FULL STRONG EXCEPTIONAL COLLECTIONS ON RANK 2 LINEAR GIT QUOTIENTS

Semi-Orthogonal decompositions

X variety over a field k / orbifold V/G $\rightsquigarrow$ $\mathcal{D}^bX$ bounded derived category of coherent sheaves on X.

- A semi-orthogonal decomposition (SOD)

$$\mathcal{D}^bX = \langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$$

full triangulated subcategories that

① generate $\mathcal{D}^bX$: every $\mathcal{F} \in \mathcal{D}^bX$ has

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_0 \arrow[r, "\varphi_1"] & \mathcal{F}_1 \arrow[r, "\varphi_2"] \arrow[dl] & \mathcal{F}_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & \mathcal{F}_{n-1} \arrow[r, "\varphi_n"] & \mathcal{F}_n = \mathcal{F} \arrow[dl] \\
& \mathrm{Cone}\,\varphi_1 \arrow[u, "+"] & \mathrm{Cone}\,\varphi_2 \arrow[u, "+"] & & & \mathrm{Cone}\,\varphi_n \arrow[u, "+"] & \\
& \mathcal{C}_1 \arrow[u] & \mathcal{C}_2 \arrow[u] & & & \mathcal{C}_n \arrow[u] &
\end{tikzcd}$$

② are semi-orthogonal : For $\mathcal{F} \in \mathcal{C}_j$, $G \in \mathcal{C}_i$

$$\mathrm{Hom}_X(\mathcal{F}, G[m]) = 0 \text{ for } j > i \text{ and all } m$$

- A sequence of objects in $\mathcal{D}^bX$ $E_1, \dots, E_n$ is a full exceptional collection (FEC) if

① Each E_i is exceptional : $\langle E_i \rangle \cong \mathcal{D}^b(k)$

② $\mathcal{D}^bX \cong \langle E_1, \dots, E_n \rangle$ is a SOD.

- A FEC is strong if

$$\mathrm{Hom}(E_i, E_j[m]) = 0 \text{ for } i < j \text{ and } m > 0.$$

- Consequences :

① Splitting of additive invariants : K_0, HH.

$\Rightarrow$ If $E_1, \dots, E_n$ is a FEC on X, then $[E_1], \dots, [E_n]$ forms a basis for $K_0(X)$.

② Relation to tilting theory :

If $E_1, \dots, E_n$ is a FSEC of vector bundles on X, then $\mathcal{D}^bX \cong \mathcal{D}^b(A\text{-Mod})$, where

$$A = \mathrm{End}(\underbrace{E_1 \oplus \cdots \oplus E_n}_{\text{tilting bundle}})$$

Example: (Beilinson)

The line bundles $\mathcal{O}(x), \mathcal{O}(x+1), \ldots, \mathcal{O}(x+n)$ are a FSEC on $\mathbb{P}^n$ for any $x \in \mathbb{Z}$.

$\mathbb{P}^n$

As a homogeneous Space

For what semisimple G and parabolic $P \subset G$ does G/P have a FEC of vector bundles?

- (Kapranov) $Gr(r,n)$ has a FSEC consisting of vector bundles:

$$\left\{ \Sigma^{\alpha} \mathcal{U}^{*} : \alpha \subset \underbrace{\boxed{}}_{n-r} {}^{r} \right\}$$

tautological bundle; Young diag

- Other homogeneous spaces with FEC:

① Quadrics (Kapranov)

② Lagrangian Grassmannians $SGr(n,2n)$
  - $n = 3,4,5$ (Samokhin, Polishchuk)
  - all n (Fonarev)

③ Grassmannian of isotropic planes $OGr(2,n)$ (Kuznetsov)

④ Examples for exceptional groups (Faenzi, Manivel)

As a GIT quotient

X linear representation of a reductive G
X^{ss} with finite stabilizers; X^{ss}/G proper
When does X^{ss}/G have a FSEC of vector bundles?

- In the setting of toric varieties:

King's conjecture: Smooth projective toric varieties have FSEC of line bundles.

- Many counterexamples:

  Eg. Hirzebruch surface iteratively blown up 3 times (Hille - Perling)

- Many examples:

  Eg. ① toric varieties with Picard number ≤ 2 (Costa, Miró-Roig)
  ② toric orbifolds with Picard rank ≤ 2 (Borisov - Hua)
  ③ toric Fano 3-folds (Bernadi-Tirabassi + Uehara)
  ④ toric Fano 4-folds (Prabhu - Naik)
  ⑤ $(\mathbb{P}^1)^n /\!/ \mathbb{G}_m$ (Castravet - Tevelev)

Setup

G reductive group $/k$ $\quad$ $\mathrm{char}(k) = 0$

V $\quad$ G representation

$X = \mathrm{Spec}(\mathrm{Sym}\, V^*)$

ℓ = Weyl-invariant character

$\rightsquigarrow$ GIT

$X^{ss}(\ell) \subset X$ $\quad$ semi-stable locus

$X^{us}(\ell) = X \setminus X^{ss}(\ell)$ $\quad$ unstable locus

$$= \bigcup_{\substack{\lambda \text{ s.t } \langle \lambda, \ell \rangle < 0}} G \cdot X^{\lambda \geq 0}$$

cocharacter

attracting locus of λ: $\{x \in X : \lim_{t \to 0} \lambda(t) \cdot x \text{ exists}\}$

Assume: $X^{ss}(\ell)$ has finite G-stabilizers

good quotient $\quad$ quotient stack

$X^{ss}(\ell) /\!/ G$ $\quad$ $X^{ss}(\ell)/G$

can have bad singularities

Smooth Deligne-Mumford (DM) stack

- Reasonable to ask if it has a finite FSEC

Problem: Find examples of pairs (G, X, ℓ) such that $X^{ss}(\ell)/G$ has a FSEC consisting of vector bundles.

Theorem (Borisov-Hua)

Smooth toric Fano DM stacks s.t either

- Picard rank ≤ 2
- Quotient by $\mathbb{G}_m^2$

have a FSEC consisting of line bundles.

- gives many examples (G, X, w^*)

torus $\quad$ anti-canonical character

Theorem (Halpern-Leistner, K.)

G has rank 2

Assume: All weights of X pair strictly negatively with some cocharacter λ_0 $\quad$ $(\Rightarrow \mathcal{O}_X^G = k)$

$\ell = w^*$ (or "close" to w^*),

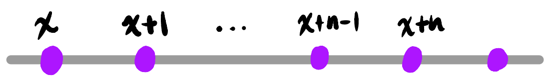
$X^{ss}(\ell)$ with finite stabilizers

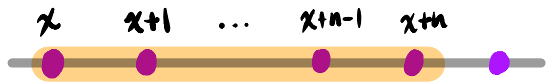
Then $X^{ss}(\ell)/G$ has a FSEC consisting of vector bundles.

An idea of what these vector bundles are :

For $\mathbb{P}^n$

$$\begin{tikzcd} \mathbb{C}^{n+1}/\mathbb{C}^* = X/G \\ \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^* = X^{ss}/G \arrow[u, hook, "\iota"] \end{tikzcd}$$





$$\begin{tikzcd} \mathcal{D}^b(X/G) = \langle \mathcal{O}_X(x) : x \in \mathbb{Z} \rangle \arrow[d, "\iota^*"] & \supset & W(\nabla) := \langle \mathcal{O}_X(x) : x \in \nabla \cap M \rangle \arrow[dll, bend left] \\ \mathcal{D}^b(X^{ss}/G) & & \end{tikzcd}$$

restricting gives a FSEC on X^{ss}/G

To find ∇ : Look at complexes in $\mathcal{D}^b(X/G)$ whose homology is supported in X^{us}.

$$0 \to \mathcal{O}_X(-n-1) \to \cdots \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to \mathcal{O}_{\{0\}} \to 0$$

Restricting

$$0 \to \mathcal{O}_{X^{ss}}(-n-1) \to \cdots \to \mathcal{O}_{X^{ss}}(-1)^{\oplus n+1} \to \mathcal{O}_{X^{ss}} \to 0$$

$|\nabla \cap M| = n+1$

$\mathcal{O}_{X^{ss}}(x) \rightsquigarrow$ inductively push to ∇ $\leftsquigarrow \mathcal{O}_{X^{ss}}(x)$

Windows in rank 2

Assume G is connected

$$X^{ss}/G \overset{\iota}{\hookrightarrow} X/G$$

$$D^b(X/G) = \langle \mathcal{O}_X \otimes U : U \text{ irrep of } G \rangle = \langle \mathcal{O}_X(\mu) : \mu \in M^+ \rangle$$

Want: $\nabla \subset M_{\mathbb{R}}$ such that

$$\langle \mathcal{O}_X(\mu) : \mu \in \nabla \cap M^+ \rangle \xrightarrow{\iota^*} D^b(X^{ss}/G)$$

is an equivalence.

■ $\exists$ a description by Van den Bergh of highest weights of irreps in $H^i_{X^{us}}(X, \mathcal{O}_X)$

$\Rightarrow$ can give conditions on U for

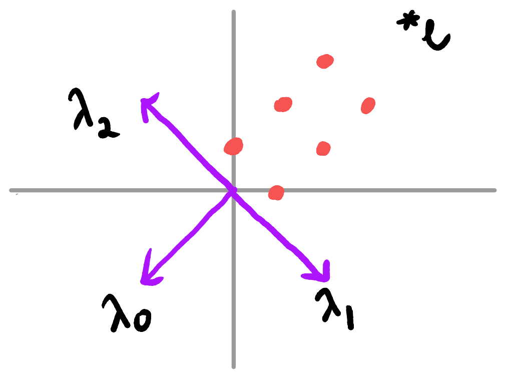
$$\boxed{R\Gamma_{X^{us}}(\mathcal{O}_X \otimes U)^G = 0}$$

gives conditions on $U, W \in \mathrm{Rep}(G)$ such that

$$R\mathrm{Hom}_{X^{ss}}(\mathcal{O}_{X^{ss}} \otimes U, \mathcal{O}_{X^{ss}} \otimes W)^G \cong R\mathrm{Hom}_X(\mathcal{O}_X \otimes U, \mathcal{O}_X \otimes W)^G$$

$$= (\mathrm{Sym}(X^*) \otimes W \otimes U^*)^G$$

A condition for vanishing of $R\Gamma_{X^{us}}(\mathcal{O}_X \otimes U)$:



$X^{ss}(\ell) = X^s(\ell)$

$G = GL_2$ over $\mathbb{C}$

To each $\lambda_i \rightsquigarrow \eta_{\lambda_i} := \langle \lambda_i, -\det X^{\lambda_i \le 0} + \det(\mathfrak{g}^{\lambda_i < 0}) \rangle$

Lemma: Let U be a G-rep.

If every weight χ appearing in U satisfies for $i = 0, 1, 2$ either

$$\begin{cases} \langle \lambda_i, \chi \rangle < \eta_{\lambda_i} \\ \langle \lambda_i, \chi \rangle = \eta_{\lambda_i} \text{ and } \langle \lambda_0, \chi \rangle < \eta_{\lambda_0} \end{cases}$$

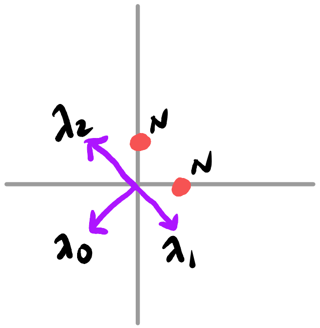
Then $R\Gamma_{X^{us}}(\mathcal{O}_X \otimes U) = 0$

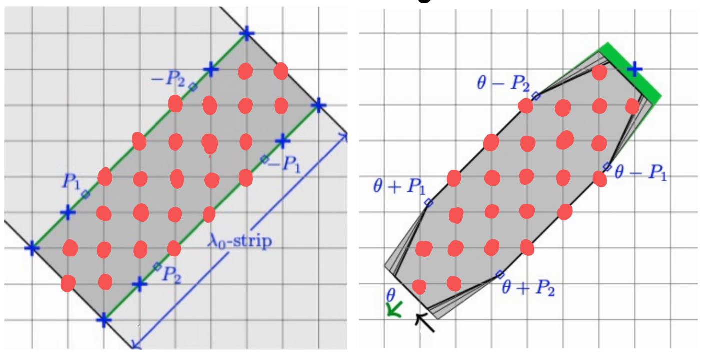
$\Rightarrow$ gives a window $\nabla \subseteq M_{\mathbb{R}}$

s.t. $\nabla = \{ x \in M_{\mathbb{R}} : |\langle \lambda_i, x \rangle| \le \eta_{\lambda_i}/2 \;\; \forall\, i = 0, 1, 2 \}$

by construction $\{ \mathcal{O}_{X^{ss}}(\mu) : \mu \in \nabla \cap M^+ \}$ is a strong exceptional collection.

Example $\quad Gr(2,N) \cong X^{ss}(w^*)/Gl_2 \; ; \; X = (\mathbb{C}^2)^{\oplus N}$





$\overline{\nabla}$ $\qquad\qquad$ $\theta + \nabla$

θ generic shift

How to show this collection is full :

- λ, χ $\rightsquigarrow$ $\exists$ minimal G-equivariant complex $C_{\lambda,\chi}$ in $\mathcal{D}^b(X/G)$

  character $\qquad$ supported in $G \cdot X^{\lambda \geq 0}$

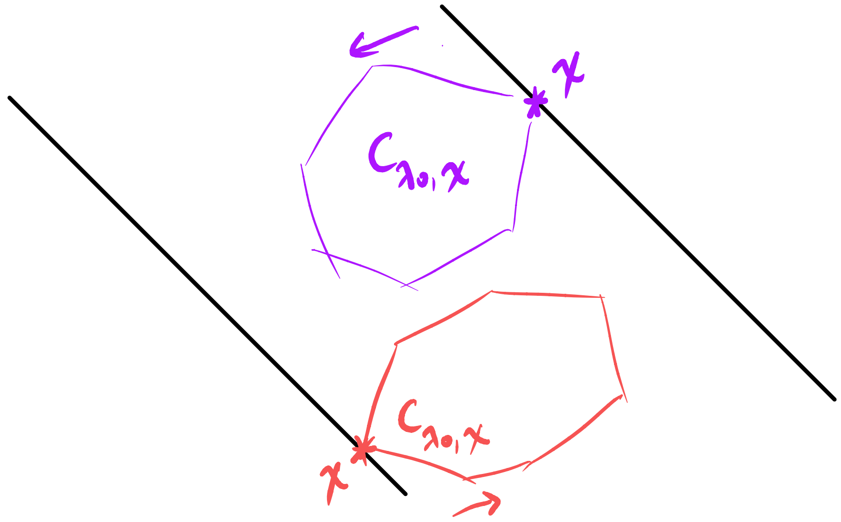
  Weights μ appearing in $C_{\lambda,\chi}$ have

  either
  - $\langle \lambda, \mu \rangle < \langle \lambda, \chi \rangle$, or
  - $\langle \lambda, \mu \rangle > \langle \lambda, \chi \rangle$

  and one term $\mathcal{O}_X(\chi)$

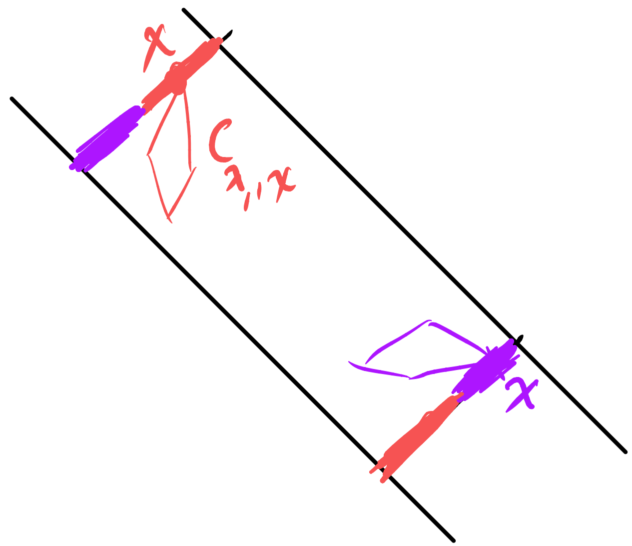
- For $\lambda_0, \lambda_1, \lambda_2$ $\rightsquigarrow$ C_λ unstably supported

Step 1: Use $C_{\lambda_0,\chi}$



Will stop in a strip with width η_0.

Step 2: Use $C_{\lambda_1,\chi}$, $C_{\lambda_2,\chi}$



Will stop in ∇