Linear Algebra

Vector Spaces and Subspaces

The Rank and the Row Reduced Form

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Rank of Matrix A

- Definition : the rank of A is the number of pivots.

$$\text{rank}(A) = r$$

- Let A be mxn matrix.
  - A will eventually be reduced to r nonzero rows.
  - Every free column is a combination of earlier pivot columns.
    - We can get the special solution from the combination of pivot columns.
  - Number of pivot variables = r
  - Number of free variables = $n - r$
  - Number of special solutions = $n - r$
  - $\boldsymbol{N}(A)$ is spanned by $(n - r)$ special solutions.
  - Dimension of $\boldsymbol{C}(A) = r$

Rank of Matrix A – Example

$$A=\begin{bmatrix}1&1&2&4\\1&2&2&5\\1&3&2&6\end{bmatrix}\Rightarrow U=\begin{bmatrix}1&1&2&4\\0&1&0&1\\0&0&0&0\end{bmatrix}\Rightarrow R=\begin{bmatrix}1&0&2&3\\0&1&0&1\\0&0&0&0\end{bmatrix}$$

$$\operatorname{rank}(A)=\operatorname{rank}\left(A^{\mathrm{T}}\right)=2$$

$$\text{In } A:(\text{col}3)=2(\text{col}1),\qquad (\text{col}4)=3(\text{col}1)+(\text{col}2)$$

$$\text{In } R:(\text{col}3)=2(\text{col}1),\qquad (\text{col}4)=3(\text{col}1)+(\text{col}2)$$

$$\Rightarrow -2(\text{col}1)+(\text{col}3)=\mathbf{0},\qquad -3(\text{col}1)-(\text{col}2)+(\text{col}4)=\mathbf{0}$$

$$\Rightarrow \boldsymbol{s}_1=\begin{bmatrix}-2\\0\\1\\0\end{bmatrix},\qquad \boldsymbol{s}_2=\begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}$$

$$\Rightarrow A\boldsymbol{s}_1=\mathbf{0}, R\boldsymbol{s}_1=\mathbf{0},\qquad A\boldsymbol{s}_2=\mathbf{0}, R\boldsymbol{s}_2=\mathbf{0}$$

Matrix of Rank One

- Matrix of rank one
  - <u>Matrix of rank one</u> have only one pivot.
  - Every row is a multiple of the pivot row.
  - Every column is a multiple of the pivot column.
- Example

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - Pivot row is (row1), only one.
  - In A, (col2) = 3(col1), (col3) = 10(col1), (row2) = 2(row1), (row3) = 3(row1)
  - In R, (col2) = 3(col1), (col3) = 10(col1)

$A\boldsymbol{x} = \mathbf{0}$, rank($A$) = 1

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- rank(A) = 1
- Dimension of $\boldsymbol{C}(A)$ = 1
  - All columns are <u>on the line through $\boldsymbol{u}$ = (1, 2, 3)</u>

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} & 3\boldsymbol{u} & 10\boldsymbol{u} \end{bmatrix} = \boldsymbol{u}\begin{bmatrix} 1 & 3 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

Let $\boldsymbol{v}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$, then $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$

$A\boldsymbol{x} = \mathbf{0}$, rank$(A) = 1$ *cont.*

$$A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \Rightarrow \quad R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row space: combinations of the rows

$$\Rightarrow \quad A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}, \quad \text{where} \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

$$A\boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad \left(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\right)\boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{u}\left(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{x}\right) = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{v}^{\mathrm{T}}\boldsymbol{x} = 0 \ \left(\because \boldsymbol{u} \neq \mathbf{0}\right)$$

Inner product is 0 $\Rightarrow$ All $\boldsymbol{x} \in N(A)$ are orthogonal to $\boldsymbol{v}$ in the row space.

$\Rightarrow$ Nullspace(plane) is perpendicular to row space(line).

Pivot variable : x_1 Free variables : x_2, x_3

$$\text{Special solutions}: \boldsymbol{s}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}_2 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

$N(A)$: span of $\{\boldsymbol{s}_1, \boldsymbol{s}_2\}$: plane produced by $x_1 + 3x_2 + 10x_3 = 0$

$A\boldsymbol{x} = \mathbf{0}$, rank$(A) = 1$ – Examples

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot variable : x_1

Free variables : x_2, x_3

Special solutions : $\boldsymbol{s}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{s}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

$N(A)$: span of $\{\boldsymbol{s}_1, \boldsymbol{s}_2\}$: linear combinations of $\boldsymbol{s}_1, \boldsymbol{s}_2$

plane produced by $x_1 + 3x_2 + 4x_3 = 0$

$N(A)$ is perpendicular to $\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}^{\mathrm{T}}$

$$A = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Pivot variable : x_2

Free variable : x_1

Special solutions : $\boldsymbol{s} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$N(A)$: span of $\{\boldsymbol{s}\}$: linear combination of $\boldsymbol{s}$. (i.e. $(c, 0)$)

line produced by $x_2 = 0$

$N(A)$ is perpendicular to $\begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$

Another Definitions of rank(A)

- Definitions of rank(A)
  - rank(A) = number of pivots
  - rank(A) = number of independent rows(columns)
  - rank(A) = dimension of the column space $\boldsymbol{C}(A)$

Pivot rows(columns) are not combinations of other rows.
$\Rightarrow$ Pivot rows(columns) are independent rows(columns).

The Pivot Columns

- Pivot columns of R
  - 1's in the pivots
  - 0's everywhere else

$$A=\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\1&3&1&6&-4\end{bmatrix}\Rightarrow\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\0&0&0&0&0\end{bmatrix}=R \qquad \begin{cases}\boldsymbol{C}(A)\neq\boldsymbol{C}(R)\\ \text{Pivot columns}:1,3\\ r=\text{rank}(A)=\text{rank}(R)=2\end{cases}$$

$$E=\begin{bmatrix}1&0&0\\0&1&0\\-1&-1&1\end{bmatrix}, E^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}\Rightarrow EA=R\Rightarrow A=E^{-1}R$$

$$A\boldsymbol{x}=\boldsymbol{0}\Rightarrow EA\boldsymbol{x}=\boldsymbol{0}\Rightarrow R\boldsymbol{x}=\boldsymbol{0}$$

- r rows of pivot columns of R is I_{rxr} .
- r rows of the 1st r columns of E^{-1} is I_{rxr} .

- The pivot columns of R are not combinations of earlier columns.
- The free columns are combinations of earlier columns.
  - These combinations are the special solutions.

The Special Solutions

$$A=\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\1&3&1&6&-4\end{bmatrix}\Rightarrow\begin{bmatrix}1&3&0&2&-1\\0&0&1&4&-3\\0&0&0&0&0\end{bmatrix}=R \qquad \text{Identity matrix(2x2) in pivot columns}$$

$$\begin{cases}\text{pivot variables}: x_1, x_3\\ \text{free variables}: x_2, x_4, x_5\end{cases}$$

Special solutions : set one free variable to 1, others 0.

$$\begin{cases}\boldsymbol{s}_1 : x_2=1, x_4=0, x_5=0 \Rightarrow x_1=-3, x_3=0 \Rightarrow \boldsymbol{s}_1=\begin{bmatrix}-\mathbf{3} & 1 & \mathbf{0} & 0 & 0\end{bmatrix}^{\mathrm{T}} : -(\text{col2 of } R)\\ \boldsymbol{s}_2 : x_2=0, x_4=1, x_5=0 \Rightarrow x_1=-2, x_3=-4 \Rightarrow \boldsymbol{s}_2=\begin{bmatrix}-\mathbf{2} & 0 & -\mathbf{4} & 1 & 0\end{bmatrix}^{\mathrm{T}} : -(\text{col4 of } R)\\ \boldsymbol{s}_3 : x_2=0, x_4=0, x_5=1 \Rightarrow x_1=1, x_3=3 \Rightarrow \boldsymbol{s}_3=\begin{bmatrix}\mathbf{1} & 0 & \mathbf{3} & 0 & 1\end{bmatrix}^{\mathrm{T}} : -(\text{col5 of } R)\end{cases}$$

$\boldsymbol{N}(A)$: span of $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$ (linear combinations of $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3$)

complete solution to $A\boldsymbol{x}=\mathbf{0}$ $(R\boldsymbol{x}=\mathbf{0})$

$$\text{Let } N=\begin{bmatrix}\boldsymbol{s}_1 & \boldsymbol{s}_2 & \boldsymbol{s}_3\end{bmatrix}=\begin{array}{c}\begin{bmatrix}-\mathbf{3} & -\mathbf{2} & \mathbf{1}\\ 1 & 0 & 0\\ \mathbf{0} & -\mathbf{4} & \mathbf{3}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array}\begin{array}{l}\text{not free}: -(\text{row1 of } R)\\ \text{free}\\ \text{not free}: -(\text{row2 of } R)\\ \text{free}\\ \text{free}\end{array}$$

N : nullspace matrix
Identity matrix(3x3) in free rows

$AN=0$: zero matrix

Numbers in $A\mathbf{x} = \mathbf{0}$: r, $n - r$

- A: mxn matrix, rank(A) = r

r	$n - r$
Pivot columns	Free columns
Pivot variables	Free variables
	Special solutions
Dimension of $\boldsymbol{C}(A)$	Dimension of $\boldsymbol{N}(A)$
Independent equations	Independent solutions

Special Solutions for $R\boldsymbol{x} = \mathbf{0}$

$A\boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad R\boldsymbol{x} = \mathbf{0}$ where $A, R : m \mathrm{x} n$

Suppose that the first r columns are the pivot columns.

$$\begin{array}{cc} & \\ R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} & \begin{array}{r} r \text{ pivot rows} \\ m-r \quad \text{zero rows} \end{array} \\ r \text{ pivot columns} \quad n-r \text{ free columns} & \end{array}$$

The pivot variables in the $n-r$ special solutions come by changing F to $-F$:

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{array}{r} r \text{ pivot variables} \\ n-r \quad \text{free variables} \end{array}$$

$$RN = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$R\boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \text{pivot variables} \\ \text{free variables} \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \quad I[\text{pivot variables}] + F[\text{free variables}] = \mathbf{0}$$

$$\Rightarrow \quad I[\text{pivot variables}] = -F[\text{free variables}]$$

Special Solutions for $R\mathbf{x} = \mathbf{0}$ – Example

$$R = \left[\begin{array}{cc|ccc} 1 & 0 & 3 & 2 & -1 \\ 0 & 1 & 0 & 4 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$$N = \left[\begin{array}{ccc} -3 & -2 & 1 \\ 0 & -4 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$RN = \left[\begin{array}{cc|ccc} 1 & 0 & 3 & 2 & -1 \\ 0 & 1 & 0 & 4 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} -3 & -2 & 1 \\ 0 & -4 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Question?