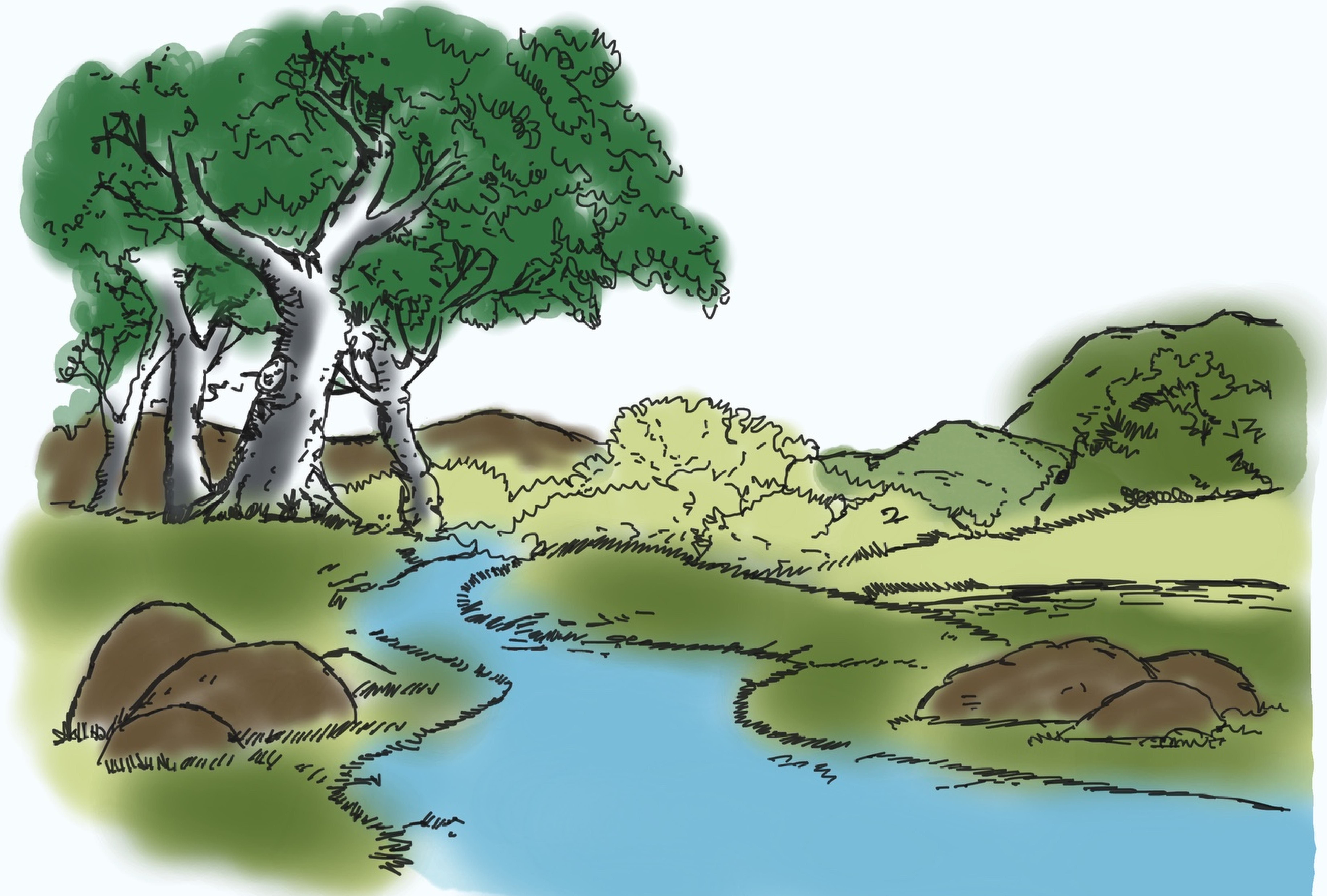
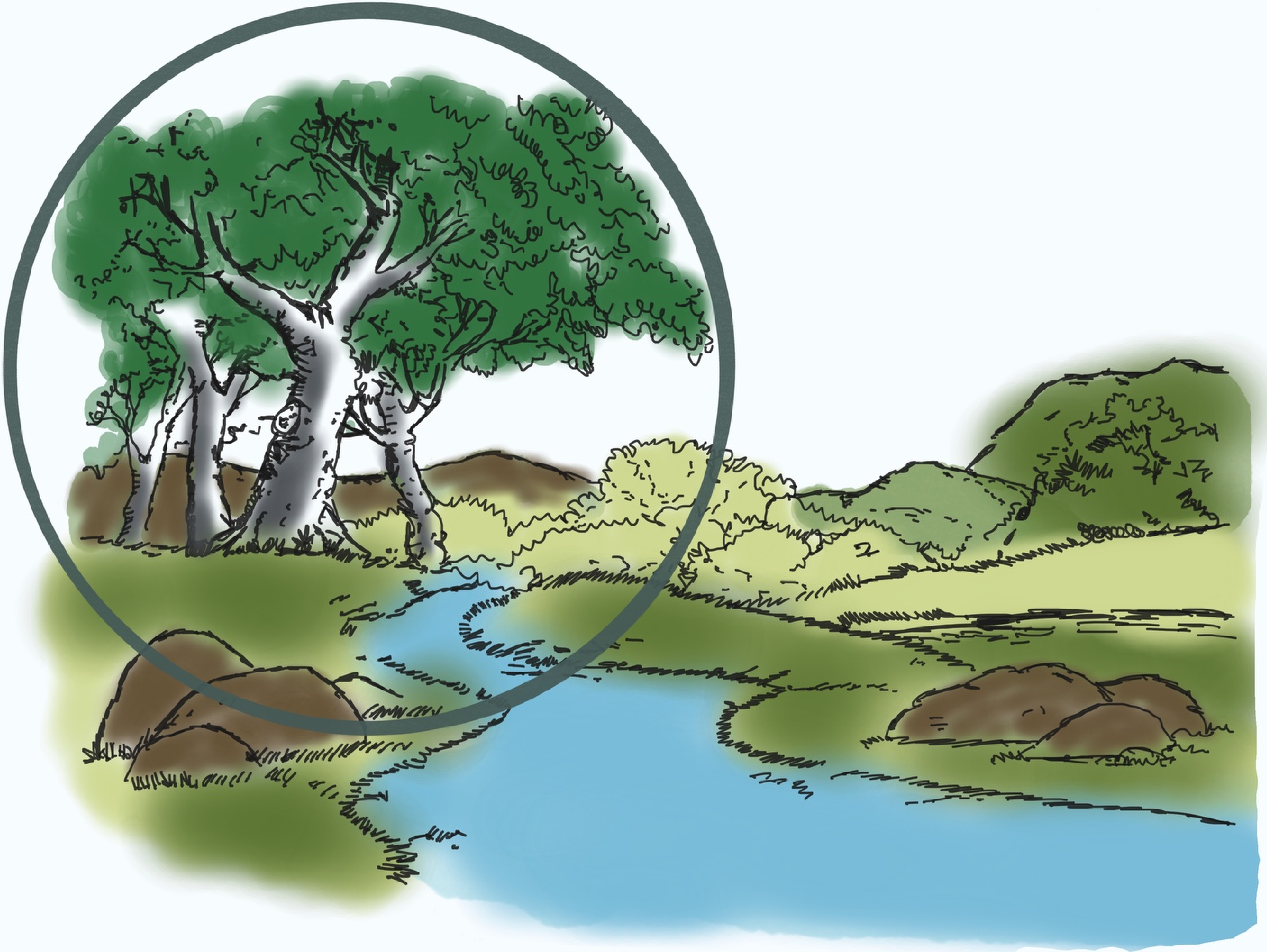
Decompositions of Derived Categories and connections to geometry

KIMOI KEMBOI

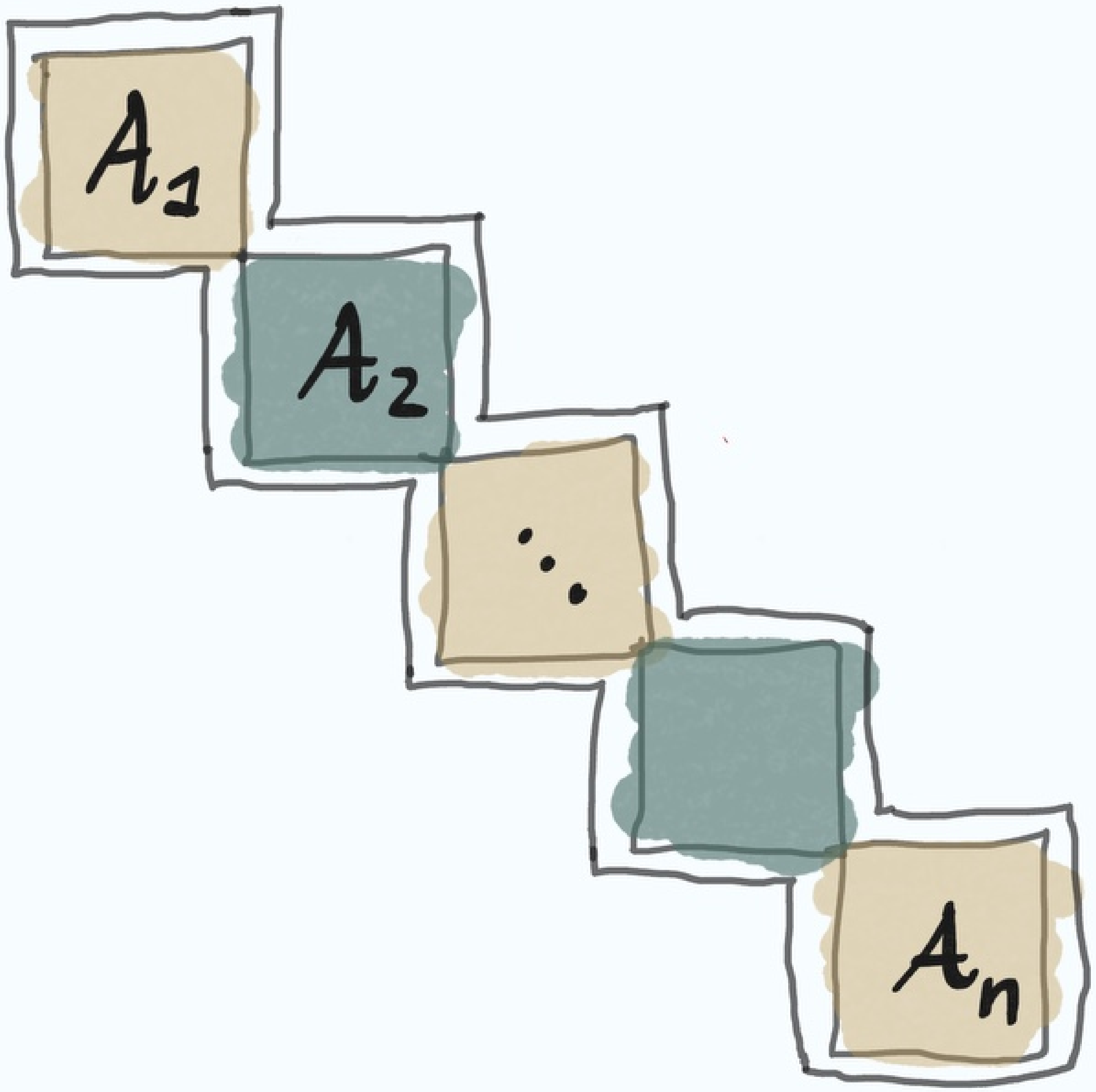
SLMath Spring 2024

Non-commutative algebraic geometry





Semi-orthogonal decompositions



Semi-orthogonal decompositions

$$\mathcal{D}^b(X) = \langle A_1, \ldots, A_n \rangle$$

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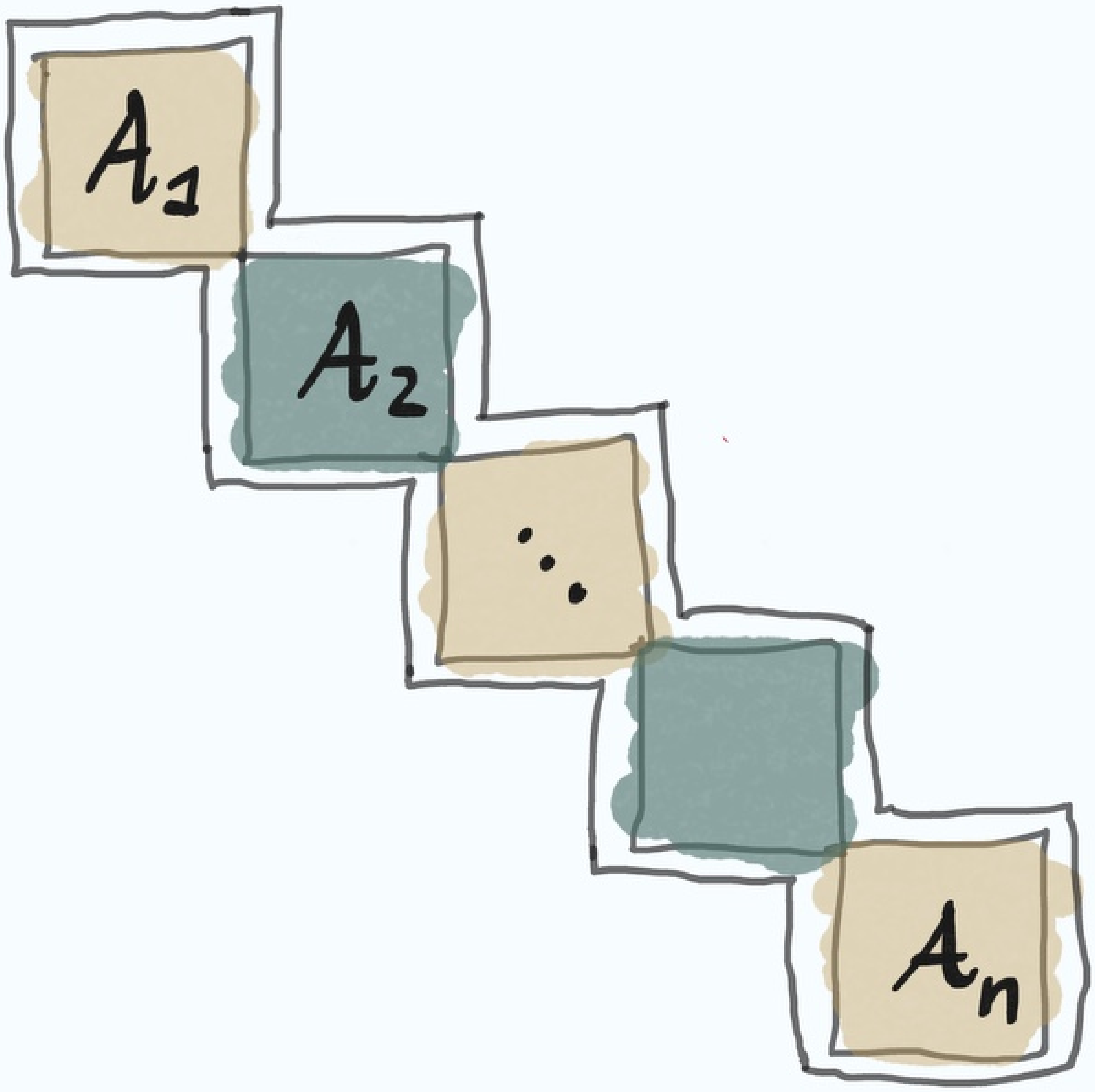

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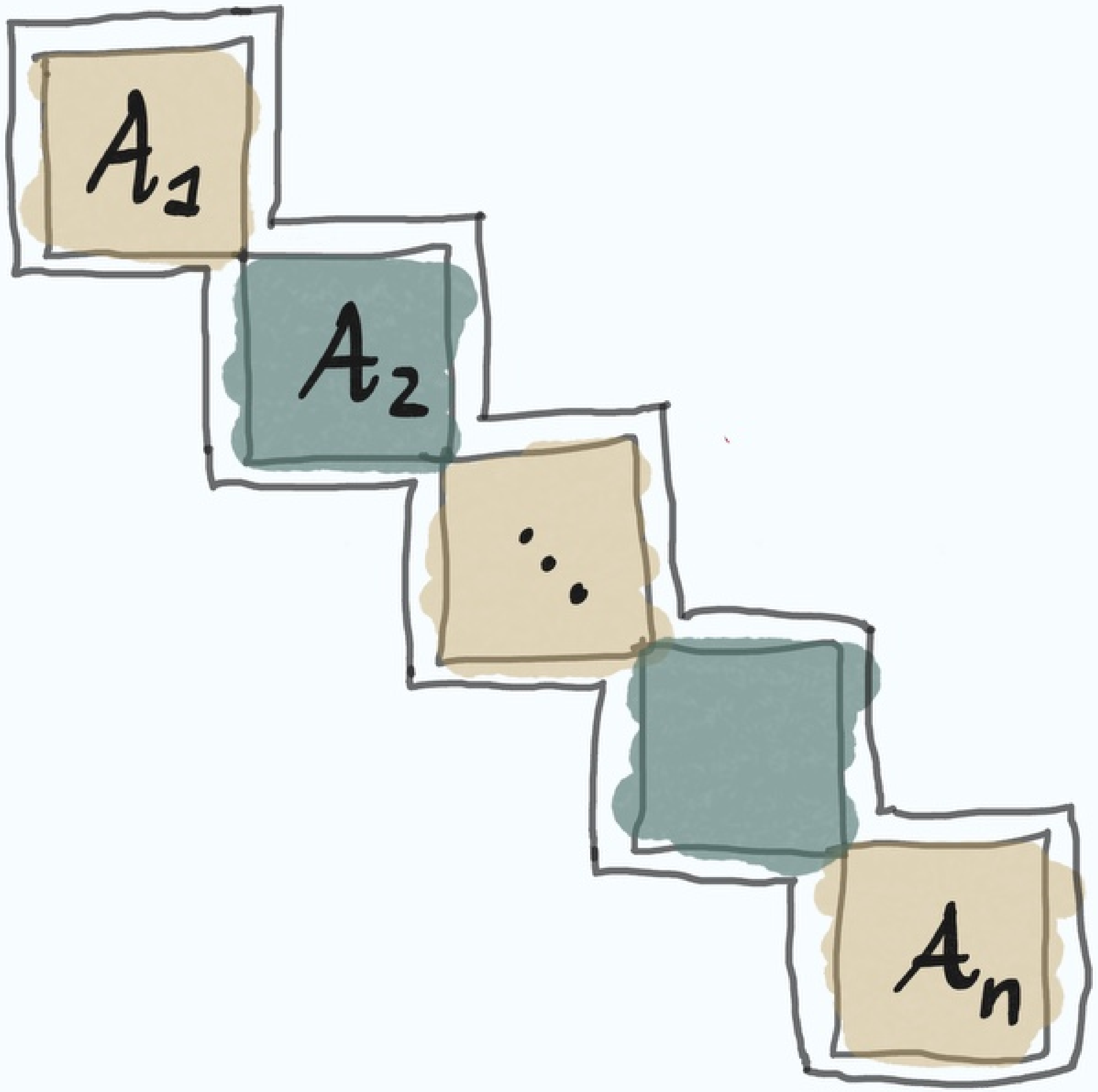

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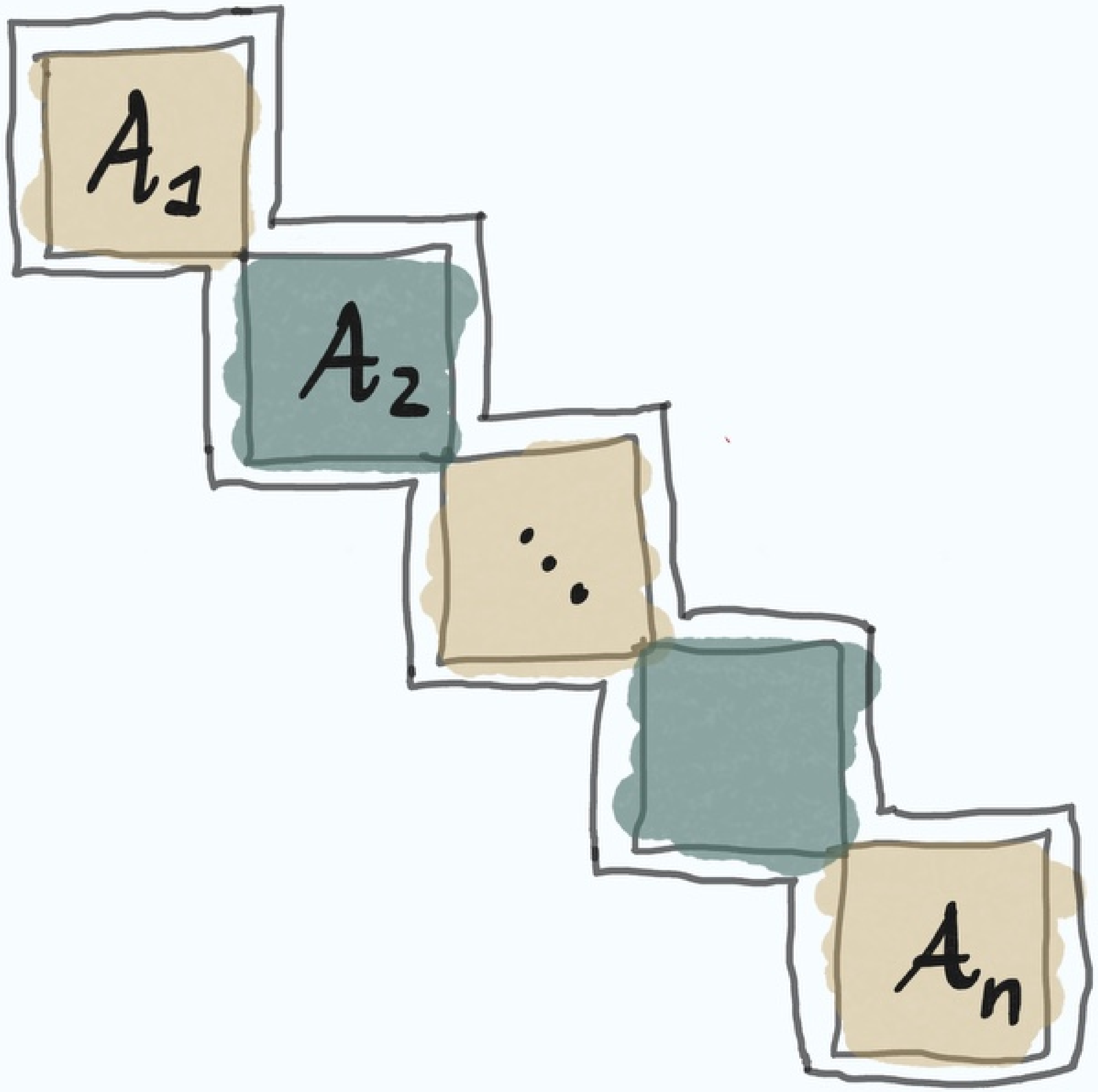

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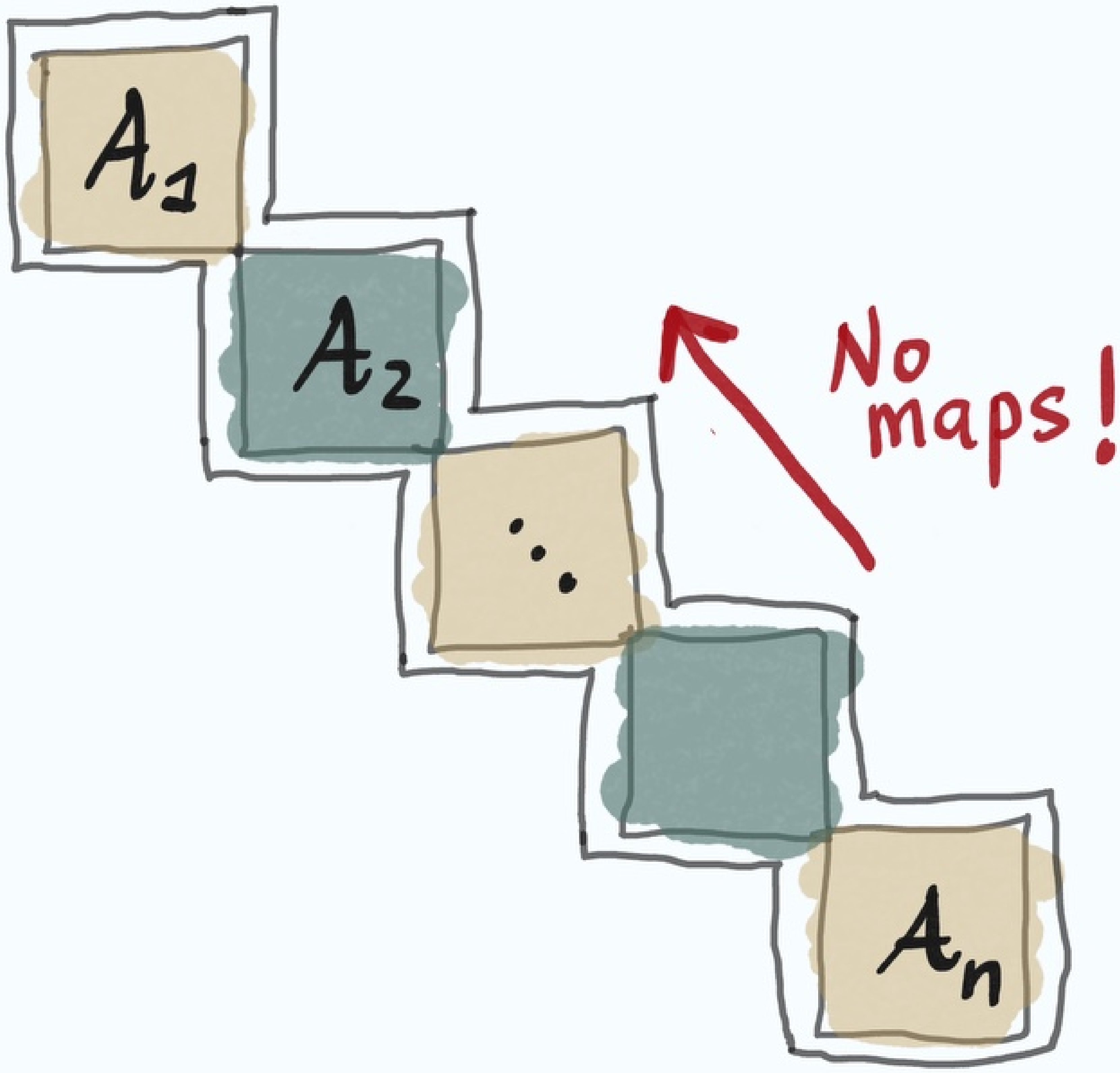
→ $D^b(X)$ is built from the pieces A_i



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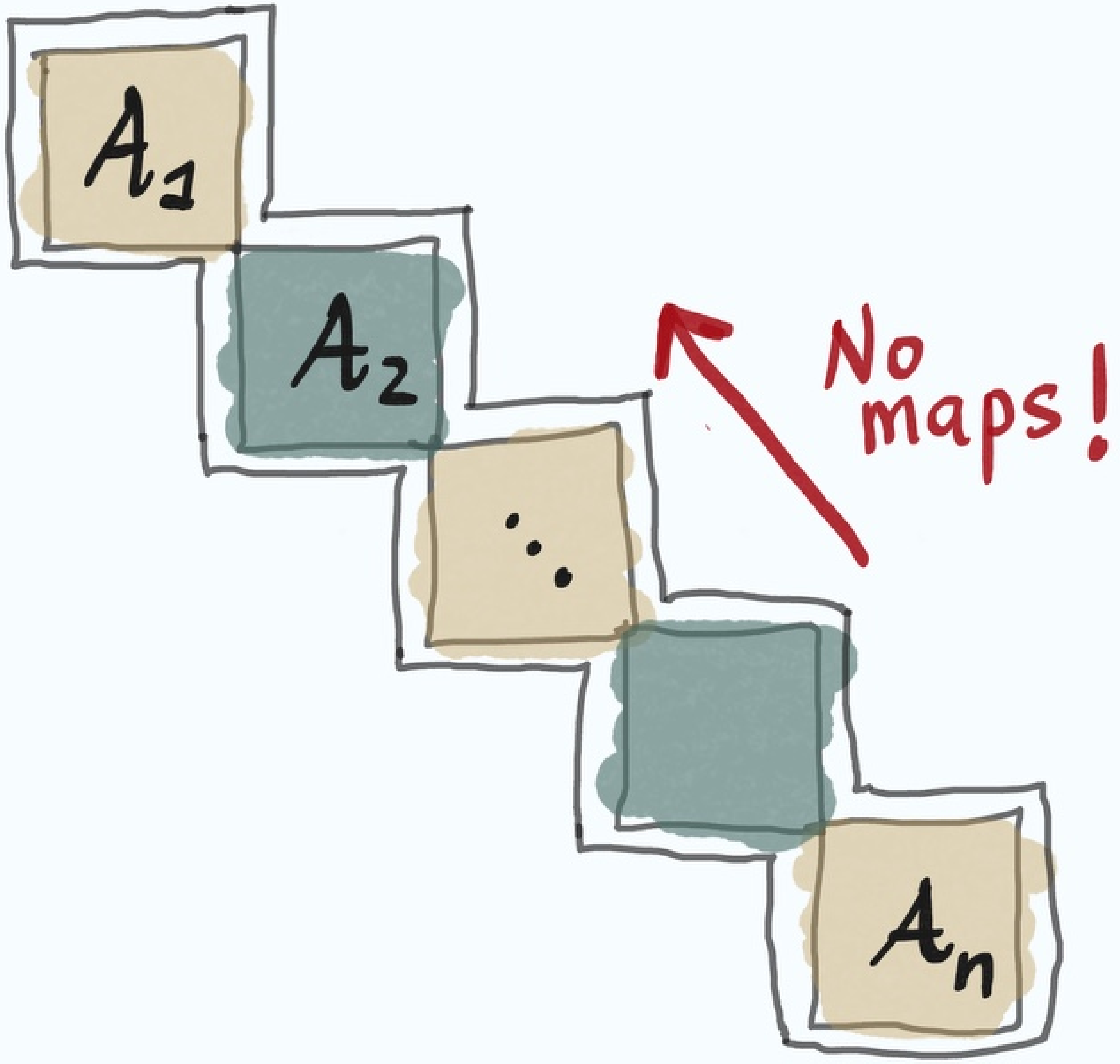

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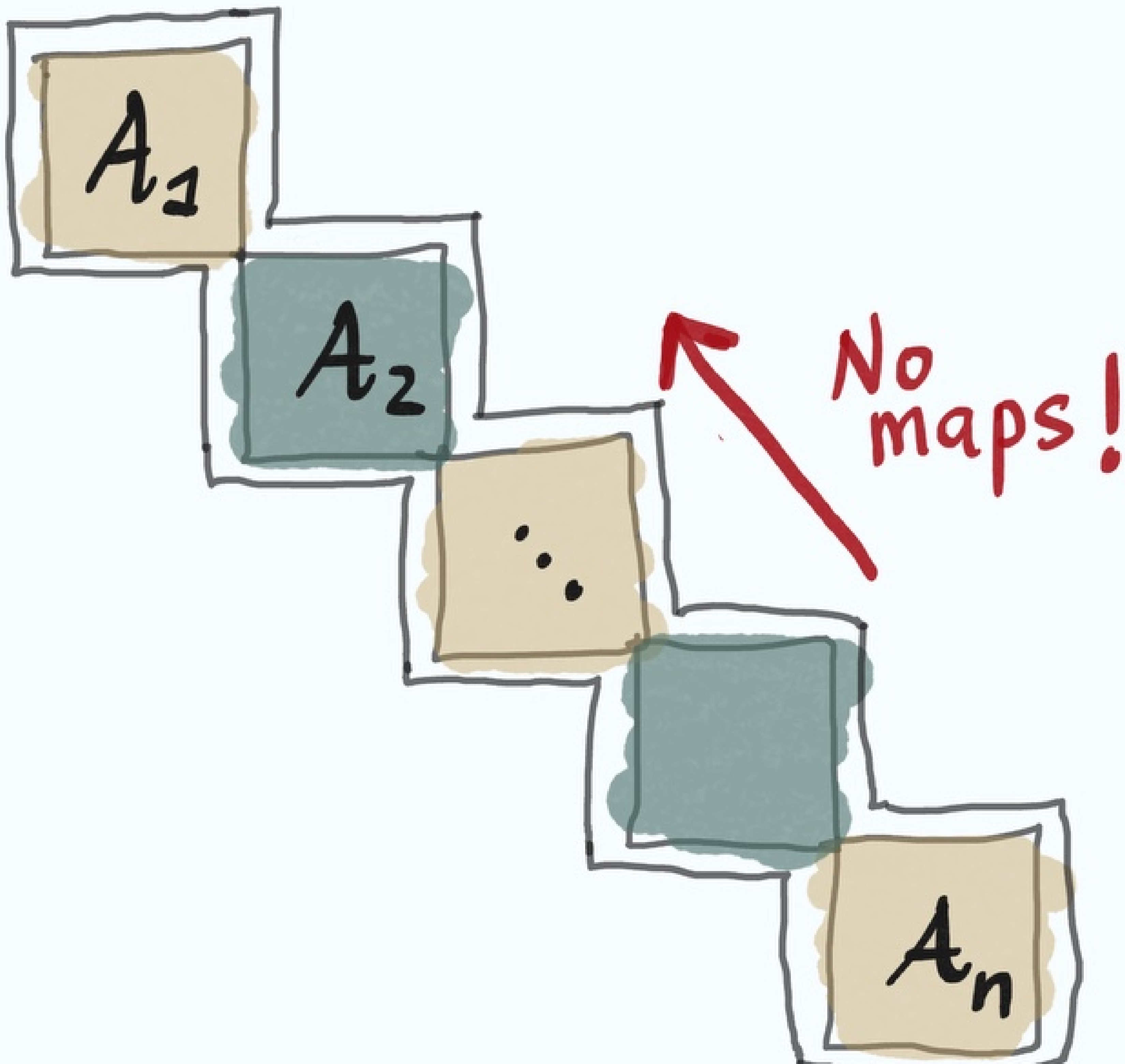
"Nice": Image of Fourier-Mukai transforms



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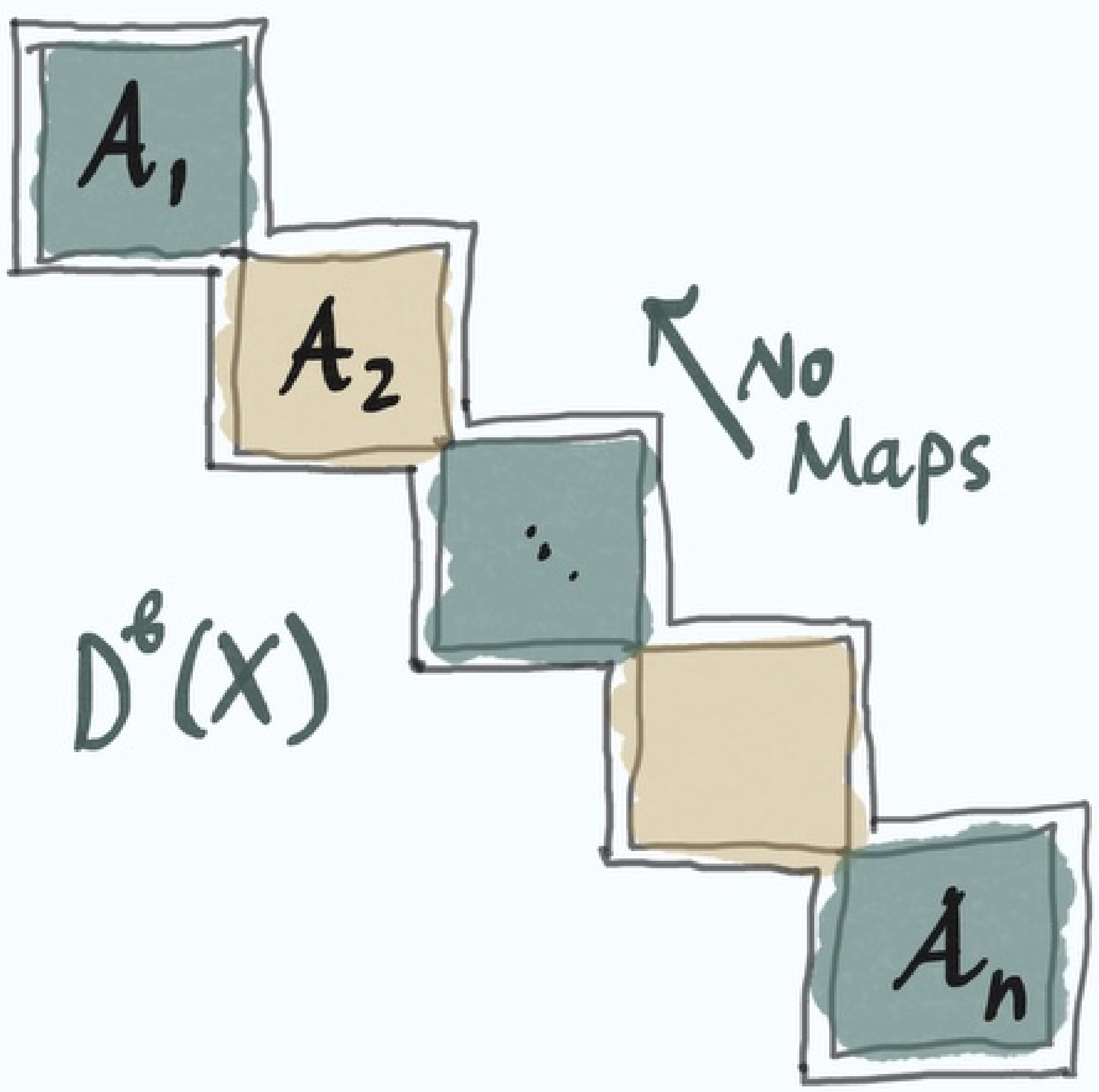
"Nice": Image of Fourier-Mukai transforms

$$\begin{tikzcd} & X \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X \end{tikzcd}$$

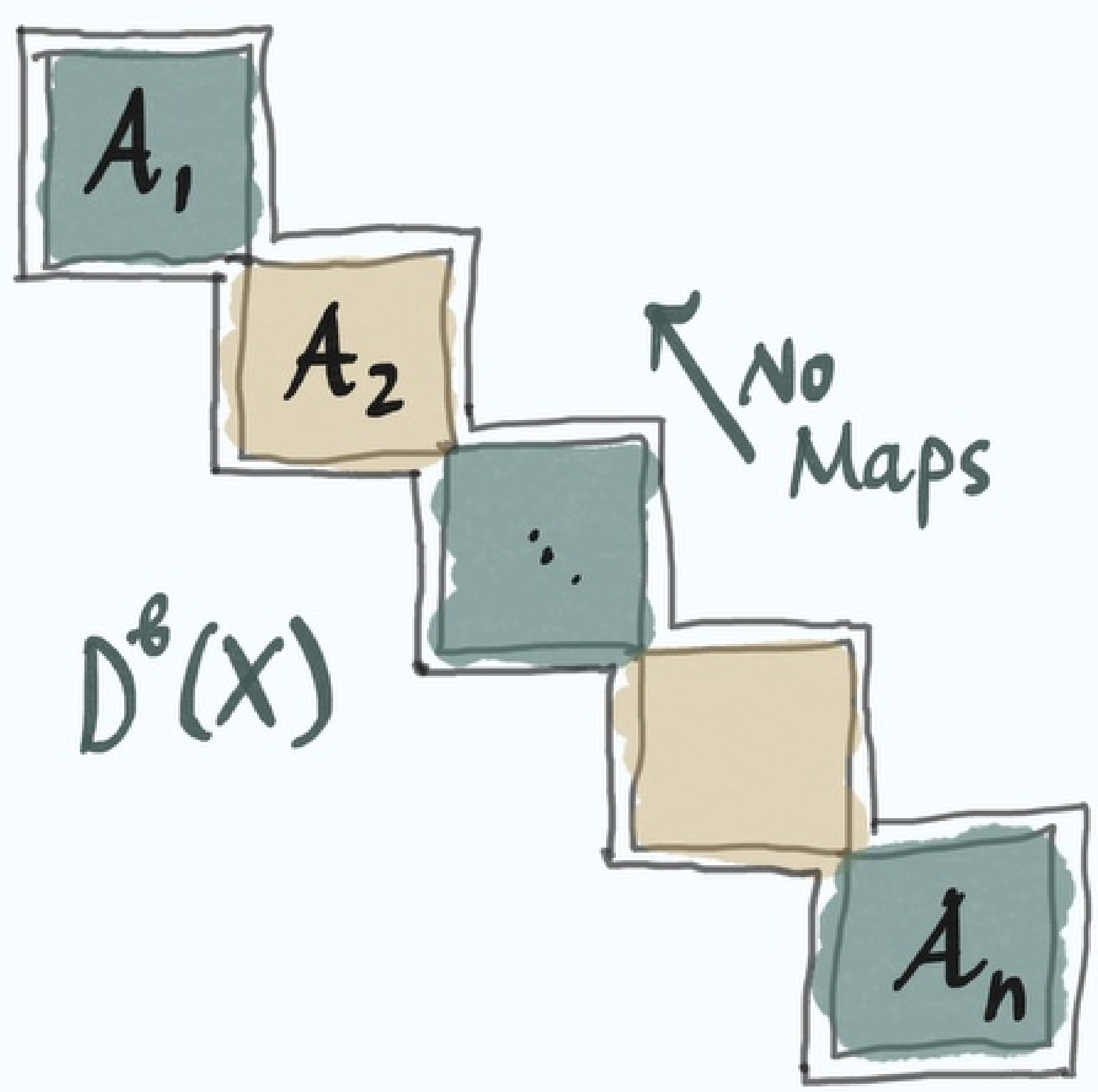
$$\Phi_K(-) = q_*\left(p^*(-) \otimes K\right) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

Splitting "additive" invariants



Splitting "additive" invariants

Example: Grothendieck group $K_0(X)$



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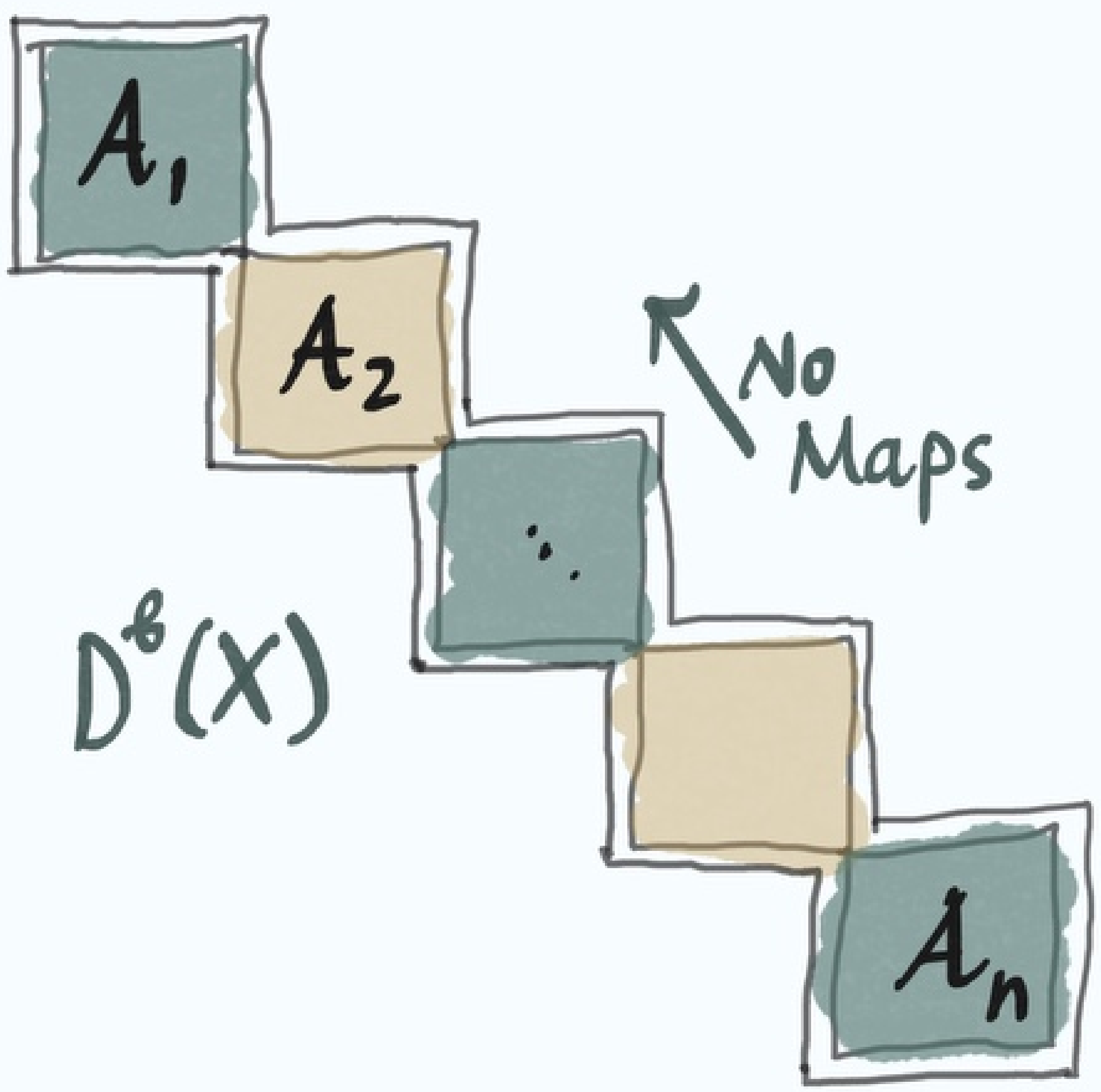

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$\rightsquigarrow$ captures information about extensions of vector bundles

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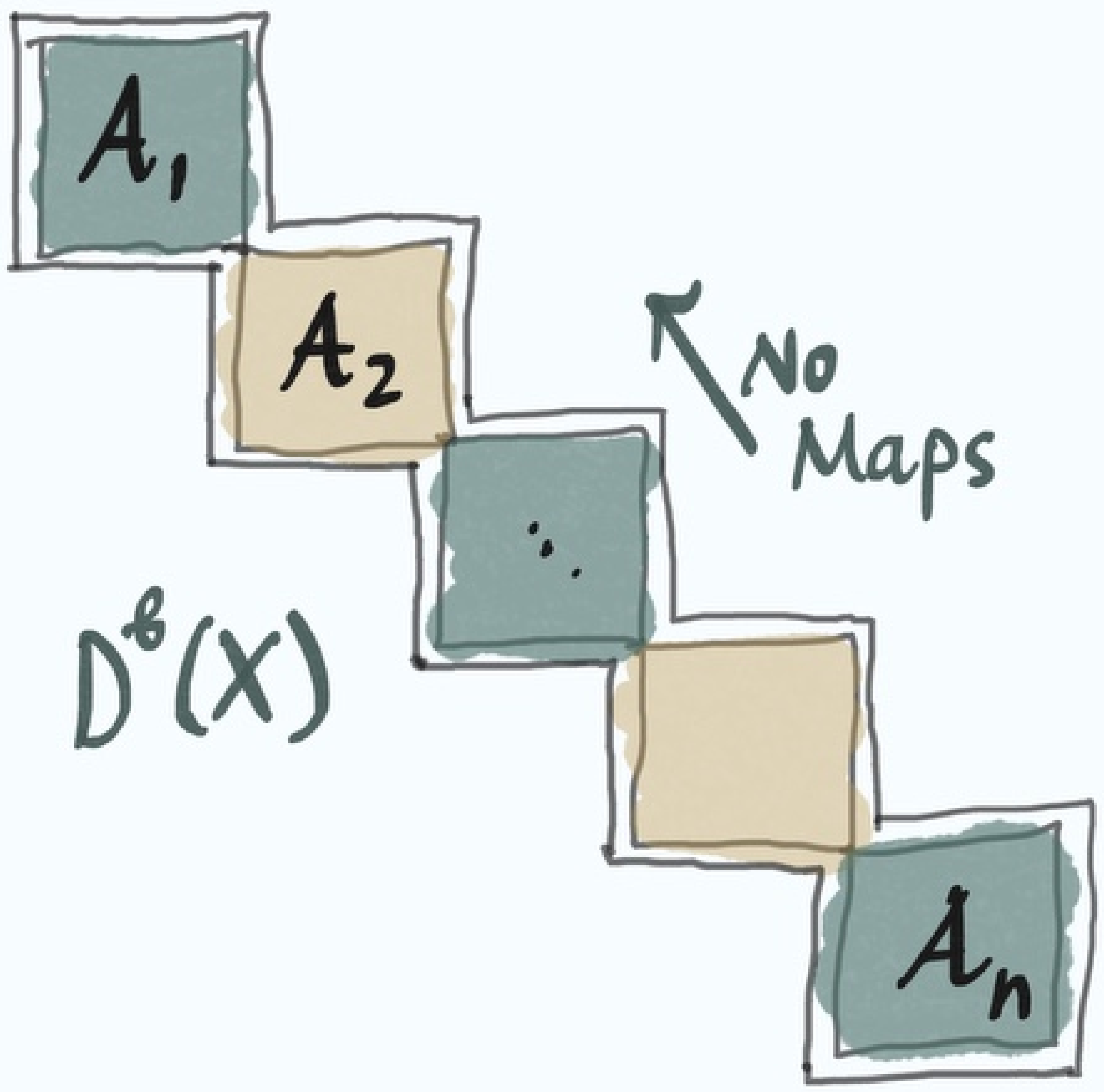

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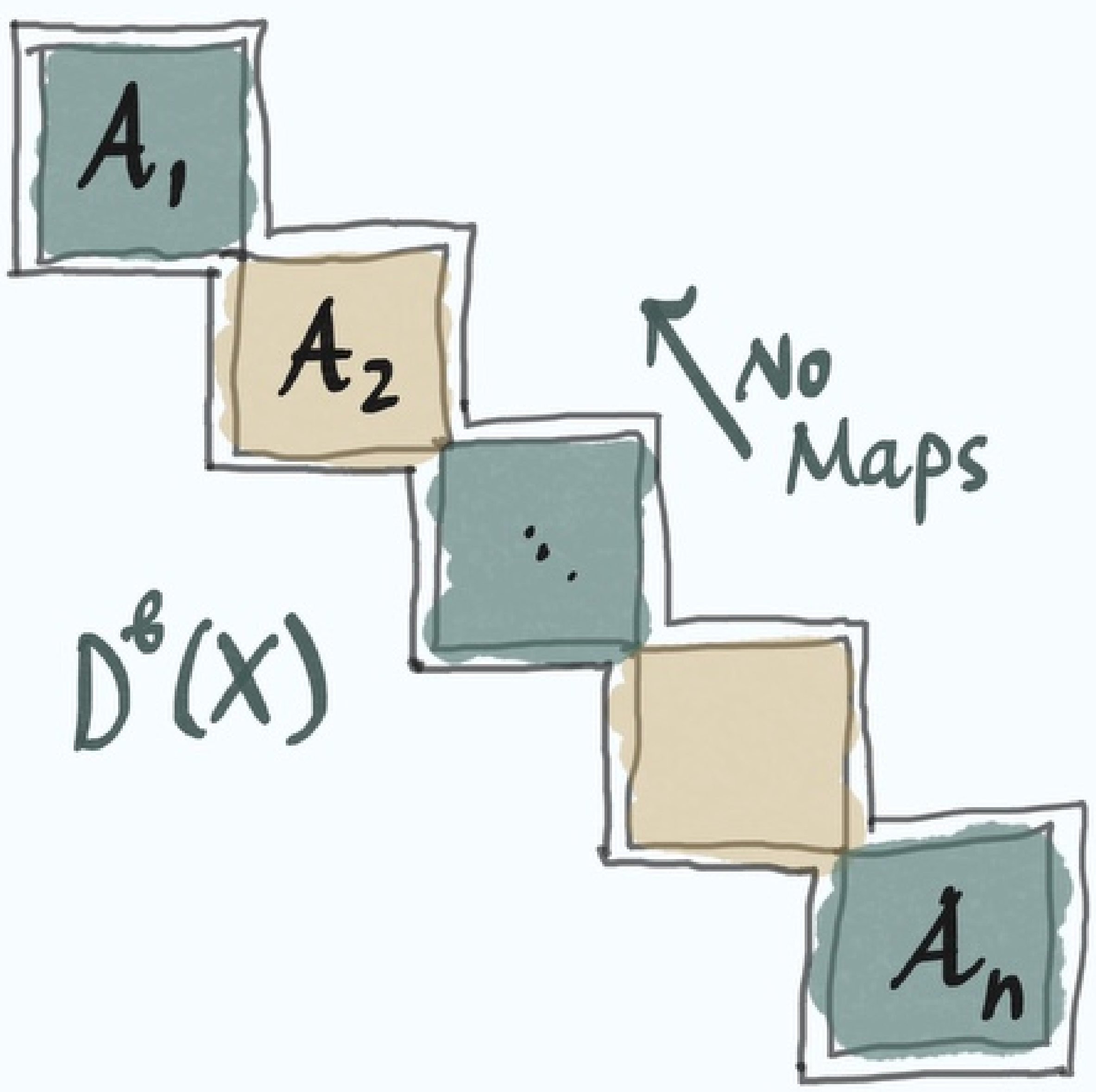

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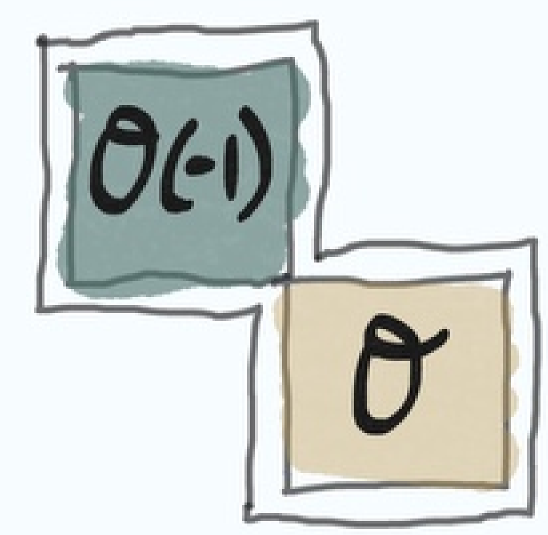

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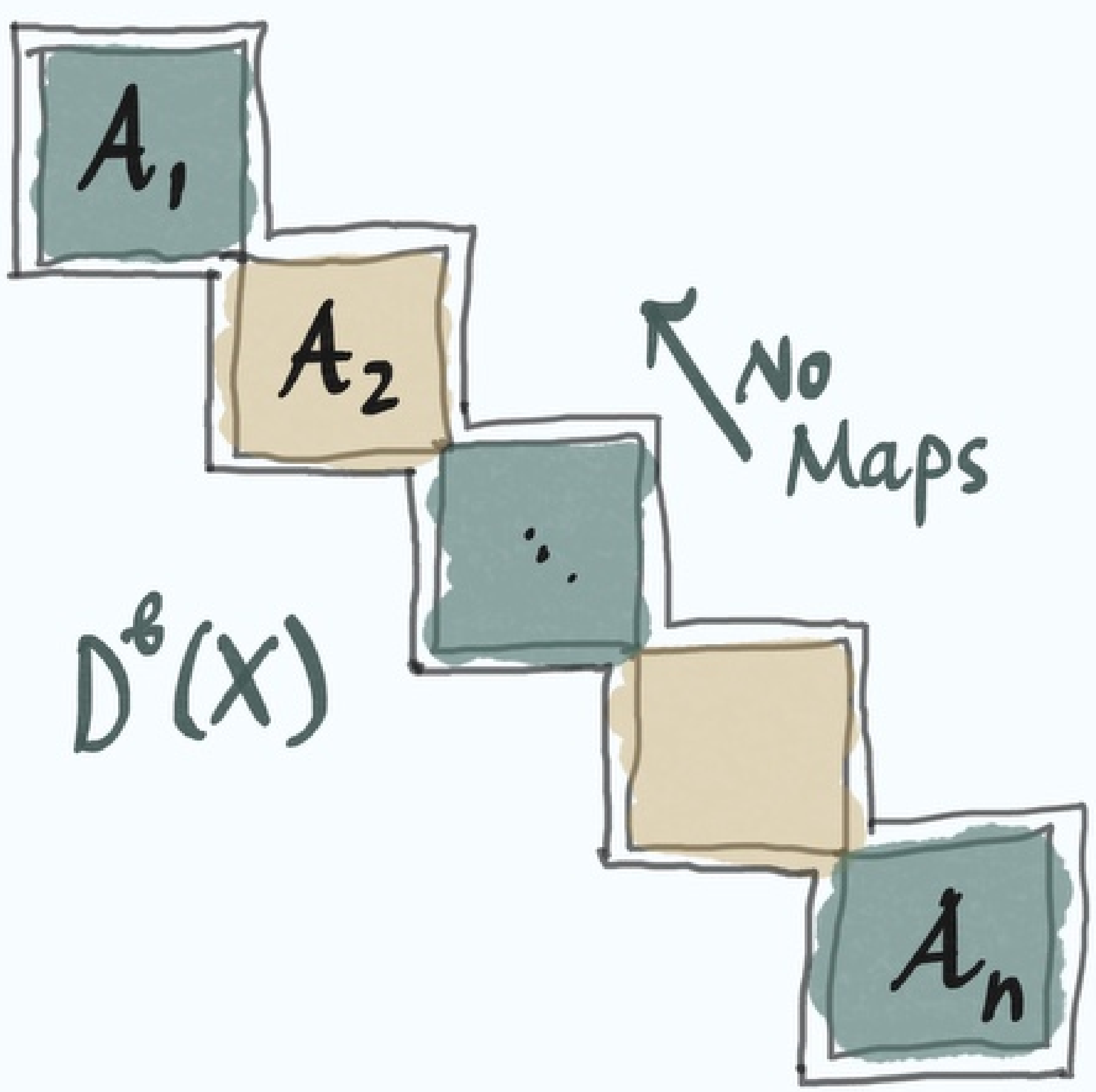

$D^b(X)$

Example: Grothendieck group $K_0(X)$

⇝ captures information about extensions of vector bundles

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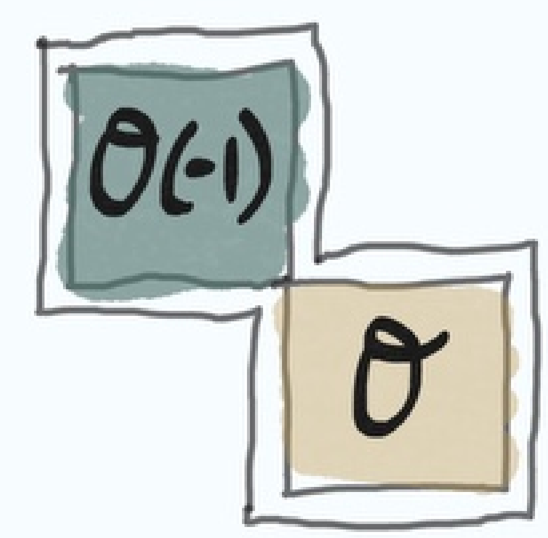

$\Rightarrow$ $[\mathcal{O}(-1)]$, $[\mathcal{O}]$ basis for $K_0(\mathbb{CP}^1)$

Categorical perspectives on non-commutative geometry

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Classical picture

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Classical picture

$$
\begin{array}{cl}
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Y & \text{singular}
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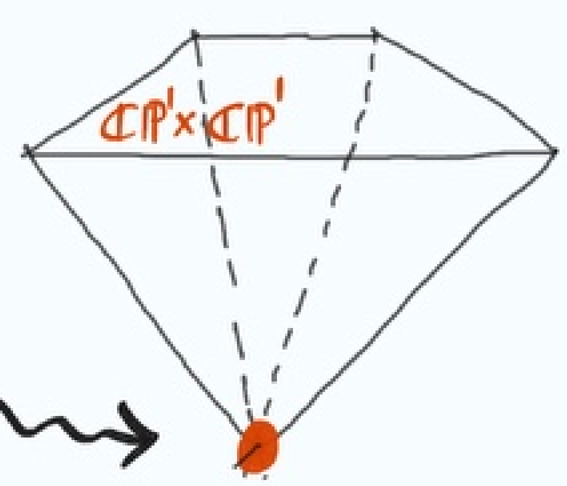
Categorical perspectives on non-commutative geometry

Classical picture

$$\begin{array}{ll} X & \text{smooth replacement} \\ \downarrow & \\ Y & \text{singular} \end{array}$$



singular point

$$\{xw = yz\} = Y \subseteq \mathbb{C}^4_{x,y,z,w}$$

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Classical picture

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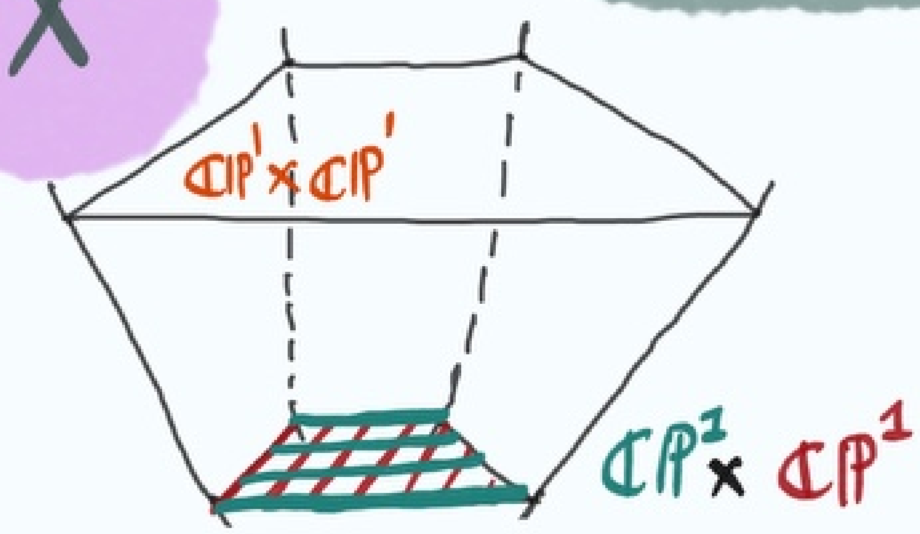

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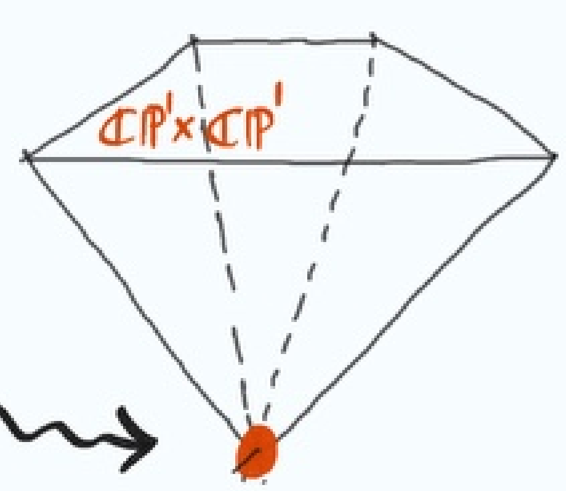

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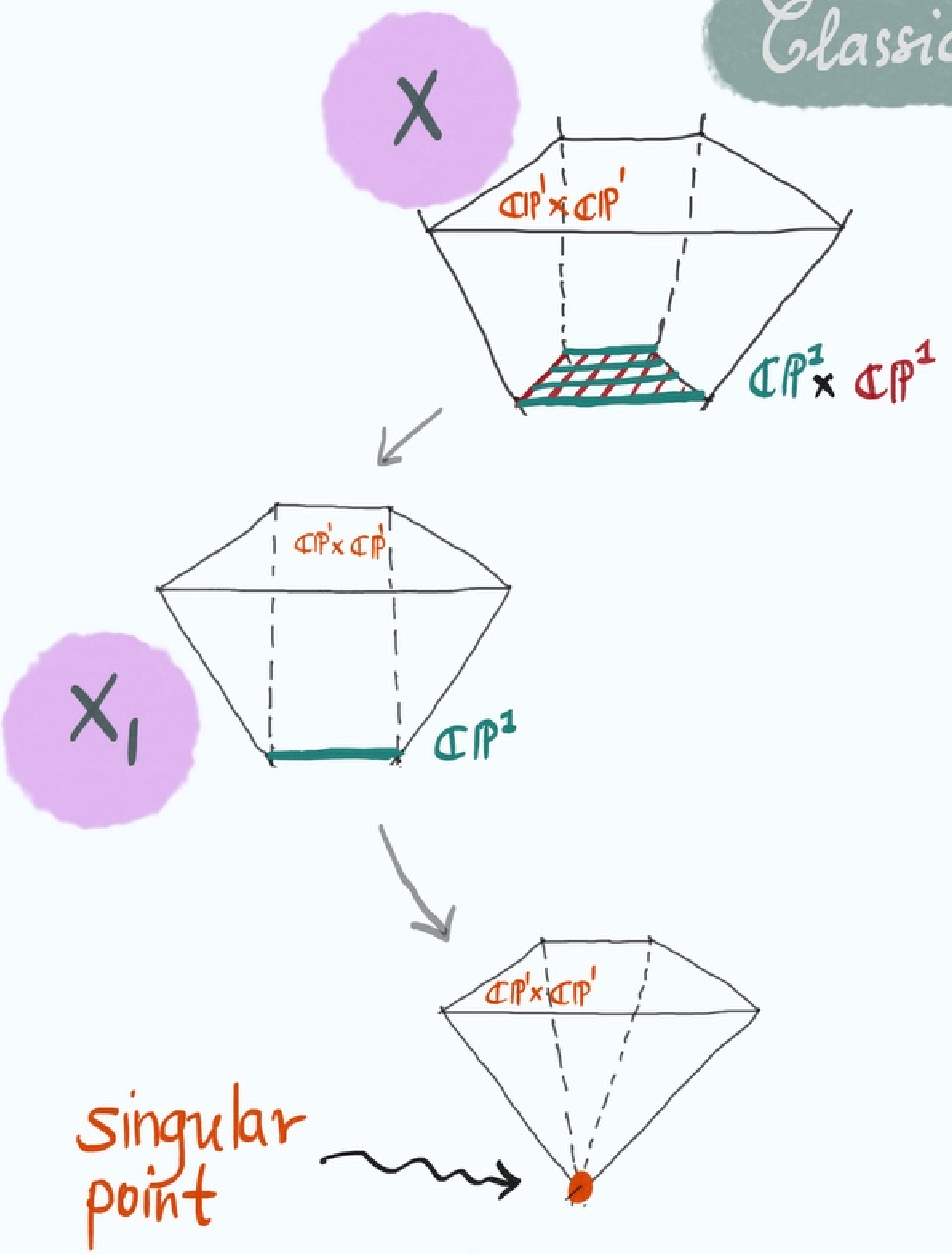

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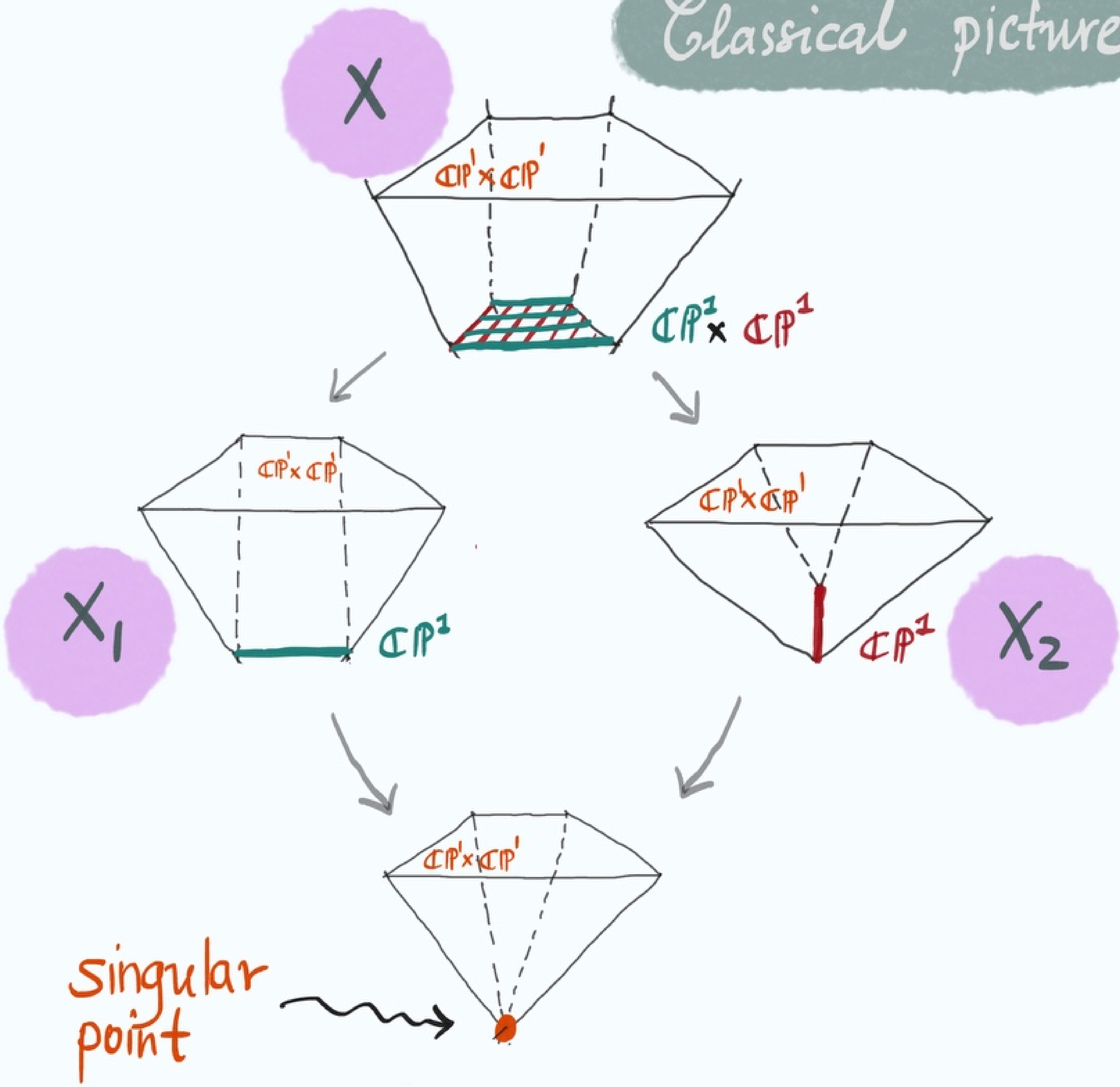

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"smallest" replacements

<u>i.e.</u> does not change det (cotangent bundle)

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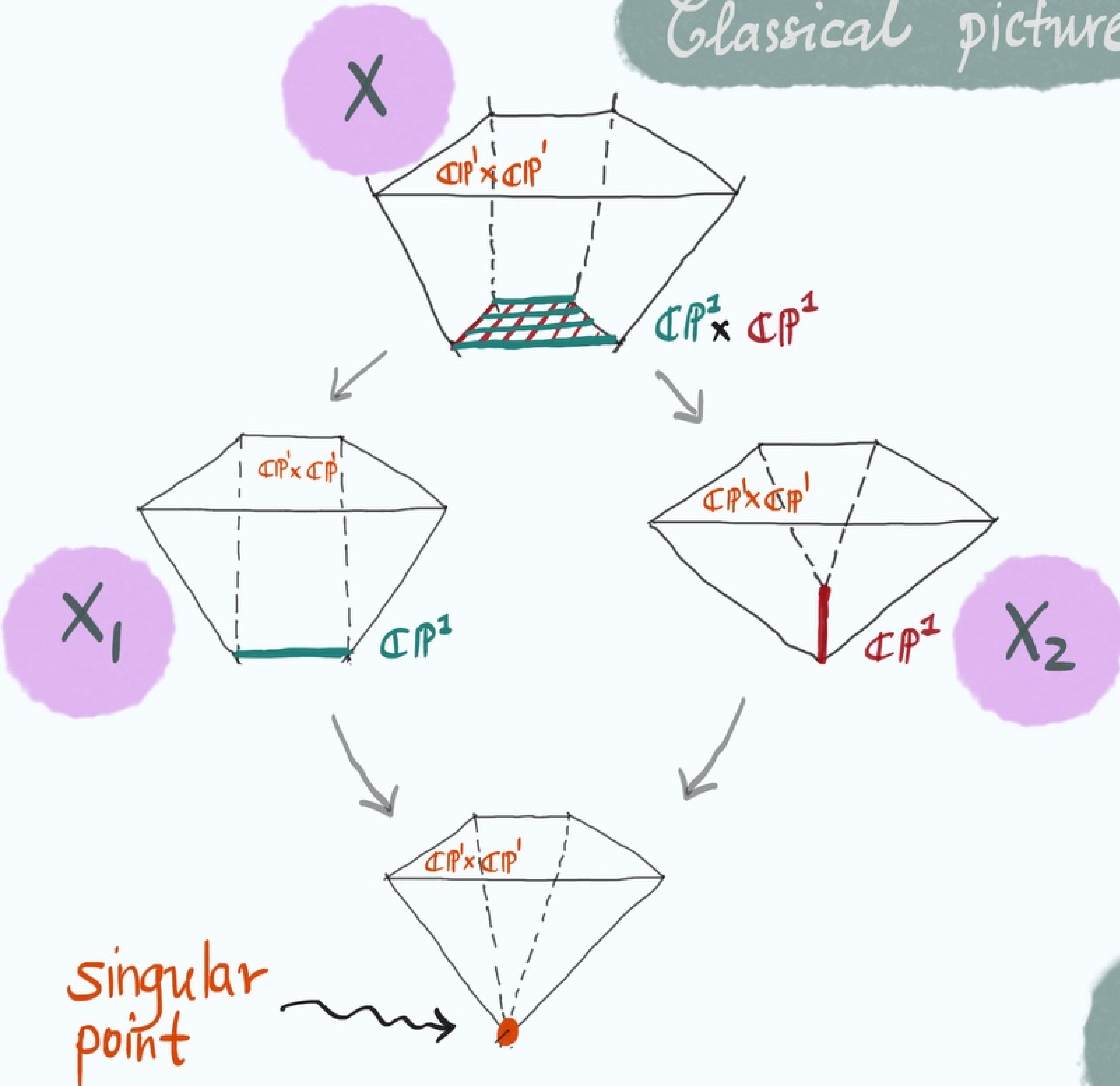

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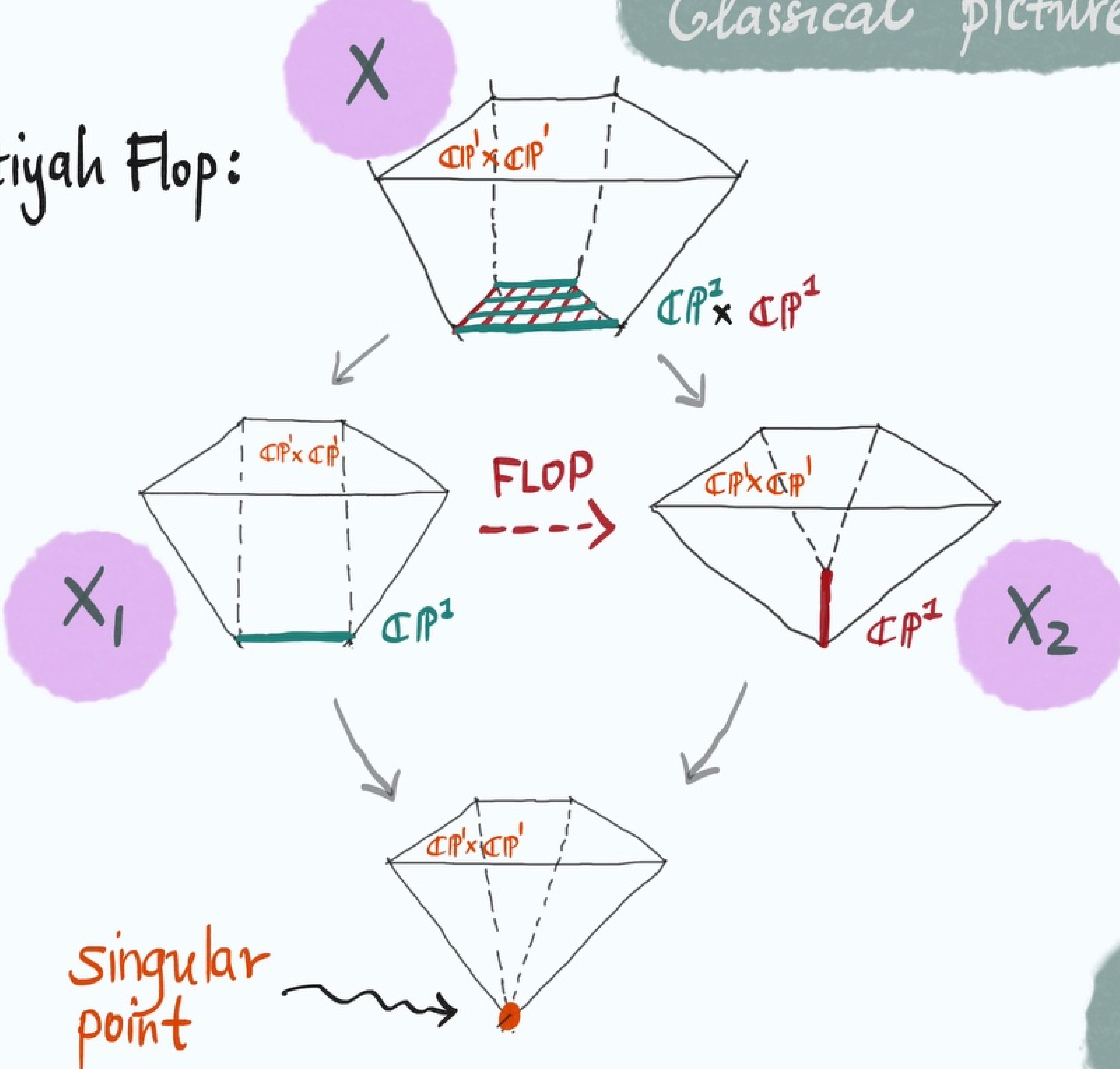
Atiyah Flop:



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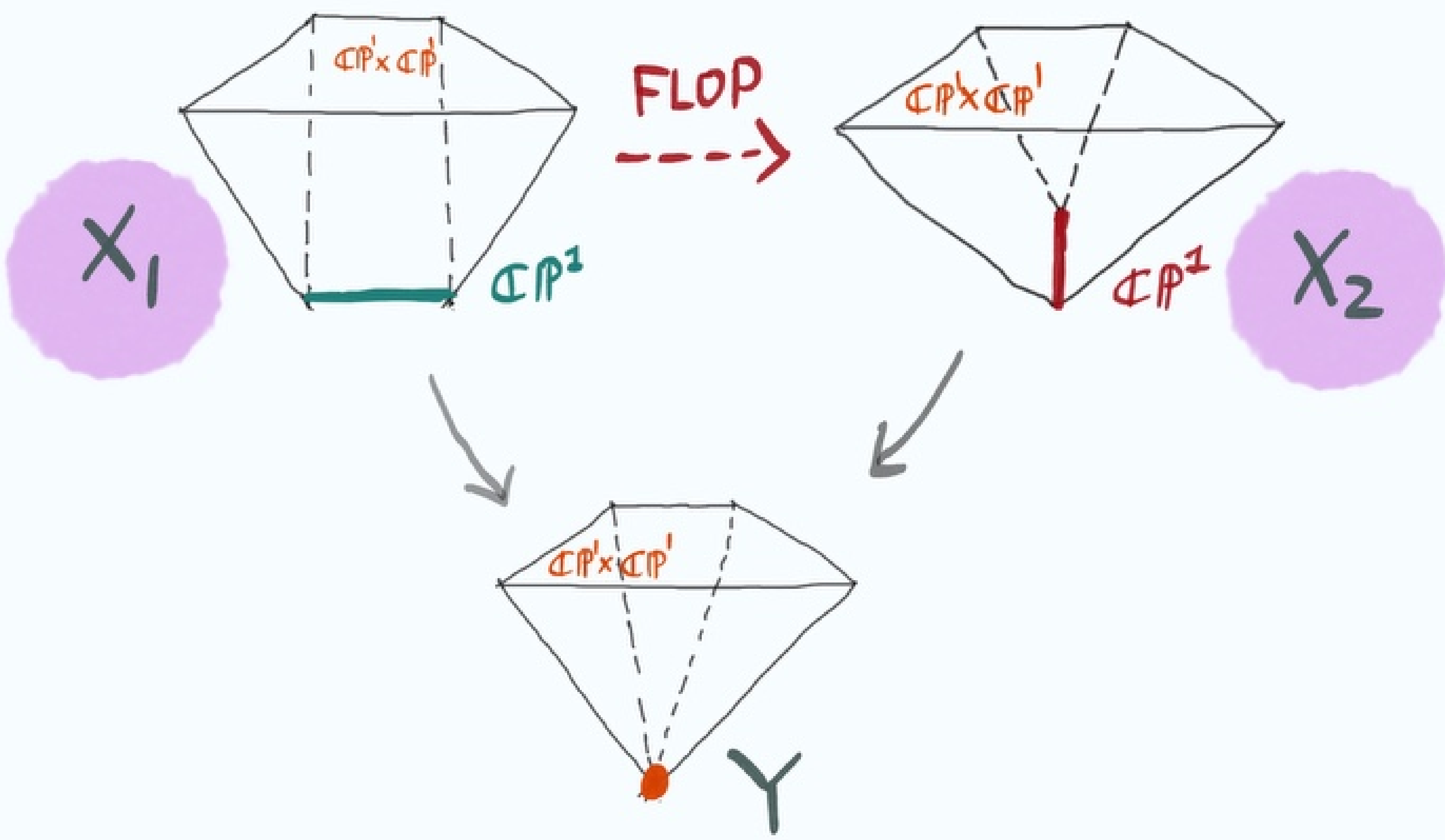

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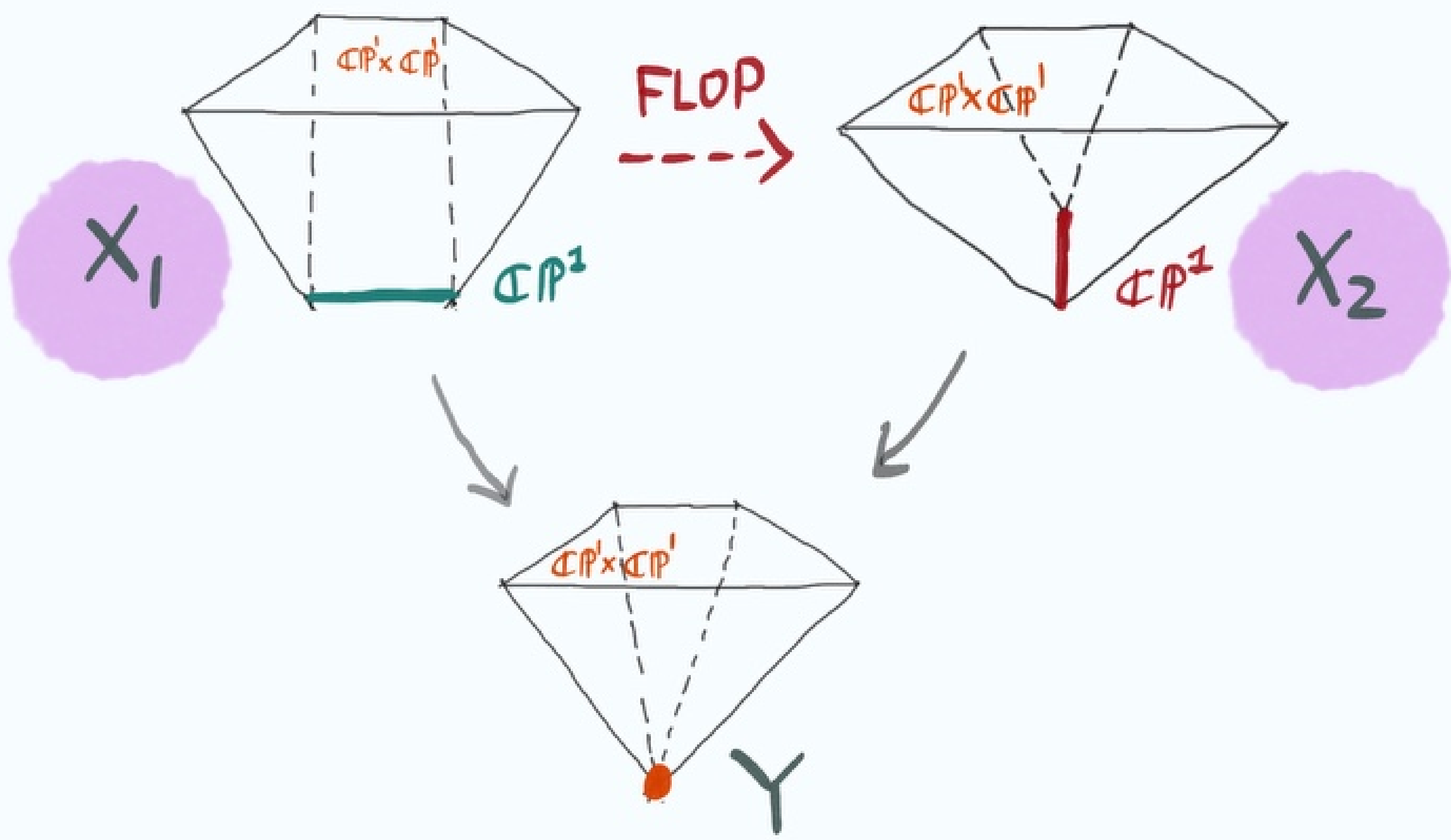

Bondal-Orlov '95 :

① $\mathcal{D}^b(X_1) \simeq \mathcal{D}^b(X_2)$

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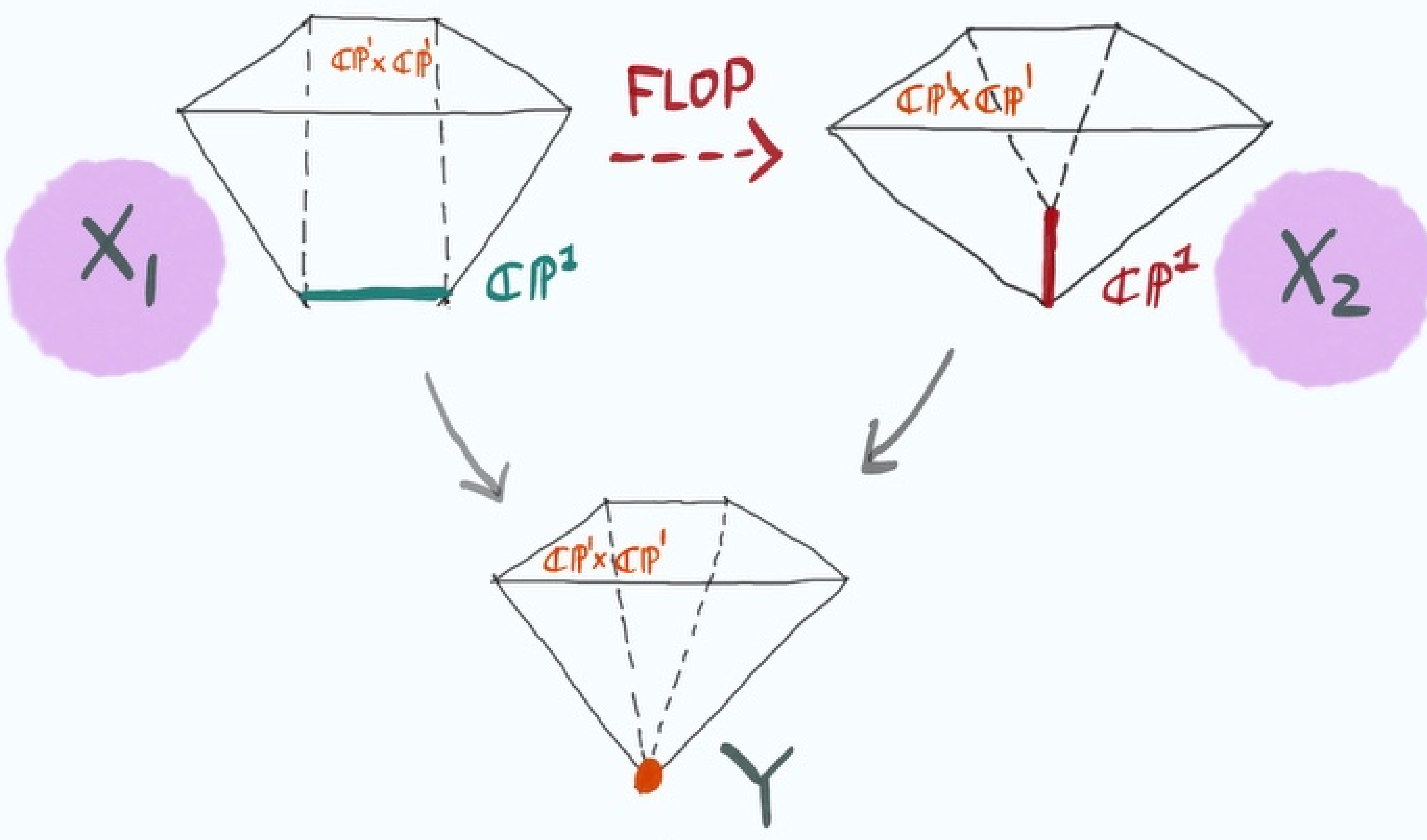

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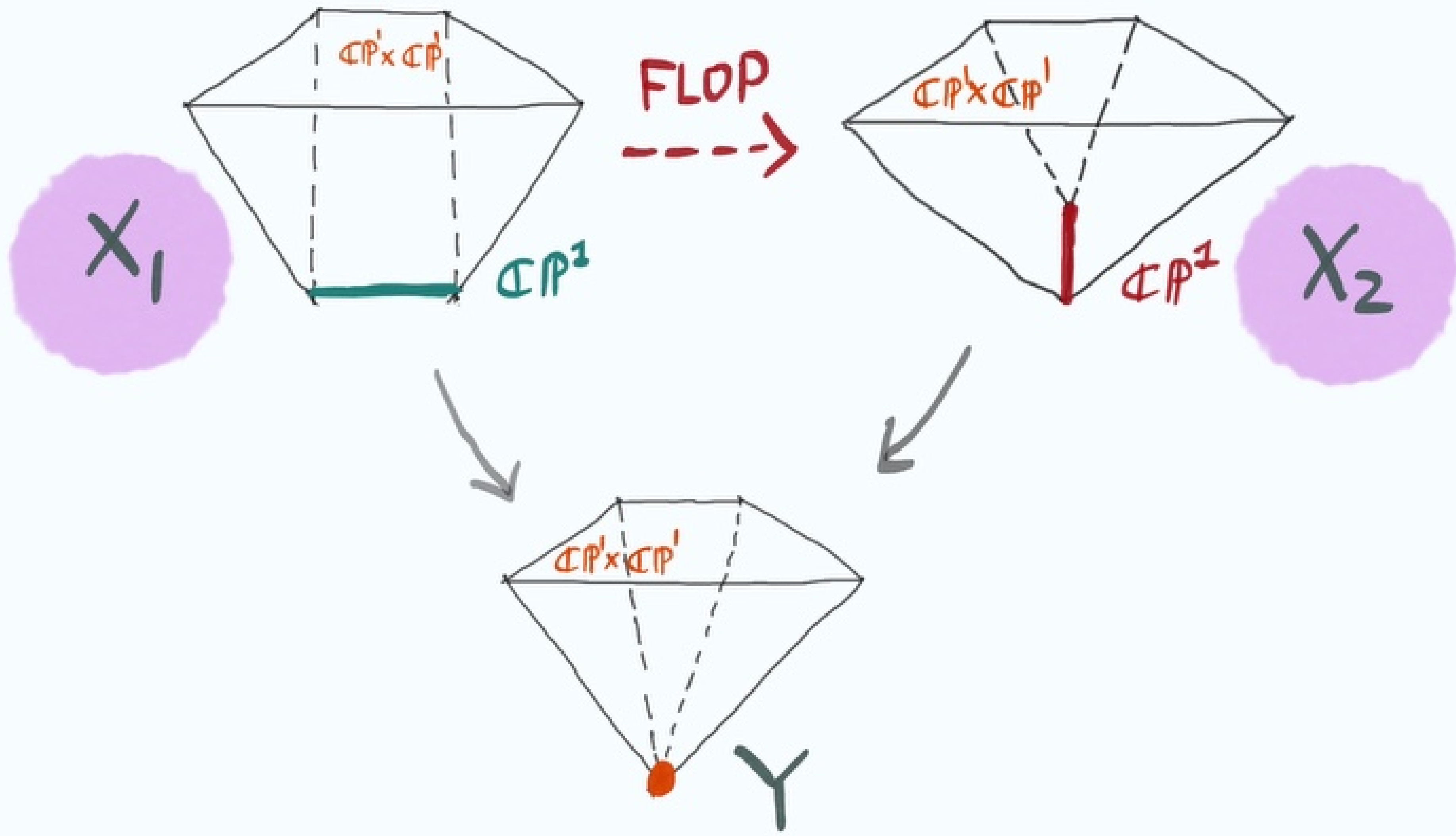

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[Refined by: Kuznetsov, Luntz]

Replace $\mathcal{D}^b(X)$ by $\mathcal{C}$ a "smooth" category s.t.

$$\mathcal{C} = \langle A, \underset{\substack{\cap \\ \mathcal{D}^b(X)}}{\mathrm{Perf}(X)} \rangle$$

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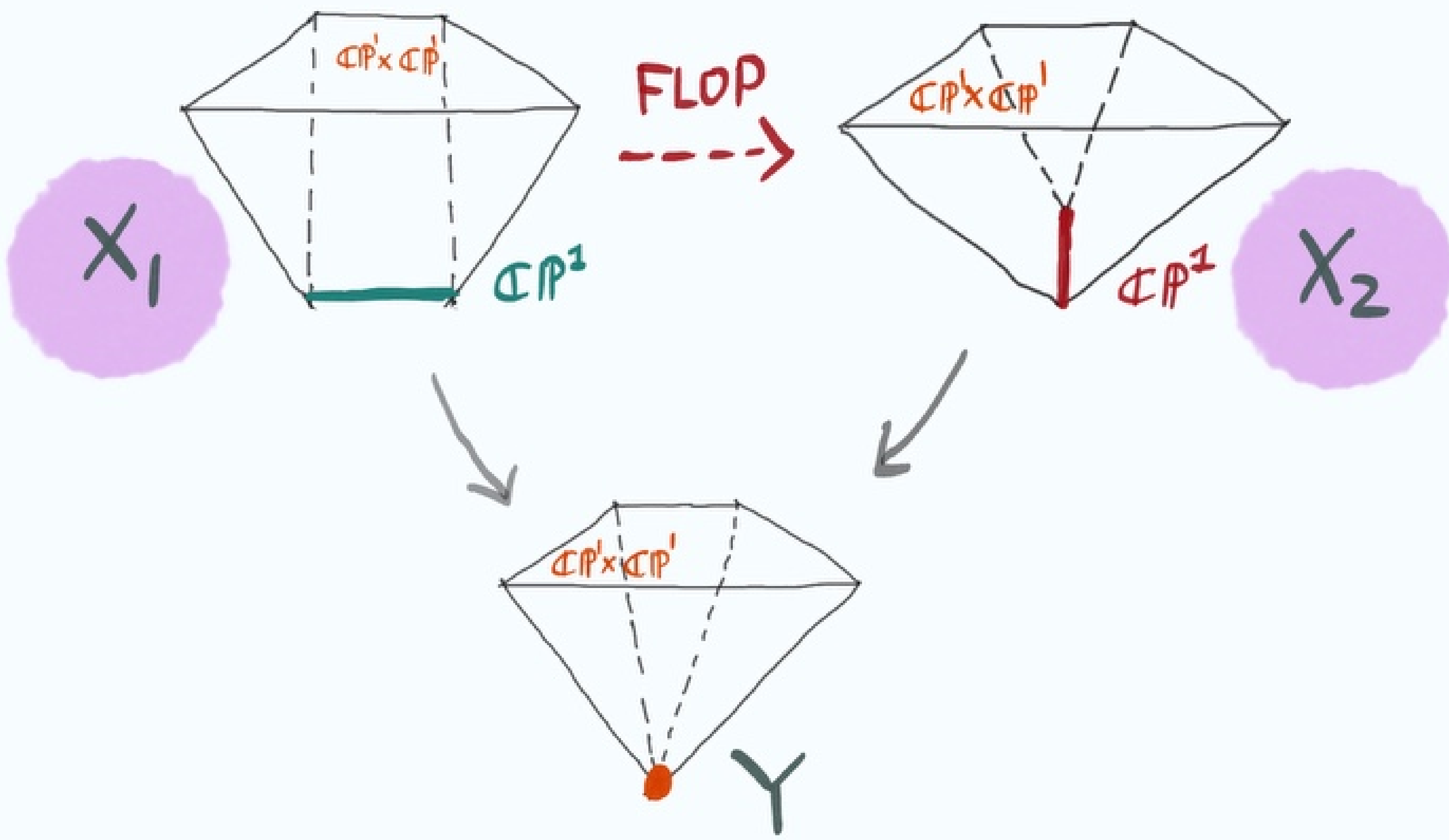

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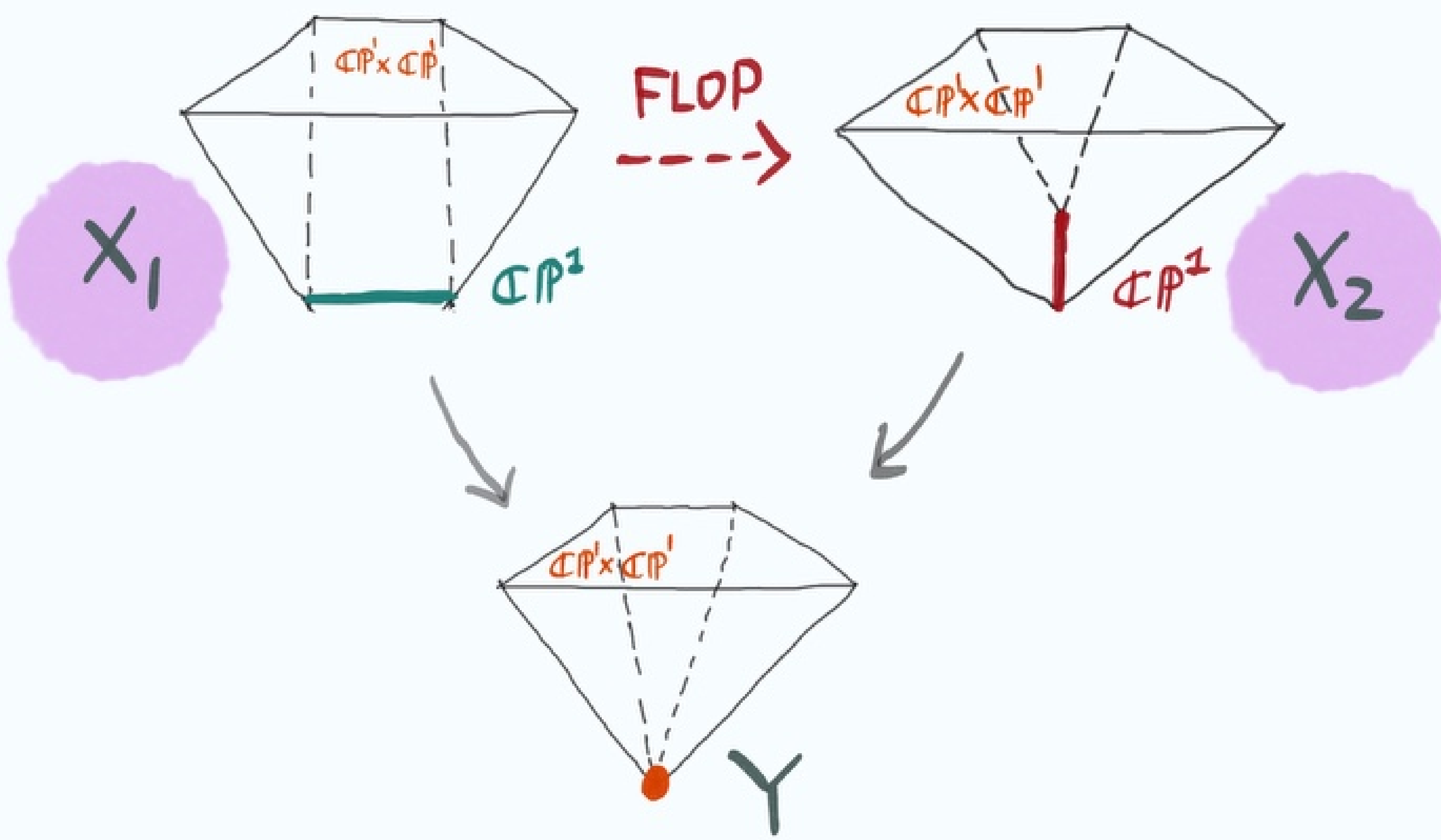

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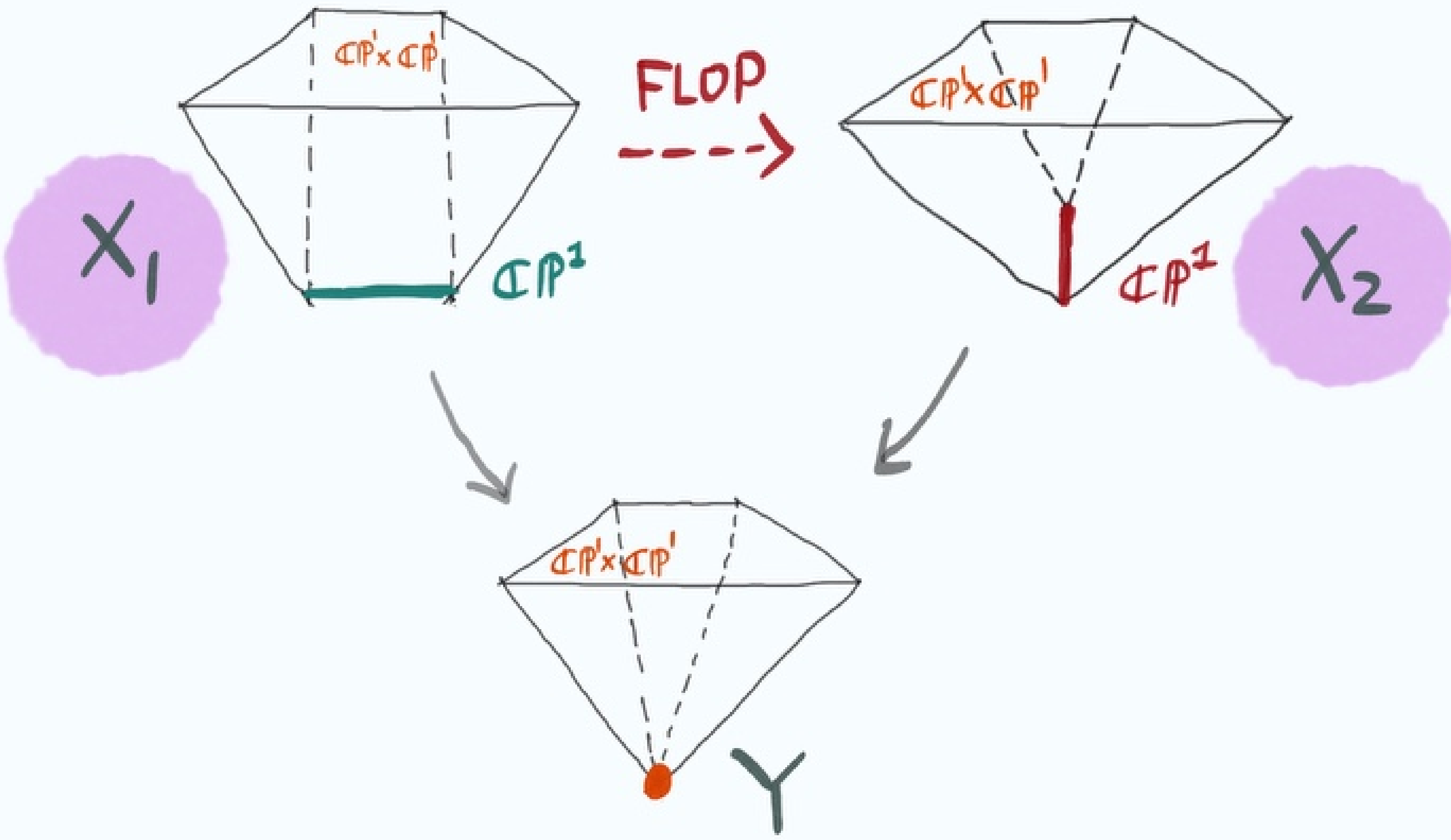

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$X_1 \dashrightarrow X_2$ (flop), X_1, X_2 Calabi-Yau

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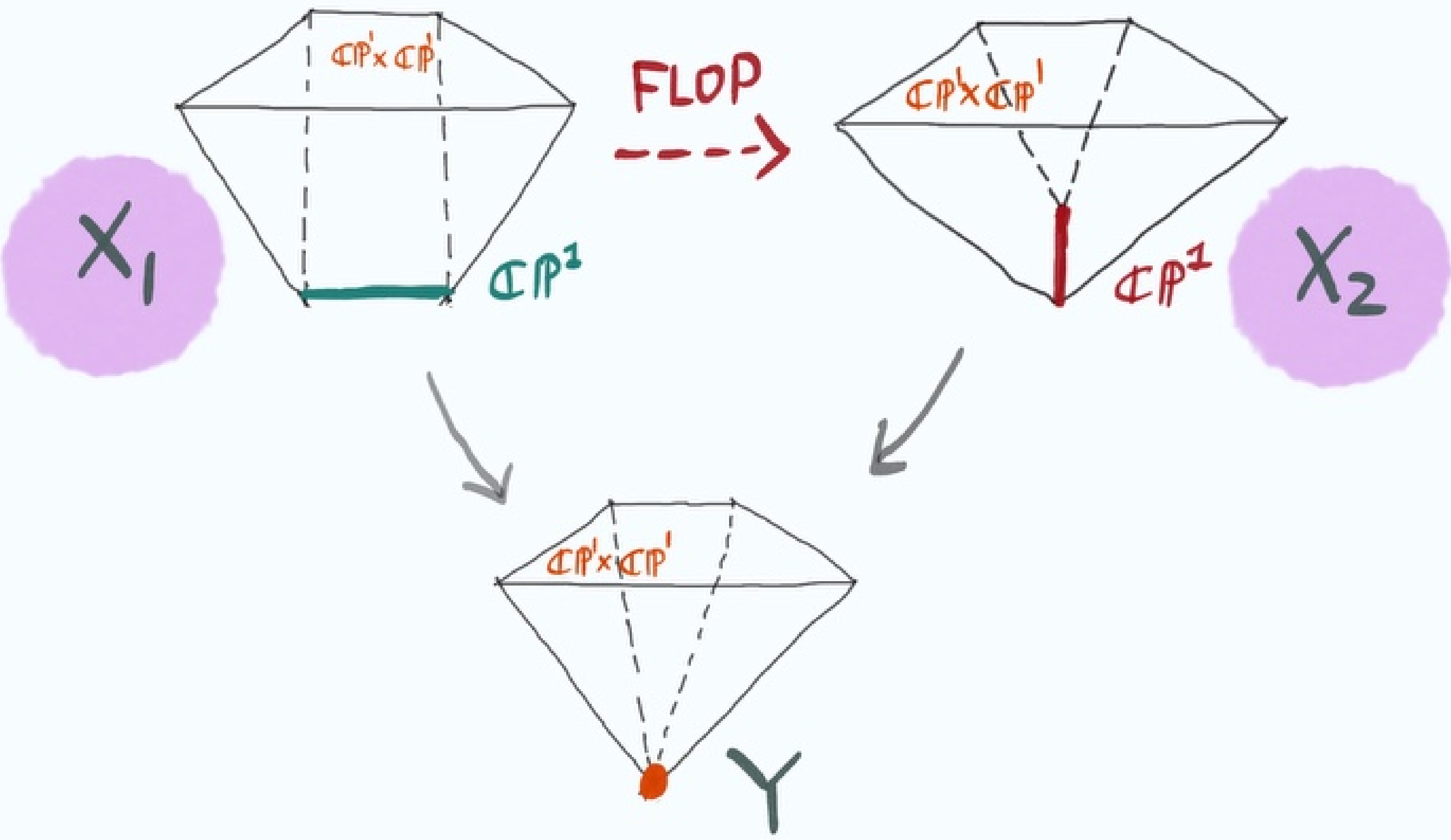

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Bridgeland: True in dim 3

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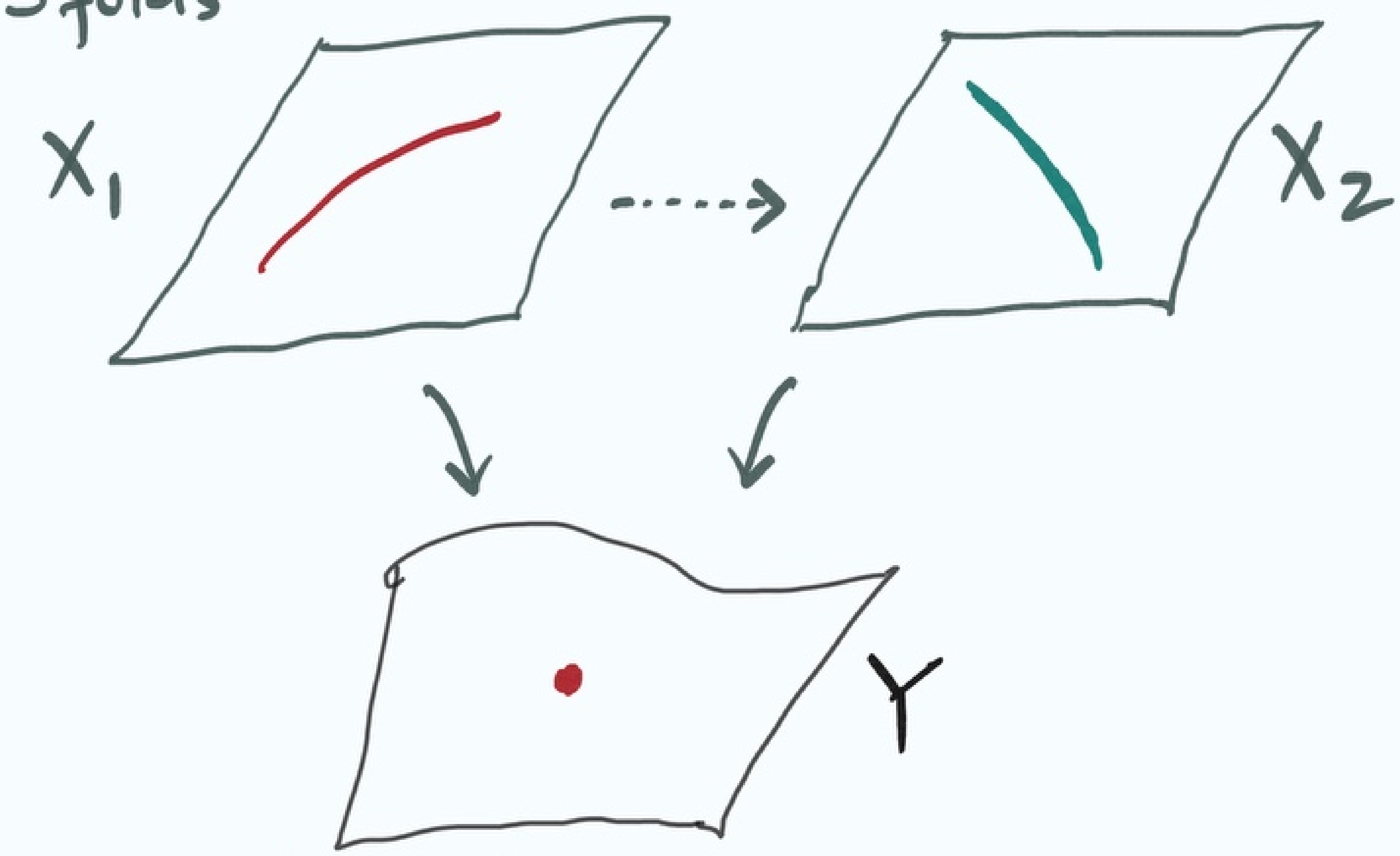

"mild" singularities

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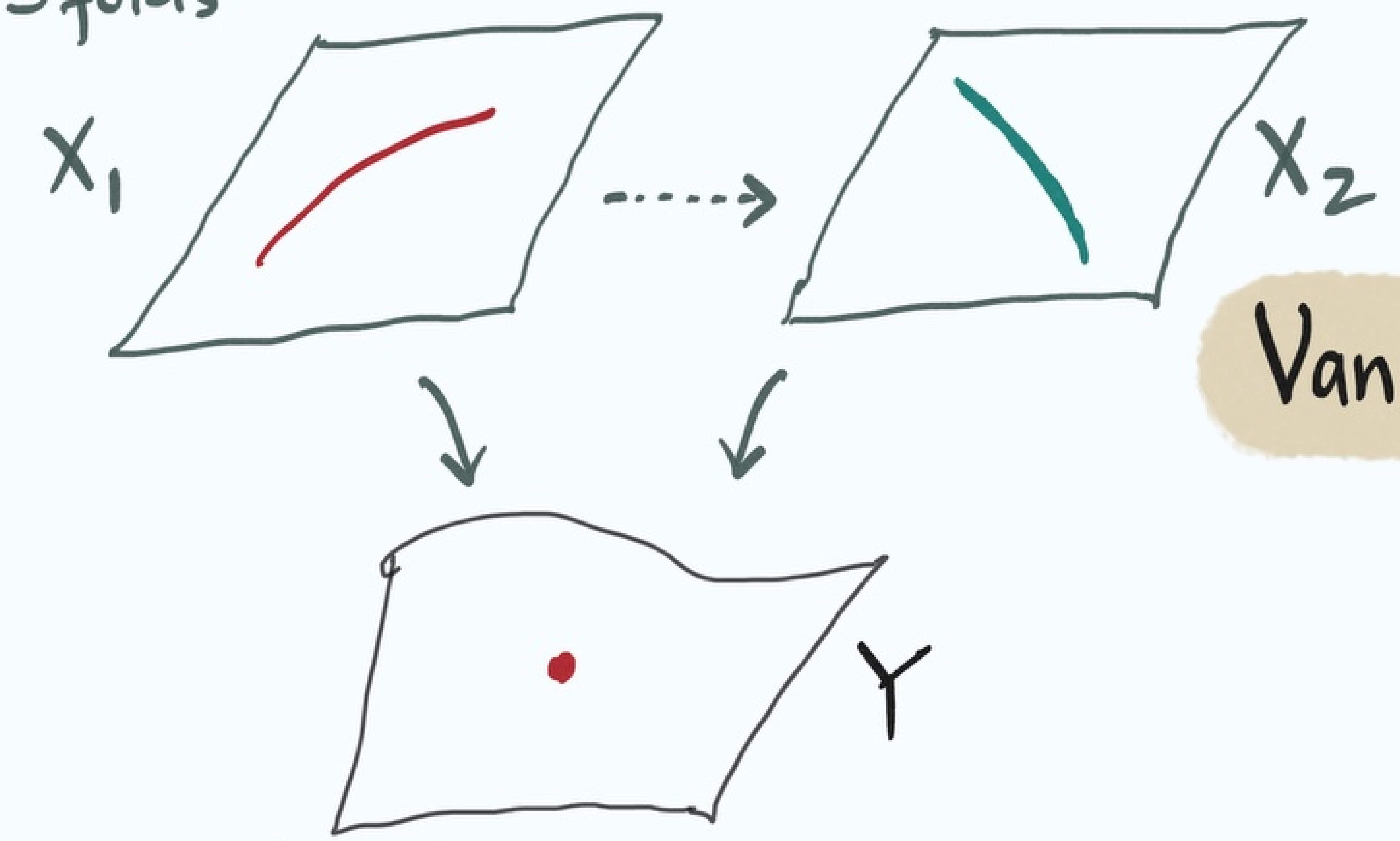

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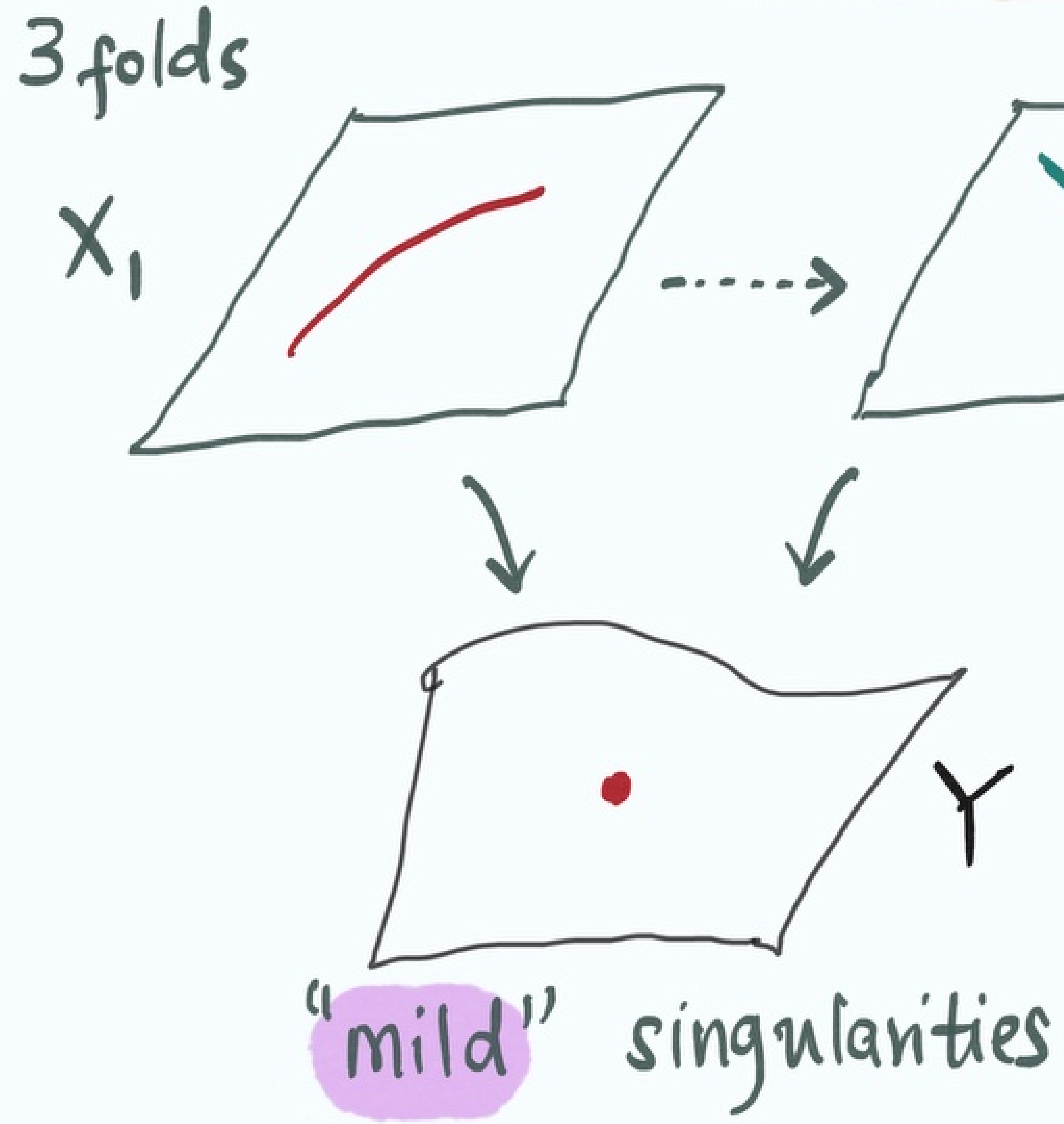

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$\mathcal{D}^b(X_1)$ $\qquad$ $\mathcal{D}^b(X_2)$

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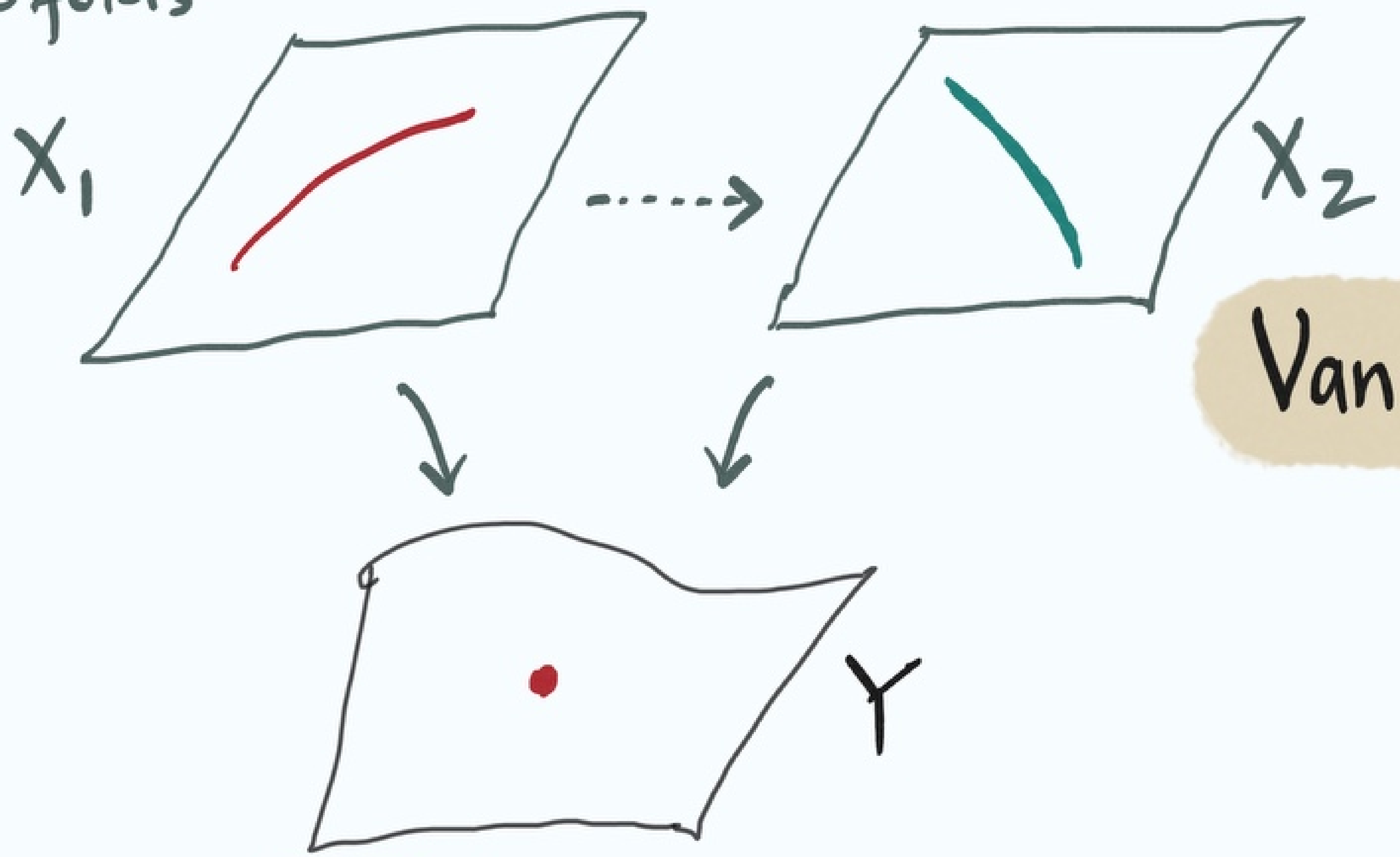

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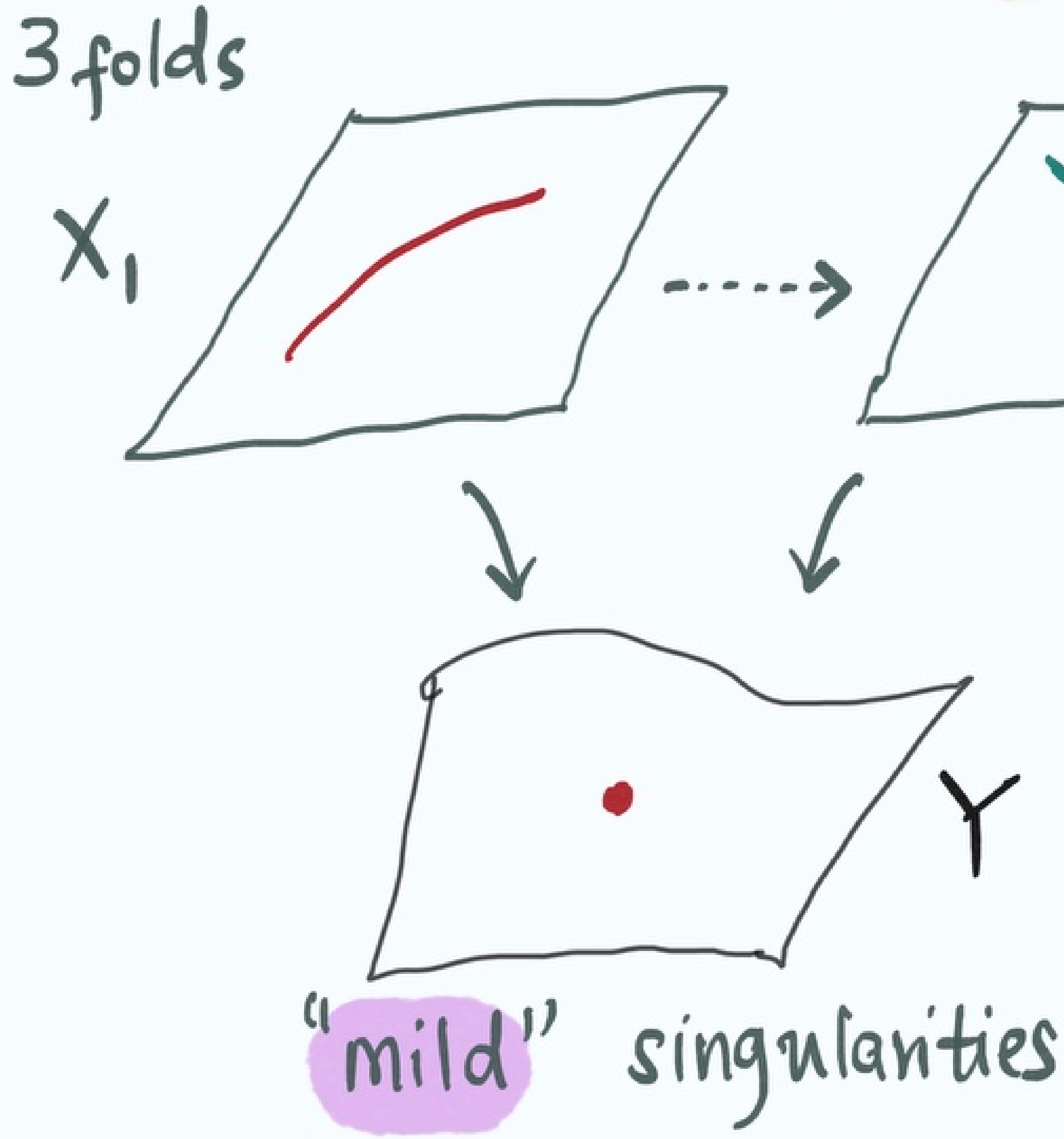

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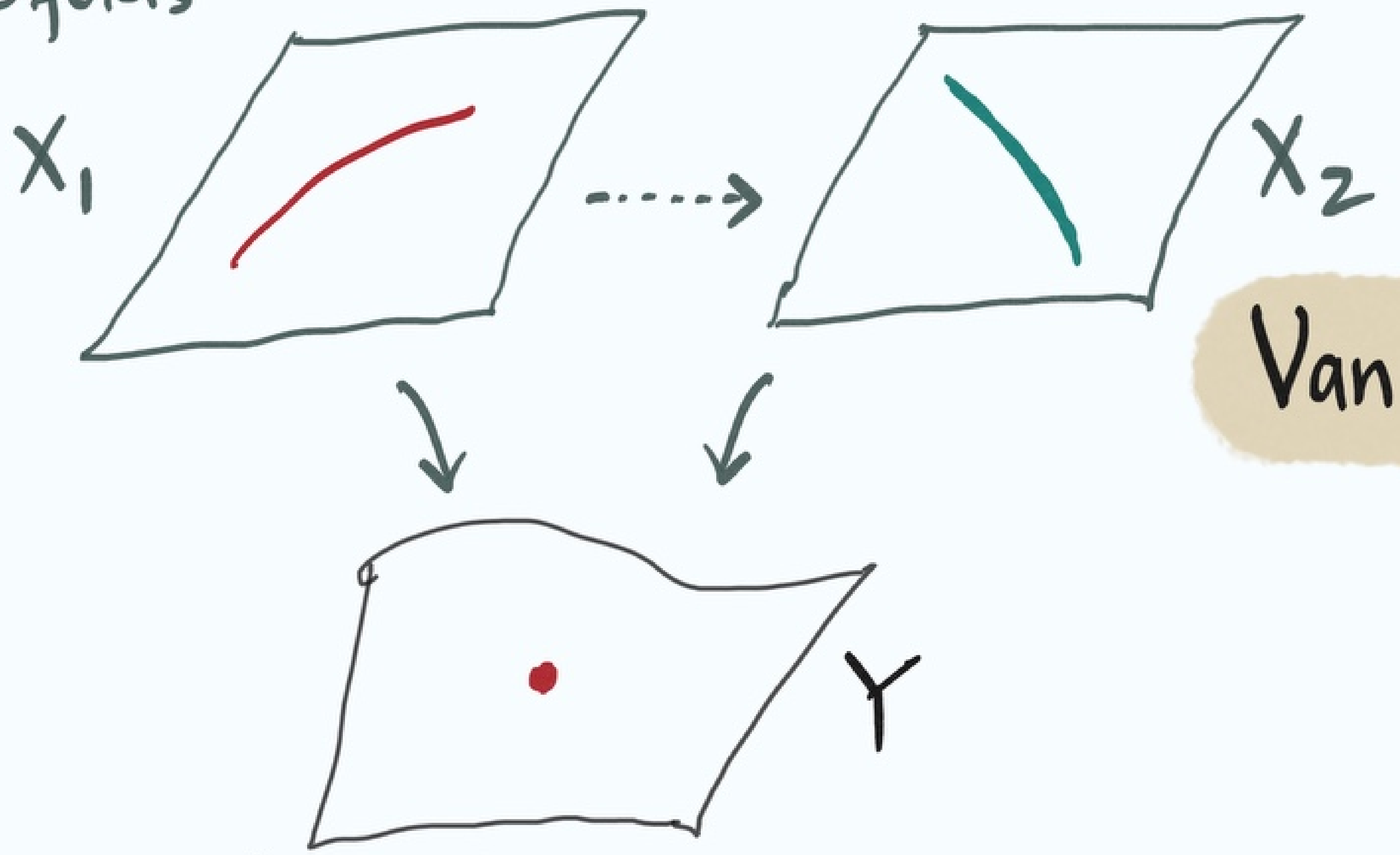

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- Existence?
- Connections to (classical) crepant resolutions, categorical resolutions
- Are NCCRs derived equivalent?

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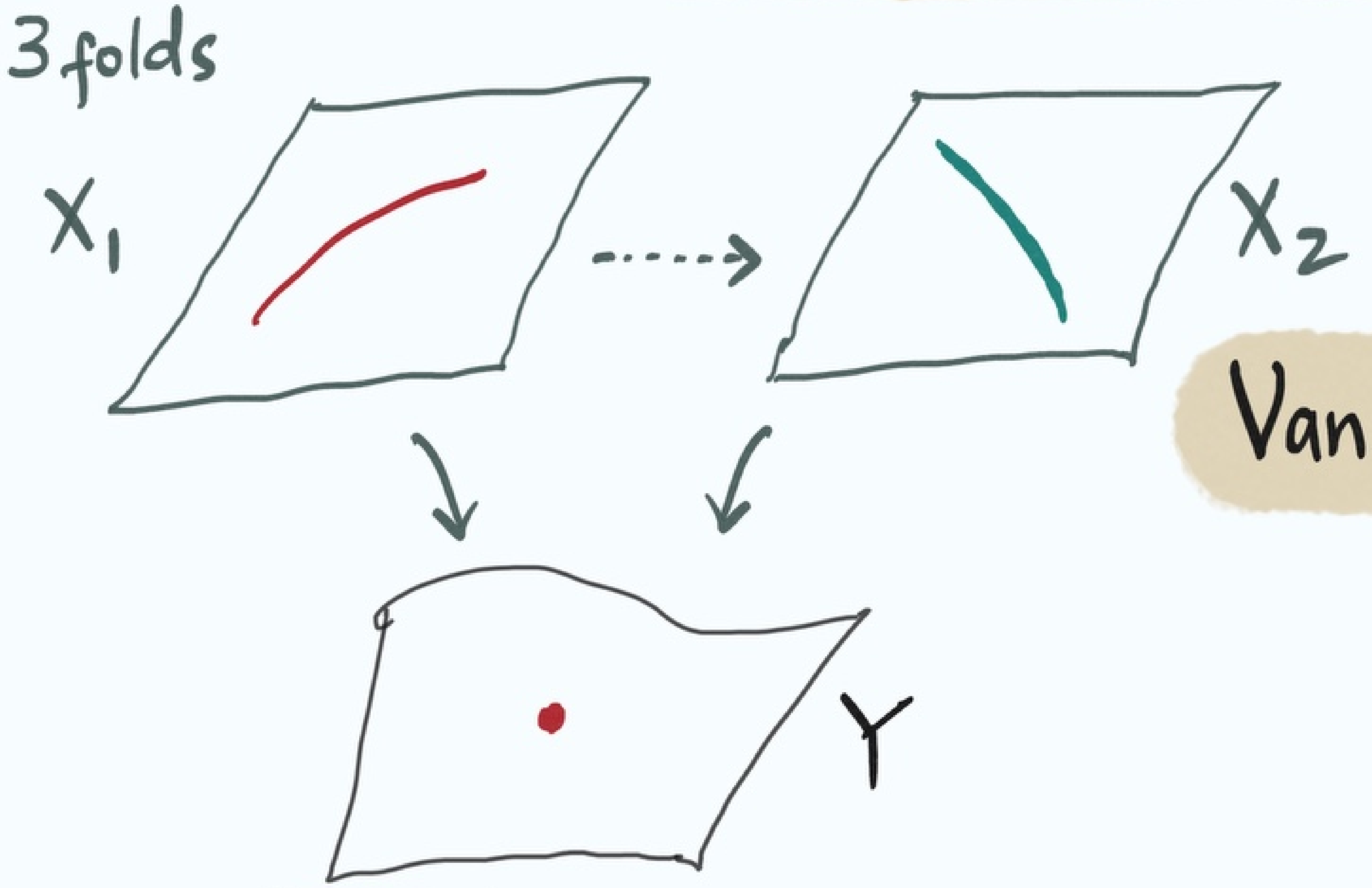

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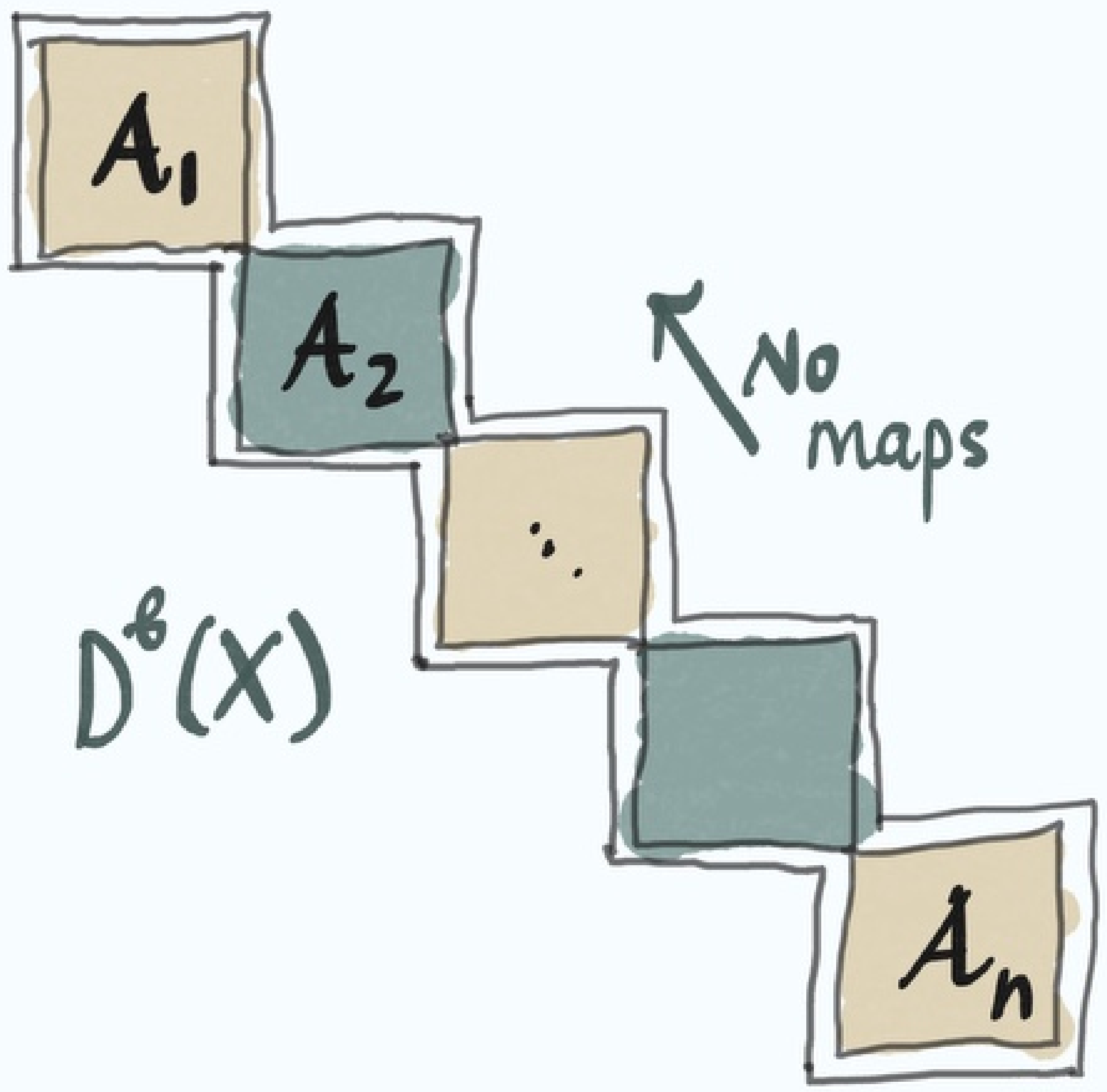
<u>Example:</u> Špenko - Van den Bergh

G reductive group $\curvearrowright W$

$W /\!/ G = (\operatorname{Sym} W)^G$

has an NCCR

Full exceptional collections



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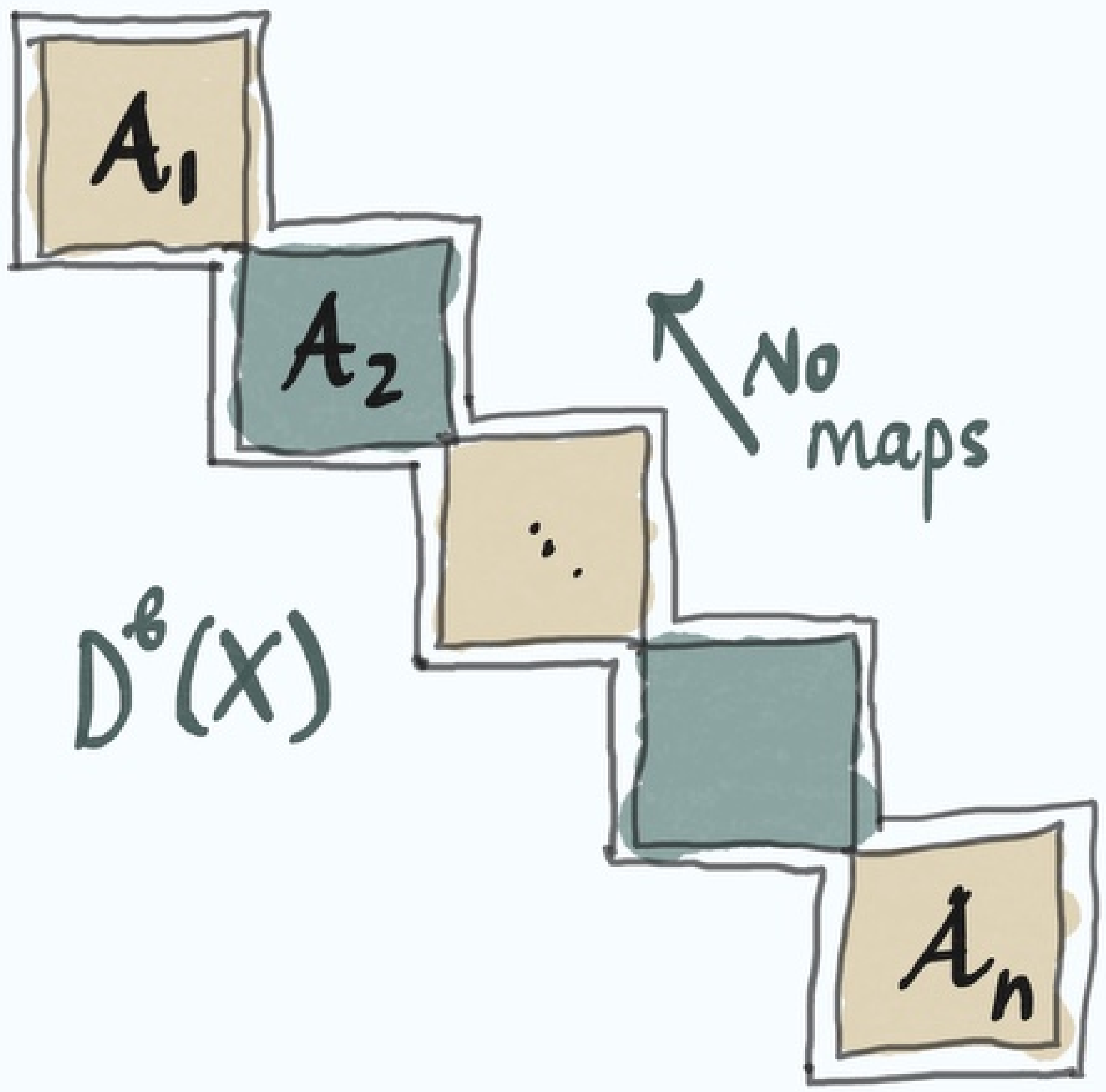
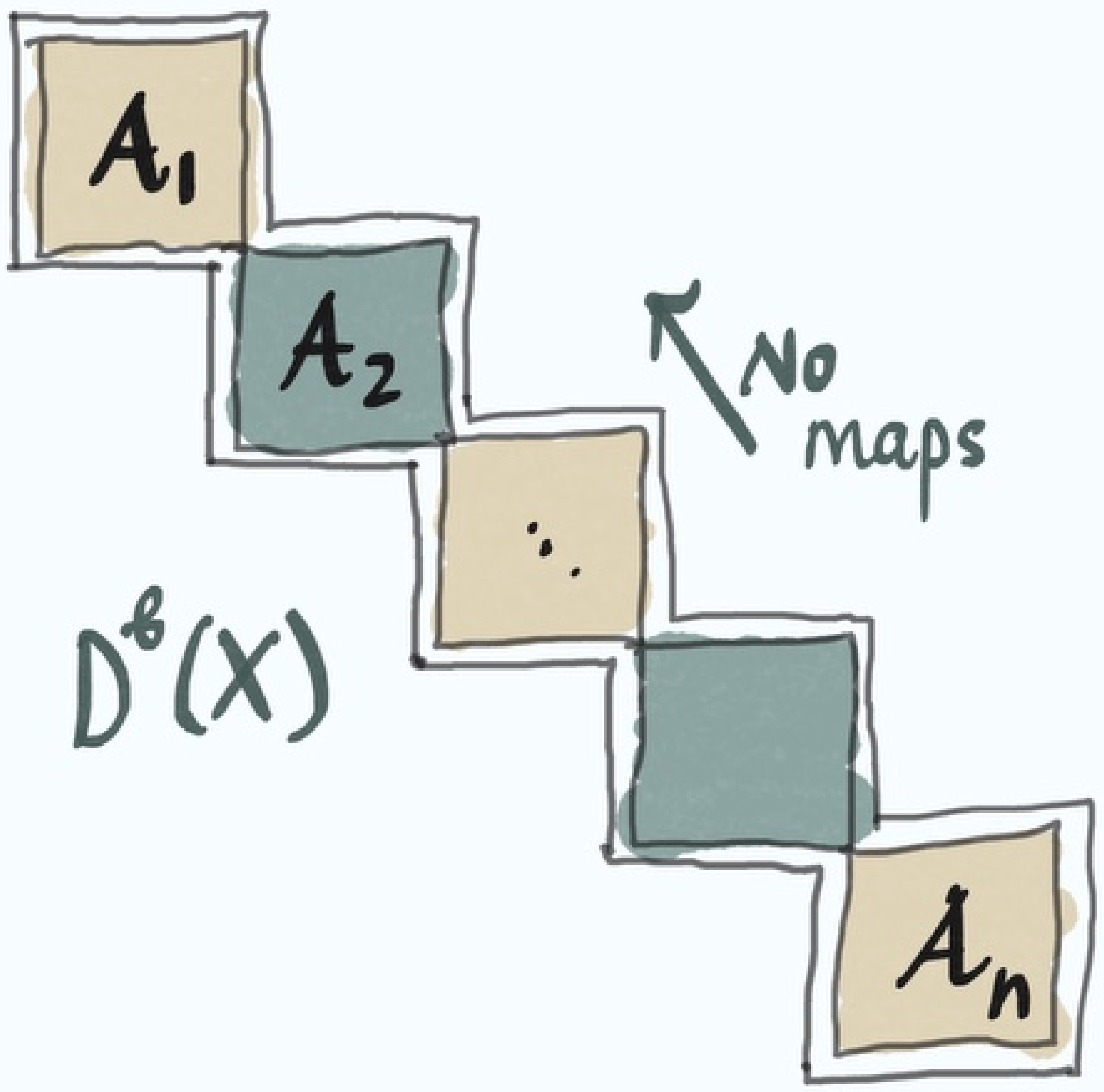
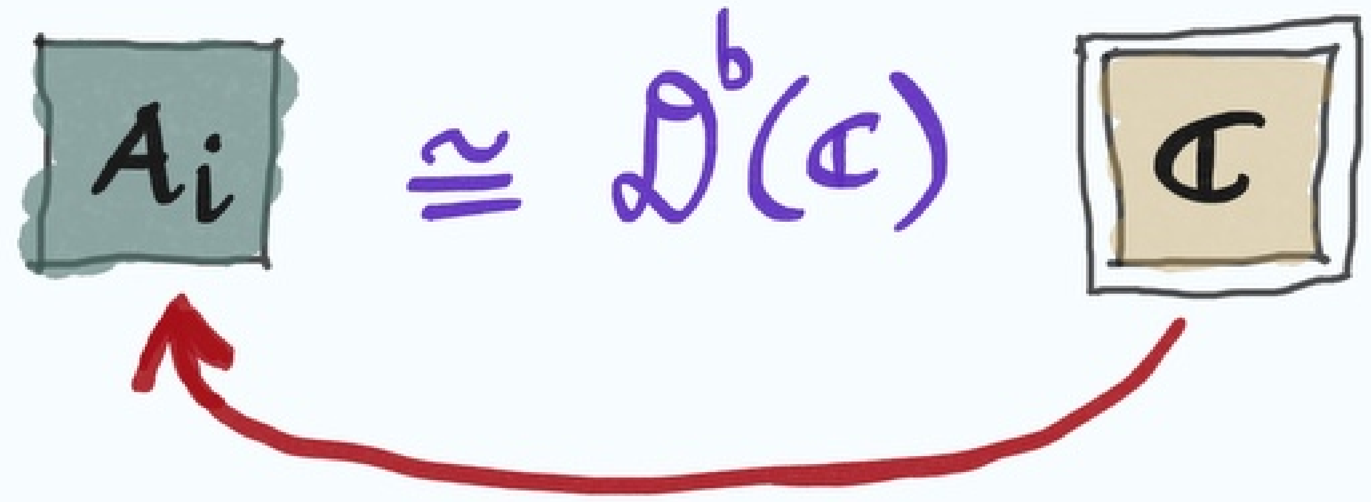


$$\boxed{A_i} \cong D^b(\mathbb{C})$$

Full exceptional collections





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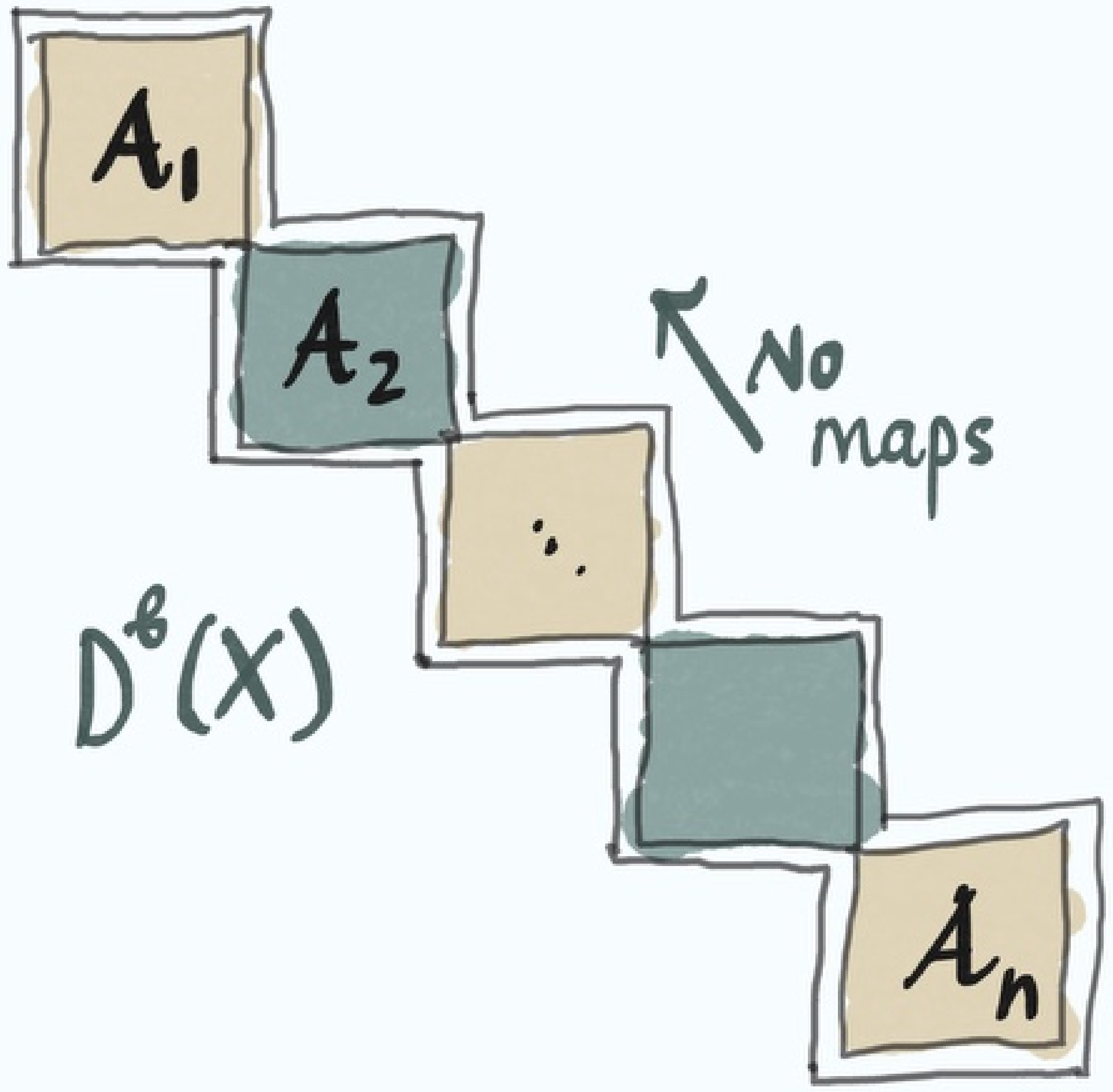

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Fourier- Mukai transform

Kernel $= E_i$

$$
\begin{tikzcd}
 & \mathbb{C} \times X \arrow[dl] \arrow[dr] & \\
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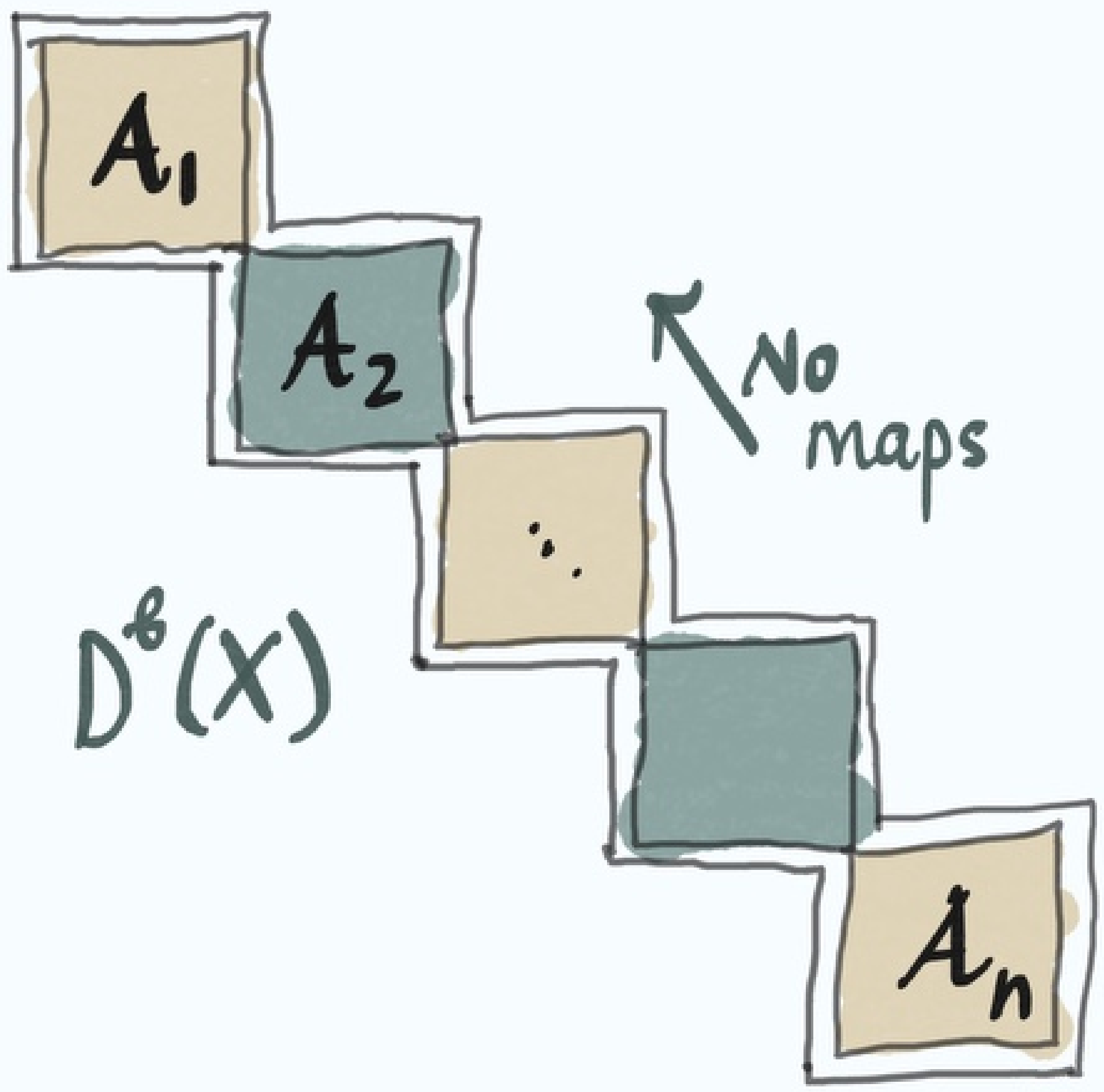

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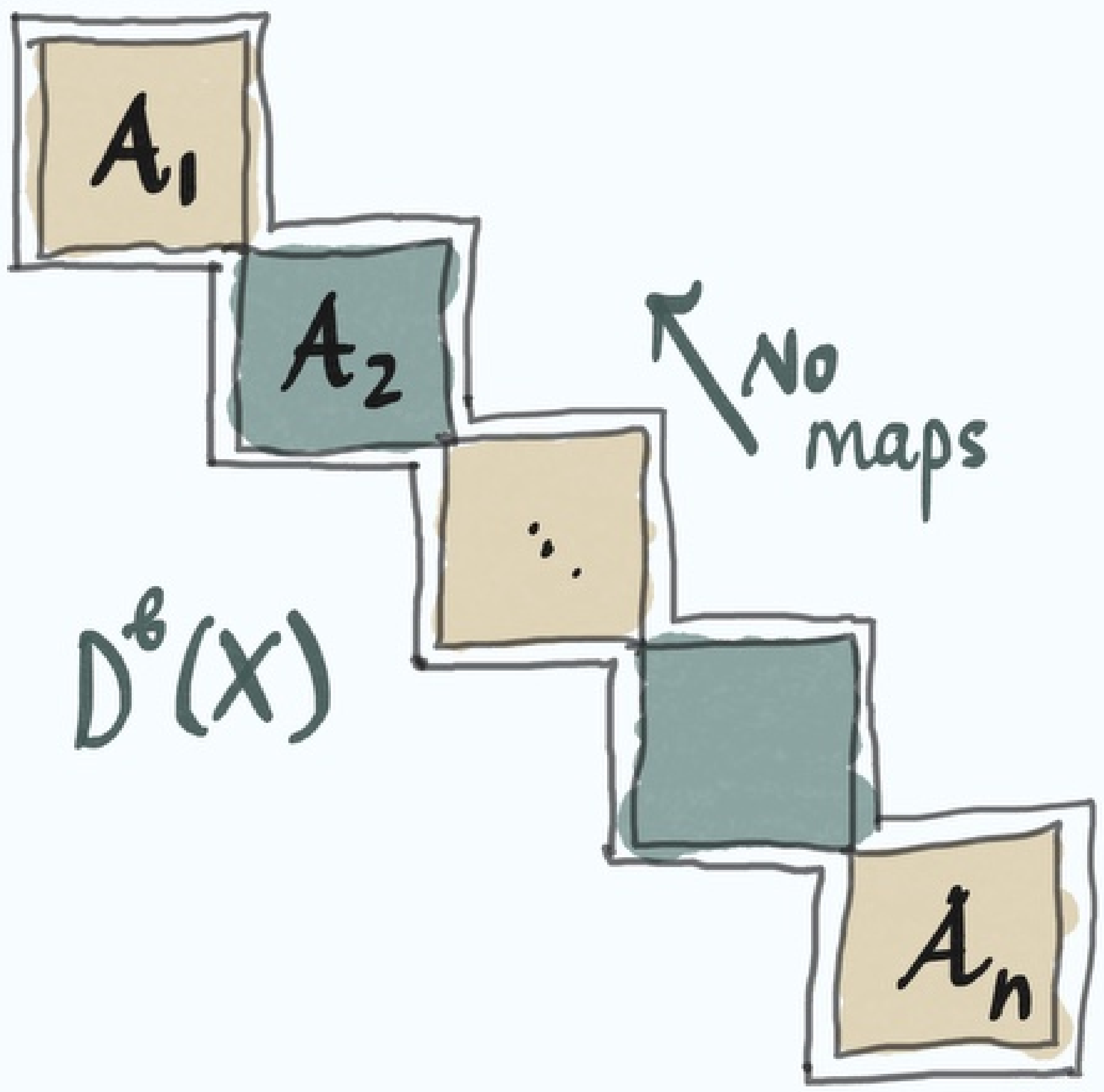
$E_1, E_2, \ldots, E_n$ full exceptional collection.

Full exceptional collections



$A_i \cong D^b(\mathbb{C})$ $\quad \mathbb{C}$

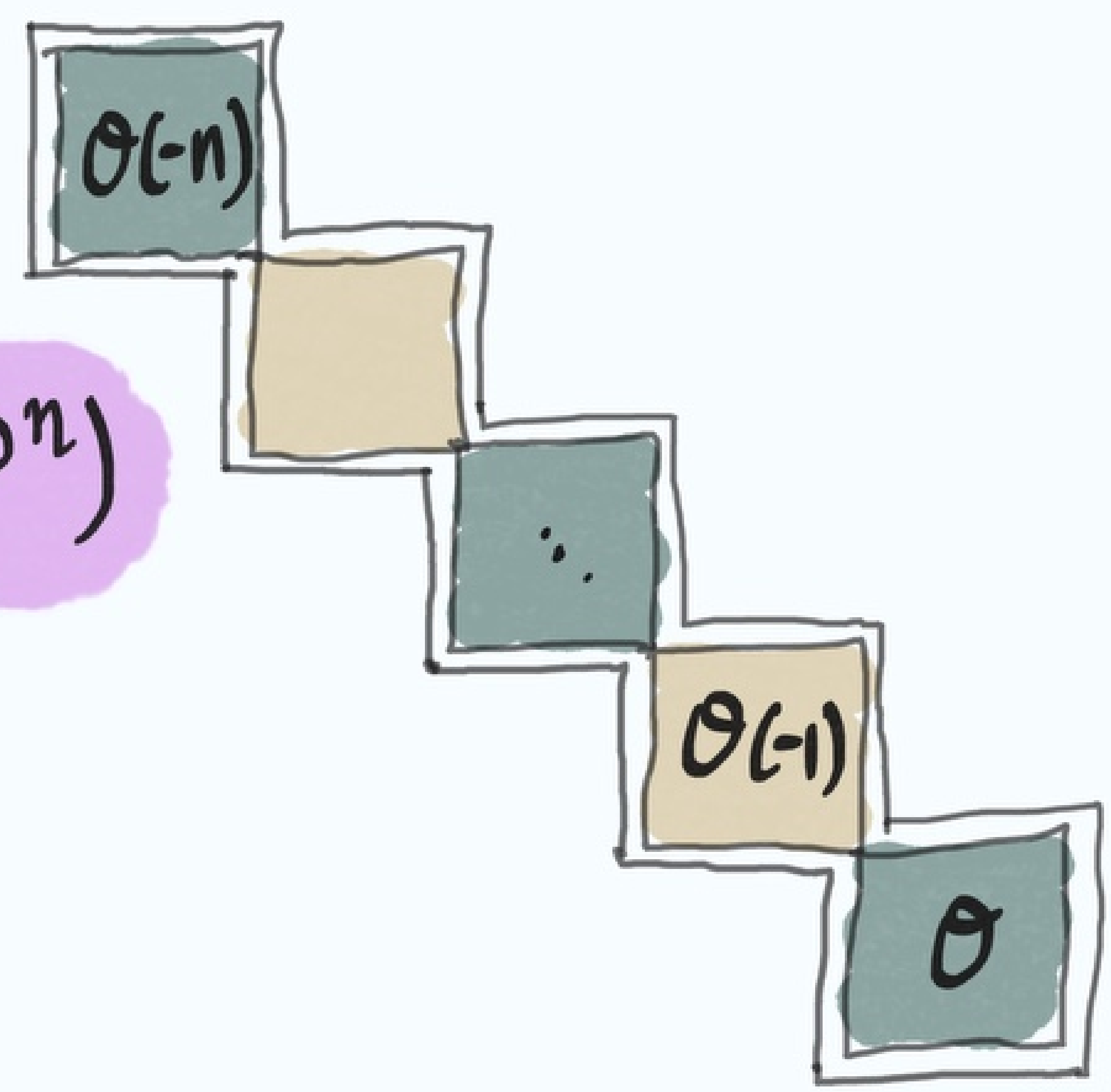
$$\begin{tikzcd} & \mathbb{C} \times X \arrow[dl] \arrow[dr] & \\ \mathbb{C} & & X \end{tikzcd}$$

Fourier-Mukai transform
Kernel = E_i

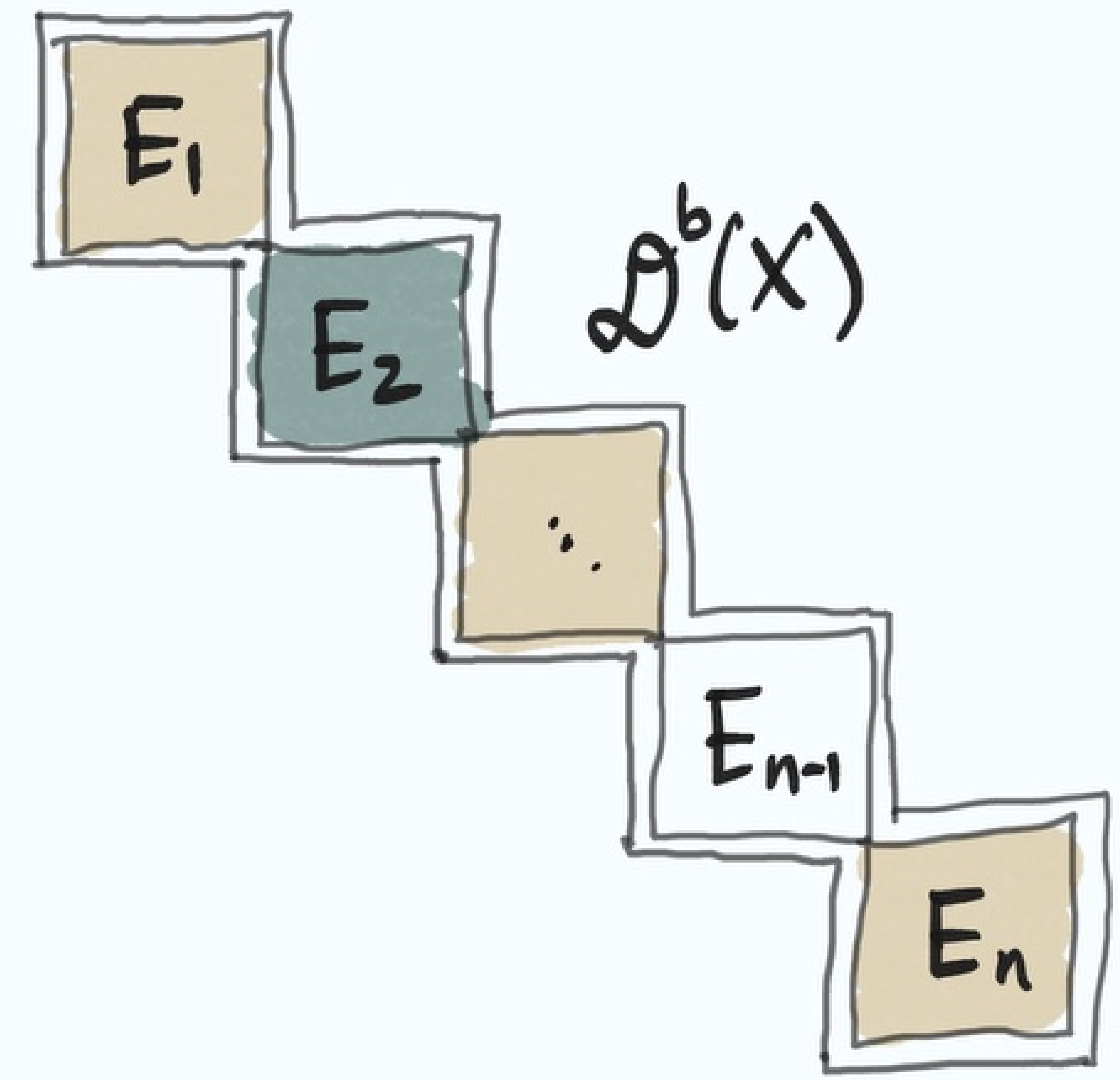
$E_1, E_2, \ldots, E_n$ full exceptional collection.

Example (Beilinson)

$D^b(\mathbb{C}P^n)$

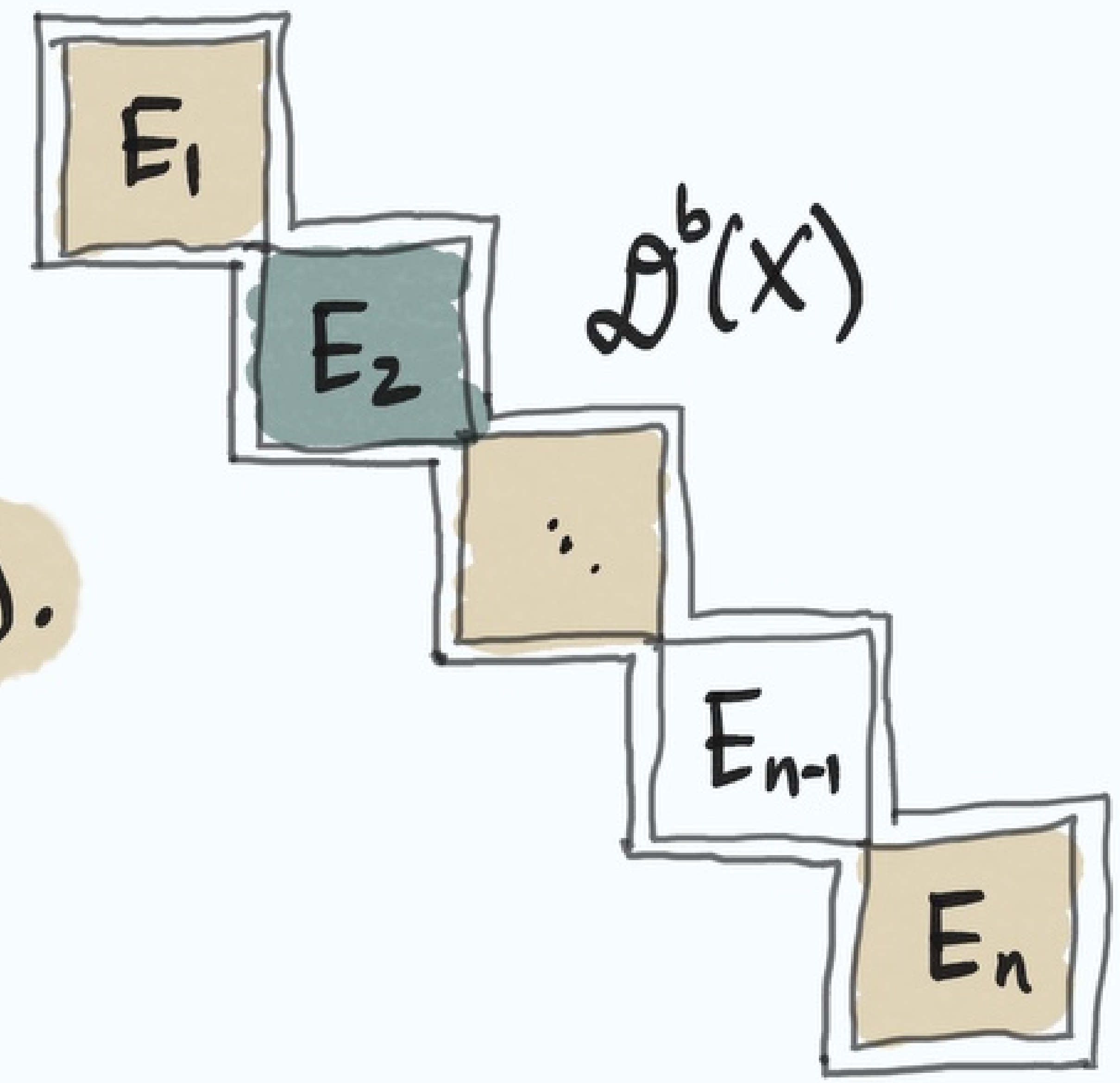


Applications



Applications

① $E_1, \ldots, E_n$ give a basis for $K_0(X)$.



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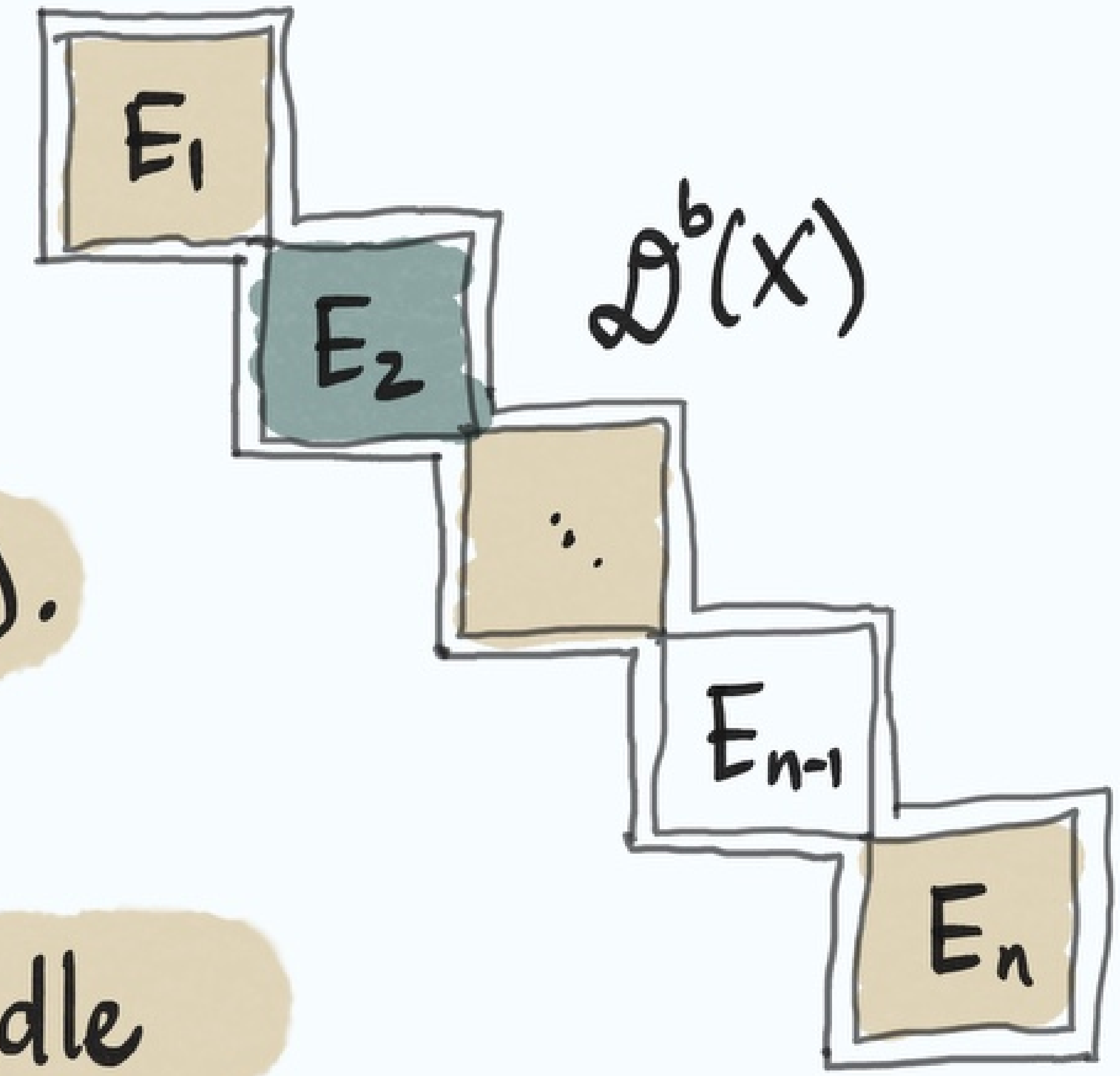


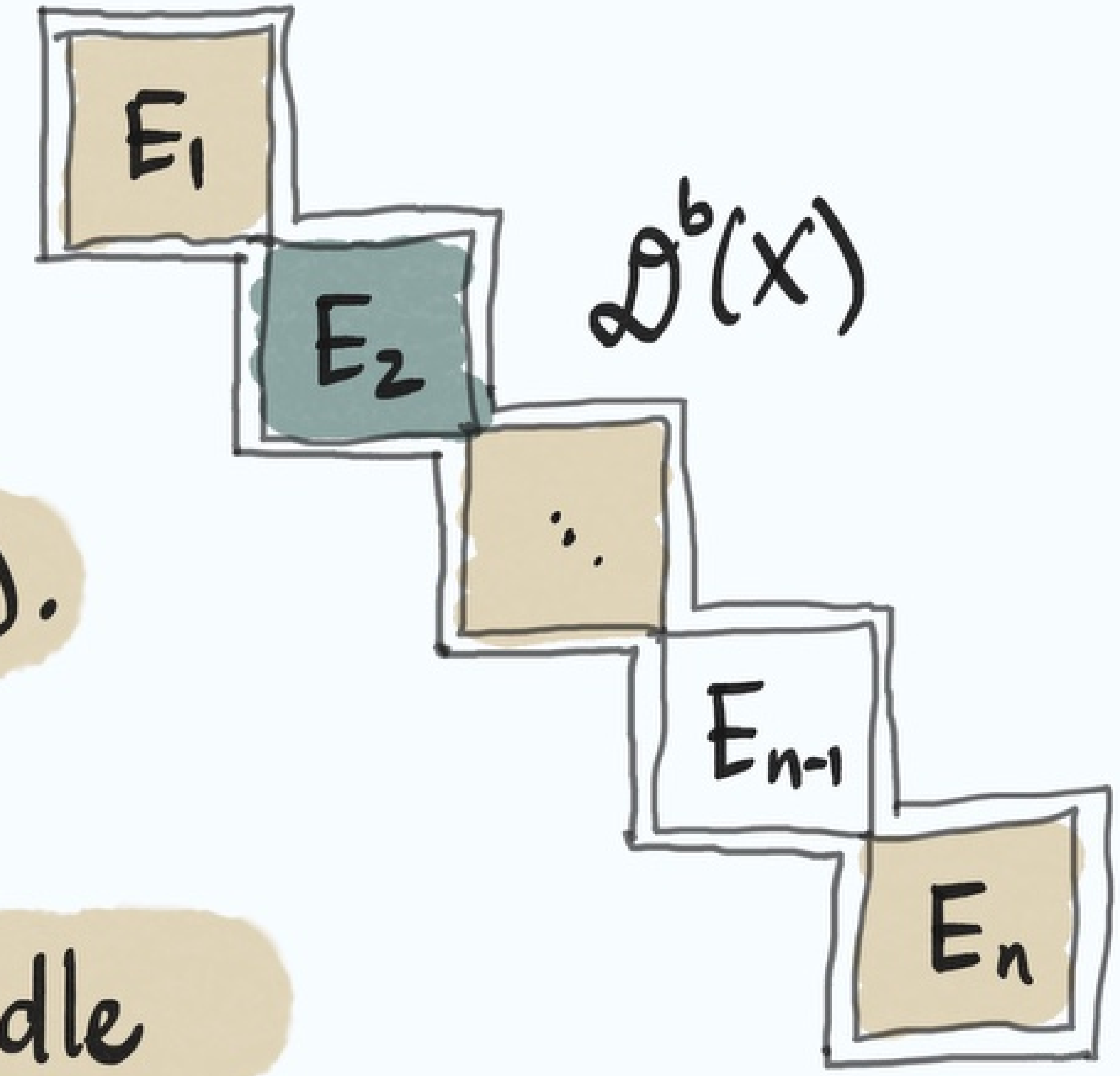
① $E_1, \ldots, E_n$ give a basis for $K_0(X)$.

② For "nice" collections: tilting bundle

$$\rightsquigarrow \mathcal{D}^b(X) \cong \mathcal{D}^b_{f.g}(R)$$

finite dimensional algebra

Applications



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big Quantum cohomology

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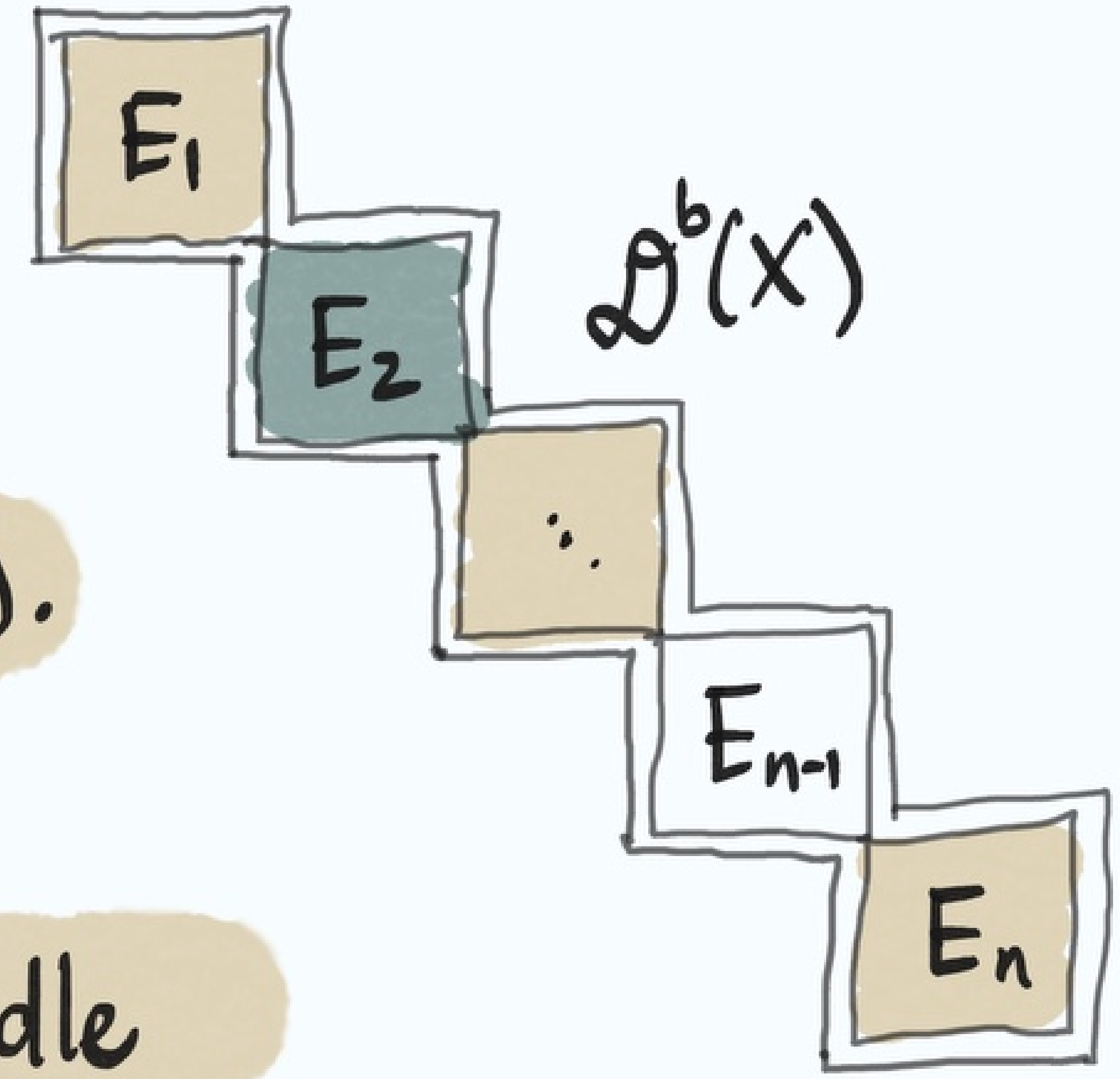

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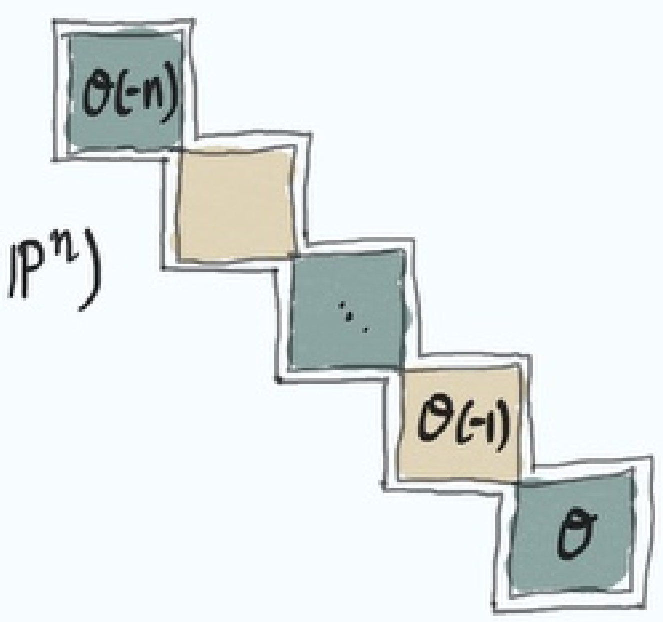
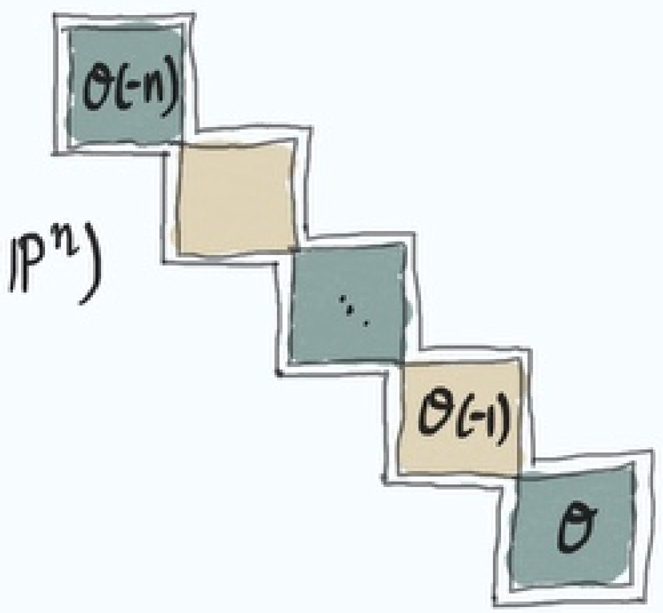
④ Orlov's rationality conjecture: FEC $\Rightarrow$ rational

Atomic examples		
Homogeneous spaces	$\mathcal{D}^b(\mathbb{CP}^n)$ 	$\mathcal{D}^b(Gr(r,n))$ Quadrics Some isotropic Grassmannians

Atomic examples	Toric Varieties	
Homogeneous spaces	$\mathcal{D}^b(\mathbb{CP}^n)$ 	$\mathcal{D}^b(Gr(r,n))$ Quadrics Some isotropic Grassmannians
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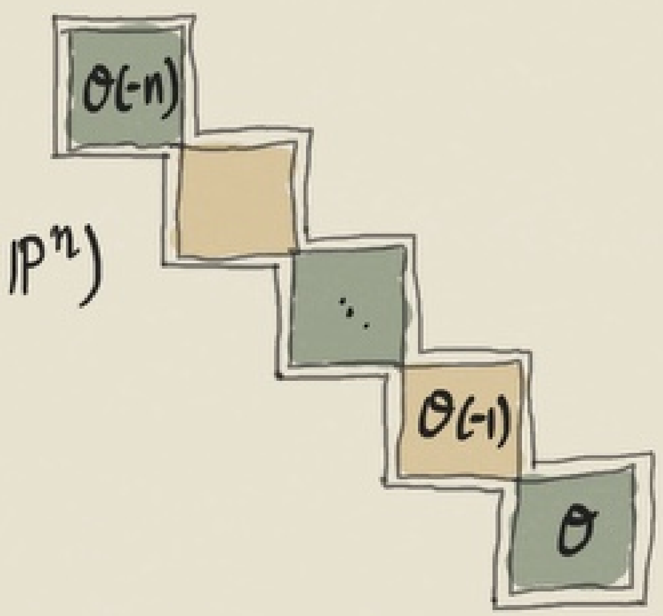
Homogeneous spaces	$\mathcal{D}^b(\mathbb{CP}^n)$ 	$\mathcal{D}^b(Gr(r,n))$ Quadrics Some isotropic Grassmannians
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