Data Structure and Algorithm

Class 9

Seongjin Lee

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DSA_2017_09

insight@gnu.ac.kr
http://resourceful.github.io
Systems Research Lab.
GNU

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Graph Operations

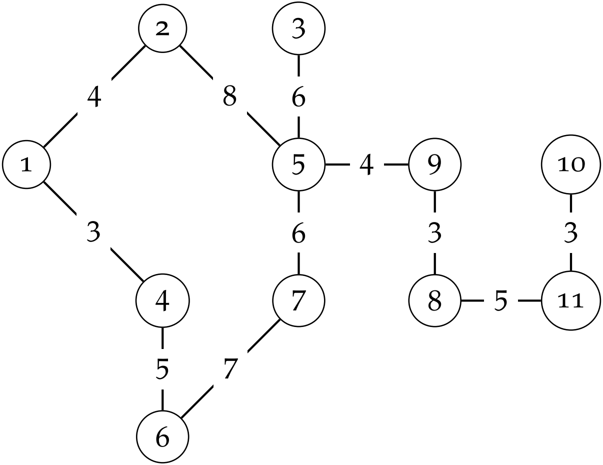
Some of the Graph Problems are

- Path Finding
- Connectedness
- Spanning tree

Graph Operations : Path Finding

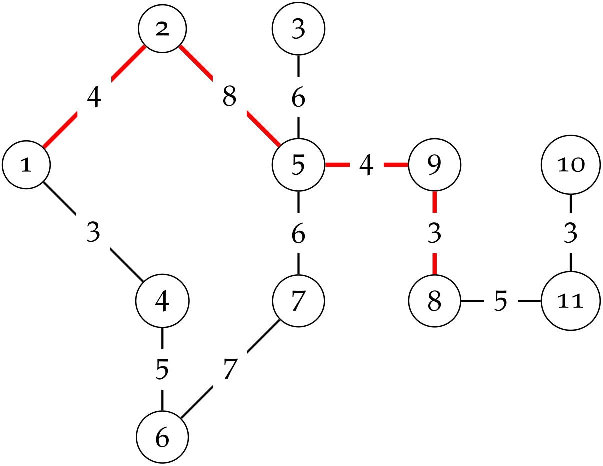
Path Finding

- Path length between 1 and 8



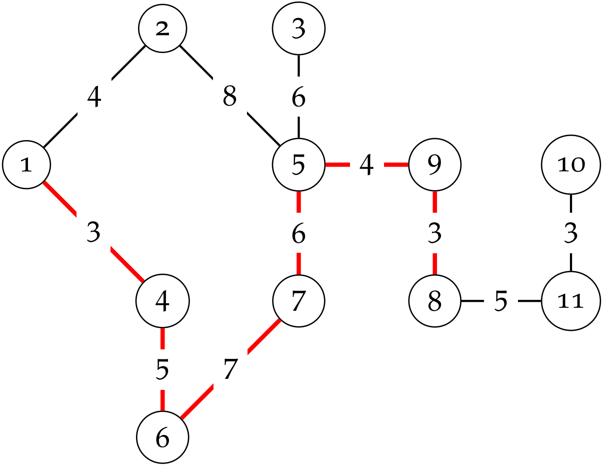
Path Finding

- Edges (1, 2), (2, 5), (5, 9), and (9, 8) length = 19



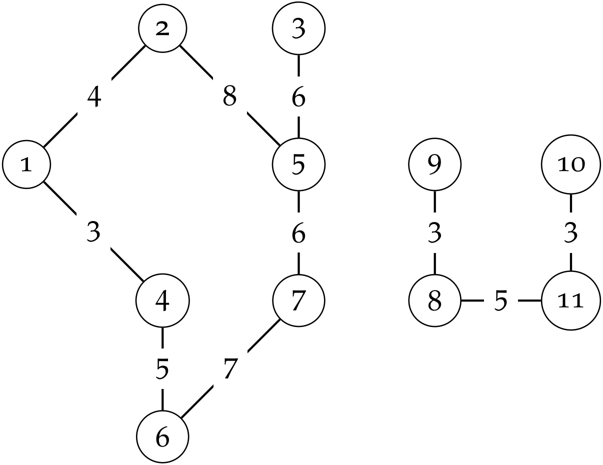
Path Finding

- Edges (1, 4), (4, 6), (6, 7), (5, 9) and (9, 8) length = 28



Example of No Path

- No path between 4 to 11

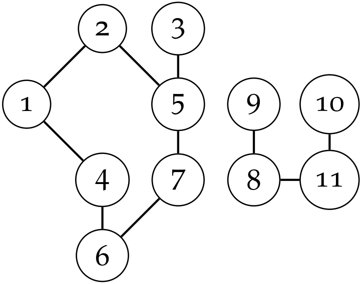


Graph Operations : Connected Graph

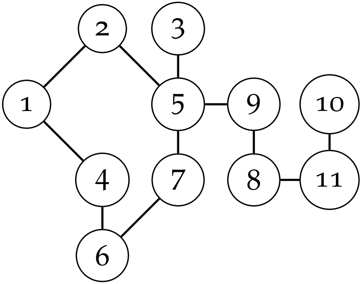
Connected Graph

- Undirected graph
- There is a path between every pair of vertices

- A directed graph $G = (V, E)$ is **strongly connected** if, for every pair of vertices u, v in V, there is a directed path from u to v and also from v to u

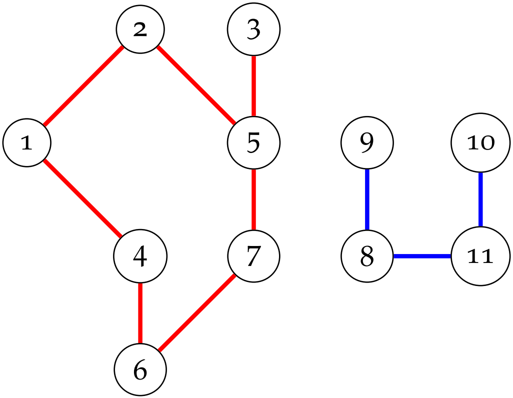


Not connected graph



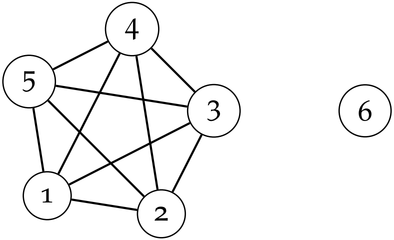
Connected Graph

Connected Components



Connected Component

- A connected component is a *maximal subgraph* in which all vertices are reachable from every other vertices.
  - *maximal* means that it is the largest possible subgraph
  - Cannot add vertices and edges from original graph and retain connectedness.
  - A connected graph has exactly 1 component.



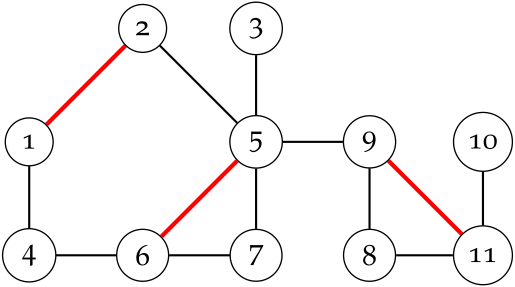
Connectedness

There are two types of connected components in digraphs

- Strong Components
  - maximal subgraph in which there is a path from every vertex to every vertex following all the edges in the direction they are pointing
- Weak Components
  - maximal subgraph which would be connected if we ignore the direction of the edges

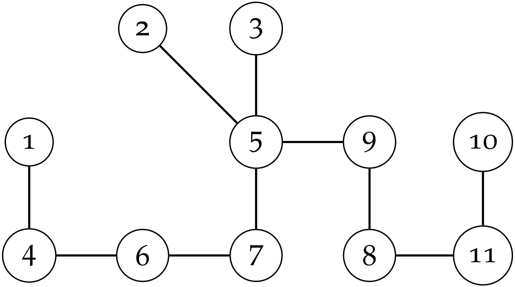
Cycles and Connectedness

Removal of an edge that is on a cycle does not affect connectedness



Cycles and Connectedness

Connected subgraph with all vertices and minimum number of edges hs no cycles



Tree

A tree can be thought of as connected graph that has no cycles

- n vertex connected graph with $n-1$ edges

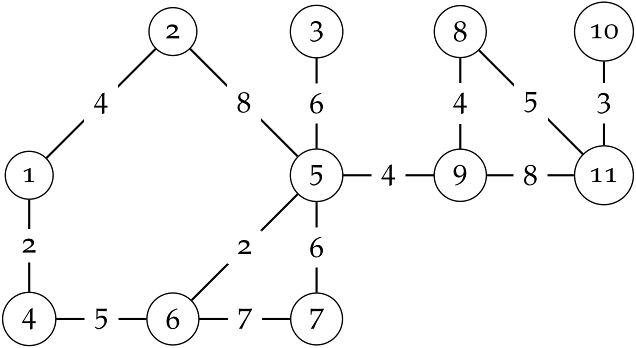
Graph Operations : Spanning Tree

Spanning Tree

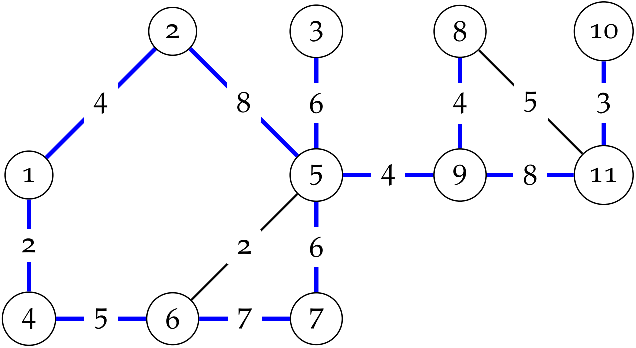
- Subgraph that includes all vertices of the original graph.
- Subgraph is a tree.
  - If original graph has n vertices, the spanning tree has n vertices and $n-1$ edges.

Minimum Cost Spanning Tree : MST

- Tree cost is sum of edge weights/costs

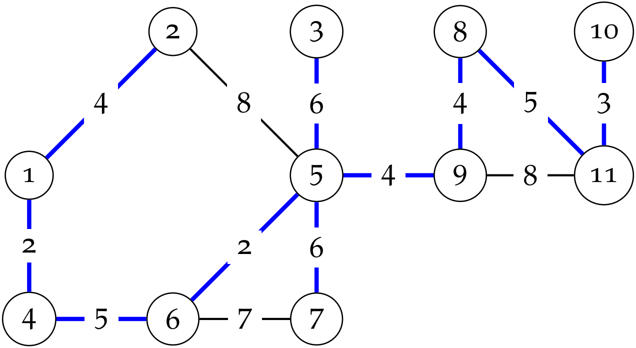


- Spanning Tree cost is 51



A Spanning Tree

- Spanning Tree cost is 41 (= MST)

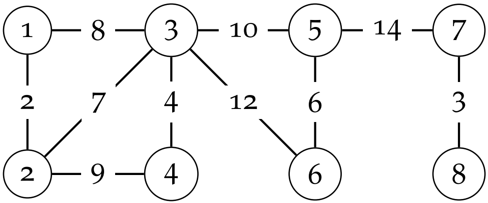


Minimum-Cost Spanning Tree

- Weighted connected undirected graph
- Spanning tree
- Cost of spanning tree is sum of edge costs
- Find spanning tree that has minimum cost

Example

- Network has 10 edges
- Spanning tree has only $n - 1 = 7$ edges
- Need to either select 7 edges or discard 3
- Which edges should be discarded to become MST ?



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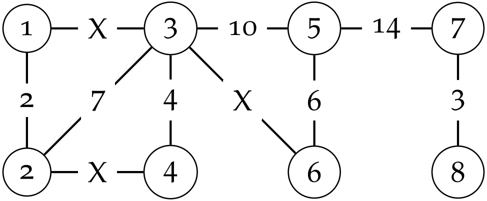
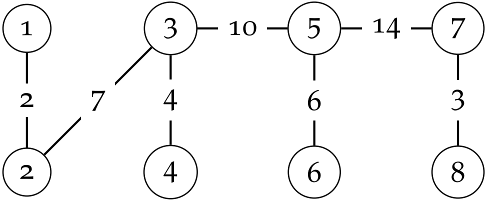


Example : Answer

- Which edges should be discarded to become MST ?



Graph Operations : Greedy Strategy

Edge Selection Greedy Strategies

- Start with an $n - vertex, 0 - edge$ forest. Consider edges in ascending order of cost. Select edge if it does not form a cycle together with already selected edges.
  - Kruskal's algorithm
- Start with a $1 - vertex$ tree and grow it into an $n - vertex$ tree by repeatedly adding a vertex and an edge. When there is a choice, add a least cost edge.
  - Prim's algorithm
- Start with an $n - vertex$ forest. Each component/tree selects a least cost edge to connect to another component/tree. Eliminate duplicate selections and possible cycles. Repeat until only 1 component/tree is left.
  - Sollin's algorithm

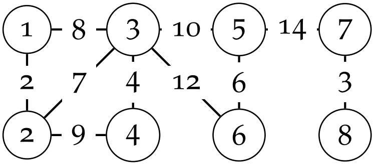
Edge Rejection Greedy Strategies

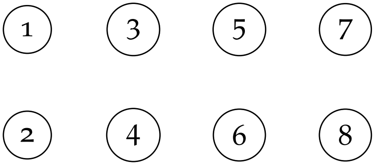
- Start with the connected graph. Repeatedly find a cycle and eliminate the highest cost edge on this cycle. Stop when no cycles remain.
- Consider edges in descending order of cost. Eliminate an edge provided this leaves behind a connected graph.

Graph Operations : Kruskal's Algorithm

Kruskal's Algorithm

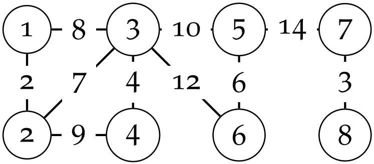
- Start with a forest that has no edges
- Consider edges in ascending order of cost.

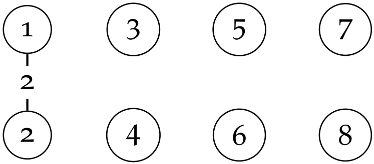




Kruskal's Algorithm

- Start with a forest that has no edges
- Consider edges in ascending order of cost.
- Edge $(1, 2)$ is considered first and added to the forest.





Kruskal's Algorithm

- Start with a forest that has no edges
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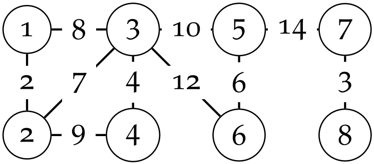
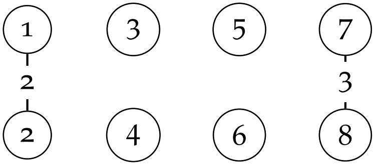
- Edge $(1, 2)$ is considered first and added to the forest.

- Edge $(7, 8)$





Kruskal's Algorithm

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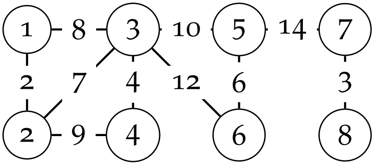
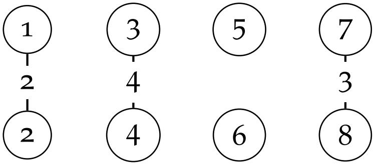
- Consider edges in ascending order of cost.
- Edge $(1,2)$ is considered first and added to the forest.

- Edge $(7,8)$
- Edge $(3,4)$





Kruskal's Algorithm

- Start with a forest that has no edges
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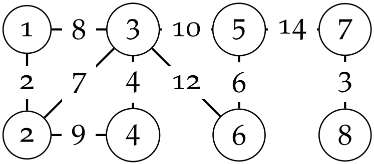
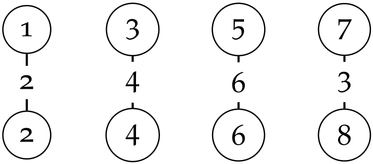
- Edge $(1, 2)$ is considered first and added to the forest.

- Edge $(7, 8)$
- Edge $(3, 4)$
- Edge $(5, 6)$





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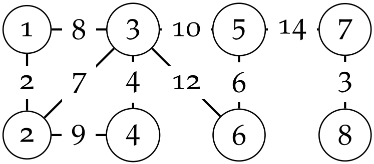
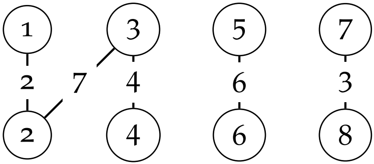
- Edge $(7, 8)$
- Edge $(3, 4)$
- Edge $(5, 6)$
- Edge $(2, 3)$





Kruskal's Algorithm

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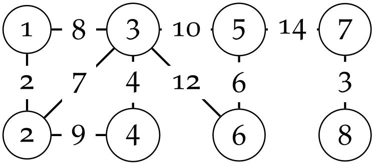
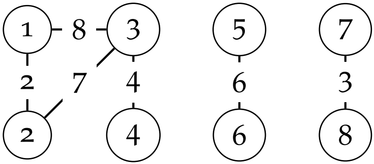
- Edge $(1, 2)$ is considered first and added to the forest.

- Edge $(7, 8)$
- Edge $(3, 4)$
- Edge $(5, 6)$
- Edge $(2, 3)$
- Edge $(1, 3)$ creates cycle (`rejected`)





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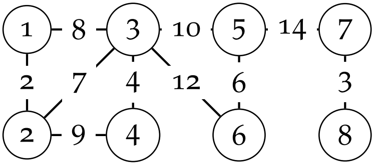
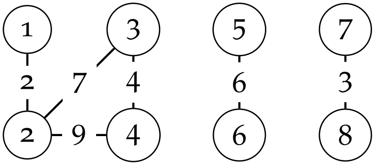
- Edge $(2,4)$ creates cycle





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- Edge $(7, 8)$
- Edge $(2, 3)$
- Edge $(3, 5)$
- Edge $(3, 4)$
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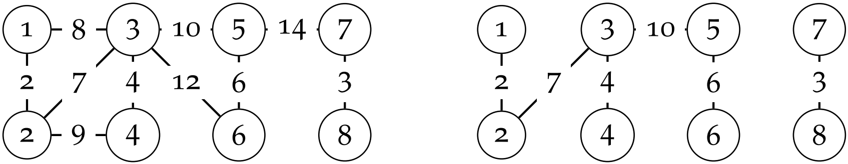
- Edge $(5, 6)$
- Edge $(2, 4)$ creates cycle



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- Edge $(1, 3)$ creates cycle (`rejected`)
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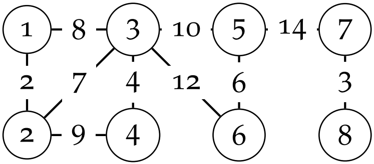
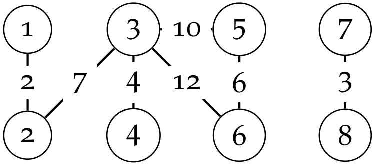
- Edge $(5, 6)$
- Edge $(2, 4)$ creates cycle





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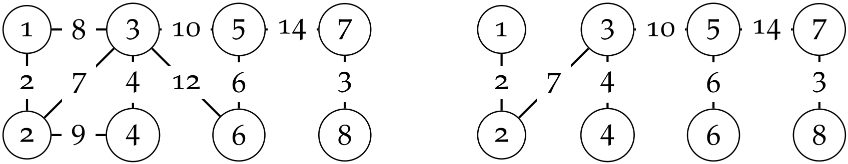
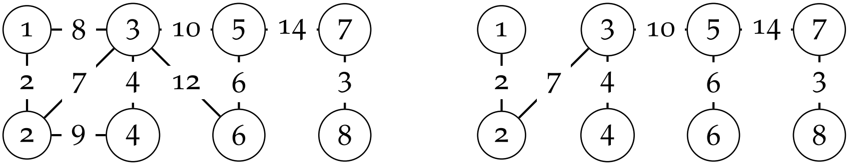
- Edge $(5, 6)$
- Edge $(2, 4)$ creates cycle
- Edge $(5, 7)$



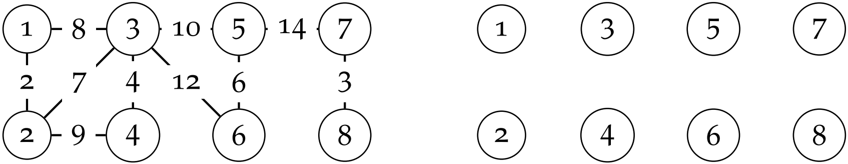
Kruskal's Algorithm

- $n-1$ edges have been selected and no cycle formed, so we must have a spanning tree
  - The cost is 46
- The minimum cost spanning tree is unique when all edge costs are different



Prim's Algorithm

- Start with any single vertex tree
- Get a 2-vertex tree by adding a cheapest edge
- Get a 3-vertex tree by adding a cheapest edge
- Grow the tree one edge at a time until the tree has $n - 1$ edges (and hence has all n vertices)



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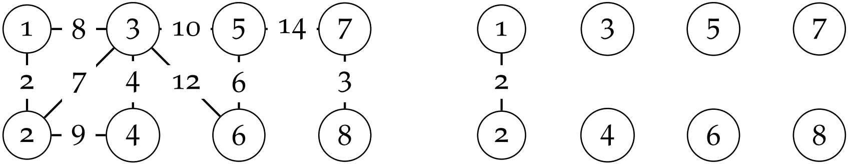

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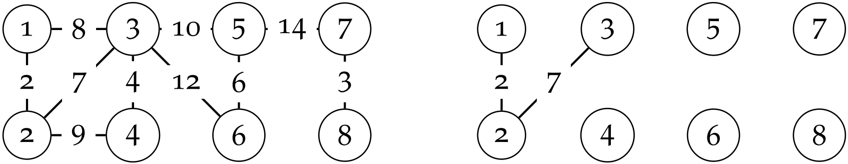

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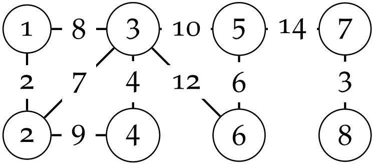

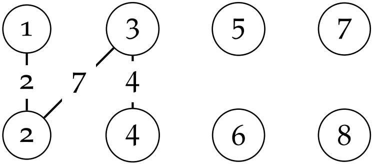

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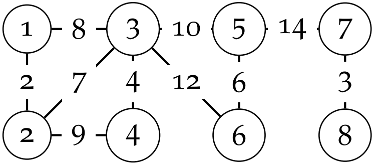

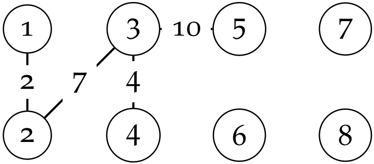

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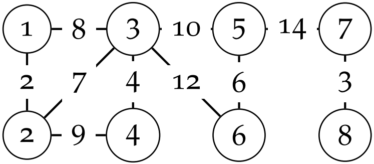

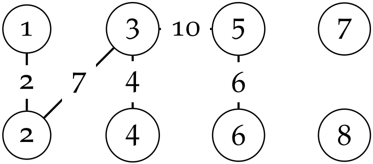

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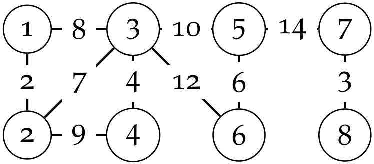

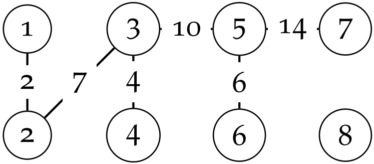

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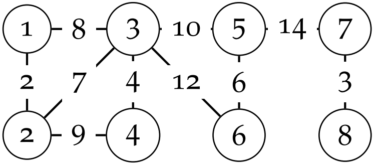
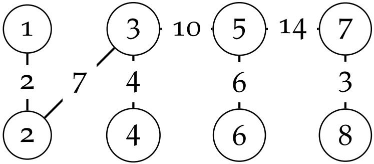
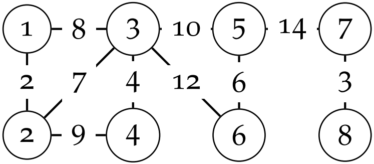
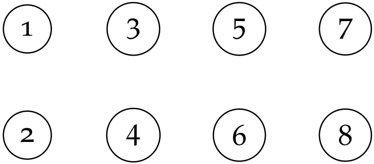




- Start with a forest that has no edges.
- Each component selects a least cost edge with which to connect to another component.
- Duplicate selections are eliminated.
- Cycles are possible when the graph has some edges that have the same cost.
- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.





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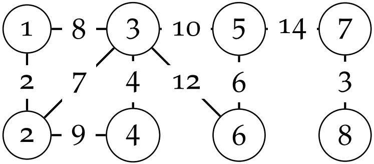
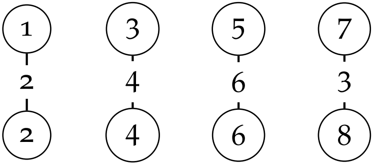
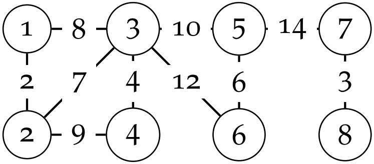
- Duplicate selections are eliminated.
- Cycles are possible when the graph has some edges that have the same cost.
- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.

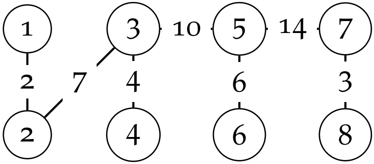




Sollin's Algorithm

- Start with a forest that has no edges.
- Each component selects a least cost edge with which to connect to another component.
- Duplicate selections are eliminated.
- Cycles are possible when the graph has some edges that have the same cost.
- Each component that remains selects a least cost edge with which to connect to another component.
- Beware of duplicate selections and cycles.





Greedy Minimum-Cost Spanning Tree Algorithms

- Can prove that all result in a minimum-cost spanning tree.
- Prim's Algorithm is the fastest
  - $O(n^2)$ using an implementation similar to that of Dijkstra's shortest-path algorithm
  - $O(e + n \log n)$ using a Fibonacci heap
- Kruskal's algorithm uses **union-find trees** to run in $O(n + e \log e)$ time
  - `union(x,y)` joins two subsets containing `x` and `y` into a single subset
  - `find(x)` determines the subset with the element `x`

Exmple: Union-find

Assume the following set $S = \{1, 2, 3, 4, 5, 6\}$ and create a six independent sets: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

After performing `union(1, 4)` and `union(2, 5)`, then we have $\{1, 4\}, \{5, 2\}, \{3\}, \{4\}$

After running `union(2, 1)` and `union(3, 6)`, then we have $\{1, 2, 4, 5\}, \{3, 6\}$