Graph Theory - Trees

Trees are graphs that do not contain even a single cycle. They represent hierarchical structure in a graphical form. Trees belong to the simplest class of graphs. Despite their simplicity, they have a rich structure.

Trees provide a range of useful applications as simple as a family tree to as complex as trees in data structures of computer science.

Tree

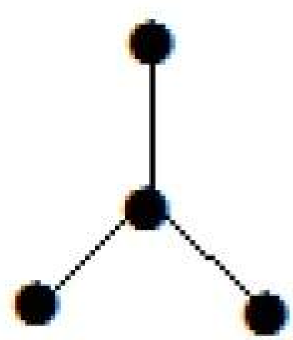
A **connected acyclic graph** is called a tree. In other words, a connected graph with no cycles is called a tree.

The edges of a tree are known as **branches**. Elements of trees are called their **nodes**. The nodes without child nodes are called **leaf nodes**.

A tree with ‘n’ vertices has ‘n-1’ edges. If it has one more edge extra than ‘n-1’, then the extra edge should obviously has to pair up with two vertices which leads to form a cycle. Then, it becomes a cyclic graph which is a violation for the tree graph.

Example 1

The graph shown here is a tree because it has no cycles and it is connected. It has four vertices and three edges, i.e., for ‘n’ vertices ‘n-1’ edges as mentioned in the definition.



Note − Every tree has at least two vertices of degree one.

Example 2

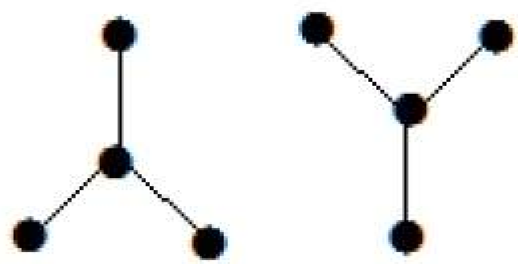
In the above example, the vertices ‘a’ and ‘d’ has degree one. And the other two vertices ‘b’ and ‘c’ has degree two. This is possible because for not forming a cycle, there should be at least two single edges anywhere in the graph. It is nothing but two edges with a degree of one.

Forest

A **disconnected acyclic graph** is called a forest. In other words, a disjoint collection of trees is called a forest.

Example

The following graph looks like two sub-graphs; but it is a single disconnected graph. There are no cycles in this graph. Hence, clearly it is a forest.



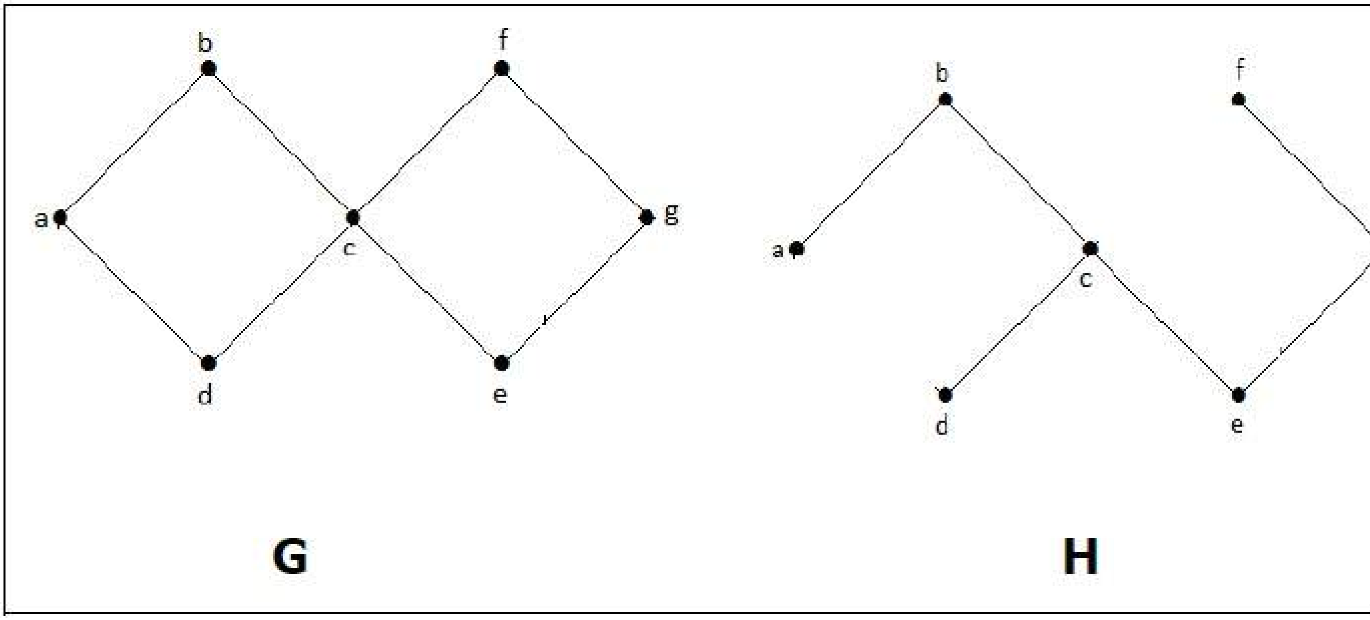
Spanning Trees

Let G be a connected graph, then the sub-graph H of G is called a spanning tree of G if −

- H is a tree
- H contains all vertices of G.

A spanning tree T of an undirected graph G is a subgraph that includes all of the vertices of G.

Example



In the above example, G is a connected graph and H is a sub-graph of G.

Clearly, the graph H has no cycles, it is a tree with six edges which is one less than the total number of vertices. Hence H is the Spanning tree of G.

Circuit Rank

Let ‘G’ be a connected graph with ‘n’ vertices and ‘m’ edges. A spanning tree ‘T’ of G contains (n-1) edges.

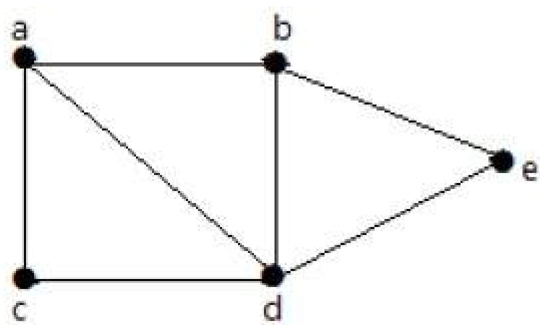
Therefore, the number of edges you need to delete from ‘G’ in order to get a spanning tree = m-(n-1), which is called the circuit rank of G.

This formula is true, because in a spanning tree you need to have ‘n-1’ edges. Out of ‘m’ edges, you need to keep ‘n–1’ edges in the graph.

Hence, deleting ‘n–1’ edges from ‘m’ gives the edges to be removed from the graph in order to get a spanning tree, which should not form a cycle.

Example

Take a look at the following graph −



For the graph given in the above example, you have m=7 edges and n=5 vertices.

Then the circuit rank is

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G = m – (n – 1)
  = 7 – (5 – 1)
  = 3
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Example

Let ‘G’ be a connected graph with six vertices and the degree of each vertex is three. Find the circuit rank of ‘G’.

By the sum of degree of vertices theorem,

n ∑ i=1 deg(V_i) = 2|E|

6 × 3 = 2|E|

|E| = 9

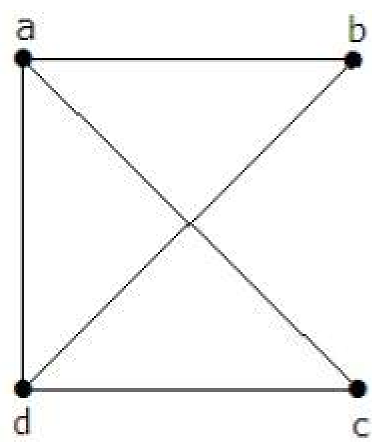
Circuit rank = |E| – (|V| – 1)

= 9 – (6 – 1) = 4

Kirchoff’s Theorem

Kirchoff’s theorem is useful in finding the number of spanning trees that can be formed from a connected graph.

Example



The matrix ‘A’ be filled as, if there is an edge between two vertices, then it should be given as ‘1’, else ‘0’.

$$A = \begin{vmatrix} 0 & a & b & c & d \\ a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

By using kirchoff's theorem, it should be changed as replacing the principle diagonal values with the degree of vertices and all other elements with -1.A

$$= \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix} = M$$

$$M = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{vmatrix}$$

Cofactor of m1 = $\begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 8$

Thus, the number of spanning trees = 8.