Introduction to Vectors

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Linear Equation

- A linear equation in *n unknowns* is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where a_i and b are real numbers and x_i are variables.

Linear System

- A linear system of <u>*m* equations in *n* unknowns</u> is a system of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- How to solve this system?

Vectors & linear Combinations

n-Dimensional Vectors

$$\text{2 - Dim. Vector : } \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (u_1, u_2)$$

$$\text{3 - Dim. Vector : } \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (u_1, u_2, u_3)$$

$$\text{n - Dim. Vector : } \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = (u_1, u_2, \cdots, u_n)$$

$$\text{where } u_i \text{ : components}$$

Two Important Operations of Linear Algebra

1. Vector Addition : $\boldsymbol{v} + \boldsymbol{w}$
2. Scalar Multiplication : $c\boldsymbol{v}$, $d\boldsymbol{w}$ where c, d are scalar

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$*\text{Vector Addition}: \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

$$*\text{Scalar Multiplication}: \quad c\boldsymbol{v} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

$$\text{Properties}: \ -\boldsymbol{v} + \boldsymbol{v} = \boldsymbol{v} - \boldsymbol{v} = 0$$

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$$

$$\text{Ex)} \ \boldsymbol{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \boldsymbol{v} + \boldsymbol{w} = ? \quad 6\boldsymbol{v} = ?$$

Linear Combination

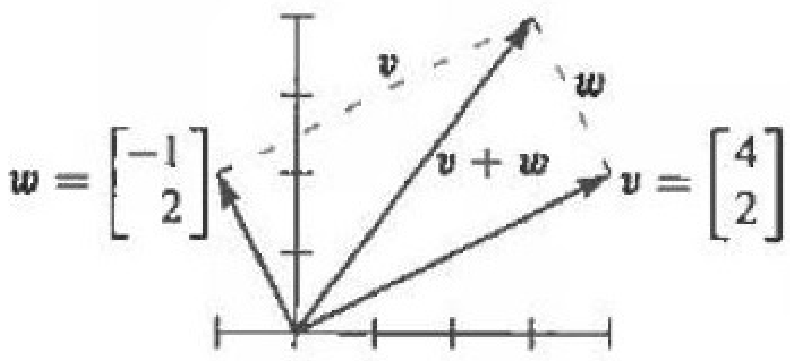
- Definition : $c\boldsymbol{v} + d\boldsymbol{w}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$
- Ex)

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

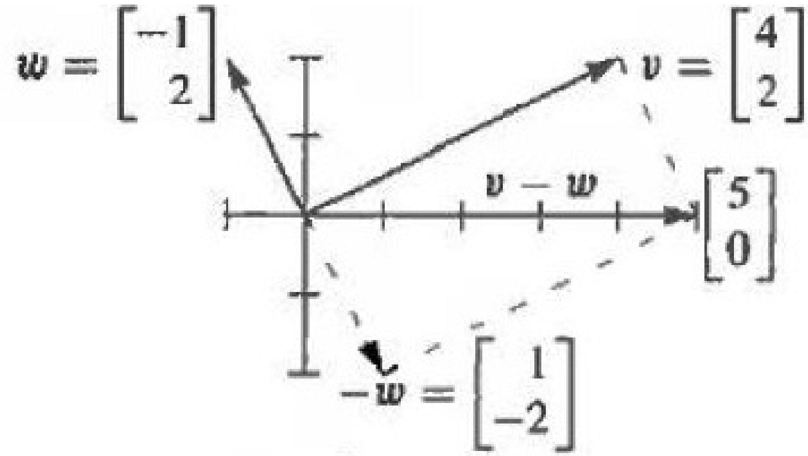
$$c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c \\ c \end{bmatrix} + \begin{bmatrix} 2d \\ 3d \end{bmatrix} = \begin{bmatrix} c+2d \\ c+3d \end{bmatrix}$$

- 4 special cases of linear combination
  ① Sum : $1\boldsymbol{v} + 1\boldsymbol{w}$
  ② Difference : $1\boldsymbol{v} - 1\boldsymbol{w}$
  ③ Zero vector : $0\boldsymbol{v} + 0\boldsymbol{w}$
  ④ Scalar multiple : $c\boldsymbol{v} + 0\boldsymbol{w}$
- See figures in the next slide

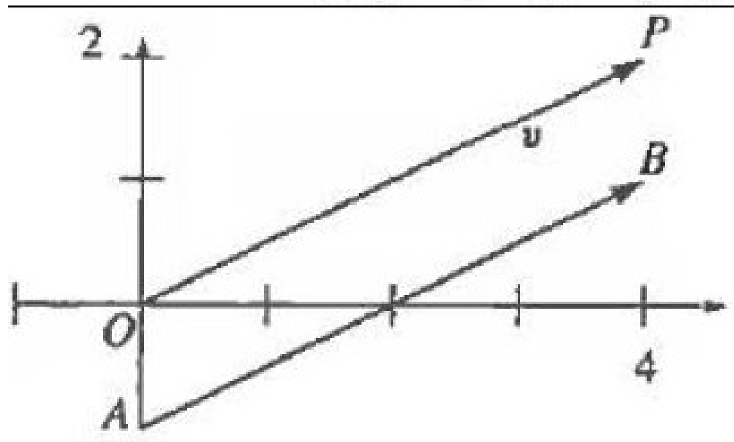
Linear Combination – Example

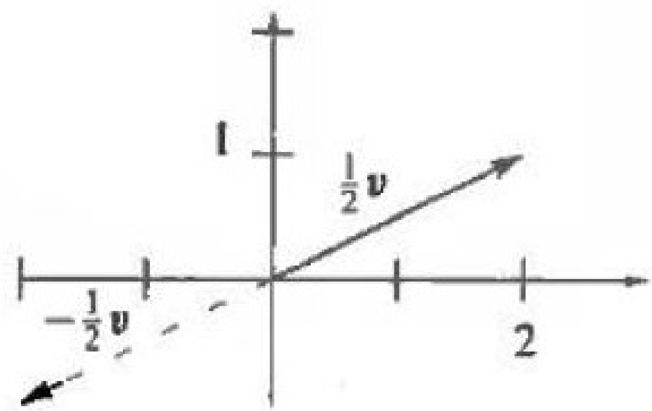


$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

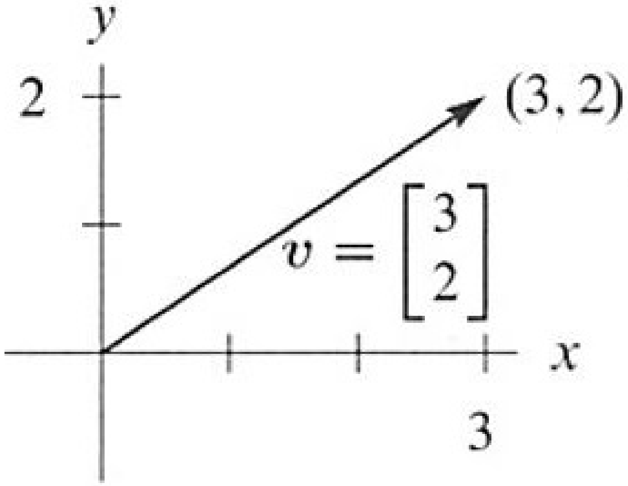


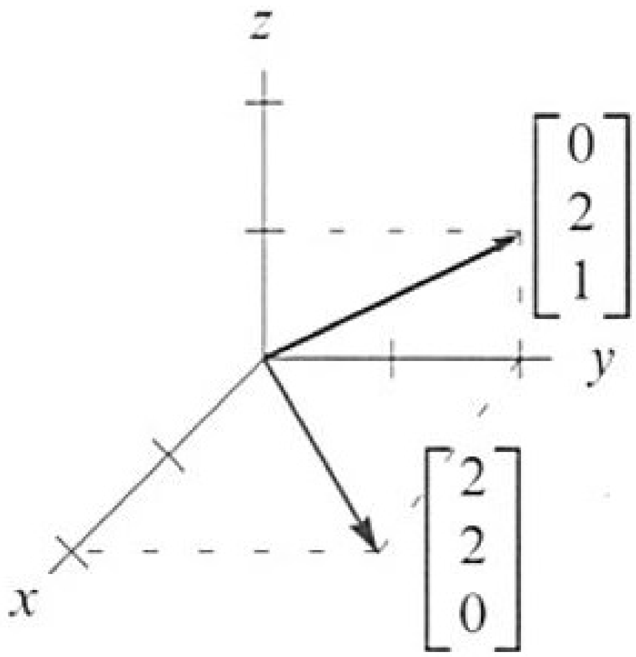
$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$





Vectors in 3 Dimensions





Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z)

Two Operations in 3-dim. Space

1. Vector Addition : ***v* + *w***
2. Scalar Multiplication : *c**v***, *d**w*** where *c*, *d* are scalar

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$* \text{Vector Addition}: \quad \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

$$* \text{Scalar Multiplication}: \quad c\mathbf{v} = c\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix}$$

$$\text{Ex)} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{v} + \boldsymbol{w} = ? \quad 6\mathbf{v} = ?$$

Linear Combination in 3 Dimensions

- Linear combination of 2 vectors in 3 dimensions
- Linear combination of 3 vectors in 3 dimensions

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

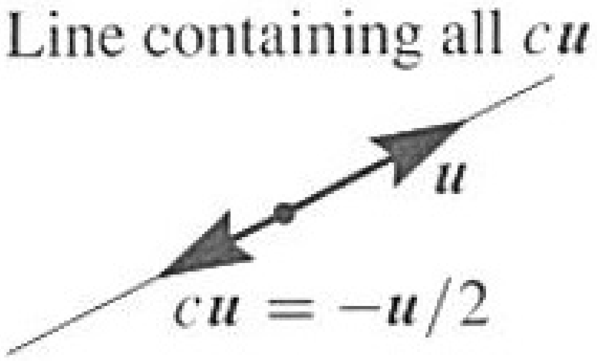
$$2\boldsymbol{w} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

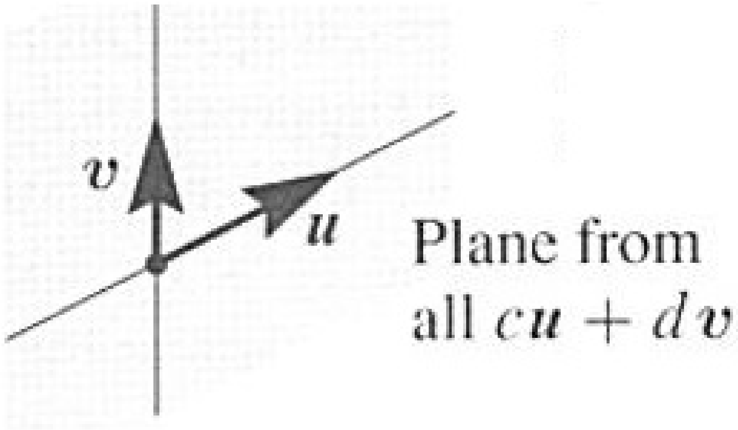
$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\boldsymbol{u} + 4\boldsymbol{v} - 2\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$$

Linear Combination and Dimension

- All combinations c***u*** : 1-dim. space, line
- All combinations c***u*** + d***v*** : 2-dim. space, plane
- All combinations c***u*** + d***v*** + e***w*** : 3-dim. space





Lengths and Dot Products

Dot Product

- **Definition:** The ***dot product*** or inner product of $\boldsymbol{v}=(v_1,v_2,\ldots,v_\mathrm{n})$ and $\boldsymbol{w}=(w_1,w_2,\ldots,w_\mathrm{n})$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

  - Example: $\boldsymbol{v} = (4,2)$, $\boldsymbol{w} = (-1,2)$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 4 \cdot (-1) + 2 \cdot 2 = -4 + 4 = 0$$

- **Properties**
  - $\boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{w}$
  - Perpendicular vectors : angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 90°.

Dot Product – Examples

- Example : In Engineering & Science (2-D)

  Put a weight of 4 at the point x = -1 and a weight of 2 at the point x = 2.

  - Vector of weight : $\boldsymbol{w} = (w_1, w_2) = (4,2)$
  - Vector of distances : $\boldsymbol{v} = (v_1, v_2) = (-1,2)$
  - Moment of each item: $w_1 \cdot v_1,\ \ w_2 \cdot v_2$
  - The equation for the see-saw to balance: $\boldsymbol{w} \cdot \boldsymbol{v} = w_1 \cdot v_1 + w_2 \cdot v_2 = 0$

- Example : In Economics & Business(3-D)

  There are three products to buy or sell.

  - Price vector : $\boldsymbol{p} = (p_1, p_2, p_3)$
  - Quantity vector : $\boldsymbol{q} = (q_1, q_2, q_3)$
  - Total price to buy or sell: $\boldsymbol{p} \cdot \boldsymbol{q} = p_1 \cdot q_1 + p_2 \cdot q_2 + p_3 \cdot q_3$

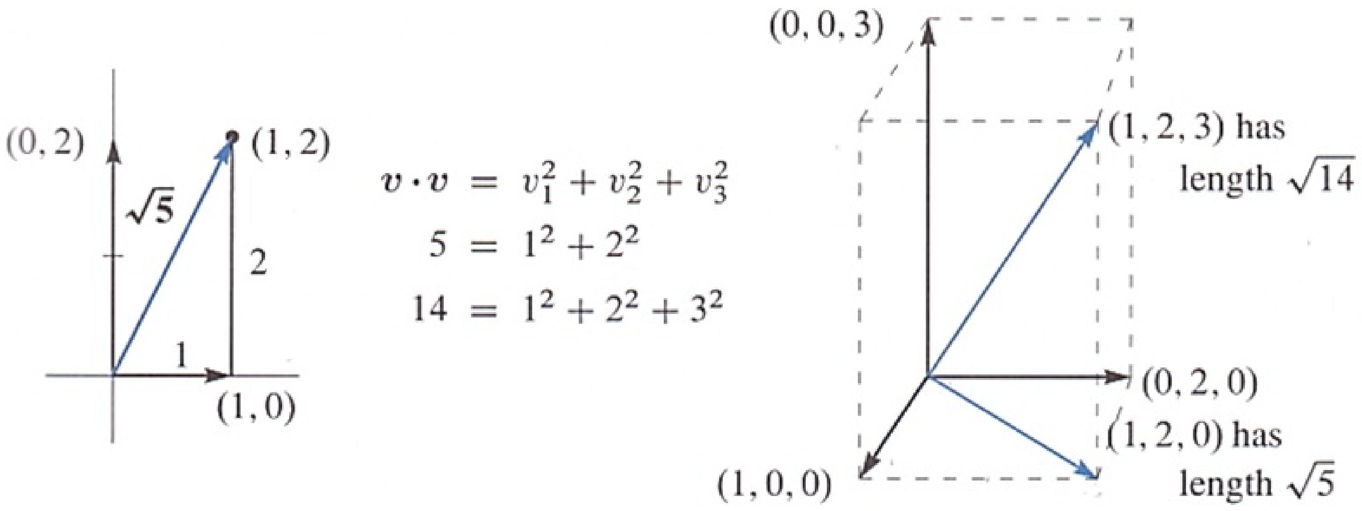
Lengths of Vectors

- **Definition :** The ***length*** (or **norm**) of a vector $\boldsymbol{v}$ is the square root of $\boldsymbol{v} \cdot \boldsymbol{v}$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1 v_1 + v_2 v_2 + \cdots + v_n v_n}$$

- Example: $\boldsymbol{v} = (1,2,3)$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3} = \sqrt{14}$$

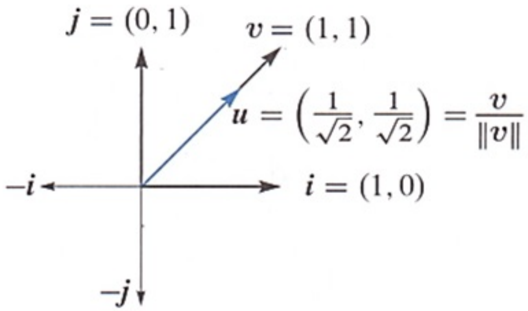


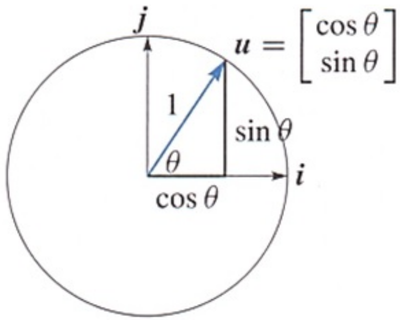
Unit Vector

- **Definition :** A ***Unit vector*** is a vector whose length equals one.

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = 1$$

- Examples:
  - $\boldsymbol{i} = (1,0)$
  - $\boldsymbol{j} = (0,1)$
  - $\boldsymbol{u} = (\cos\theta, \sin\theta)$





- For any non-zero vector $\boldsymbol{v}$,
  - $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ is a unit vector in <u>the same direction as $\boldsymbol{v}$</u>.
- Zero vector: $\boldsymbol{v}=(0,0,\ldots,0)$

The Angle between Two Vectors

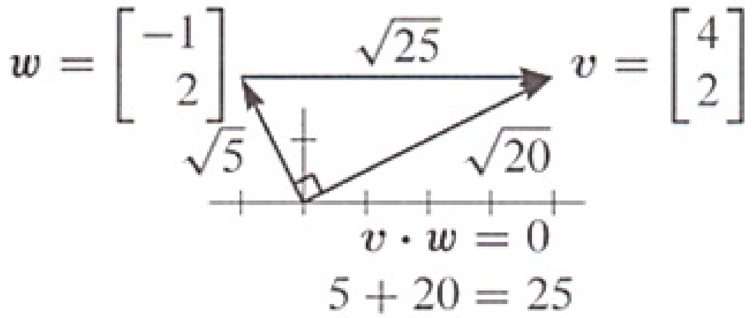
- **Prop :** $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ when $\boldsymbol{v}$ is perpendicular to $\boldsymbol{w}$

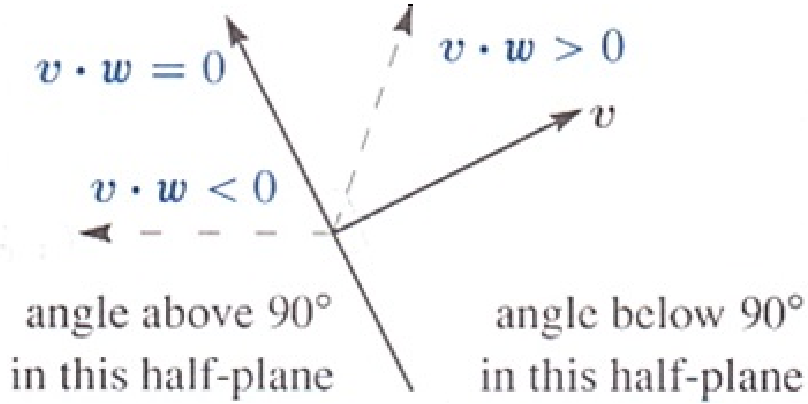
  Perpendicular → $\|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = \|\boldsymbol{v} - \boldsymbol{w}\|^2$ by Pythagoras

  – $\boldsymbol{v} \cdot \boldsymbol{w} = 0$: 90°.

  – $\boldsymbol{v} \cdot \boldsymbol{w} > 0$: below 90°

  – $\boldsymbol{v} \cdot \boldsymbol{w} < 0$: above 90°





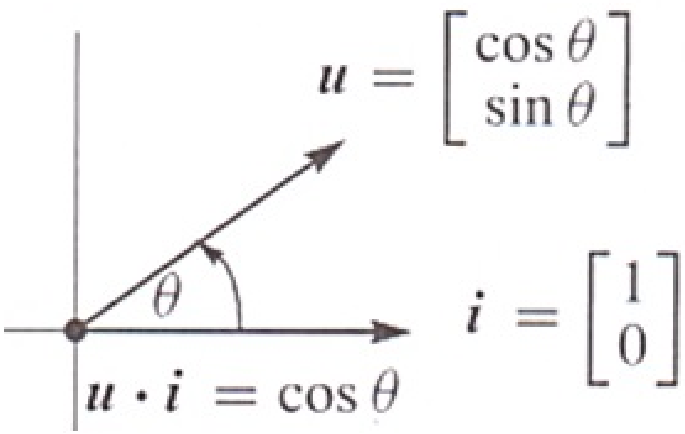
Cosine of θ

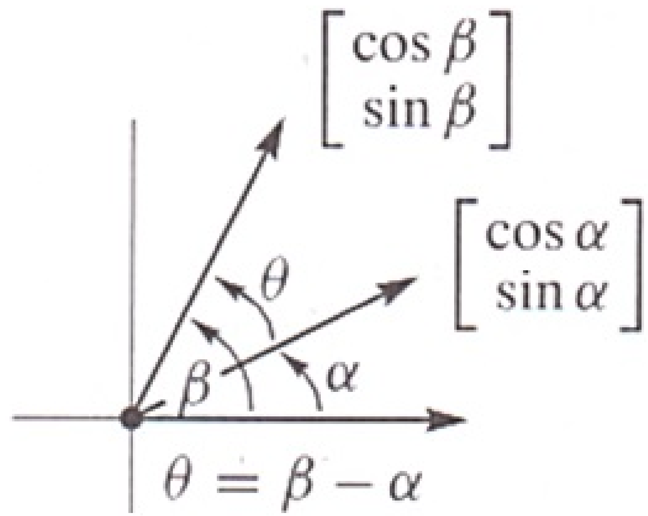
- If $\boldsymbol{u}$ and $\boldsymbol{U}$ are unit vectors at angle θ,

  $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$. Certainly $|\boldsymbol{u} \cdot \boldsymbol{U}| \le 1$.

- What if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not unit vector?

  Divide by their lengths to get $\boldsymbol{u}= \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$ and $\boldsymbol{w} = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}$





Remarks

1. Zero vector is perpendicular to every vector $\boldsymbol{w}$
2. **Cosine Formula** : if $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors, then

$$\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \cos\theta \quad (\text{i.e. } \boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\,\|\boldsymbol{w}\| \cos\theta)$$

3. Unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ at angle θ have

$$\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta \rightarrow |\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$$

4. **Schwarz Inequality** : $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$
5. **Triangle Inequality** : $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

Cosine of θ – Example

- Example : $\boldsymbol{v} = (a, b)$, $\boldsymbol{w} = (b, a)$
  - $\boldsymbol{v} \cdot \boldsymbol{w} = ab + ba = 2ab$
  - Schwarz Inequality ($|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|$) :

$$|2ab| \leq \sqrt{a^2 + b^2} \cdot \sqrt{a^2 + b^2} = a^2 + b^2$$

Let $x = a^2$, $y = b^2$, $\quad 2\sqrt{xy} \leq x + y \rightarrow \sqrt{xy} \leq \frac{x+y}{2}$

- Example : $\boldsymbol{v} = (2,1)$ and $\boldsymbol{w} = (1,2)$
  - $\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\| \|\boldsymbol{w}\|} = \frac{\mathbf{2 \cdot 1 + 1 \cdot 2}}{\sqrt{\mathbf{5}} \cdot \sqrt{\mathbf{5}}} = \frac{\mathbf{4}}{\mathbf{5}}$
  - Schwarz Inequality ($|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|$) : $4 \leq 5$
  - Triangle Inequality ($\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$) :

$$\|(\mathbf{3}, \mathbf{3})\| \leq \mathbf{2}\sqrt{\mathbf{5}}$$

Matrices

Matrix Form of Equations

- Consider 3 unknowns are *x*, *y* and *z* and have 3 linear equations as follows:

$$\begin{cases} x+2y+3z=6 \\ 2x+5y+2z=4 \\ 6x-3y+z=2 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}}_{\boldsymbol{b}}$$

$$\Rightarrow \quad A\boldsymbol{x} = \boldsymbol{b} \quad \text{where } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} : \text{Coefficient matrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

Combination Using Matrix

- Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$

  Linear combination of columns of A : $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$

$$c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{c}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + d\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + e\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} & \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} & \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix}$$

$$= [\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}] \begin{bmatrix} c \\ d \\ e \end{bmatrix}$$: Matrix times vector

$$c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \begin{bmatrix} cu_1 + dv_1 + ew_1 \\ cu_2 + dv_2 + ew_2 \\ cu_3 + dv_3 + ew_3 \end{bmatrix}$$

$$= \begin{bmatrix} (u_1, v_1, w_1) \cdot (c, d, e) \\ (u_2, v_2, w_2) \cdot (c, d, e) \\ (u_3, v_3, w_3) \cdot (c, d, e) \end{bmatrix}$$: Dot product with rows

$$A\boldsymbol{x} = \boldsymbol{b}$$

A: transformation

$\boldsymbol{x}$: input

$\boldsymbol{b}$: output

Two Special Matrices

Difference matrix:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Cyclic difference matrix:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Difference Matrix

$$\boldsymbol{u}=\begin{bmatrix}1\\-1\\0\end{bmatrix},\quad \boldsymbol{v}=\begin{bmatrix}0\\1\\-1\end{bmatrix},\quad \boldsymbol{w}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

– Linear combinations in 3-D space :

$$c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}=c\begin{bmatrix}1\\-1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\-1\end{bmatrix}+e\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}c\\d-c\\e-d\end{bmatrix}$$

– Using a Matrix

$$\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}c\\d\\e\end{bmatrix}}_{\boldsymbol{x}}=\begin{bmatrix}\boldsymbol{u}&\boldsymbol{v}&\boldsymbol{w}\end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix}=\underbrace{\begin{bmatrix}c\\d-c\\e-d\end{bmatrix}}_{\boldsymbol{b}}$$

A: Difference matrix

b contains differences of the input vector ***x***.

Difference Matrix – Example 1

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\boldsymbol{b}}$$

- Input $\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$ is known, output $\boldsymbol{b}$ is not known

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 4-1 \\ 9-4 \end{bmatrix}}_{\boldsymbol{b}} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

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Difference Matrix – Example 2

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\boldsymbol{b}}$$

- Output $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is known, intput $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is not known

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\boldsymbol{b}} \Rightarrow \begin{cases} x_1 = b_1 \\ x_2 = b_1 + b_2 \\ x_3 = b_1 + b_2 + b_3 \end{cases}$$

i) When $\boldsymbol{b} = \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{x} = \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ For any matrix C, does $C\boldsymbol{x} = \boldsymbol{0}$ implies $\boldsymbol{x} = \boldsymbol{0}$?

ii) When $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\begin{cases} x_1 = b_1 = 1 \\ x_2 = b_1 + b_2 = 1 + 3 = 4 \\ x_3 = b_1 + b_2 + b_3 = 1 + 3 + 5 = 9 \end{cases}$

$\boxed{A \text{ is invertible.}}$

Cyclic Difference Matrix

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}$$

C : Cyclic difference matrix

Cyclic Difference Matrix – Example

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{b}$$

i) When $\boldsymbol{b} = \boldsymbol{0}$ is known, $\boldsymbol{x}$ is unknown

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \boldsymbol{0} \Rightarrow \quad x_1 = x_2 = x_3 = \text{Any constant}$$

Infinitely many solutions

$\boxed{C \text{ is singular.}}$

ii) When $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is known, $\boldsymbol{x}$ is unknown

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \Rightarrow$$

No solution

C is not invertible

The Inverse Matrix

$$A=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}:\text{Difference matrix}$$

$$S=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}:\text{Sum matrix}$$

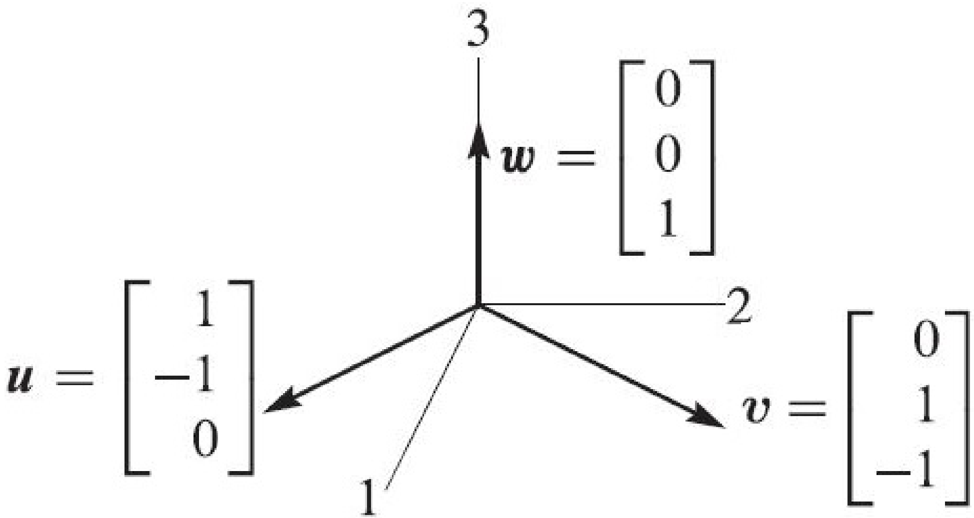
$$AS=SA=I\ \Rightarrow\ S=A^{-1}$$

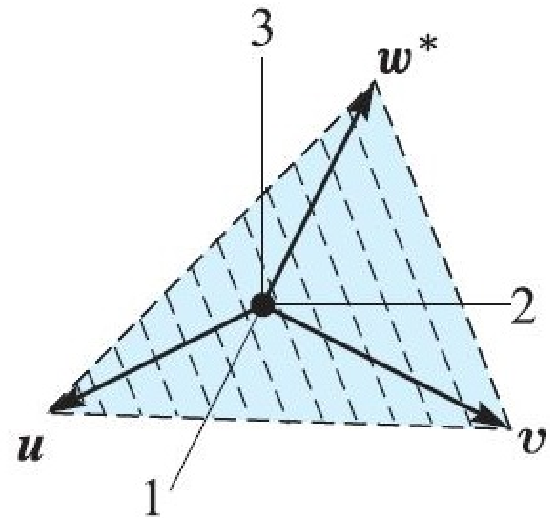
$$A\boldsymbol{x}=\boldsymbol{b}\ \Rightarrow\ \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix}=\boldsymbol{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$$\Rightarrow\ SA\boldsymbol{x}=S\boldsymbol{b}\ \Rightarrow\ \boldsymbol{x}=S\boldsymbol{b}$$

$$\boldsymbol{x}=S\boldsymbol{b}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix}$$

Independence and Dependence





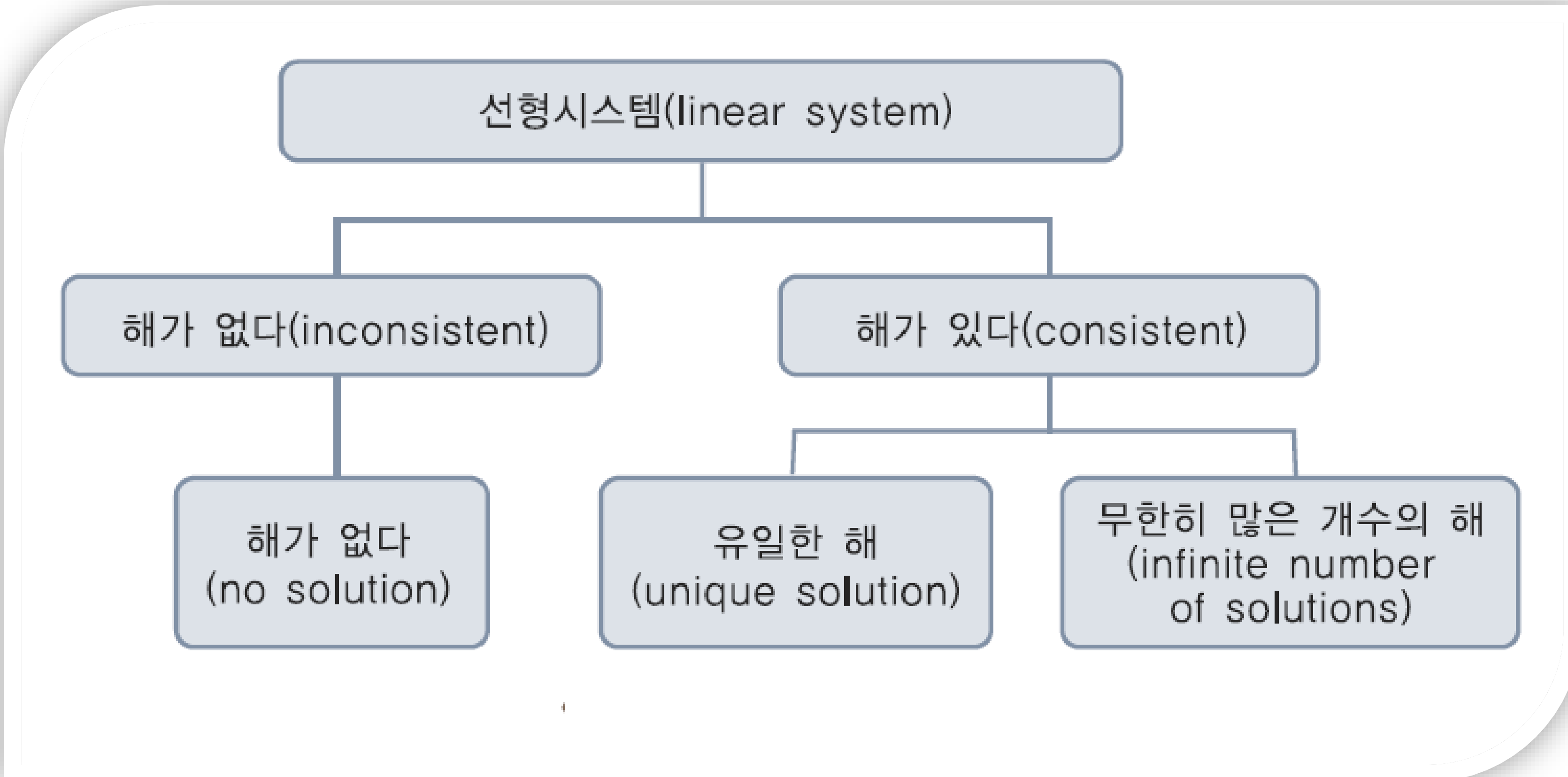
- Independence : $\boldsymbol{w}$ is not in the plane of $\boldsymbol{u}$ & $\boldsymbol{v}$
- Dependence : $\boldsymbol{w^*}$ is in the plane of $\boldsymbol{u}$ & $\boldsymbol{v}$
- $\boldsymbol{w^*}$: linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w^*} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{w^*} = -\boldsymbol{u} - \boldsymbol{v}$$

- Independent columns
  - $Ax = \mathbf{0}$ has one solution.
  - A is an invertible matrix.

- Dependent columns
  - $Ax = \mathbf{0}$ has many solutions.
  - A is a singular matrix.

- Does $A\boldsymbol{x}=\boldsymbol{b}$ has a solution?



Question?