Gibbs Sampling

November 3, 2017

Because $p(x|y) \propto ye^{-yx}, 0 < x < B < \infty$, we can write $p(x|y) = a \times ye^{-yx}$.

We know \$ 0 < x < B \$, we get $a = \frac{1}{\int_0^B f(x)\,\mathrm{d}x} = \frac{1}{1-e^{-By}}$.

Now we can write \$ p(x | y) = ye^{-yx} $\frac{}{1-e^{-By}\$;similarly,\$p(y|x)=\frac{xe^{-yx}}{1-e^{-Bx}}\$}$

To use the inversion sampling, we need to first drawn a random variable u $U(0,1)$ and then inverting this draw using F^{-1}.

Therefore, we calculate $F(x|y) = \int_0^x f(x|y)dx = \frac{1-e^{-yx}}{1-e^{-By}}$.

Let $u = F(z)$

$z = F^{-1}(u) = \frac{\log\left(1-u\times(1-e^{-yB})\right)}{-y}$

Given a draw of u and y, z is a draw from $p(x|y)$. The same can be done for $p(y|x)$ and we obtain that the function is \$ $\log\left(1-u\times(1-e^{-xB})\right)\frac{}{-x\$ for drawing from the \$(p(y|x)\$).}$

We first choose an initial value for x and y as x0, y0 respectively within the range of 0 to B. Then, we generate a u from the Uniform distribution [0,1], plug the value of u and y0 into the inversed function and get an updated x. Similarly, we will update y accordingly. Repeating the steps will generate a series of x and y's which forms our new distribution.

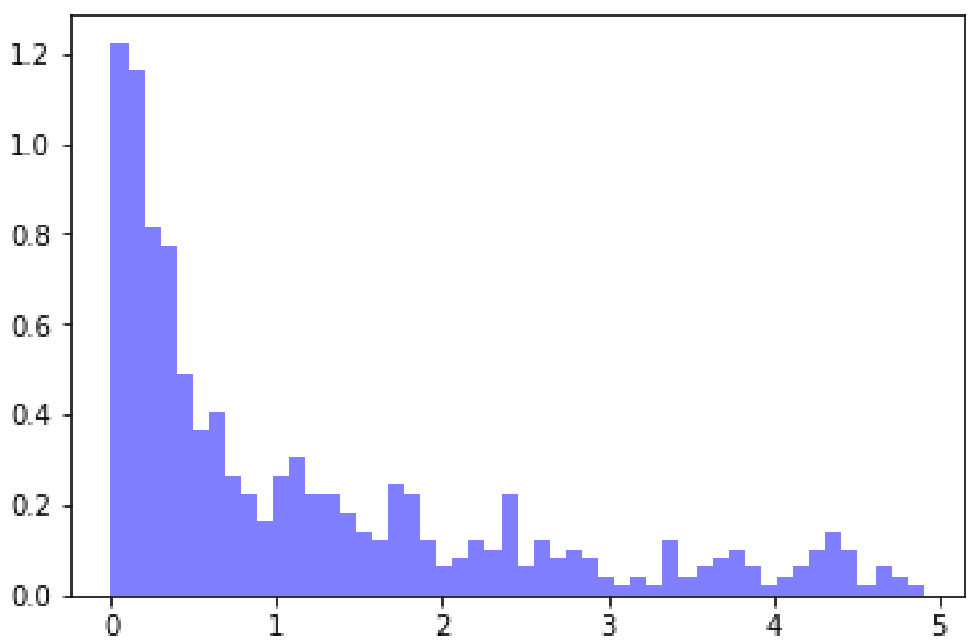
See the code below:

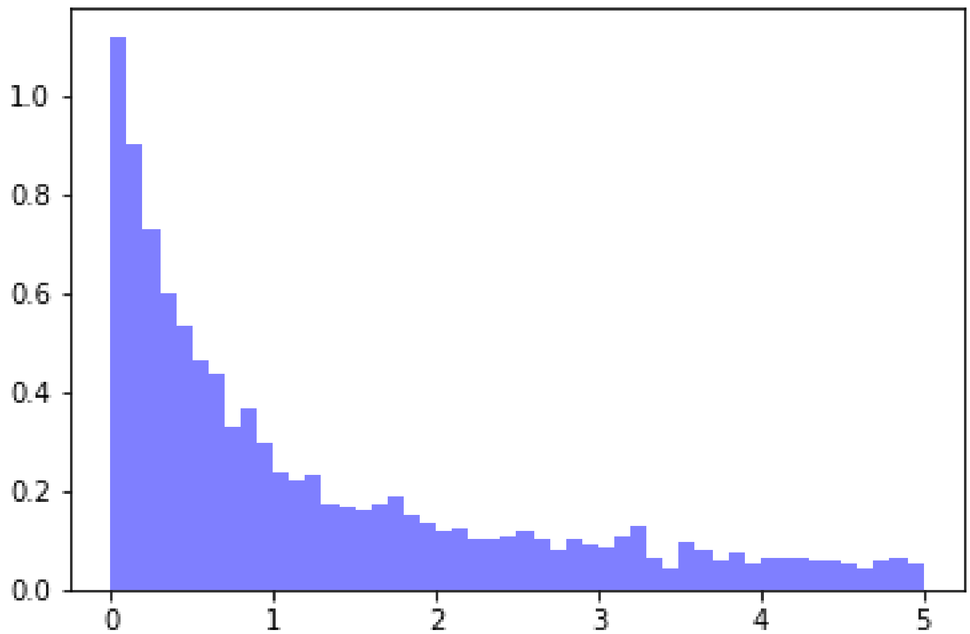
```
In [5]: import numpy as np
        import matplotlib.pyplot as plt
        def gibbs(x0,y0,B,n):
            #setting empty arrays for storing values later
            x = np.empty(n)
            y = np.empty(n)
            #a is simply a matrix form of x,y
            a = np.empty([n,2])
            #take x0, y0 as initial values for x, y
            x[0] = x0
            y[0] = y0
            a[0] = (x0,y0)
            for i in range(1,n):
                #sample from f(x|y) using inversion method
                #u1 is a random number generated from the uniform distribution
                u1 = np.random.uniform(0,1)
                x[i] = np.log(1-u1*(1-np.exp(-y[i-1]*B)))/(-y[i-1])
                #sample from f(y|x) using inversion method
                #u2 is a random number generated from the uniform distribution
```

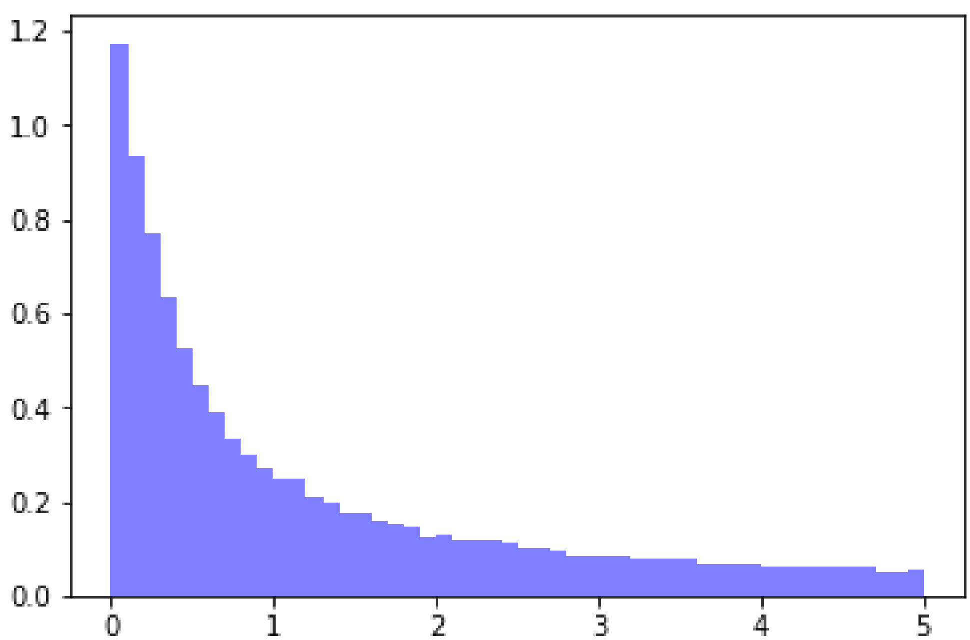
```
        u2 = np.random.uniform(0,1)
        y[i] = np.log(1-u2*(1-np.exp(-x[i]*B)))/(-x[i])
        #output the matrix formed by x and y
        a[i] = (x[i],y[i])
    return a
```

Now, by choosing x=1, y=1 as starting point, B=5, and 500, 5000, 50000 as the number of iterations, we obtain matrices a1, a2, a3 which stored all values of x and y's we generated from each condition. Use matplotlib.pyplot package to plot the histograms below. As we can see, the more iterations we have done, the smoother the histogram. I think the values of x, y fluctuates the most at the beginning of the processes, but stablizes after it runs repeatedly. Therefore, if I have thrown out the first n x,y pairs we got, the estimation may even be better (built a burnIn part which throw out 0.1n numbers of the beginning iterations for example). But I choose not to do it here, to just maintain the full distribution of all 500 iterations.

```
In [6]: #histogram of X
        #Since B is 5, choose a number from 0 to 5 as a starting point
        a1 = gibbs(1,1,5,500)
        a2 = gibbs(1,1,5,5000)
        a3 = gibbs(1,1,5,50000)
        plt.hist(a1[:,[1]], 50, normed=1, facecolor='blue', alpha=0.5)
        plt.show()
        plt.hist(a2[:,[1]], 50, normed=1, facecolor='blue', alpha=0.5)
        plt.show()
        plt.hist(a3[:,[1]], 50, normed=1, facecolor='blue', alpha=0.5)
        plt.show()
```







As we can see below, the means the x's generated by different sample sizes are close together but much closer between 500 and 5000 iterations. I use the mean for 50000 samples to estimate E(x) = 1.26

```
In [7]: np.mean(a1[:,[1]])
```

```
Out[7]: 1.1905003657315947

In [8]: np.mean(a2[:,[1]])

Out[8]: 1.2721319566586644

In [9]: np.mean(a3[:,[1]])

Out[9]: 1.2562601478846753
```