = \begin{bmatrix} A\boldsymbol{x}_1 & A\boldsymbol{x}_2 & \cdots \end{bmatrix} \Rightarrow (AB)^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} A^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} A^{\mathrm{T}} \\ \vdots \end{bmatrix} = B^{\mathrm{T}} A^{\mathrm{T}}$$

Meaning of Inner Products

Let $\boldsymbol{x}, \boldsymbol{y}$: vector of n elements

$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$: dot product or inner product ($^{\mathrm{T}}$ is inside) $(1\mathrm{x}n)(n\mathrm{x}1) = (1\mathrm{x}1)$

$\boldsymbol{x}\boldsymbol{y}^{\mathrm{T}}$: rank one product or outer product ($^{\mathrm{T}}$ is outside) $(n\mathrm{x}1)(1\mathrm{x}n) = (n\mathrm{x}n)$

$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y}$: inner product of $A\boldsymbol{x}$ with $\boldsymbol{y}$

$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\left(A^{\mathrm{T}}\boldsymbol{y}\right)$: inner product of $\boldsymbol{x}$ with $A^{\mathrm{T}}\boldsymbol{y}$

Example :

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \left(\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)^{\mathrm{T}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (x_2 - x_1)y_1 + (x_3 - x_2)y_2$$

$$\boldsymbol{x}^{\mathrm{T}}\left(A^{\mathrm{T}}\boldsymbol{y}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^{\mathrm{T}}\left(\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} -y_1 \\ y_1 - y_2 \\ y_2 \end{bmatrix} = -x_1y_1 + x_2(y_1 - y_2) + x_3y_2$$

$$\Rightarrow \quad (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\left(A^{\mathrm{T}}\boldsymbol{y}\right)$$

Symmetric Matrices

- A is symmetric matrix iff

  $A^{\mathrm{T}} = A$ (i.e. $a_{ij} = a_{ji}$)

- Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^{\mathrm{T}}, \qquad A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^{\mathrm{T}}, \qquad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

- Note
  - A : symmetric → $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = A^{-1}$ → A^{-1} : symmetric
  - R : mxn matrix → R^{T}: nxm matrix

    → $R^{\mathrm{T}}R$: nxn matrix

$R^T R$, RR^T

- R : any matrix, probably rectangular mxn matrix
  - $R^T R$: symmetric, positive diagonal, mxm matrix
  - RR^T : symmetric, positive diagonal, nxn matrix
  - $R^T R \neq RR^T$

Symmetric $A = LDU = LDL^{\mathrm{T}}$

- Let A : symmetric, and $A = LDU$

$$A = LDU \;\Rightarrow\; A = A^{\mathrm{T}} = (LDU)^{\mathrm{T}} = U^{\mathrm{T}}DL^{\mathrm{T}} = LDU \;\Rightarrow\; U = L^{\mathrm{T}}$$

$$\Rightarrow\; A = LDL^{\mathrm{T}}$$

- Example

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{L^{\mathrm{T}}}$$

- Note
  - Elimination is faster.
  - Factorization can save storage in half.

Permutation Matrices

- A permutation matrix P has a single "1" in every row and every column.
  - P: square matrix
  - P^T also has a single "1" in every row and every column
  - P^T: permutation matrix
  - There are $n!$ permutation matrices of order n
- Meaning of permutation matrix

  Let $P = (p_{ij})$

  If $p_{ij} = 1$, then PA moves j^{th} row of A to i^{th} row.

Permutation Matrices – Example

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}, \quad P_{31}P_{32} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$P_{21}A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}, \quad P_{31}A = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}, \quad P_{32}A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

$$P_{32}P_{21}A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}, \quad P_{21}P_{32}A = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad P_{31}P_{32}A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

Transpose of Permutation Matrices

Let $P = \left(p_{ij}\right),\ P^{\mathrm{T}} = \left(p_{ij}^{t}\right)$

$$p_{ij} = 1 \rightarrow p_{ji}^{t} = 1$$

$$\begin{cases} P^{\mathrm{T}}PA: \text{move } j^{\text{th}} \text{ row of } A \text{ to } i^{\text{th}} \text{ row, and then move } i^{\text{th}} \text{ row to } j^{\text{th}} \text{ row.} \\ PP^{\mathrm{T}}A: \text{move } i^{\text{th}} \text{ row of } A \text{ to } j^{\text{th}} \text{ row, and then move } j^{\text{th}} \text{ row to } i^{\text{th}} \text{ row.} \end{cases}$$

$$\Rightarrow P^{\mathrm{T}}PA = PP^{\mathrm{T}}A = A$$

$$\Rightarrow P^{\mathrm{T}}P = PP^{\mathrm{T}} = I$$

$$\Rightarrow P^{\mathrm{T}} = P^{-1}$$

Factorization with/without Row Exchanges

- Factorization without row exchange:

Elimination :

$$\left(E_{n(n-1)}\right)\cdots\left(E_{n2}\cdots E_{32}\right)\left(E_{n1}\cdots E_{21}\right)A=U$$

$$\Rightarrow A=\left(E_{21}^{-1}\cdots E_{n1}^{-1}\right)\left(E_{32}^{-1}\cdots E_{n2}^{-1}\right)\cdots\left(E_{n(n-1)}^{-1}\right)U=LU$$

- When row exchanges are needed due to zero pivot

Permutation and Elimination :

$$\left(E_{n(n-1)}\right)\cdots\left(E_{n2}\cdots E_{32}P_{k_2 2}\right)\left(E_{n1}\cdots E_{21}P_{k_1 1}\right)A=U$$

$$\Rightarrow A=\left(P_{k_1 1}^{-1}E_{21}^{-1}\cdots E_{n1}^{-1}\right)\left(P_{k_2 2}^{-1}E_{32}^{-1}\cdots E_{n2}^{-1}\right)\cdots\left(E_{n(n-1)}^{-1}\right)U=LU$$

Factorization with Row Exchanges $PA = LU$

- In the elimination process, pivot can be zero.
  - Row exchange is required.
  - Row exchange can be done <u>in advance</u>.

$$
\begin{aligned}
&\text{Permutation and Elimination:}\\
&\left(E_{n(n-1)}\right)\cdots\left(E_{n2}\cdots E_{32}P_{k_2 2}\right)\left(E_{n1}\cdots E_{21}P_{k_1 1}\right)A = U\\
&\Rightarrow \left(E'_{n(n-1)}\right)\cdots\left(E'_{n2}\cdots E'_{32}\right)\left(E'_{n1}\cdots E'_{21}\right)\left(\cdots P_{k_2 2}P_{k_2 2}\right)A = U\\
&\Rightarrow \left(E'_{n(n-1)}\right)\cdots\left(E'_{n2}\cdots E'_{32}\right)\left(E'_{n1}\cdots E_{21}\right)PA = U,\\
&\qquad\qquad \text{where } P = \cdots P_{k_2 2}P_{k_2 2}\\
&\Rightarrow PA = \left(E'^{-1}_{21}\cdots E'^{-1}_{n1}\right)\left(E'^{-1}_{32}\cdots E'^{-1}_{n2}\right)\cdots\left(E'^{-1}_{n(n-1)}\right)U = LU,\\
&\qquad\qquad \text{where } L = \left(E'^{-1}_{21}\cdots E'^{-1}_{n1}\right)\left(E'^{-1}_{32}\cdots E'^{-1}_{n2}\right)\cdots\left(E'^{-1}_{n(n-1)}\right)
\end{aligned}
$$

$PA = LU$ – Example

Elimination process

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{31}A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 0 & 3 & 1 \end{bmatrix} = A_1$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{21}A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix} = A_2$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_{32}A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

Factorization

$$P_{32}E_{21}P_{31}A = U$$

$$\Rightarrow A = \left(P_{32}E_{21}P_{31}\right)^{-1}U = \left(P_{31}^{-1}E_{21}^{-1}P_{32}^{-1}\right)U = LU$$

$$\Rightarrow L = P_{31}^{-1}E_{21}^{-1}P_{32}^{-1}$$

$$L = P_{31}^{-1}E_{21}^{-1}P_{32}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Factorization

$$P_{32}E_{21}P_{31}A = U \;\Rightarrow E_{21}P_{32}P_{31}A = U$$

$$\Rightarrow E_{21}PA = U, \quad \text{where } P = P_{32}P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 1 \\ 2 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 2 & 2 & 4 \end{bmatrix} = A_1$$

Elimination

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_{31}A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$\Rightarrow E_{31}PA = U \;\Rightarrow\; PA = E_{31}^{-1}U = LU$$

$$\Rightarrow L = E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Example1: linear equation system
linear Algebra with Application, Leon, 한티미디어

다음의 광합성 작용을 하는 화학방정식의 균형을 맞추어 보자.

$$CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$$

풀이 $(x_1)CO_2 + (x_2)H_2O \rightarrow (x_3)C_6H_{12}O_6 + (x_4)O_2$가 균형 잡힌 화학방정식이 되도록 양의 정수 x_1, x_2, x_3, x_4를 구해야 한다. 먼저 각 원소의 수는 방정식의 양변에서 같아야 하므로

탄소(C) : $x_1 = 6x_3$

수소(H) : $2x_2 = 12x_3$

산소(O) : $2x_1 + x_2 = 6x_3 + 2x_4$

이것을 선형시스템으로 나타내면 다음과 같다.

$$\begin{aligned} x_1 \qquad\quad - 6x_3 \qquad\quad &= 0 \\ 2x_2 - 12x_3 \qquad\quad &= 0 \\ 2x_1 + x_2 - 6x_3 - 2x_4 &= 0 \end{aligned}$$

이것을 풀기 위해 가우스-조단 소거법을 적용하면 다음과 같다.

$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 2 & -12 & 0 & 0 \\ \boxed{2} & 1 & -6 & -2 & 0 \end{array}\right] \qquad (-2)\times R_1 + R_3 \rightarrow R_3, \quad \frac{1}{2}\times R_2 \rightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & -6 & 0 & 0 \\ 0 & \boxed{1} & 6 & -2 & 0 \end{array}\right] \qquad (-1)\times R_2 + R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 12 & -2 & 0 \end{array}\right] \qquad \frac{1}{2}\times R_3 \rightarrow R_3$$

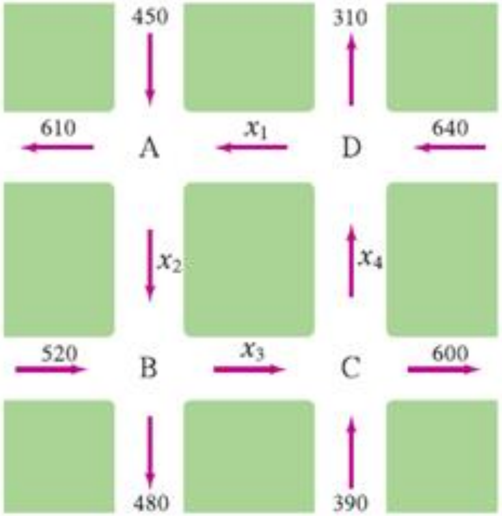
$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right]$$

그 결과 $x_3=\frac{1}{6}x_4$, $x_2=x_4$, $x_1=x_4$의 값들을 얻을 수 있다. 여기서는 모든 변수들이 정수가 되어야 하므로 $x_4=6$이라면, $x_1=6$, $x_2=6$, $x_3=1$, $x_4=6$이 된다. 따라서 구하고자 하는 최종 화학방정식은

$6CO_2+6H_2O \longrightarrow C_6H_{12}O_6+6O_2$이다. ■

Example 2: Traffic

어떤 도시의 중심가에 그림 와 같은 4개의 일방통행 길이 있다고 한다. 각 교차점에 한 시간당 유입되는 교통량과 빠져나가는 교통량이 그림과 같이 주어졌을 경우 각 네거리에서의 교통량을 결정해 보자.



4개의 일방통행 길

풀이 각 교차점에 유입되는 차량의 숫자와 빠져나가는 차량의 숫자가 같으므로, 교차점 A에 유입되는 차량의 수는 $x_1 + 450$이고 빠져나가는 차량의 수는 $x_2 + 610$이다.

따라서

$$x_1 + 450 = x_2 + 610 \quad (\text{교차점 A})$$

이와 같은 방법으로,

$$x_2 + 520 = x_3 + 480 \quad (\text{교차점 B})$$
$$x_3 + 390 = x_4 + 600 \quad (\text{교차점 C})$$
$$x_4 + 640 = x_1 + 310 \quad (\text{교차점 D})$$

과 같은 4개의 선형방정식을 만들 수 있다. 이것을 첨가행렬로 만들면 다음과 같다.

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right]$$

이 행렬을 기약 행 사다리꼴로 변환시키면 다음과 같이 된다.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

이 시스템은 하나의 자유변수를 가지므로 여러 가지 해를 가질 수 있다. 만약 교차로 C와 D 사이의 평균 교통량이 한 시간당 200대라고 가정하면, $x_4 = 200$일 것이고 x_1, x_2, x_3을 x_4에 대해 풀면 다음과 같다.

$$\begin{aligned} x_1 &= x_4 + 330 = 530 \\ x_2 &= x_4 + 170 = 370 \\ x_3 &= x_4 + 210 = 410 \quad \blacksquare \end{aligned}$$

Example 3. 암호 해독

암호 해독에 있어서 코드화된 메시지를 주고받는 가장 일반적인 방법은 각 알파벳 문자에다 정수 값을 부여하고 그 메시지를 정수의 열(string)로 보내는 것이다 예를 들어, SEND MONEY라는 메시지는 5, 8, 10, 21, 7, 2, 10, 8, 3과 같이 코드화될 수 있다.

여기서 S는 5에 해당하고 Y는 3에 해당한다. 나머지도 순서에 따라 그 값을 가진다. 일반적으로 이런 형식의 문장은 해독하기가 비교적 쉬우므로 행렬의 곱을 이용하여 암호화하는 것이 좋다. 따라서 앞의 메시지를 다음 행렬 A와의 곱으로 변환시키면 해독하기가 매우 어렵게 될 것이다.

$$A=\begin{bmatrix}1 & 2 & 1\\ 2 & 5 & 3\\ 2 & 3 & 2\end{bmatrix}$$

주어진 원래의 메시지(SEND MONEY)를 차례로 적으면 행렬 B와 같다.

$$B=\begin{bmatrix}5 & 21 & 10\\ 8 & 7 & 8\\ 10 & 2 & 3\end{bmatrix}$$

암호화를 위해 행렬 A에다 원래의 메시지 행렬인 B를 곱하면

$$AB=\begin{bmatrix}1 & 2 & 1\\ 2 & 5 & 3\\ 2 & 3 & 2\end{bmatrix}\begin{bmatrix}5 & 21 & 10\\ 8 & 7 & 8\\ 10 & 2 & 3\end{bmatrix}=\begin{bmatrix}31 & 37 & 29\\ 80 & 83 & 69\\ 54 & 67 & 50\end{bmatrix}$$

그러면 A와 B의 곱은 다음과 같이 코드화된 메시지로 나올 것이다.

31, 80, 54, 37, 83, 67, 29, 69, 50

그 메시지를 받는 사람은 받은 값의 행렬에다 미리 약속된 원래의 행렬 A의 역행렬인 A^{-1}를 곱함으로써 그 문자를 해독할 수 있다.

$$\begin{bmatrix}1 & -1 & 1\\ 2 & 0 & -1\\ -4 & 1 & 1\end{bmatrix}\begin{bmatrix}31 & 37 & 29\\ 80 & 83 & 69\\ 54 & 67 & 50\end{bmatrix}=\begin{bmatrix}5 & 21 & 10\\ 8 & 7 & 8\\ 10 & 2 & 3\end{bmatrix}$$

이 결과는 원래 보낸 행렬 B의 값과 같으므로 동일한 메시지로 해독하는 셈이다.

Question?