DIAGONAL CYCLES AND ANTICYCLOTOMIC IWASAWA THEORY OF MODULAR FORMS

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ABSTRACT. We construct a new anticyclotomic Euler system (in the sense of Jetchev–Nekovář–Skinner) for the Galois representation $V_{f,\chi}$ attached to a newform f of weight $2r \geq 2$ twisted by an anticyclotomic Hecke character χ. We then show some arithmetic applications of the constructed Euler system, including new results on the Bloch–Kato conjecture in ranks zero and one, and a divisibility towards the Iwasawa–Greenberg main conjecture for $V_{f,\chi}$.

In particular, in the case where the base-change of f to our imaginary quadratic field has root number $+1$ and χ has higher weight (which implies that the complex L-function $L(V_{f,\chi}, s)$ vanishes at the center), our results show that the Bloch–Kato Selmer group of $V_{f,\chi}$ is nonzero, and if a certain distinguished class $\kappa_{f,\chi}$ is nonzero, then the Selmer group is one-dimensional. Such applications to the Bloch–Kato conjecture for $V_{f,\chi}$ were left wide open by the earlier approaches using Heegner cycles and/or Beilinson–Flach classes. Our construction is based instead on a generalisation of the Gross–Kudla–Schoen diagonal cycles.

CONTENTS

Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_{2r}(\Gamma_0(N_f))$ be an elliptic newform of even weight $2r \geq 2$, and let $p \nmid 6N_f$ be a prime. Let K be an imaginary quadratic field in which p splits. Let L be a number field containing K and the Fourier coefficients of f, and let $\mathfrak{P}$ be a prime of L above p at which f is ordinary, i.e. $v_{\mathfrak{P}}(a_p) = 0$. Let χ be an anticyclotomic Hecke character of K, and consider the conjugate self-dual $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}/K)$-representation

$$V_{f,\chi} := V_f^{\vee}(1-r) \otimes \chi^{-1},$$

where $V_f^{\vee}$ is the contragredient of Deligne's $\mathfrak{P}$-adic Galois representation associated to f.

We prove the following applications to the Bloch–Kato conjecture for $V_{f,\chi}$. Under mild hypotheses on f and χ, the nonvanishing of the Rankin–Selberg L-function $L(f/K, \chi, s)$ at the center $s = r$ implies that the associated Bloch–Kato Selmer group is 0; and when this central L-value vanishes, the nonvanishing of a distinguished class $\kappa_{f,\chi}$ implies that the dimension of the associated Bloch–Kato Selmer group is 1. In addition, we also prove a divisibility in the Iwasawa main conjecture for $V_{f,\chi}$, both in the definite and in the indefinite settings. These results are deduced as applications of the main contribution of this paper, which is the construction of a new anticyclotomic Euler system for $V_{f,\chi}$. By exploiting the decomposition of certain triple products, our construction is based on a generalisation of the diagonal cycles introduced by Gross–Kudla [GK92] and Gross–Schoen [GS95], and studied more recently by Darmon–Rotger and Bertolini–Seveso–Venerucci (see [BDR$^+$22]).

0.1. **Main results.** Fix once and for all complex and p-adic embeddings $i_{\infty} : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{C}}$ and $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{C}}_p$. Assume that the discriminant D_K of K satisfies $(D_K, N_f) = 1$, and writing $N_f = N^+ N^-$ with N^+ (resp. N^-) divisible only by primes that are split (resp. inert) in K, assume that

(sq) N^- is squarefree.

Let Γ^- be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K. We consider anticyclotomic Hecke characters of K of the form $\chi = \chi_0 \phi$, with χ_0 a ring character such that

(cond) χ_0 has conductor $c\mathcal{O}_K$ with $(c, pN_f) = 1$,

and ϕ an anticyclotomic Hecke character of K whose p-adic avatar (still denoted ϕ) factors through Γ^-. Denote by $\nu(N^-)$ the number of prime factors of N^-. Under hypotheses (sq) and (cond), the sign $\epsilon(f, \chi)$ in the functional equation for $L(f/K, \chi, s)$ (relating its values at s and $2r - s$) depends only on the global root number of the base-change of f to K, given by

$$\epsilon(f/K) = (-1)^{\nu(N^-)+1},$$

and the infinity type, say $(-j, j)$ with $j \in \mathbf{Z}$, of χ. Because $L(f/K, \chi, s) = L(f/K, \chi^{\mathbf{c}}, s)$, where $\chi^{\mathbf{c}}$ is the composition of χ with the action of complex conjugation, without loss of generality we may assume $j \geq 0$. Accordingly, the values of $\epsilon(f, \chi)$ are as in the following table.

	$\epsilon(f/K) = -1$	$\epsilon(f/K) = +1$
$0 \leq j < k/2$	-1	$+1$
$j \geq k/2$	$+1$	-1

0.1.1. *The Euler system.* Assume that f and K satisfy the following hypotheses:

(ord) $$f \text{ is ordinary at } \mathfrak{P},$$

(spl) $$p = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits in } K,$$

with $\mathfrak{P} \mid \mathfrak{p}$ the primes of L/K above p induced by i_p, and that

(cn) $$p \nmid h_K, \text{ where } h_K \text{ is the class number of } K.$$

For every positive integer n, let $K[n]$ denote the maximal p-subextension of the ring class field of K of conductor n. Denote by $\mathcal{L}$ the set of rational primes $\ell \neq p$ split in K. For each $\ell \in \mathcal{L}$, we fix a prime $\mathfrak{l}$ of K lying above it, and let $\mathcal{N}$ be the set of squarefree products of primes $\ell \in \mathcal{L}$ coprime to $pN_f c$ (with $1 \in \mathcal{N}$ by convention, corresponding to the empty product). Let $\mathcal{O}$ be the ring of integers in the completion $L_{\mathfrak{P}}$.

Theorem A (Theorem 2.4.2). *Assume* (cond), (ord), (spl), *and* (cn). *There exists a family of cohomology classes*

$$\left\{ z_{f,\chi,m,s} \in H^1(K[mp^s], T_{f,\chi}) \mid m \in \mathcal{N}, \, s \geq 0 \right\},$$

where $T_{f,\chi}$ is a certain G_K-stable $\mathcal{O}$-lattice inside $V_{f,\chi}$, such that

$$\mathrm{Norm}_{K[mp^s]}^{K[mp^{s+1}]}(z_{f,\chi,m,s+1}) = z_{f,\chi,m,s}$$

for all $s \geq 0$, and for every $m \in \mathcal{N}$ and $\ell \in \mathcal{L}$ with $m\ell \in \mathcal{N}$, we have the tame norm relation

$$\mathrm{Norm}_{K[mp^s]}^{K[m\ell p^s]}(z_{f,\chi,m\ell,s}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) \, z_{f,\chi,m,s},$$

where $P_{\mathfrak{l}}(X) = \det(1 - \mathrm{Frob}_{\mathfrak{l}} X \mid V_{f,\chi}^{\vee}(1))$, and $\mathrm{Frob}_{\mathfrak{l}}$ is a geometric Frobenius.

The collection of classes of Theorem A defines an *anticyclotomic Euler system* in the sense of Jetchev–Nekovář–Skinner [JNS] for the conjugate self-dual representation $V_{f,\chi}$. Significantly extending Kolyvagin's methods, the general theory developed in *op. cit.* provides a machinery that bounds Selmer groups attached to conjugate self-dual representations V from the input of a non-trivial anticyclotomic Euler system. The Selmer group being bounded depends on the local conditions at the primes $w \mid p$ satisfied by the Euler system classes, and in this paper we produce in fact *two different* anticyclotomic Euler systems for $V_{f,\chi}$, differing in their local conditions at the primes above p.

To describe this further, recall that by p-ordinarity of f, the Galois representation $V_f^{\vee}$ restricted to a decomposition group $G_{\mathbf{Q}_p} \subset G_{\mathbf{Q}}$ fits into a short exact sequence

$$0 \to V_f^{\vee,+} \to V_f^{\vee} \to V_f^{\vee,-} \to 0,$$

with each $V_f^{\vee,\pm}$ 1-dimensional over $L_{\mathfrak{P}}$, and with the $G_{\mathbf{Q}_p}$-action on $V_f^{\vee,-}$ given by the unramified character sending an arithmetic Frobenius Frob_p^{-1} to α_p, the unit root of $x^2 - a_p x + p^{k-1}$. Put

$$V_{f,\chi}^{\pm} := V_f^{\vee,\pm}(1 - k/2) \otimes \chi^{-1}.$$

In terms of this, the construction in Theorem 2.4.2 yields in fact:

- An anticyclotomic Euler system $\{z_{f,\chi,m,s}^{\mathrm{ord,ord}}\}_{m,s}$ with local conditions at the primes $w \mid p$ given by

$$H_{\mathrm{ord}}^1(K[mp^s]_w, V_{f,\chi}) := \ker\left(H^1(K[mp^s]_w, V_{f,\chi}) \to H^1(K[mp^s]_w, V_{f,\chi}^-)\right).$$

- An anticyclotomic Euler system $\{z_{f,\chi,m,s}^{\mathrm{rel,str}}\}_{m,s}$ with local conditions at the primes $w \mid p$ given by

$$\begin{cases} H^1(K[mp^s]_w, V_{f,\chi}) & \text{if } w \mid \mathfrak{p}, \\ 0 & \text{if } w \mid \bar{\mathfrak{p}}. \end{cases}$$

Depending on the infinity type of χ, we show that one of these classes always lands in the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K[mp^s], V_{f,\chi})$, namely the class

$$\kappa_{f,\chi,m,s} := \begin{cases} z_{f,\chi,m,s}^{\mathrm{rel,str}} & \text{if } j \geq r, \\ z_{f,\chi,m,s}^{\mathrm{ord,ord}} & \text{if } 0 \leq j < r. \end{cases}$$

0.1.2. *Applications to the Bloch–Kato conjecture in rank* 1. Let $\kappa_{\mathfrak{P}}$ denote the residue field of $L_{\mathfrak{P}}$, and let

$$\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa_{\mathfrak{P}})$$

be the residual representation associated to f. By p-ordinarity, the restriction $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is reducible, and we say that $\bar{\rho}_f$ is p-*distinguished* when the semi-simplification of $\bar{\rho}_f|_{G_{\mathbf{Q}_p}}$ is non-scalar. Put

$$\kappa_{f,\chi} := \kappa_{f,\chi,1,0} \in \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}),$$

using that $K[1] = K$ as a consequence of (cn). From the general results of [JNS] applied to the construction of Theorem A we deduce in particular the following result.

Theorem B (Theorem 5.7.1). *Under the hypotheses of Theorem A, assume in addition that*

- (sq) *holds;*
- $\bar{\rho}_f$ *is absolutely irreducible and p-distinguished;*
- $p > 2r - 2$;
- f *is not of CM-type.*

Assume also that

$$\epsilon(f/K) = +1 \quad and \quad j \geq r,$$

which implies $L(f/K, \chi, r) = 0$. Then

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \geq 1.$$

Moreover, if the class $\kappa_{f,\chi}$ is nonzero, then

$$\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = L_{\mathfrak{P}} \cdot \kappa_{f,\chi}.$$

By the Gross–Zagier formula for the modified diagonal cycles introduced in [GK92, GS95] (a special case of the arithmetic Gan–Gross–Prasad conjecture for $\mathrm{SO}(3) \times \mathrm{SO}(4)$) proved by Yuan–Zhang–Zhang [YZZ] in certain cases, the non-triviality of $\kappa_{f,\chi}$ is expected to be governed by the nonvanishing of $L'(f/K, \chi, r)$, and hence Theorem B yields evidence towards the Bloch–Kato conjecture for $V_{f,\chi}$ in analytic rank 1.

Our methods also yield an analogue of Theorem B in the "indefinite case" $\epsilon(f/K) = -1$ and $0 \leq j < r$ (see Theorem 6.7.1), but we note that in this case such result can also be obtained from the Euler system of (generalised) Heegner cycles [Nek92, CH18a].

0.1.3. *Applications to the Bloch–Kato conjecture in rank* 0. We now turn our attention to the cases where $\epsilon(f, \chi) = +1$, so the central value $L(f/K, \chi, r)$ is expected to be generically nonzero. Put

$$\kappa'_{f,\chi,m,r} := \begin{cases} z_{f,\chi,m,r}^{\mathrm{ord,ord}} & \text{if } j \geq r, \\ z_{f,\chi,m,r}^{\mathrm{rel,str}} & \text{if } 0 \leq j < r, \end{cases} \qquad \kappa'_{f,\chi} := \kappa'_{f,\chi,1,0}.$$

Building on the explicit reciprocity law for diagonal cycles by Bertolini–Seveso–Venerucci [BSV22], we show the equivalence

$$\kappa'_{f,\chi} \in \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \quad \Longleftrightarrow \quad L(f/K, \chi, r) = 0.$$

Hence when $L(f/K, \chi, r) \neq 0$, the classes $\kappa'_{f,\chi}$ provide non-trivial annihilators of classes in $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ via global duality. Together with the general results of [JNS] applied to the construction of Theorem A extending $\kappa'_{f,\chi}$, this leads in particular to the following cases of the Bloch–Kato conjecture for $V_{f,\chi}$.

Theorem C (Theorem 5.5.1). *Under the hypotheses of Theorem A, assume in addition that*

- (sq) *holds;*
- $\bar{\rho}_f$ *is absolutely irreducible and p-distinguished;*
- $p > 2r - 2$;
- f *is not of CM-type.*

Assume also that

$$\epsilon(f/K) = +1 \quad and \quad 0 \leq j < r,$$

which implies $\epsilon(f, \chi) = +1$. Then

$$L(f/K, \chi, r) \neq 0 \quad \implies \quad \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture holds in this case.

We also obtain an analogue of Theorem C for $\epsilon(f/K) = -1$ and $j \geq r$ (see Theorem 6.5.1), but in these cases the result was previously known using generalised Heegner cycles [CH18a]. Finally, we note that our results also include the proof of a divisibility in the anticyclotomic Iwasawa main conjecture for $V_{f,\chi}$, both in the definite and in the indefinite settings, giving in particular a new proof of the main result of [BD05] (see Theorem 5.6.1) dispensing with their "level-raising" ramification hypotheses.

0.2. **Relation to previous works.** Starting with the landmark results by Gross–Zagier and Kolyvagin [GZ86, Kol88] (see also [BD90]), and followed by their vast generalisations by Zhang [Zha97], Tian [Tia03], Nekovář [Nek07], Yuan–Zhang–Zhang [YZZ13] and others, the Euler system of Heegner points and Heegner cycles has been a key ingredient in the study of the arithmetic of $V_{f,\chi}$ under the Heegner hypothesis

$$\epsilon(f/K) = -1.$$

Classical Heegner cycles account for the cases where the anticyclotomic character χ has finite order (i.e., $j = 0$), but using their new variant by Bertolini–Darmon–Prasanna [BDP13], one obtains classes controlling the arithmetic of $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ in the following cases:

(1st quadrant) $$\epsilon(f/K) = -1, \quad 0 \leq j < r.$$

In another major advance, Bertolini–Darmon [BD05] exploited congruences between modular forms on different quaternion algebras and the Cerednik–Drinfeld theory of interchange of invariants to realise the Galois representation (on finite quotients of) $T_{f,\chi}$ in the torsion of the Jacobian of certain Shimura curves. This allowed them to still use the Heegner point construction in situations where $\epsilon(f/K) = +1$. Together with the extension to higher weights by Chida–Hsieh [CH15], these methods yielded a proof of many cases of the Bloch–Kato conjecture in analytic rank 0 when

(2nd quadrant) $$\epsilon(f/K) = +1, \quad j = 0$$

under certain 'level-raising' hypotheses. More recently, the Euler system of Beilinson–Flach elements constructed by Lei–Loeffler–Zerbes [LLZ14, LLZ15] and Kings–Loeffler–Zerbes [KLZ17, KLZ20] (inspired in part by earlier results of Bertolini–Darmon–Rotger [BDR15a, BDR15b]) provided an alternative approach to similar rank 0 results (among other applications) under different mild hypotheses.

On the other hand, exploiting the variation of Heegner cycles in p-adic families, the first-named author and Hsieh [CH18a, Cas20] (see also related work by Magrone [Mag22] and Kobayashi [Kob23]), obtained results on the Bloch–Kato conjecture for $V_{f,\chi}$ in rank 0 in the cases

(3rd quadrant) $$\epsilon(f/K) = -1, \quad j \geq r.$$

Contrastingly, in the cases where

(4th quadrant) $$\epsilon(f/K) = +1, \quad j \geq r,$$

the conjectures of Beilinson–Bloch and Bloch–Kato predict the existence of nonzero classes in $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ coming from geometry (since $\epsilon(f, \chi) = -1$ and therefore $L(f/K, \chi, r) = 0$), but the construction of such classes seems to fall outside of all the aforementioned methods.

The 'degenerate' diagonal cycle Euler system constructed in this paper allows us to fill this gap, yielding new evidence towards the Bloch–Kato conjecture for $V_{f,\chi}$ in analytic rank 1 in this case, while also providing a new approach to the aforementioned results in the other cases:

	$\epsilon(f/K) = -1$	$\epsilon(f/K) = +1$
$0 \leq j < r$	1$^{\text{st}}$ quadrant [Kol88], [BD90], [Tia03], [Nek07], etc. Theorem 6.7.1	2$^{\text{nd}}$ quadrant [BD05], [CH15], [KLZ17], [KLZ17], etc. Theorem 5.5.1
$j \geq r$	3$^{\text{rd}}$ quadrant [CH18a], [Cas20], [Mag22], [Kob23], etc. Theorem 6.5.1	4$^{\text{th}}$ quadrant – Theorem 5.7.1

To obtain our anticyclotomic Euler system classes, we exploit diagonal cycles attached to triple products of modular forms with two of the factors having CM by K. This setting is also considered in the work of the first-named author with Hsieh [CH22] on the conjectures of Darmon–Rotger [DR16] in the 'adjoint CM case'. The method of construction in this paper has recently been adapted by the second-named author [Do24] to the case where two of the factors have CM by *different* imaginary quadratic fields, resulting in an anticyclotomic Euler system for modular forms based-changed to a biquadratic CM field, together with results on the Bloch–Kato conjecture, and a divisibility towards the Iwasawa–Greenberg main conjecture.

In future work, we intend to generalise our construction to totally real fields, a setting in which one finds even more cases where the arithmetic of Rankin–Selberg convolutions falls outside the scope of Heegner cycles and/or Beilinson–Flach classes.

Part 1. The Euler system

1. PRELIMINARIES

1.1. **Modular curves and Hecke operators.** We give a precise description of the modular curves and Hecke operators that will appear in our construction. The main references for this section are [Kat04, §2], [BSV22, §2], and [ACR23, §2], where more details can be found.

1.1.1. *Modular curves.* Let M, N, u, v be positive integers such that $M + N \geq 5$. Define $Y(M, N)$ to be the affine modular curve over $\mathbf{Z}[1/MN]$ representing the functor

$$S \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of triples } (E, P, Q) \text{ where } E \text{ is an elliptic curve over } S, \\ P, Q \text{ are sections of } E \text{ over } S \text{ such that } M \cdot P = N \cdot Q = 0; \text{ and the map} \\ \mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to E, \text{ sending } (a, b) \mapsto a \cdot P + b \cdot Q \text{ is injective} \end{array} \right\}$$

for $\mathbf{Z}[1/MN]$-schemes S. More generally, define the affine modular curve $Y(M(u), N(v))$ over $\mathbf{Z}[1/MNuv]$ representing the functor

$$S \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of quintuples } (E, P, Q, C, D) \text{ where } (E, P, Q) \text{ is as above,} \\ P \in C \text{ is a cyclic subgroup of } E \text{ of order } Mu, \\ Q \in D \text{ is a cyclic subgroup of } E \text{ of order } Nv \text{ such that} \\ C \text{ is complementary to } Q \text{ and } D \text{ is complementary to } P \end{array} \right\}$$

for $\mathbf{Z}[1/MNuv]$-schemes S. When either $u = 1$ or $v = 1$, we drop them from the notation.

Let $\mathbf{H}$ be the Poincaré upper half-plane and define the modular group:

$$\Gamma(M(u), N(v)) = \left\{ \gamma \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \begin{pmatrix} M & Mu \\ Nv & N \end{pmatrix} \right\}.$$

The Riemann surface $Y(M, N)(\mathbf{C})$ admits a complex uniformisation:

$$\begin{array}{ccc} (\mathbf{Z}/M\mathbf{Z})^\times \times \Gamma(M, N)\backslash\mathbf{H} & \xrightarrow{\sim} & Y(M, N)(\mathbf{C}) \\ (m, z) & \mapsto & (\mathbf{C}/\mathbf{Z} + \mathbf{Z}z, mz/M, 1/N), \end{array}$$

and similarly for $Y(M(u), N(v))(\mathbf{C})$.

Let ℓ be a prime. There is an isomorphism of $\mathbf{Z}[1/\ell MN]$-schemes:

$$\begin{array}{rcl} \varphi_\ell : Y(M, N(\ell)) & \to & Y(M(\ell), N) \\ (E, P, Q, C) & \mapsto & (E/NC, P + NC, \ell^{-1}(Q) \cap C + NC, (\ell^{-1}(\mathbf{Z} \cdot P + NC)/NC)), \end{array}$$

which under the complex uniformisation is induced by the map $(m, z) \mapsto (m, \ell \cdot z)$.

1.1.2. *Degeneracy maps.* We have the natural degeneracy maps

$$Y(M, N\ell) \xrightarrow{\mu_\ell} Y(M, N(\ell)) \xrightarrow{\nu_\ell} Y(M, N)$$
$$\downarrow{\varphi_\ell}$$
$$Y(M\ell, N) \xrightarrow{\check{\mu}_\ell} Y(M(\ell), N) \xrightarrow{\check{\nu}_\ell} Y(M, N),$$

where $\mu_\ell(E, P, Q) = (E, P, \ell \cdot Q, \mathbf{Z} \cdot Q)$, $\nu_\ell(E, P, Q, C) = (E, P, Q)$, and $\check{\mu}_\ell$, $\check{\nu}_\ell$ are defined similarly. Put

$$\mathrm{pr}_1 := \nu_\ell \circ \mu_\ell : Y(M, N\ell) \to Y(M, N),$$
$$(E, P, Q) \mapsto (E, P, \ell \cdot Q)$$

and

$$\mathrm{pr}_\ell := \check{\nu}_\ell \circ \varphi_\ell \circ \mu_\ell : Y(M, N\ell) \to Y(M, N)$$
$$(E, P, Q) \mapsto (E/N\mathbf{Z} \cdot Q, P + N\mathbf{Z} \cdot Q, Q + N\mathbf{Z} \cdot Q).$$

On the complex upper half plane $\mathbf{H}$, the map pr_1 (resp. pr_ℓ) is induced by the identity (resp. multiplication by ℓ). Moreover, $\mu_\ell, \check{\mu}_\ell, \nu_\ell, \check{\nu}_\ell, \mathrm{pr}_1, \mathrm{pr}_\ell$ are all finite étale morphisms of $\mathbf{Z}[1/MN\ell]$-schemes.

1.1.3. *Relative Tate modules and Hecke operators.* Let S be a $\mathbf{Z}[1/MN\ell p]$-scheme where p is a fixed prime. For each $\mathbf{Z}[1/MN\ell p]$-scheme X, denote the base change $X_S = X \times_{\mathbf{Z}[1/MN\ell p]} S$. Notate $A = A_X$ to be either the locally constant sheaf $\mathbf{Z}/p^m\mathbf{Z}(j)$ or the locally constant p-adic sheaf $\mathbf{Z}_p(j)$ on $X_{\text{ét}}$ for some fixed $j \in \mathbf{Z}$ and $m \geq 1$.

For the ease of notation, we may write $\cdot$ for $M(u), N(v)$ (i.e. $Y(\cdot) = Y(M(u), N(v))$). Denote by $E(\cdot)$ the universal elliptic curve over $Y(\cdot)$. Then one obtains a natural degree ℓ isogeny of universal elliptic curves under the base change by $\varphi_\ell^* E(M(\ell), N) \to Y(M, N(\ell))$:

$$\lambda_\ell : E(M, N(\ell)) \to \varphi_\ell^*(E(M(\ell), N)).$$

Denote by $v. : E(\cdot)_S \to Y(\cdot)_S$ the structure map. We also use $\nu_\ell, \check{\nu}_\ell$ and λ_ℓ for the base change to S of the corresponding degeneracy maps. Set:

$$\mathscr{T}.(A) = R^1 v_{.*}\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} A \text{ and } \mathscr{T}^*.(A) = \mathrm{Hom}_A(\mathscr{T}.(A), A)$$

where $R^q v_{.*}$ is the q-th right derivative of $v_{.*} : E(\cdot)_{\text{ét}} \to Y(\cdot)_{\text{ét}}$. When $A = \mathbf{Z}_p$, this gives the relative Tate module of the universal elliptic curve, in which case we will drop the $\mathbf{Z}_p$ from the notation.

Fix an integer $r \geq 0$. The (perfect) cup product pairing combined with the relative trace

$$\mathscr{T}. \otimes_{\mathbf{Z}_p} \mathscr{T}. \to R^2 v_{.*}\mathbf{Z}_p(2) \cong \mathbf{Z}_p(1)$$

allows us to identify $\mathscr{T}.(-1)$ with $\mathscr{T}^*.$. Put

$$\mathscr{L}_{.,r}(A) = \mathrm{TSym}_A^r \mathscr{T}.(A), \quad \mathscr{S}_{.,r}(A) = \mathrm{Sym}_A^r \mathscr{T}^*.(A),$$

where $\mathrm{TSym}_R^r M$ is the R-submodule of the symmetric tensors in $M^{\otimes r}$, $\mathrm{Sym}_R^r M$ is the maximal symmetric quotient of $M^{\otimes r}$, and M is any finite free module over a profinite $\mathbf{Z}_p$-algebra R. When the level is clear, we shall simplify the notations, e.g. writing:

$$(1.1) \qquad \mathscr{L}_r(A) = \mathscr{L}_{M(u),N(v),r}(A), \quad \mathscr{L}_r = \mathscr{L}_r(\mathbf{Z}_p), \quad \mathscr{S}_r(A) = \mathscr{S}_{M(u),N(v),r}(A), \quad \mathscr{S}_r = \mathscr{S}_r(\mathbf{Z}_p).$$

Let $\mathscr{F}_\cdot^r$ be either $\mathscr{L}_{\cdot,r}(A)$ or $\mathscr{S}_{\cdot,r}(A)$. Then there are natural isomorphisms of sheaves

$$\nu_\ell^*(\mathscr{F}_{M,N}^r) \cong \mathscr{F}_{M,N(\ell)}^r, \quad \check{\nu}_\ell^*(\mathscr{F}_{M,N}^r), \cong \mathscr{F}_{M(\ell),N}^r,$$

and these induce pullback maps

$$H_{\text{ét}}^i(Y(M,N(\ell))_S, \mathscr{F}_{M,N(\ell)}^r) \xleftarrow{\nu_\ell^*} H_{\text{ét}}^i(Y(M,N)_S, \mathscr{F}_{M,N}^r) \xrightarrow{\check{\nu}_\ell^*} H_{\text{ét}}^i(Y(M(\ell),N)_S, \mathscr{F}_{M(\ell),N}^r)$$

and traces

$$H_{\text{ét}}^i(Y(M,N(\ell))_S, \mathscr{F}_{M,N(\ell)}^r) \xrightarrow{\nu_{\ell*}} H_{\text{ét}}^i(Y(M,N)_S, \mathscr{F}_{M,N}^r) \xleftarrow{\check{\nu}_{\ell*}} H_{\text{ét}}^i(Y(M(\ell),N)_S, \mathscr{F}_{M(\ell),N}^r).$$

The finite étale isogeny λ_ℓ induces morphisms

$$\lambda_{\ell*} : \mathscr{F}_{M,N(\ell)}^r \to \varphi_\ell^*(\mathscr{F}_{M(\ell),N}^r), \quad \lambda_\ell^* : \varphi_\ell^*(\mathscr{F}_{M(\ell),N}^r) \to \mathscr{F}_{M,N(\ell)}^r$$

and this allows us to define a pushforward

$$\Phi_{\ell*} := \varphi_{\ell*} \circ \lambda_{\ell*} : H_{\text{ét}}^i(Y(M,N(\ell))_S, \mathscr{F}_{M,N(\ell)}^r) \to H_{\text{ét}}^i(Y(M(\ell),N)_S, \mathscr{F}_{M(\ell),N}^r)$$

and a pullback

$$\Phi_\ell^* := \lambda_\ell^* \circ \varphi_\ell^* : H_{\text{ét}}^i(Y(M(\ell),N)_S, \mathscr{F}_{M(\ell),N}^r) \to H_{\text{ét}}^i(Y(M,N(\ell))_S, \mathscr{F}_{M,N(\ell)}^r).$$

The Hecke operator T_ℓ and the dual Hecke operator T_ℓ' acting on $H_{\text{ét}}^i(Y(M,N)_S, \mathscr{F}_{M,N}^r)$ are defined by

$$T_\ell := \check{\nu}_{\ell*} \circ \Phi_{\ell*} \circ \nu_\ell^*, \quad T_\ell' := \nu_{\ell*} \circ \Phi_\ell^* \circ \check{\nu}_\ell^*.$$

Remark 1.1.1. Note the relations

$$\deg(\mu_\ell) T_\ell = \mathrm{pr}_{\ell*} \circ \mathrm{pr}_1^*, \quad \deg(\mu_\ell) T_\ell' = \mathrm{pr}_{1*} \circ \mathrm{pr}_\ell^*,$$

as it follows immediately from the definitions.

For $d \in (\mathbf{Z}/MN\mathbf{Z})^*$, the diamond operator $\langle d \rangle$ on $Y(\cdot)$ is defined in terms of moduli by

$$(E, P, Q, C, D) \mapsto (E, d^{-1} \cdot P, d \cdot Q, C, D).$$

There is also a unique diamond operator $\langle d \rangle$ on the universal elliptic curve making the following diagram cartesian:

$$
\begin{array}{ccc}
E(\cdot)_S & \xrightarrow{\langle d \rangle} & E(\cdot)_S \\
\Big\downarrow{\scriptstyle v.} & & \Big\downarrow{\scriptstyle v.} \\
Y(\cdot)_S & \xrightarrow{\langle d \rangle} & Y(\cdot)_S,
\end{array}
$$

and this induces automorphisms $\langle d \rangle = \langle d \rangle^*$ and $\langle d \rangle' = \langle d \rangle_*$ on $H_{\text{ét}}^i(Y(\cdot)_S, \mathscr{F}_\cdot)$.

For any profinite $\mathbf{Z}_p$-algebra R and finite free R-module M, the evaluation map induces a perfect pairing

$$\mathrm{TSym}_R^r M \otimes_R \mathrm{Sym}_R^r M^* \to R,$$

where $M^* = \mathrm{Hom}_R(M, \mathbf{Z}_p)$. This gives a perfect pairing $\mathscr{L}_r \otimes_{\mathbf{Z}_p} \mathscr{S}_r \to \mathbf{Z}_p$, and therefore a cup product

$$\langle \cdot, \cdot \rangle : H_{\text{ét}}^1(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{L}_r(1)) \otimes_{\mathbf{Z}_p} H_{\text{ét},c}^1(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{S}_r) \to H_{\text{ét}}^2(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p,$$

which is perfect after inverting p. Moreover, the Hecke operators $T_\ell, T_\ell', \langle d \rangle, \langle d \rangle'$ induce endomorphisms on the compactly supported cohomology groups $H_{\text{ét},c}^1(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{S}_r)$, and by construction, (T_ℓ, T_ℓ') and $(\langle d \rangle, \langle d \rangle')$ are adjoint pairs under $\langle \cdot, \cdot \rangle$. The Eichler–Shimura isomorphism [Shi94]

$$H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbf{Q}}}, \mathscr{L}_r) \otimes_{\mathbf{Z}_p} \mathbf{C} \cong M_{r+2}(N, \mathbf{C}) \oplus \overline{S_{r+2}(N, \mathbf{C})}$$

commutes with the action of the Hecke operators on both sides.

1.2. **Galois representations associated to newforms.** Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a normalised newform of weight $k \geq 2$, level $\Gamma_1(N_f)$, and nebentype χ_f. Let $p \nmid N_f$ be an odd prime. Fix embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and let $L/\mathbf{Q}$ be a finite extension containing all values $i_\infty^{-1}(a_n)$ and $i_\infty^{-1} \circ \chi_f$. Let $\mathfrak{P}$ be the prime of L above p with respect to i_p. Then Eichler–Shimura and Deligne construct a p-adic Galois representation associated to f:

$$\rho_f = \rho_{f,\mathfrak{P}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(L_{\mathfrak{P}})$$

which is unramified outside pN_f, and characterised by the property for all finite primes $\ell \nmid pN_f$,

$$\mathrm{trace}(\rho_f(\mathrm{Frob}_\ell)) = i_p(a_\ell), \quad \det(\rho_f(\mathrm{Frob}_\ell)) = i_p(\chi_f(\ell)l^{k-1}),$$

where Frob_ℓ is a geometric Frobenius. Moreover, $\rho_{f,\mathfrak{P}}$ is known to be irreducible [Rib77], hence absolutely irreducible since the image of the complex conjugation has eigenvalues 1 and -1.

1.2.1. *Geometric realisations.* The representation $\rho_{f,\mathfrak{P}}$ can be realised geometrically as the largest subspace V_f of

$$H^1_{\text{ét},c}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{P}}$$

on which T_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d \rangle = \langle d \rangle^*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbf{Z}/N_f\mathbf{Z})^\times$. If N is any multiple of N_f, then the above subspace with N_f replaced by N gives rise to a representation $V_f(N)$ isomorphic (non-canonically) to a finite number of copies of V_f.

The dual $V_f^\vee = \mathrm{Hom}(V_f, L_{\mathfrak{P}})$ can be interpreted as the maximal quotient of

$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes L_{\mathfrak{P}}$$

on which the dual Hecke operator T'_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d \rangle' = \langle d \rangle_*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbf{Z}/N_f\mathbf{Z})^\times$.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. In this paper we shall be mostly working with $V_f^\vee$ and the $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$ defined as the image of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes \mathcal{O}$ in $V_f^\vee$.

1.2.2. *The p-ordinary case.* If f is ordinary at p, i.e. $i_p(a_p) \in \mathcal{O}^\times$, then the restriction of V_f to $G_{\mathbf{Q}_p} \subset G_{\mathbf{Q}}$ is reducible, fitting into an exact sequence of $L_{\mathfrak{P}}[G_{\mathbf{Q}_p}]$-modules

$$0 \to V_f^+ \to V_f \to V_f^- \to 0$$

with $\dim_{L_{\mathfrak{P}}} V_f^\pm = 1$, and with $G_{\mathbf{Q}_p}$-action on the subspace V_f^+ given by the unramified character sending Frob_p to α_p, the unit root of $x^2 - a_p x + \chi_f(p)p^{k-1}$. By duality, we also obtain an exact sequence for $V_f^\vee$ restricted to $G_{\mathbf{Q}_p}$

(1.2) $$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0$$

with $V_f^{\vee,+} \simeq (V_f^-)^\vee(1-k)(\chi_f^{-1})$, and with the $G_{\mathbf{Q}_p}$-action on the quotient $V_f^{\vee,-}$ given by the unramified character sending arithmetic Frobenius Frob_p^{-1} to α_p.

1.3. **Patched CM Hecke modules.** In this section, after explaining our conventions on Hecke characters, we recall the construction of certain patched CM Hecke modules from [LLZ15, §5.2].

1.3.1. *Hecke characters and theta series.* Let K be an imaginary quadratic field of discriminant $-D_K < 0$ in which $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits, with $\mathfrak{p}$ the prime above p induced by i_p.

We say that a Hecke character $\psi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ has infinity type $(a, b) \in \mathbf{Z}^2$ if $\psi_\infty(x_\infty) = x_\infty^a \bar{x}_\infty^b$ for all $x_\infty \in K \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathbb{C}$ under the identification induced by i_∞. Then the character sending $x \mapsto \psi(x)x_\infty^{-a}\bar{x}_\infty^{-b}$ is a ray class character, hence it takes value in a finite extension L/K. For $\mathfrak{P} \mid \mathfrak{p}$ the prime of L above p induced by i_p, we define the *p-adic avatar* $\psi_{\mathfrak{P}} : G_K \to L_{\mathfrak{P}}^\times$ of ψ as follows. Let $\mathrm{rec}_K : \mathbb{A}_K^\times/K^\times \to G_K^{\mathrm{ab}}$ be the *geometrically* normalised Artin reciprocity map. For $g \in G_K$, we take $x \in \mathbb{A}_K^\times$ such that $\mathrm{rec}_K(x) = g|_{K^{\mathrm{ab}}}$ and define

$$\psi_{\mathfrak{P}}(g) = i_p \circ i_\infty^{-1}(\psi(x)x_\infty^{-a}\bar{x}_\infty^{-b})x_{\mathfrak{p}}^a x_{\bar{\mathfrak{p}}}^b.$$

Since there should be no confusion, in the following we shall also use ψ to denote its p-adic avatar.

Let ψ be a Hecke character of K of infinity type $(-1, 0)$, conductor dividing $\mathfrak{f}$, and with $x \mapsto \psi(x)x_\infty$ taking values in a finite extension L/K. Viewing ψ as an L-valued character on the group of fractional ideals of K coprime to $\mathfrak{f}$, the theta series attached to ψ is

$$\theta_\psi = \sum_{(\mathfrak{a}, \mathfrak{f}) = 1} \psi(\mathfrak{a}) q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in S_2(\Gamma_1(N_\psi), \chi_\psi \epsilon_K)$$

where $N_\psi = N_{K/\mathbf{Q}}(\mathfrak{f})D_K$, χ_ψ is the unique Dirichlet character modulo $N_{K/\mathbf{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi_\psi(n)$ for all $n \in \mathbf{Z}$ with $(n, N_{K/\mathbf{Q}}(\mathfrak{f})) = 1$, and ϵ_K is the quadratic Dirichlet character attached to $K/\mathbf{Q}$. The cuspform θ_ψ is new of level N_ψ if $\mathfrak{f}$ is the exact conductor of ψ, and its p-adic $G_\mathbf{Q}$-representation satisfies

$$V_{\theta_\psi} \cong \operatorname{Ind}_K^\mathbf{Q} L_\mathfrak{P}(\psi), \qquad V_{\theta_\psi}^\vee \cong \operatorname{Ind}_K^\mathbf{Q} L_\mathfrak{P}(\psi^{-1}).$$

1.3.2. *Hecke algebras and norm maps.* Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal divisible by $\mathfrak{f}$. Put $N = N_{K/\mathbf{Q}}(\mathfrak{n})D_K$. Let $K_\mathfrak{n}$ be the ray class field of K with conductor $\mathfrak{n}$, and $H_\mathfrak{n} = \operatorname{Gal}(K_\mathfrak{n}/K)$ be the ray class group of K modulo $\mathfrak{n}$. For an ideal $\mathfrak{k}$ of K coprime to $\mathfrak{n}$, let $[\mathfrak{k}]$ be the class of $\mathfrak{k}$ in $H_\mathfrak{n}$.

We denote by $K(\mathfrak{n})$ the largest p-extension of K contained in $K_\mathfrak{n}$, so $\operatorname{Gal}(K(\mathfrak{n})/K) \cong H_\mathfrak{n}^{(p)}$, where $H_\mathfrak{n}^{(p)}$ is the largest p-power quotient of $H_\mathfrak{n}$.

Proposition 1.3.1 ([LLZ15, Prop. 3.2.1]). *Let $\mathbb{T}'(N)$ be the subalgebra of $\operatorname{End}_\mathbf{Z}(H^1(Y_1(N)(\mathbf{C}), \mathbf{Z}))$ generated by $\langle d \rangle'$ and T_ℓ' for all primes ℓ. There exists a homomorphism $\phi_\mathfrak{n} : \mathbb{T}'(N) \to \mathcal{O}[H_\mathfrak{n}]$ defined by*

$$\phi_\mathfrak{n}(T_\ell') = \sum_\mathfrak{l} [\mathfrak{l}]\psi(\mathfrak{l}),$$

$$\phi_\mathfrak{n}(\langle d \rangle') = \chi_\psi(d)\epsilon_K(d)[(d)],$$

where the sum is over the ideals $\mathfrak{l} \subset \mathcal{O}_K$ with $\mathfrak{l} \nmid \mathfrak{n}$ and $N_{K/\mathbf{Q}}(\mathfrak{l}) = \ell$.

For $\mathfrak{n}' = \mathfrak{nl}$, with $\mathfrak{l}$ a prime, put $N' = N_{K/\mathbf{Q}}(\mathfrak{n}')D_K$. As in [LLZ15, §3.3], we consider the norm maps

$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} : \mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N') \otimes \mathbf{Z}_p, \phi_{\mathfrak{n}'}} H_{\text{ét}}^1(Y_1(N')_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \to \mathcal{O}[H_\mathfrak{n}^{(p)}] \otimes_{\mathbb{T}'(N) \otimes \mathbf{Z}_p, \phi_\mathfrak{n}} H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))$$

defined by the following formulae:

- If $\mathfrak{l} \mid \mathfrak{n}$ then

$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \operatorname{pr}_{1*};$$

- If $\mathfrak{l} \nmid \mathfrak{n}$ is split or ramified in K, then

$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \operatorname{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \operatorname{pr}_{\ell*};$$

- If $\mathfrak{l} \nmid \mathfrak{n}$ is inert in K, say $\mathfrak{l} = (\ell)$, then

$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \operatorname{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell^2} \otimes \operatorname{pr}_{\ell\ell*},$$

  where $\operatorname{pr}_{\ell\ell} : Y(N') \to Y(N)$ denotes the degeneracy map induced by $z \mapsto \ell^2 z$, for z on the complex upper half plane $\mathbf{H}$.

By composition, the definition of $\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'}$ is extended to any pair of ideals $\mathfrak{n} \mid \mathfrak{n}'$.

1.3.3. *Patching.* Note that since p splits in K, the restriction $V_{\theta_\psi}|_{G_{\mathbf{Q}_p}}$ is reducible.

Definition 1.3.2. We say that ψ is *p-distinguished* if the residual representation $\bar{\rho}_{\theta_\psi}$ is such that $\bar{\rho}_{\theta_\psi}|_{G_{\mathbf{Q}_p}}$ has non-scalar semi-simplification. And we say that $(\psi, \mathfrak{f})$ satisfies *Condition* ♠ is the following hold:

- ψ is p-distinghished;
- if $\mathfrak{p} \mid \mathfrak{f}$ then $\bar{\mathfrak{p}} \nmid \mathfrak{f}$.

It follows from the proof of [LLZ15, Prop. 5.1.2] that if $(\psi, \mathfrak{f})$ satisfies Condition ♠, then for any ideal $\mathfrak{n} \subset \mathcal{O}_K$ divisible by $\mathfrak{f}$ and $\mathfrak{p}$ and with $(\mathfrak{n}, \bar{\mathfrak{p}}) = 1$, the maximal ideal of $\mathbb{T}'(N)$ defined by the kernel of the composite map

$$\mathbb{T}'(N) \xrightarrow{\phi_{\mathfrak{n}}} \mathcal{O}[H_{\mathfrak{n}}] \xrightarrow{\mathrm{aug}} \mathcal{O} \to \mathcal{O}/\mathfrak{P},$$

where $\phi_{\mathfrak{n}}$ is the map from Proposition 1.3.1, is non-Eisenstein, $\mathfrak{P}$-ordinary, and p-distinguished in the sense of Definitions 4.1.2 and 4.3.3 of [LLZ15]. Through its use in [*op. cit.*, Prop. 5.2.5], this condition is to allow $\mathfrak{f}$ and $\mathfrak{m}$ to be possibly divisible by $\mathfrak{p}$ in the following result.

Theorem 1.3.3. *Let $\mathcal{A}^{(\bar{\mathfrak{p}})}$ be the set of ideals $\mathfrak{m} \subset \mathcal{O}_K$ with $\bar{\mathfrak{p}} \nmid \mathfrak{m}$, and put*

$$\mathcal{A} := \begin{cases} \mathcal{A}^{(\bar{\mathfrak{p}})} & \text{if } (\psi, \mathfrak{f}) \text{ satisfies Condition ♠,} \\ \{\mathfrak{m} \in \mathcal{A}^{(\bar{\mathfrak{p}})} : \mathfrak{p} \nmid \mathfrak{m}\} & \text{otherwise,} \end{cases}$$

and $\mathcal{A}_{\mathfrak{f}} = \{\mathfrak{fm} : \mathfrak{m} \in \mathcal{A}\}$. Then there is a family of $G_{\mathbb{Q}}$-equivariant isomorphisms of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$-modules

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\cong} \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

indexed by $\mathfrak{n} \in \mathcal{A}_{\mathfrak{f}}$, such that for any $\mathfrak{n}, \mathfrak{n}' \in \mathcal{A}_{\mathfrak{f}}$ with $\mathfrak{n} \mid \mathfrak{n}'$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N') \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}'}} H^1_{\text{ét}}(Y_1(N')_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\nu_{\mathfrak{n}'}} & \mathrm{Ind}_{K(\mathfrak{n}')}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} \downarrow & & \mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'} \downarrow \\ \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N) \otimes \mathbb{Z}_p, \phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\nu_{\mathfrak{n}}} & \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}), \end{array}$$

where $\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'}$ is the natural norm map.

Proof. This is a reformulation of Corollary 5.2.6 in [LLZ15] in the case where p splits in K. ☐

1.4. Diagonal classes.

We sketch the construction of the diagonal classes in the triple product of modular curves $Y_1(N)$ using classical invariant theory, following Section 3 in [BSV22].

With the notations of §1.1.3, we put $Y_1(N) = Y_1(N)_{\mathbb{Q}}$, and let $E_1(N) = E_1(N)_{\mathbb{Q}}$ denote the universal elliptic curve over $Y_1(N)$, together with the structural map $v : E_1(N) \to Y_1(N)$. Let $\mathscr{T} = R^1 v_* \mathbb{Z}_p(1)$ be the relative Tate module of the universal elliptic curve, and let $\mathscr{T}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{T}, \mathbb{Z}_p)$ be its dual. The cup product pairing combined with the relative trace

$$\mathscr{T} \otimes_{\mathbb{Z}_p} \mathscr{T} \to R^2 v_* \mathbb{Z}_p(2) \cong \mathbb{Z}_p(1)$$

gives a perfect relative Weil pairing

$$\langle -, - \rangle_{E_1(N)_{p^\infty}} : \mathscr{T} \otimes_{\mathbb{Z}_p} \mathscr{T} \to \mathbb{Z}_p(1),$$

which allows $\mathscr{T}(-1)$ to be identified with $\mathscr{T}^*$.

For a fixed geometric point $\eta : \mathrm{Spec}(\overline{\mathbb{Q}}) \to Y_1(N)$, denote by $\mathcal{G}_\eta = \pi_1^{\text{ét}}(Y_1(N), \eta)$ the fundamental group of $Y_1(N)$ with base point η. The stalk of $\mathscr{T}$ at η, denoted $\mathscr{T}_\eta$, is a free $\mathbb{Z}_p$-module of rank 2, equipped with a continuous action of $\mathcal{G}_\eta$. Fix a choice of $\mathbb{Z}_p$-module isomorphism $\zeta : \mathscr{T}_\eta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $\langle x, y \rangle_{E_1(N)_{p^\infty}} = \zeta(x) \wedge \zeta(y)$ (where we identify $\bigwedge^2 \mathbb{Z}_p^2$ with $\mathbb{Z}_p$ via $(1,0) \wedge (0,1) = 1$). One then obtains a continuous group homomorphism:

$$\rho_\eta : \mathcal{G}_\eta \to \mathrm{Aut}_{\mathbb{Z}_p}(\mathscr{T}_\eta) \cong \mathrm{GL}_2(\mathbb{Z}_p).$$

By [FK88, Prop. A I.8], the category of locally constant p-adic sheaves on $Y_1(N)_{\text{ét}}$ is equivalent to the category of p-adic representations of $\mathcal{G}_\eta$ via $\mathscr{F} \mapsto \mathscr{F}_\eta$. Using ρ_η, one can associate with every continuous representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ on a free finite $\mathbb{Z}_p$-module M a smooth sheaf $M^{\text{ét}}$ on $Y_1(N)$ such that $M_\eta^{\text{ét}} = M$.

Let $S_i(A)$ be the set of 2-variable homogeneous polynomials of degree i in $A[x_1, x_2]$ equipped with the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ by $gP(x_1, x_2) = P((x_1, x_2) \cdot g)$ for all $g \in \mathrm{GL}_2(\mathbb{Z}_p)$ and $P \in S_i(A)$. Its A-linear dual

$L_i(A)$ is also equipped with a $\mathrm{GL}_2(\mathbf{Z}_p)$-action by $g\tau(P(x_1, x_2)) = \tau(g^{-1}P(x_1, x_2))$ for all $g \in \mathrm{GL}_2(\mathbf{Z}_p)$, $P \in S_i(A)$, and $\tau \in L_i(A)$. As sheaves on $Y_1(N)_\mathbf{Q}$, one has

$$L_i(A)^{\text{ét}} = \mathscr{L}_i(A) \quad \text{and} \quad S_i(A)^{\text{ét}} = \mathscr{S}_i(A).$$

Hence $\mathscr{T}_\eta \cong L_1(\mathbf{Z}_p)$ and $\mathbf{Z}_p(1)_\eta \cong \bigwedge^2 \mathscr{T}_\eta \cong \det^{-1}$. This implies that for any $j \in \mathbf{Z}$, and any p-adic representation M of $\mathrm{GL}_2(\mathbf{Z}_p)$:

$$(1.3) \qquad H^0(\mathrm{GL}_2(\mathbf{Z}_p), M \otimes \det^{-j}) \hookrightarrow H^0(\mathcal{G}_\eta, M \otimes \det^{-j}) \cong H^0_{\text{ét}}(Y_1(N), M^{\text{ét}}(j)).$$

Assumption 1.1. Let $\mathbf{r} = (r_1, r_2, r_3)$ be such that $r_i \in \mathbf{Z}_{\geq 0}$, $(r_1 + r_2 + r_3)/2 = r \in \mathbf{Z}_{\geq 0}$, and $r_i + r_j \geq r_k$ for all permutations (i, j, k) of $(1, 2, 3)$. We call this a *balanced triple*.

Under the Assumption 1.1, let

$$S_\mathbf{r} = S_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_3}(\mathbf{Z}_p)$$

with its natural $\mathrm{GL}_2(\mathbf{Z}_p)$-representation from above, and let

$$\mathscr{S}_\mathbf{r} = S_\mathbf{r}^{\text{ét}} = \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}(\mathbf{Z}_p).$$

We identify $S_\mathbf{r}$ with the module of 6-variable polynomials $\mathbf{Z}_p[x_1, x_2, y_1, y_2, z_1, z_2]$ which are homogeneous of degree r_1, r_2, and r_3 in the variables (x_1, x_2), (y_1, y_2), and (z_1, z_2) respectively. By the Clebsch–Gordan decomposition of classical invariant theory, the following is a $\mathrm{GL}_2(\mathbf{Z}_p)$-invariant of $S_\mathbf{r} \otimes \det^{-r}$:

$$\mathrm{Det}_N^\mathbf{r} := \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1},$$

i.e. $\mathrm{Det}_N^\mathbf{r} \in H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_\mathbf{r} \otimes \det^{-r})$; we denote its image under (1.3) as

$$(1.4) \qquad \mathtt{Det}_N^\mathbf{r} \in H^0_{\text{ét}}(Y_1(N), \mathscr{S}_\mathbf{r}(r)).$$

Let $p_j : Y_1(N)^3 \to Y_1(N)$ for $j \in \{1, 2, 3\}$ be the natural projections and denote

$$\mathscr{S}_{[\mathbf{r}]} := p_1^* \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_2^* \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_3^* \mathscr{S}_{r_3}(\mathbf{Z}_p),$$

and

$$\mathtt{W}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}^3, \mathscr{S}_{[\mathbf{r}]}(r+2)), \quad W_{N,\mathbf{r}} = \mathtt{W}_{N,\mathbf{r}} \otimes \mathbf{Q}_p.$$

Because $Y_1(N)_{\overline{\mathbf{Q}}}$ is a smooth affine curve over $\overline{\mathbf{Q}}$, we have $H^4_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}^3, \mathscr{S}_{[\mathbf{r}]}(r+2)) = 0$. Hence by the Hochschild–Serre spectral sequence,

$$H^p(\mathbf{Q}, H^q_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}^3, \mathscr{S}_{[\mathbf{r}]}(r+2))) \implies H^{p+q}_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}^3, \mathscr{S}_{[\mathbf{r}]}(r+2))$$

one obtains

$$\mathtt{HS} : H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\mathbf{r}]}(r+2)) \to H^1(\mathbf{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

If we let $d : Y_1(N) \to Y_1(N)^3$ be the diagonal embedding, then there is a natural isomorphism $d^* \mathscr{S}_{[\mathbf{r}]} \cong \mathscr{S}_\mathbf{r}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$. As d is an embedding of codimension 2, there is a pushforward map

$$d_* : H^0_{\text{ét}}(Y_1(N), \mathscr{S}_\mathbf{r}(r)) \to H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_\mathbf{r}(r+2)),$$

and we can consider the class

$$(\mathtt{HS} \circ d_*)(\mathtt{Det}_N^\mathbf{r}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

Dually, by the bilinear form $\det^* : L_i(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} L_i(\mathbf{Z}_p) \to \mathbf{Z}_p \otimes \det^{-i}$ defined by $\det^*(\tau \otimes \sigma) = \tau \otimes \sigma((x_1 y_2 - x_2 y_1)^i)$ that becomes perfect after inverting p, we obtain an isomorphism of $\mathrm{GL}_2(\mathbf{Z}_p)$-modules

$$(1.5) \qquad \mathbf{s}_i : S_i(\mathbf{Q}_p) \cong L_i(\mathbf{Q}_p) \otimes \det^i,$$

and so $\mathbf{s}_i : \mathscr{S}_i(\mathbf{Q}_p) \cong \mathscr{L}_i(\mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-i)$ by the above equivalence of categories. We similarly define the sheaves $\mathscr{L}_\mathbf{r}$ on $Y_1(N)$ and $\mathscr{L}_{[\mathbf{r}]}$ on $Y_1(N)^3$, and set

$$\mathtt{V}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}^3, \mathscr{L}_{[\mathbf{r}]}(2-r)), \quad V_{N,\mathbf{r}} = \mathtt{V}_{N,\mathbf{r}} \otimes \mathbf{Q}_p.$$

Let $\mathbf{s_r} = \mathbf{s_{r_1}} \otimes \mathbf{s_{r_2}} \otimes \mathbf{s_{r_3}}$, which gives an isomorphism $W_{N,\mathbf{r}} \to V_{N,\mathbf{r}}$, and finally as in [BSV22] put

(1.6) $$\kappa_{N,\mathbf{r}} := (\mathbf{s_{r*}} \circ \mathtt{HS} \circ d_*)(\mathtt{Det}_N^{\mathbf{r}}) \in H^1(\mathbf{Q}, V_{N,\mathbf{r}}).$$

As explained in detail in [*loc. cit.*, §3.2], the class $\kappa_{N,\mathbf{r}}$ is closely related to the p-adic étale Abel–Jacobi image of the generalised Gross–Kudla–Schoen diagonal cycles on Kuga–Sato varieties studied in [DR14].

Proposition 1.4.1. *For a prime number ℓ and a positive integer m, if $(m\ell, N) = 1$ then*

$$(\mathrm{pr}_{i*}, \mathrm{pr}_{j*}, \mathrm{pr}_{k*})\kappa_{Nm\ell,\mathbf{r}} = (\bigstar)\kappa_{Nm,\mathbf{r}}$$

where

(i,j,k)	$\bigstar$
$(\ell,1,1)$	$(\ell-1)(T_\ell, 1, 1)$
$(1,\ell,1)$	$(\ell-1)(1, T_\ell, 1)$
$(1,1,\ell)$	$(\ell-1)(1, 1, T_\ell)$
$(1,\ell,\ell)$	$\ell^{r-r_1}(\ell-1)(T'_\ell, 1, 1)$
$(\ell,1,\ell)$	$\ell^{r-r_2}(\ell-1)(1, T'_\ell, 1)$
$(\ell,\ell,1)$	$\ell^{r-r_3}(\ell-1)(1, 1, T'_\ell)$

If $(\ell, m) = 1$ then we also have

(i,j,k)	$\bigstar$
$(1,1,1)$	$(\ell^2 - 1)$
(ℓ,ℓ,ℓ)	$(\ell^2 - 1)\ell^r$

Proof. See equations (174) and (176) in [BSV22]. (The proof of the above relations in *op. cit.* is given for $\ell = p$, but the same argument applies for any prime ℓ as above.) $\square$

2. Main theorems

In this section, for a newform $f \in S_{2r}(\Gamma_0(N_f))$ of weight $2r \geq 2$ and a family of anticyclotomic Hecke characters χ of K, we construct a family of cohomology classes for the conjugate self-dual representation

$$V_{f,\chi} := V_f^\vee(1-r) \otimes \chi^{-1}$$

defined over ring class extensions of K, and prove that they satisfy the Euler system norm relations.

The construction builds on the results from [BSV22] and [LLZ15] recalled in the preceding section, and is done in two steps: we consider (suitable modifications of) diagonal classes attached to triples $(f, \theta_{\psi_1}, \theta_{\psi_2})$ consisting of our fixed newform f and a pair of theta series $\theta_{\psi_1}, \theta_{\psi_2}$ attached to Hecke characters ψ_1, ψ_2 of K satisfying the self-duality condition

$$\chi_{\psi_1}\chi_{\psi_2} = 1,$$

and first give the construction in the case where $(f, \theta_{\psi_1}, \theta_{\psi_2})$ have weights $(2, 2, 2)$, resulting in the construction of classes

$$z_{f,\psi,\psi_2,m} \in H^1(K[m], T_{f,\psi_1\psi_1\mathbf{N}^{-1}}), \qquad {}^{\mathbf{c}}z_{f,\psi,\psi_2,m} \in H^1(K[m], T_{f,\psi_1\psi_2^{\mathbf{c}}\mathbf{N}^{-1}}),$$

where $\psi_2^{\mathbf{c}}$ is the composition of ψ_2 with the action of the non-trivial automorphism of $K/\mathbf{Q}$, for which we prove the tame norm relations (see Theorem 2.2.6 and Corollary 2.3.1).

Replacing $(\theta_{\psi_1}, \theta_{\psi_2})$ by a pair of CM Hida families $(\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2))$ attached to ray class characters ξ_1, ξ_2 of K satisfying $\chi_{\xi_1}\chi_{\xi_2} = 1$, and considering (suitable modifications of) their associated 'big' diagonal classes, we extend the construction to all weights $2r \geq 2$ and more general anticyclotomic Hecke characters, and deduce the proof of the wild norm relations (see Theorem 2.4.2).

Throughout we use the notations introduced in the preceding sections, so in particular K is an imaginary quadratic field such that

(spl) $$p = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits in } K,$$

with $\mathfrak{p}$ the prime of K above p induced by i_p. Further, we assume that $p \nmid 6N_f$ and that

(cn) $$p \nmid h_K,$$

where $h_K = |H_1|$ is the class number of K

2.1. Construction for weight $(2, 2, 2)$.

Suppose f has weight 2, and ψ_1, ψ_2 are Hecke characters of K of infinity type $(-1, 0)$ and modulus $\mathfrak{f}_1, \mathfrak{f}_2 \subset \mathcal{O}_K$ with

$$(p, \mathfrak{f}_i) = 1$$

for $i = 1, 2$, and satisfying $\chi_{\psi_1} \chi_{\psi_2} = 1$. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$ and for every positive integer m put

$$Y(m) := Y(1, Nm) = Y_1(Nm).$$

When $m = 1$, we drop it from the notation, so $Y := Y_1(N)$. We begin with the cohomology class

$$(2.1) \qquad \tilde{\kappa}_m^{(1)} := \kappa_{Nm, \mathbf{r}} \in H^1\big(\mathbf{Q}, H^3_{\text{ét}}(Y(m)^3_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(2))\big)$$

of (1.6), where $\mathbf{r} = (0, 0, 0)$, and put

$$\tilde{\kappa}_m^{(2)} = (\mathrm{pr}_{m*}, 1, 1)\tilde{\kappa}_m^{(1)} \in H^1\big(\mathbf{Q}, H^3_{\text{ét}}(Y_{\overline{\mathbf{Q}}} \times Y(m)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(2))\big),$$

where, writing $m = \prod_i \ell_i$ as a product of primes ℓ_i, pr_{m*} is defined as the composition of the pushforward by the degeneracy maps pr_{ℓ_i}. Applying the Künneth decomposition theorem [Mil13, Thm. 22.4], the class $\tilde{\kappa}_m^{(2)}$ is projected to

$$(2.2) \qquad \tilde{\kappa}_m^{(3)} \in H^1\big(\mathbf{Q}, H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y(m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y(m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))(-1)\big).$$

Now we restrict to squarefree integers $m > 0$ divisible only by primes split in K with $(m, pN) = 1$, and write

$$m = \mathfrak{m}\overline{\mathfrak{m}}$$

according to a *fixed* choice of splitting $\ell = \mathfrak{l}\overline{\mathfrak{l}}$ for each prime $\ell \mid m$. We also fix a triple of level-N test vectors

$$(\breve{f}, \breve{\theta}_{\psi_1}, \breve{\theta}_{\psi_2}) \in S_2(\Gamma_0(N))[f] \times S_2(\Gamma_1(N))[\theta_{\psi_1}] \times S_2(\Gamma_1(N))[\theta_{\psi_2}].$$

(Even though it will not be reflected in the notation, our construction will depend on this choice; in later applications we shall specify the choice of test vectors when needed.)

Since the maps used in the construction $\tilde{\kappa}_m^{(3)}$ are compatible with correspondences, after tensoring with $\mathcal{O}$ and taking $(f, \theta_{\psi_1}, \theta_{\psi_2})$-isotypic components, the above process and the choice of test vectors give rise to a class

$$\tilde{\kappa}^{(4)}_{f, \psi_1, \psi_2, m} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1} m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_1 \mathfrak{m}}]$$

$$\otimes H^1_{\text{ét}}(Y_1(N_{\psi_2} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2} m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_2 \overline{\mathfrak{m}}}](-1)\big),$$

where the labeled tensor products are with respect to the Hecke algebra homomorphisms

$$(2.3) \qquad \phi_{\mathfrak{f}_1 \mathfrak{m}} : \mathbb{T}'(N_{\psi_1} m) \to \mathcal{O}[H^{(p)}_{\mathfrak{f}_1 \mathfrak{m}}], \quad \phi_{\mathfrak{f}_2 \overline{\mathfrak{m}}} : \mathbb{T}'(N_{\psi_2} m) \to \mathcal{O}[H^{(p)}_{\mathfrak{f}_2 \overline{\mathfrak{m}}}]$$

of Proposition $1.3.1$, and we used $\breve{f}$ to take the image under the projection $H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))[f] \to T_f^\vee$ in the first factor, and similarly for $H^1_{\text{ét}}(Y(m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))[\theta_{\psi_i}] \to H^1_{\text{ét}}(Y_1(N_{\psi_i} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))$ using $\breve{\theta}_{\psi_i}$, $i = 1, 2$.

By the isomorphisms from Proposition $1.3.3$:

$$(2.4) \qquad \begin{aligned} &\nu_{\mathfrak{f}_1 \mathfrak{m}} : H^1_{\text{ét}}(Y_1(N_{\psi_1} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1} m), \phi_{\mathfrak{f}_1 \mathfrak{m}}} \mathcal{O}[H^{(p)}_{\mathfrak{f}_1 \mathfrak{m}}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1 \mathfrak{m})} \mathcal{O}(\psi_1^{-1}), \\ &\nu_{\mathfrak{f}_2 \overline{\mathfrak{m}}} : H^1_{\text{ét}}(Y_1(N_{\psi_2} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2} m), \phi_{\mathfrak{f}_2 \overline{\mathfrak{m}}}} \mathcal{O}[H^{(p)}_{\mathfrak{f}_2 \overline{\mathfrak{m}}}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2 \overline{\mathfrak{m}})} \mathcal{O}(\psi_2^{-1}), \end{aligned}$$

the class $\tilde{\kappa}^{(4)}_{f, \psi_1, \psi_2, m}$ defines an element in

$$H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1 \mathfrak{m})} \mathcal{O}(\psi_1^{-1}) \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2 \overline{\mathfrak{m}})} \mathcal{O}(\psi_2^{-1})(-1)\big)$$

$$\cong H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_K \mathcal{O}_{\psi_1^{-1}}[H^{(p)}_{\mathfrak{f}_1 \mathfrak{m}}] \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_K \mathcal{O}_{\psi_2^{-1}}[H^{(p)}_{\mathfrak{f}_2 \overline{\mathfrak{m}}}](-1)\big),$$

where for the second factor in the triple product we use the G_K-module isomorphism $\operatorname{Ind}_{K(\mathfrak{f}_1\mathfrak{m})}^{K}\mathcal{O}(\psi_1^{-1}) \cong \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{f}_1\mathfrak{m}}^{(p)}]$, with $\mathcal{O}_{\psi_1^{-1}}$ standing for the free $\mathcal{O}$-module of rank one on which G_K acts via ψ_1^{-1} and with $H_{\mathfrak{f}_1\mathfrak{m}}$ being equipped with the G_K-action arising from the projection $G_K \twoheadrightarrow H_{\mathfrak{f}_1\mathfrak{m}}^{(p)}$; and likewise for the last factor in the triple product. Taking the image under the maps induced by $H_{\mathfrak{f}_1\mathfrak{m}} \twoheadrightarrow H_\mathfrak{m}$ and $H_{\mathfrak{f}_2\overline{\mathfrak{m}}} \twoheadrightarrow H_{\overline{\mathfrak{m}}}$, from $\tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(4)}$ we obtain

$$(2.5) \qquad \tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(5)} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_\mathcal{O} \operatorname{Ind}_K^\mathbf{Q}\mathcal{O}_{\psi_1^{-1}}[H_\mathfrak{m}^{(p)}] \otimes_\mathcal{O} \operatorname{Ind}_K^\mathbf{Q}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}](-1)\big).$$

2.1.1. *Projection to ring class groups.* Directly from the definitions of the class groups involved, we deduce the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
\mathcal{O}_K^\times \times \mathcal{O}_K^\times & \longrightarrow & (\mathcal{O}_K/\mathfrak{m})^\times \times (\mathcal{O}_K/\overline{\mathfrak{m}})^\times & \longrightarrow & H_\mathfrak{m} \times H_{\overline{\mathfrak{m}}} & \longrightarrow & H_1 \times H_1 & \longrightarrow & 1 \\
\Delta\uparrow & & \simeq\uparrow & & \uparrow & & \Delta\uparrow & & \\
\mathcal{O}_K^\times & \longrightarrow & (\mathcal{O}_K/m\mathcal{O}_K)^\times & \longrightarrow & H_m & \longrightarrow & H_1 & \longrightarrow & 1,
\end{array}
$$

where the unlabelled vertical arrow is given by the restriction map

$$\sigma \mapsto (\sigma|_{K_\mathfrak{m}}, \sigma|_{K_{\overline{\mathfrak{m}}}}).$$

Since we assume $p \nmid 6h_K$, taking p-primary parts this map induces an isomorphism

$$(2.6) \qquad H_m^{(p)} \xrightarrow{\simeq} H_\mathfrak{m}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}.$$

Given an integer $n > 0$, let $H[n]$ be the ring class group of K of conductor n, so $H[n] \simeq \operatorname{Pic}(\mathcal{O}_n)$ under the Artin reciprocity map, where $\mathcal{O}_n = \mathbf{Z} + n\mathcal{O}_K$ is the order of K of conductor n. Let $H[n]^{(p)}$ be the maximal p-power quotient of $H[n]$, and denote by $K[n]$ be the maximal p-extension inside the ring class field of K of conductor n, so $H[n]^{(p)} = \operatorname{Gal}(K[n]/K)$.

Proposition 2.1.1. *Suppose $p \nmid 6h_K$ and $m > 0$ is an integer divisible only by primes split in K. Then writing $m = \mathfrak{m}\overline{\mathfrak{m}}$ and identifying $H_m^{(p)}$ with $H_\mathfrak{m}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}$ as in (2.6), we have an exact sequence*

$$1 \longrightarrow (\mathbf{Z}/m\mathbf{Z})^{\times,(p)} \xrightarrow{\Delta} H_\mathfrak{m}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)} \xrightarrow{\pi_\Delta} H[m]^{(p)} \longrightarrow 1,$$

where the map Δ sends $a \mapsto ([a], [a])$ for every integer a coprime to m. Moreover, if $\ell \nmid m$ is a prime that splits in K, the projection π_Δ sends

$$[\lambda] \times [\lambda] \mapsto \operatorname{Frob}_\lambda$$

for every prime λ of K above ℓ, where $\operatorname{Frob}_\lambda$ is the geometric Frobenius element of λ in $H[m]^{(p)}$.

Proof. The first part is clear from the above discussion together with the commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
\mathcal{O}_K^\times & \longrightarrow & (\mathcal{O}_K/m\mathcal{O}_K)^\times & \longrightarrow & H_m & \longrightarrow & H_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_K^\times/\mathbf{Z}^\times & \longrightarrow & (\mathcal{O}_K/m\mathcal{O}_K)^\times/(\mathbf{Z}/m\mathbf{Z})^\times & \longrightarrow & H[m] & \longrightarrow & H_1 & \longrightarrow & 1,
\end{array}
$$

where the vertical arrows are given by the natural projections. The second part follows from the functoriality properties of Frobenii. $\qquad\square$

Now we can consider the image of $\tilde{\kappa}^{(5)}_{f,\psi_1,\psi_2,m}$ under the composite map

(2.7)

$$\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}] \xrightarrow{\;\xi\;} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}]$$

$$\xi_{\Delta} \searrow \qquad \downarrow \pi_{\Delta}$$

$$\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}],$$

where the horizontal arrow is the map determined by $\phi_1 \otimes \phi_2 \mapsto \xi(\phi_1 \otimes \phi_2)$ with $\xi(\phi_1 \otimes \phi_2)(g) = \phi_1(g_1) \otimes \phi_2(g_2)$ if $g = (g_1, g_2) \in H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}$, resulting in the class

$$\tilde{\kappa}^{(6)}_{f,\psi_1,\psi_2,m} \in H^1\big(\mathbf{Q}, T_f^{\vee} \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}](-1)\big).$$

Definition 2.1.2. For $m > 0$ any squarefree integer divisible only by primes $\ell \nmid pN$ split in K, we define

$$\tilde{\kappa}_{f,\psi_1,\psi_2,m} \in H^1\big(K[m], T_f^{\vee}(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

to be the image of $\tilde{\kappa}^{(6)}_{f,\psi_1,\psi_2,m}$ under the isomorphism

$$H^1\big(\mathbf{Q}, T_f^{\vee} \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}](-1)\big) \simeq H^1\big(K[m], T_f^{\vee}(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

given by Shapiro's lemma.

We finish this section by recording the following observation for our later use.

Lemma 2.1.3. *For any integer $m = \mathfrak{m}\overline{\mathfrak{m}}$ divisible only by primes split in K, and any prime $\ell = \mathfrak{l}\overline{\mathfrak{l}}$ split in K, the following diagram is commutative:*

$$\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}\mathfrak{l}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}\overline{\mathfrak{l}}}^{(p)}] \longrightarrow \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m\ell]^{(p)}]$$

$$\downarrow{\scriptstyle \mathrm{Norm}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathrm{Norm}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}}} \qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{Norm}_{m}^{m\ell}}$$

$$\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}] \longrightarrow \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}],$$

where the horizontal arrows are given by the composition ξ_{Δ} in (2.7).

Proof. This is clear from the explicit description of the maps involved. $\qquad\square$

2.2. **Proof of the tame norm relations.** Let $m > 0$ be any integer for which we have the class $\tilde{\kappa}_{f,\psi_1,\psi_2,m}$ of Definition 2.1.2.

Proposition 2.2.1. *Let $\ell = \mathfrak{l}\overline{\mathfrak{l}}$ be a prime split in K and coprime to mpN. Then*

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(\tilde{\kappa}_{f,\psi_1,\psi_2,m\ell}) = (\ell-1)\bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}] \times [\mathfrak{l}]) - \frac{\psi_1(\overline{\mathfrak{l}})\psi_2(\overline{\mathfrak{l}})}{\ell}([\overline{\mathfrak{l}}] \times [\overline{\mathfrak{l}}])$$
$$+ (1-\ell)\frac{\psi_1(\mathfrak{l})\psi_2(\overline{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\overline{\mathfrak{l}}])\bigg)(\tilde{\kappa}_{f,\psi_1,\psi_2,m}).$$

Proof. In the notations of Theorem 1.3.3, for any $\mathfrak{n} = \mathfrak{fm} \in \mathcal{A}_{\mathfrak{f}}$ put

$$H^1(\psi, \mathfrak{fm}) := H^1_{\text{ét}}(Y_1(N_\psi m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_\psi m)} \mathcal{O}[H_{\mathfrak{fm}}^{(p)}].$$

Then from Lemma 2.1.3 we have the following commutative diagram:

(2.8)

$$H^1\big(\mathbf{Q}, T_f^{\vee} \otimes H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}\mathfrak{l}) \otimes H^1(\psi_2, \mathfrak{f}_2\overline{\mathfrak{m}}\overline{\mathfrak{l}})(-1)\big) \longrightarrow H^1\big(K[m\ell], T_f^{\vee}(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

$$\downarrow{\scriptstyle 1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}}} \qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{Norm}_{m}^{m\ell}}$$

$$H^1\big(\mathbf{Q}, T_f^{\vee} \otimes H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}) \otimes H^1(\psi_2, \mathfrak{f}_2\overline{\mathfrak{m}})(-1)\big) \longrightarrow H^1\big(K[m], T_f^{\vee}(\psi_1^{-1}\psi_2^{-1})(-1)\big),$$

where, for $\mathfrak{m}' \in \{\mathfrak{m}\mathfrak{l}, \mathfrak{m}\}$, the horizontal arrows are given by the maps induced by the projections

$$H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}') \xrightarrow{\nu_{\mathfrak{f}_1\mathfrak{m}'}} \mathrm{Ind}_{K(\mathfrak{f}_1\mathfrak{m}')}^{\mathbf{Q}} \mathcal{O}(\psi_1^{-1}) \simeq \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{f}_1\mathfrak{m}'}^{(p)}] \longrightarrow \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}'}^{(p)}]$$

$$H^1(\psi_2, \bar{\mathfrak{f}}_1\overline{\mathfrak{m}}') \xrightarrow{\nu_{\mathfrak{f}_2\overline{\mathfrak{m}}'}} \mathrm{Ind}_{K(\mathfrak{f}_2\overline{\mathfrak{m}}')}^{\mathbf{Q}} \mathcal{O}(\psi_2^{-1}) \simeq \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{f}_2\overline{\mathfrak{m}}'}^{(p)}] \longrightarrow \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}'}^{(p)}]$$

from Theorem 1.3.3 together with the map ξ_Δ in (2.7). Now, tracing through the definitions we compute:

$$(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(2)})$$

$$= (1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\mathrm{pr}_{m\ell*}, 1, 1)(\tilde{\kappa}_{m\ell}^{(1)})$$

$$= (\mathrm{pr}_{m*}, 1, 1)(\mathrm{pr}_{\ell*} \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(1)})$$

$$= (\mathrm{pr}_{m*}, 1, 1)\left(\mathrm{pr}_{\ell*} \times (1 \otimes \mathrm{pr}_{1*} - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \mathrm{pr}_{\ell*}) \times (1 \otimes \mathrm{pr}_{1*} - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell} \otimes \mathrm{pr}_{\ell*})\right)(\tilde{\kappa}_{m\ell}^{(1)})$$

$$= (\mathrm{pr}_{m*}, 1, 1)\left((\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*}) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}) - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell}(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{\ell*})\right.$$

$$\left. + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*})\right)(\tilde{\kappa}_{m\ell}^{(1)}).$$

Together with Proposition 1.4.1, we thus obtain

$$(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(2)}) = (\ell - 1)(\mathrm{pr}_{m*}, 1, 1)\left((T_\ell, 1, 1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(1, 1, T_\ell') - (1, T_\ell', 1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell}\right.$$

$$\left. + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\right)(\tilde{\kappa}_m^{(1)})$$

$$= (\ell - 1)\left((T_\ell, 1, 1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(1, 1, T_\ell') - (1, T_\ell', 1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell}\right.$$

$$\left. + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\right)(\tilde{\kappa}_m^{(2)}),$$

and from this it follows that

$$(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{f,\psi_1,\psi_2,m\ell}^{(5)})$$

$$= (\ell - 1)\left(a_\ell(f) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(\psi_2(\mathfrak{l})[\mathfrak{l}] + \psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]) - (\psi_1(\mathfrak{l})[\mathfrak{l}] + \psi_1(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}])\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell}\right.$$

$$\left. + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\right)(\tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(5)})$$

$$= (\ell - 1)\left(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}] \times [\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}])\right.$$

$$\left. + (1 - \ell)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])\right)(\tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(5)}).$$

In light of the commutative diagram (2.8), this yields the result. $\square$

Remark 2.2.2. The appearance of the factor $(\ell - 1)$ in Proposition 2.2.1 can be traced back to the relations $\deg(\mu_\ell)T_\ell = \mathrm{pr}_{\ell*} \circ \mathrm{pr}_1^*$ and $\deg(\mu_\ell)T_\ell' = \mathrm{pr}_{1*} \circ \mathrm{pr}_\ell^*$, i.e., it is caused by the degeneracy map μ_ℓ. In the next subsection we shall get rid of this extra factor.

Remark 2.2.3. We want to emphasize that Proposition 2.2.1 is the key result for the construction of our anticyclotomic Euler system for $T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)$. In fact, with the factor $(\ell - 1)$ stripped out, the term on the right-hand side of Proposition 2.2.1 can be massaged to agree with the local Euler factor at $\mathfrak{l}$ of the Galois representation $[T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)]^\vee(1) = T_f(\psi_1\psi_2)(2)$, giving the correct norm relations.

2.2.1. *Removing the extra factor* $(\ell - 1)$. Adapting an idea from [DR17, §1.4], we now introduce a modification of the classes $\tilde{\kappa}_{f,\psi_1,\psi_2,m}$ for which we can prove an analogue of Proposition 2.2.1 without the extra factor $(\ell - 1)$.

We begin by noting that for any prime $\ell \nmid N$ the degeneracy maps $\mathrm{pr}_1, \mathrm{pr}_\ell : Y_1(N\ell) \to Y_1(N)$ can be factored as

$$
\begin{array}{ccc}
Y_1(N\ell) & & Y_1(N\ell) \\
\mu_\ell \downarrow \quad \searrow \mathrm{pr}_1 & & \mu_\ell \downarrow \quad \searrow \mathrm{pr}_\ell \\
Y(1, N(\ell)) \xrightarrow{\pi_1} Y_1(N) & & Y(1, N(\ell)) \xrightarrow{\pi_\ell} Y_1(N),
\end{array}
$$

where π_1 and π_ℓ are a non-Galois coverings of degree $\ell + 1$, and we recall that μ_ℓ is a cyclic Galois covering of degree $\ell - 1$.

Denote by

$$(2.9) \qquad D_m = \{ (\langle d \rangle, \langle d \rangle) : d \in (\mathbf{Z}/Nm\mathbf{Z})^\times, \ d \equiv 1 \ (\mathrm{mod}\, N) \}$$

the set of diamond operators acting diagonally and freely on $Y_1(Nm)^2$, and set

$$W_1(Nm) = (Y_1(Nm) \times Y_1(Nm))/D_m.$$

Let $\tilde{\kappa}_m^{(1)}$ be as in (2.1), and let

$$\kappa_m^{(1)} \in H^1\big(\mathbf{Q}, H^3_{\text{ét}}(Y(1, N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)(2)\big)$$

be the image of $(\mu_{m*}, 1, 1)(\tilde{\kappa}_m^{(1)})$ under the natural map induced by the projection $Y_1(Nm)^2 \to W_1(Nm)$, which is an étale morphism of degree $\phi(m) = |(\mathbf{Z}/m\mathbf{Z})^\times|$. Thus, the class $\kappa_m^{(1)}$ is defined by the relation

$$(2.10) \qquad (\mu_{m*}, d_{m*})\tilde{\kappa}_m^{(1)} = \phi(m)\kappa_m^{(1)}.$$

Proposition 2.2.4. *For any prime number ℓ and positive integer m such that $(m, \ell) = 1$ and $(m\ell, N) = 1$, we have*

$$(\pi_{i*}, \mathrm{pr}_{j*}, \mathrm{pr}_{k*})\kappa_{m\ell}^{(1)} = (\bigstar)\kappa_m^{(1)},$$

where

(i,j,k)	$\bigstar$	(i,j,k)	$\bigstar$
$(\ell, 1, 1)$	$(T_\ell, 1, 1)$	$(\ell, 1, \ell)$	$(1, T'_\ell, 1)$
$(1, \ell, 1)$	$(1, T_\ell, 1)$	$(\ell, \ell, 1)$	$(1, 1, T'_\ell)$
$(1, 1, \ell)$	$(1, 1, T_\ell)$	$(1, 1, 1)$	$(\ell + 1)$
$(1, \ell, \ell)$	$(T'_\ell, 1, 1)$	(ℓ, ℓ, ℓ)	$(\ell + 1)$

Proof. Directly from the definitions we find

$$(\mu_{m*}, d_{m*})(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})\tilde{\kappa}_{m\ell}^{(1)} = (\pi_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})(\mu_{m\ell*}, d_{m\ell*})\tilde{\kappa}_{m\ell}^{(1)}$$

$$= \phi(m\ell)(\pi_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})\kappa_{m\ell}^{(1)},$$

while on the other hand, from Proposition 1.4.1 and (2.10) we have

$$(\mu_{m*}, d_{m*})(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})\tilde{\kappa}_{m\ell}^{(1)} = (\mu_{m*}, d_{m*})(\ell - 1)(T_\ell, 1, 1)\tilde{\kappa}_m^{(1)}$$

$$= \phi(m)(\ell - 1)(T_\ell, 1, 1)\kappa_m^{(1)}.$$

Since $\phi(m\ell) = (\ell - 1)\phi(m)$ according to our assumptions, this shows the result in the case $(i, j, k) = (\ell, 1, 1)$ and the other cases are shown in the same manner. $\square$

Now we want to proceed as above to obtain from the new $\kappa_m^{(1)}$ a construction of classes satisfying the correct norm relations (i.e., without the factor $\ell - 1$). This requires a careful study of the étale cohomology of the quotient $Y(1, N(m)) \times W_1(Nm)$.

We begin with the Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p\big(D_m, H^q_{\text{ét},c}(Y(1, N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)\big) \Rightarrow H^{p+q}_{\text{ét},c}\big(Y(1, N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big).$$

This yields the exact sequence

$$(2.11) \qquad E \longrightarrow H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big) \xrightarrow{(1,d_m^*)} E_2^{0,3} \xrightarrow{d_2^{0,3}} E_2^{2,2},$$

where E is naturally identified with a subquotient of $E_2^{1,2} \oplus E_2^{2,1}$. Thus, we see that the difference between the two middle pieces are classes coming from $H^q_{\text{ét},c}(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$ with $0 \leq q \leq 2$. From the Künneth decomposition [Mil13, Thm. 22.4], each of these classes will have a factor from either $H^0_{\text{ét},c}(Y(1,N(m))_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$ or $H^0_{\text{ét},c}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$. These vanish because étale cohomology of affine smooth curves with compact support vanishes in degree zero. Hence, we obtain an isomorphism

$$H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big) \xrightarrow{(1,d_m^*)} H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big)^{D_m}.$$

As in the proof of [DR17, Lem. 1.8], Poincaré duality implies from (2.11) that the following map

$$(2.12) \qquad H^3_{\text{ét}}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big) \xleftarrow{(1,d_{m*})} H^3_{\text{ét}}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big)_{D_m}$$

is also an isomorphism. Therefore from (2.12) and the Künneth decomposition, it follows that we get a natural map

$$(2.13) \quad H^3_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \xrightarrow{(1,d_{m*}^{-1})} H^1_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \otimes$$
$$H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \otimes_{D_m} H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)).$$

Now we put $\kappa_m^{(2)} := (\pi_{m*}, 1, 1)\kappa_m^{(1)}$, and define

$$\kappa_m^{(3)} \in H^1\big(\mathbf{Q}, H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \otimes H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p) \otimes_{D_m} H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)(-1)\big),$$

to be the image of $\kappa_m^{(2)}$ under the map (2.13).

Note that taking $D_m^{(p)}$-coinvariants (where $D_m^{(p)}$ denotes the p-part of D_m) is compatible with the map ξ_Δ in (2.7), since by Theorem 1.3.1 for $(\langle d \rangle, \langle d \rangle) \in D_m^{(p)}$ we have $\phi_{\mathfrak{m}}(\langle d \rangle') \times \phi_{\overline{\mathfrak{m}}}(\langle d \rangle') = [d] \times [d] \in H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}$, and this is in the kernel of π_Δ. Thus, applying to $\kappa_m^{(3)}$ the same process we used above to go from $\tilde{\kappa}_m^{(3)}$ to the class $\tilde{\kappa}_{f,\psi_1,\psi_2,m}$ of Definition 2.1.2, we obtain

$$\kappa_{f,\psi_1,\psi_2,m} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big).$$

Proposition 2.2.5. *Suppose m is a positive squarefree integer divisible only by primes $q \nmid pN$ split in K. Let $\ell \nmid pmN$ be a prime split in K. Then*

$$\text{Norm}_{K[m]}^{K[m\ell]}(\kappa_{f,\psi_1,\psi_2,m\ell}) = \bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}] \times [\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}])$$
$$+ (1 - \ell)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])\bigg)(\kappa_{f,\psi_1,\psi_2,m}).$$

Proof. After the above discussion, the same calculation as in the proof of Proposition 2.2.1 applies, replacing the use of Proposition 1.4.1 by Proposition 2.2.4. $\square$

Therefore, we arrive at the following theorem:

Theorem 2.2.6. *Suppose $p > 3$ is a prime split in K with $p \nmid h_K$. Let $m = \mathfrak{m}\overline{\mathfrak{m}}$ run over the squarefree integers divisible only by primes split in K and coprime to pN. Then there exists a collection of cohomology classes*

$$z_{f,\psi_1,\psi_2,m} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

such that for every prime $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ split in K with $(\ell, mpN) = 1$ we have the norm relation

$$\text{Norm}_{K[m]}^{K[m\ell]}(z_{f,\psi_1,\psi_2,m\ell}) = P_{\mathfrak{l}}(\text{Frob}_{\mathfrak{l}})(z_{f,\psi_1,\psi_2,m}),$$

where $P_{\mathfrak{l}}(X) = \det(1 - X \cdot \text{Frob}_{\mathfrak{l}} \,|\, T_f(\psi_1\psi_2)(2))$.

Proof. Denote by $Q_{\mathfrak{l}}$ the factor appearing in the right-hand side of Proposition 2.2.5. Recalling that $[\mathfrak{l}] \times [\mathfrak{l}]$ corresponds to $\mathrm{Frob}_{\mathfrak{l}} \in H[m]^{(p)}$ under the map π_{Δ} of Proposition 2.1.1, we find the following congruences as endomorphisms of $H^1(K[m], T_f^{\vee}(\psi_1^{-1}\psi_2^{-1})(-1))$:

$$- \psi_1\psi_2(\mathfrak{l})([\mathfrak{l}] \times [\mathfrak{l}]) \cdot Q_{\mathfrak{l}}$$

$$\equiv -a_\ell(f)\psi_1\psi_2(\mathfrak{l})([\mathfrak{l}] \times [\mathfrak{l}]) + \frac{\psi_1\psi_2(\mathfrak{l})^2}{\ell}([\mathfrak{l}] \times [\mathfrak{l}])^2 + \frac{\psi_1\psi_2((\ell))}{\ell}([\ell] \times [\ell])$$

$$\equiv P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) \pmod{\ell - 1},$$

using the relation $\psi_1\psi_2((\ell)) = \chi_{\psi_1}\chi_{\psi_2}(\ell)\ell^2 = \ell^2$ and the fact that $[\ell] \times [\ell]$ is in the kernel of π_{Δ} for the second congruence. Therefore, by Lemmas 9.6.1 and 9.6.3 in [Rub00] (which will not alter the bottom class of our Euler system), the existence of classes $z_{f,\psi_1,\psi_2,m}$ with $z_{f,\psi_1,\psi_2,1} = \kappa_{f,\psi_1,\psi_2,1}$ and satisfying the stated norm relations follows from Proposition 2.2.5. $\square$

Remark 2.2.7. Similar to what we did for $\mathfrak{l}$ a split prime of K, when $\mathfrak{l} = (\ell)$ is inert in K, we also obtain such a norm relation like in Theorem 2.2.6. Remember that in this case, we push forward from level $N\ell^2$ to level N. First, note that the norm map from Proposition 1.3.1 is then given by

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}(\ell)} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\ell)[(\ell)]}{\ell^2} \otimes \mathrm{pr}_{\ell\ell*}$$

Second, to calculate $(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\ell} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\ell})(\kappa_{m\ell}^{(2)})$, just like in Proposition 2.2.1, we use the table in Proposition 1.4.1 together with

$$(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})(T_\ell, 1, 1)\kappa_{m\ell}^{(2)} = \{(T_\ell^2, 1, 1) - (\ell+1)(\langle\ell\rangle, 1, 1)\}\kappa_m^{(2)},$$

$$(\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{1*})(1, 1, T_\ell')\kappa_{m\ell}^{(2)} = \{(1, 1, T_\ell'^2) - (\ell+1)(1, 1, \langle\ell\rangle')\}\kappa_m^{(2)},$$

and arrive at

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(\kappa_{f,\psi_1,\psi_2,m\ell}^{(6)}) = (\ell-1)\left(a_\ell(f)^2 - (\ell+1) - \frac{2(\ell+1)}{\ell}[\ell] \times [\ell] + (\ell+1)[\ell] \times [\ell]\right)(\kappa_{f,\psi_1,\psi_2,m}^{(6)}).$$

Instead of Proposition 2.1.1, we use the following exact sequence

$$1 \longrightarrow H_m \xrightarrow{\Delta} H_m \times H_m \longrightarrow H_m \longrightarrow 1,$$

combining with the quotient $H_m \to H[m]$, which makes $[\ell] \times [\ell]$ acting trivially on the cohomology class. After removing the extra factor $(\ell-1)$ and multiplying with -1 on the RHS factor, we obtain the correct Euler factor modulo $\ell^2 - 1$:

$$P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) = 2 + 2\ell - a_\ell(f)^2.$$

Note that $(\ell+1)/\ell \equiv (\ell+1)\ell = \ell^2 + \ell \equiv 1 + \ell \pmod{\ell^2 - 1}$ and the twist $\psi_1(\ell)\psi_2(\ell)/\ell^2 = 1$.

2.3. **A variant.** With a slight modification of the construction in the preceding sections, we can obtain a similar collection of cohomology classes for $T_f^{\vee}(\psi_1^{-1}\psi_2^{-\mathbf{c}})(-1)$ satisfying the corresponding norm relations. Here $\psi_2^{\mathbf{c}}(\sigma) = \psi_2(\mathbf{c}\sigma\mathbf{c}^{-1})$ denotes the composition of ψ_2 with the action of the non-trivial automorphism of $K/\mathbf{Q}$. Indeed, following Section 2.1, we replace the second map in (2.3) by

$$\phi_{\mathfrak{f}_2\mathfrak{m}} : \mathbb{T}'(N_{\psi_2}m) \to \mathcal{O}[H_{\mathfrak{f}_2\mathfrak{m}}^{(p)}]$$

and the second map in (2.4) by

$$\nu_{\mathfrak{f}_2\mathfrak{m}} : H_{\mathrm{\acute{e}t}}^1(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m), \phi_{\mathfrak{f}_2\mathfrak{m}}} \mathcal{O}[H_{\mathfrak{f}_2\mathfrak{m}}^{(p)}] \xrightarrow{\sim} \mathrm{Ind}_{K(\mathfrak{f}_2\mathfrak{m})}^{\mathbf{Q}}\mathcal{O}(\psi_2^{-1}).$$

Thus, similarly as in (2.5) we obtain

$${}^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(5)} \in H^1\big(\mathbf{Q}, T_f^{\vee} \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)}](-1)\big).$$

The diagram (2.7) is then replaced with

$$(2.14)$$

$$\operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)}] \xrightarrow{1 \otimes \upsilon} \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-\mathbf{c}}}[H_{\overline{\mathfrak{m}}}^{(p)}]$$

with downward map ξ to

$$\operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}\psi_2^{-\mathbf{c}}}[H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}]$$

with downward map π_Δ to

$$\operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}\psi_2^{-\mathbf{c}}}[H[m]^{(p)}],$$

and diagonal map $\xi_\Delta^{\mathbf{c}}$,

where the isomorphism $\upsilon : \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)}] \to \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-\mathbf{c}}}[H_{\overline{\mathfrak{m}}}^{(p)}]$ comes from the action of complex conjugation on the inducing representation.

Consequently, Lemma 2.1.3 now turns into the following commutative diagram:

$$(2.15)$$

$$\begin{array}{ccc}
\operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}\mathfrak{l}}^{(p)}] \otimes_{\mathcal{O}} \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{m}\mathfrak{l}}^{(p)}] & \xrightarrow{\xi_\Delta^{\mathbf{c}}} & \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}\psi_2^{-\mathbf{c}}}[H[m\ell]^{(p)}] \\
\downarrow{\scriptstyle \operatorname{Norm}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \operatorname{Norm}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}}} & & \downarrow{\scriptstyle \operatorname{Norm}_m^{m\ell}} \\
\operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)}] & \xrightarrow{\xi_\Delta^{\mathbf{c}}} & \operatorname{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}\psi_2^{-\mathbf{c}}}[H[m]^{(p)}].
\end{array}$$

The same process as above then leads to the following 'conjugate' variant of Theorem 2.2.6:

Corollary 2.3.1. *With notations as in Theorem 2.2.6, there exists a collection of cohomology classes*

$$^{\mathbf{c}}z_{f,\psi_1,\psi_2,m} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-\mathbf{c}})(-1)\big)$$

such that for every prime $\ell = \mathfrak{l}\bar{\mathfrak{l}}$ split in K with $(\ell, mpN) = 1$ we have the norm relation

$$\operatorname{Norm}_{K[m]}^{K[m\ell]}(^{\mathbf{c}}z_{f,\psi_1,\psi_2,m\ell}) = P_{\mathfrak{l}}(\operatorname{Frob}_{\mathfrak{l}})(^{\mathbf{c}}z_{f,\psi_1,\psi_2,m}),$$

where $P_{\mathfrak{l}}(X) = \det(1 - X \cdot \operatorname{Frob}_{\mathfrak{l}} \mid T_f(\psi_1\psi_2^{\mathbf{c}})(2))$.

Proof. We first follow Proposition 2.2.1 to have

$$(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}})(^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m\ell}^{(5)}) = (\ell-1)\bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell}([\mathfrak{l}] \times [\bar{\mathfrak{l}}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\mathfrak{l})}{\ell}([\bar{\mathfrak{l}}] \times [\mathfrak{l}])$$
$$+ (1-\ell)\frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell^2}([\mathfrak{l}] \times [\mathfrak{l}])\bigg)(^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m}^{(5)}).$$

Combining the new $\xi_\Delta^{\mathbf{c}}$ (this maps the second factor $[\mathfrak{l}]$ to $[\bar{\mathfrak{l}}]$, and leads to counterpart $^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m}$ of the classes $\tilde{\kappa}_{f,\psi_1,\psi_2,m}$ of Definition 2.1.2) and the diagram (2.15), one obtains

$$\operatorname{Norm}_{K[m]}^{K[m\ell]}(^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m\ell}) = (\ell-1)\bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2^{\mathbf{c}}(\mathfrak{l})}{\ell}([\mathfrak{l}] \times [\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2^{\mathbf{c}}(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}])$$
$$+ (1-\ell)\frac{\psi_1(\mathfrak{l})\psi_2^{\mathbf{c}}(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])\bigg)(^{\mathbf{c}}\tilde{\kappa}_{f,\psi_1,\psi_2,m}).$$

This formula should be treated as a replacement for Proposition 2.2.1.

Then we remove the extra factor $(\ell-1)$ as in §2.2.1 to obtain classes $^{\mathbf{c}}\kappa_{f,\psi_1,\psi_2,m}$, modify the remaining factor through multiplication by $-\psi_1\psi_2^{\mathbf{c}}(\mathfrak{l})([\mathfrak{l}] \times [\mathfrak{l}])$, and apply Lemmas 9.6.1 and 9.6.3 in [Rub00] to obtain the existence of classes $^{\mathbf{c}}z_{f,\psi_1,\psi_2,m}$ with $^{\mathbf{c}}z_{f,\psi_1,\psi_2,1} = {}^{\mathbf{c}}\kappa_{f,\psi_1,\psi_2,1}$ and the desired norm relations. $\qquad\square$

2.4. **Construction for general weights and wild norm relations.** We now extend the constructions of the preceding subsections to $f \in S_{2r}(\Gamma_0(N_f))$ of any weight $2r \geq 2$ and more general Hecke characters, assuming in addition that

(ord) $$f \text{ is ordinary at } \mathfrak{P},$$

which we shall often refer to as f being 'p-ordinary'; and prove that the resulting classes also satisfy the wild norm relations, i.e., they are universal norms in the anticyclotomic $\mathbf{Z}_p$-extension of K.

2.4.1. *CM Hida families.* We shall replace the weight 2 theta series $\theta_{\psi_1}, \theta_{\psi_2}$ by p-adic families, so we begin by recalling the explicit construction of certain CM Hida families, following the exposition in [Hsi21, §8.1].

Let $\Gamma_\infty = \mathrm{Gal}(K_\infty/K)$ be the Galois group of the $\mathbf{Z}_p^2$-extension of K, which under our hypotheses can be written as

$$\Gamma_\infty \simeq \Gamma_{\mathfrak{p}\infty} \times \Gamma_{\bar{\mathfrak{p}}\infty},$$

with $\Gamma_{\mathfrak{p}\infty} = \mathrm{Gal}(K_{\mathfrak{p}\infty}^\circ/K)$ (resp. $\Gamma_{\bar{\mathfrak{p}}\infty} = \mathrm{Gal}(K_{\bar{\mathfrak{p}}\infty}^\circ/K)$) representing the Galois group of the unique $\mathbf{Z}_p$-extension of K inside K_∞ unramified outside $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$). Recall that for every ideal $\mathfrak{c} \subset \mathcal{O}_K$ we denote by $K_\mathfrak{c}$ the ray class field of K of conductor $\mathfrak{c}$ (so in particular $K_{\mathfrak{p}\infty}^\circ$ is the maximal $\mathbf{Z}_p$-extension of K inside $K_{\mathfrak{p}\infty}$). For $\mathfrak{q} \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, denote by $\mathrm{Art}_\mathfrak{q} : K_\mathfrak{q}^\times \to G_K^{\mathrm{ab}}$ the restriction of the Artin reciprocity map to $K_\mathfrak{q}^\times$, with geometric normalisations. With the identification $\mathbf{Z}_p^\times \simeq \mathcal{O}_{K_\mathfrak{q}}^\times$, the map $\mathrm{Art}_\mathfrak{q}$ induces an isomorphism $1 + p\mathbf{Z}_p \xrightarrow{\sim} \Gamma_{\mathfrak{q}\infty}$ (note that this uses our hypothesis (cn)). Let $\mathbf{u} = 1 + p$ and let $\gamma_\mathfrak{q} \in \Gamma_{\mathfrak{q}\infty}$ be the topological generator $\gamma_\mathfrak{q} = \mathrm{Art}_\mathfrak{q}(\mathbf{u})|_{K_{\mathfrak{q}\infty}^\circ}$ with $\mathbf{c}\gamma_\mathfrak{p}\mathbf{c}^{-1} = \gamma_{\bar{\mathfrak{p}}}$, where $\mathrm{Gal}(K/\mathbf{Q}) = \{1, \mathbf{c}\}$.

For each variable Z, let $\Psi_Z : \Gamma_\infty \to \mathbf{Z}_p[\![Z]\!]^\times$ be the universal character given by

$$\Psi_Z(\sigma) = (1 + Z)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K_{\mathfrak{p}\infty}^\circ} = \gamma_\mathfrak{p}^{l(\sigma)}$. Note that the specialisation ψ_0 of Ψ_Z to $Z = \mathbf{u} - 1$ descends to an isomorphism

$$\psi_0 = \Psi_{\mathbf{u}-1} : \Gamma_{\mathfrak{p}\infty} \xrightarrow{\sim} 1 + p\mathbf{Z}_p$$

(namely, the inverse of the above isomorphism induced by $\mathrm{Art}_\mathfrak{p}$), and may be seen as the p-adic avatar of a Hecke character – still denoted ψ_0 – of K of infinity type $(1, 0)$ and conductor $\mathfrak{p}$.

For $\mathfrak{c}$ coprime to p, and for any finite order character $\xi : G_K \to \mathcal{O}^\times$ of conductor dividing $\mathfrak{c}$ put

(2.16) $$\boldsymbol{\theta}_\xi(Z)(q) = \sum_{(\mathfrak{a}, \mathfrak{pc})=1} \xi\psi_0^{-1}(\sigma_\mathfrak{a})\Psi_Z^{-1}(\sigma_\mathfrak{a})q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in \mathcal{O}[\![Z]\!][\![q]\!],$$

where $\sigma_\mathfrak{a} \in \mathrm{Gal}(K(\mathfrak{cp}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$. With conventions as in §4.1.1 below (which differ slightly from those in [Hsi21, §3.1]; our weight map is centered at weight 2 rather than 0), $\boldsymbol{\theta}_\xi(Z)(q)$ is a Hida family defined over $\mathcal{O}[\![Z]\!]$ of tame level $N_{K/\mathbf{Q}}(\mathfrak{c})D_K$ and tame character $(\xi \circ \mathscr{V})\epsilon_K\omega^{-1}$, where $\mathscr{V} : G_\mathbf{Q}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map, ϵ_K is the quadratic character corresponding to $K/\mathbf{Q}$, and $\omega : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ is the Teichmüller character.

2.4.2. *The construction.* Let ξ_1, ξ_2 be ray class characters of K of conductor prime-to-p with

(sd) $$\chi_{\xi_1}\chi_{\xi_2} = 1.$$

Let $\boldsymbol{f}$ be the Hida family passing through (the p-ordinary p-stabilisation of) f, and let

$$(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2))$$

be the CM Hida families attached to ξ_1 and ξ_2. The tame characters of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ are given by

$$(\chi_f, \chi_g, \chi_h) = (\omega^{2r-2}, \chi_{\xi_1}\epsilon_K\omega^{-1}, \chi_{\xi_2}\epsilon_K\omega^{-1}),$$

and so (sd) implies the self-duality condition in (sd-triple) below.

Let $\Lambda = \mathbf{Z}_p[\![1 + p\mathbf{Z}_p]\!]$ and let $\kappa : \mathbf{Z}_p^\times \to \Lambda^\times$ be a continuous character. Set $\mathrm{T} = \mathbf{Z}_p^\times \times \mathbf{Z}_p$, $\mathrm{T}' = p\mathbf{Z}_p \times \mathbf{Z}_p^\times$, and consider the Λ-modules

$$\mathcal{A}_\kappa = \left\{ f : \mathrm{T} \to \Lambda \mid f(1, z) \in \mathrm{Cont}(\mathbf{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa(a) \cdot f(t) \text{ for all } a \in \mathbf{Z}_p^\times,\, t \in \mathrm{T} \right\},$$
$$\mathcal{A}'_\kappa = \left\{ f : \mathrm{T}' \to \Lambda \mid f(pz, 1) \in \mathrm{Cont}(\mathbf{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa(a) \cdot f(t) \text{ for all } a \in \mathbf{Z}_p^\times,\, t \in \mathrm{T}' \right\}$$

equipped with the $\mathfrak{m}_\Lambda$-adic topology, for $\mathfrak{m}_\Lambda$ the maximal ideal of Λ. We also consider

$$\mathcal{D}_\kappa = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}_\kappa, \Lambda), \quad \mathcal{D}'_\kappa = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}'_\kappa, \Lambda)$$

equipped with the weak-$*$ topology.

For each $i \in \mathbf{Z}/(p-1)\mathbf{Z}$, let $\kappa_i : \mathbf{Z}_p^\times \to \Lambda^\times$ be the character

$$z \mapsto \omega^i(z)[\langle z \rangle],$$

where $\langle z \rangle = z\omega^{-1}(z) \in 1 + p\mathbf{Z}_p$ and $[\cdot] : 1 + p\mathbf{Z}_p \hookrightarrow \Lambda^\times$ is the inclusion as group-like elements. Put $\mathcal{A}_i, \mathcal{D}_i$ to denote $\mathcal{A}_{\kappa_i}, \mathcal{D}_{\kappa_i}$, and note that, by composing with the map $\rho_{k-2} : \Lambda^\times \to \mathbf{Z}_p^\times$ defined by $\mathbf{u} \mapsto \mathbf{u}^{k-2}$, the Λ-adic character κ_i interpolates the power maps $z \mapsto z^{k-2}$ on $\mathbf{Z}_p^\times$ for $k - 2 \equiv i \pmod{p-1}$. Replacing Λ by $\mathbf{Z}_p^\times$ and κ_i by the character $z \mapsto z^i$ for $z \in \mathbf{Z}_p^\times$ and $i \geq 0$, we define the $\mathbf{Z}_p$-modules A_i, D_i in the same manner, see also [ACR23, §5.4].

To ease notation, set $Y(m, p) = Y(1, Nm(p))$ and denote by $\Gamma(m, p)$ the associated congruence subgroup. As in [BSV22, Eq. (81) *et seq.*], the evaluation $\mathcal{A}_\kappa \otimes_\Lambda \mathcal{D}_\kappa \to \Lambda$ gives rise to a Λ-module homomorphism

$$\xi_i : H^1(\Gamma(m, p), \mathcal{A}_i) \to \mathrm{Hom}_\Lambda(H^1_c(\Gamma(m, p), \mathcal{D}_i), \Lambda).$$

Similarly, the determinant map $\det : \mathrm{T} \times \mathrm{T}' \to \mathbf{Z}_p^\times$ defined by $\det((x_1, x_2), (y_1, y_2)) = x_1 y_2 - x_2 y_1$, composed with $\kappa_i : \mathbf{Z}_p^\times \to \Lambda^\times$ gives rise to

$$\zeta_i : \mathrm{Hom}_\Lambda(H^1_c(\Gamma(m, p), \mathcal{D}_i), \Lambda) \to H^1(\Gamma(m, p), \mathcal{D}'_i)(-\kappa_i).$$

Then for any weight $k \geq 2$ with $k - 2 \equiv i \pmod{p-1}$ we have specialisation maps

$$\rho_{k-2} : H^1(\Gamma(m, p), \mathcal{A}_i) \to H^1_{\text{ét}}(Y(m, p)_{\overline{\mathbf{Q}}}, \mathscr{S}_{k-2}), \quad \rho_{k-2} : H^1(\Gamma(m, p), \mathcal{D}'_i) \to H^1_{\text{ét}}(Y(m, p)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2})$$

fitting into the commutative diagram

$$
\begin{array}{ccc}
H^1(\Gamma(m, p), \mathcal{A}_i) & \xrightarrow{\;\;\zeta_i \circ \xi_i\;\;} & H^1(\Gamma(m, p), \mathcal{D}'_i)(-\kappa_i) \\
\downarrow{\scriptstyle \rho_{k-2}} & & \downarrow{\scriptstyle \rho_{k-2}} \\
H^1_{\text{ét}}(Y(m, p)_{\overline{\mathbf{Q}}}, \mathscr{S}_{k-2}) & \xrightarrow{\;\;\mathbf{s}_{k-2}\;\;} & H^1_{\text{ét}}(Y(m, p)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2})(2 - k),
\end{array}
$$

where $\mathbf{s}_{k-2}$ is induced by (1.5).

Adopting the notations from [BSV22] (but working with the above modules of continuous functions $\mathcal{A}_i, A_i$ and their duals $\mathcal{D}_i, D_i$, rather than the analogous spaces of locally analytic functions considered in *op. cit.*, and letting $\boldsymbol{\mathcal{A}}_i, \boldsymbol{A}_i$ denote the (big) étale sheaves on $Y(m, p)$ associated with $\mathcal{A}_i, A_i$ as in [ACR23, §5.3,§5.6]), we let

$$(2.17) \qquad \tilde{\boldsymbol{\kappa}}_m^{(1)} \in H^1\big(\mathbf{Q}, H^1(\Gamma(m, p), D'_{2r-2}) \hat{\otimes}_\mathcal{O} H^1(\Gamma(m, p), \mathcal{D}'_{-1}) \hat{\otimes}_\mathcal{O} H^1(\Gamma(m, p), \mathcal{D}'_{-1})(2 - \boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big)$$

be the image of the element

$$\mathbf{Det}^{\boldsymbol{fgh}}_{Nm, p} \in H^0_{\text{ét}}(Y(m, p), \boldsymbol{A}'_{2r-2} \otimes \boldsymbol{\mathcal{A}}_{-1} \otimes \boldsymbol{\mathcal{A}}_{-1}(-\kappa^*_{\boldsymbol{fgh}}))$$

defined in [BSV22, §8.1] (which have specialised via $\rho_{2r-2} : \mathcal{A}'_{2r-2} \to A'_{2r-2}$) under the composition

$$
H^0_{\text{ét}}\big(Y(m,p), \mathbf{A}'_{2r-2} \otimes \mathbf{\mathcal{A}}_{-1} \otimes \mathbf{\mathcal{A}}_{-1}(-\kappa^*_{fgh})\big)
$$

$$
\xrightarrow{d_*} H^4_{\text{ét}}\big(Y(m,p)^3, \mathbf{A}'_{2r-2} \boxtimes \mathbf{\mathcal{A}}_{-1} \boxtimes \mathbf{\mathcal{A}}_{-1}(-\boldsymbol{\kappa}^*_{fgh}) \otimes \mathbf{Z}_p(2)\big)
$$

$$
\xrightarrow{\text{HS}} H^1(\mathbf{Q}, H^3_{\text{ét}}(Y(m,p)^3_{\overline{\mathbf{Q}}}, \mathbf{A}'_{2r-2} \boxtimes \mathbf{\mathcal{A}}_{-1} \boxtimes \mathbf{\mathcal{A}}_{-1})(2 + \kappa^*_{fgh}))
$$

$$
\xrightarrow{\text{K}} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), A'_{2r-2}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{A}_{-1}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{A}_{-1})(2 + \boldsymbol{\kappa}^*_{fgh})\big)
$$

$$
\xrightarrow{(w_p \otimes 1 \otimes 1)_*} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), A_{2r-2}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{A}_{-1}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{A}_{-1})(2 + \boldsymbol{\kappa}^*_{fgh})\big)
$$

$$
\xrightarrow{\mathbf{s}_{fgh}} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), D'_{2r-2}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{D}'_{-1}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{D}'_{-1})(2 - \boldsymbol{\kappa}^*_{fgh})\big),
$$

where $\mathbf{s}_{fgh} = \mathbf{s}_{2r-2} \otimes (\zeta_{-1} \circ \xi_{-1}) \otimes (\zeta_{-1} \circ \xi_{-1})$, $\kappa^*_{fgh} : \mathbf{Z}_p^\times \to \Lambda^\times$ is the square-root of the product of the characters

$$
(2.18) \qquad \kappa_f(z) = z^{2r-2}, \qquad \kappa_g(z) = \kappa_{-1}(z), \qquad \kappa_g(z) = \kappa_{-1}(z),
$$

and $\boldsymbol{\kappa}^*_{fgh} : G_{\mathbf{Q}} \to \Lambda^\times$ is the composition of κ^*_{fgh} with the p-adic cyclotomic character $\varepsilon_{\text{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^\times$.

Let $\tilde{\Gamma}(m,p) = \Gamma(1, N(mp))$. Similarly as in §2.2.1, replacing the second and third copies of $Y(m,p)$ in the above construction by the quotient $Y(m,p)^2/D_m$, where D_m is the group of diamond operators as in (2.9) acting diagonally on $Y(m,p)^2$, we obtain the class

$$
(2.19) \qquad \boldsymbol{\kappa}^{(1)}_m \in H^1\big(\mathbf{Q}, H^1(\tilde{\Gamma}(m,p), D'_{2r-2}) \hat{\otimes}_{\mathbf{Z}_p} H^1(\Gamma(m,p), \mathcal{D}'_{-1}) \hat{\otimes}_{\mathbf{Z}_p[D_m]} H^1(\Gamma(m,p), \mathcal{D}'_{-1})(2 - \boldsymbol{\kappa}^*_{fgh})\big)
$$

determined by the relation $\phi(m)\boldsymbol{\kappa}^{(1)}_m = (\mu_{m*}, d_{m*})\tilde{\boldsymbol{\kappa}}^{(1)}_m$, and we put

$$
(2.20) \qquad \boldsymbol{\kappa}^{(2)}_m := (\pi_{m*}, 1, 1)\boldsymbol{\kappa}^{(1)}_m.
$$

Proposition 2.4.1. *For a prime number ℓ and a positive integer m with $(m\ell, pN) = 1$ we have*

$$
(\pi_{i*}, \text{pr}_{j*}, \text{pr}_{k*})\boldsymbol{\kappa}^{(1)}_{m\ell} = (\bigstar)\boldsymbol{\kappa}^{(1)}_m,
$$

where

(i,j,k)	$\bigstar$
$(\ell,1,1)$	$(T_\ell, 1, 1)$
$(1,\ell,1)$	$(1, T_\ell, 1)$
$(1,1,\ell)$	$(1, 1, T_\ell)$
$(1,\ell,\ell)$	$\boldsymbol{\kappa}^*_{fgh}(\ell)\boldsymbol{\kappa}_f(\ell)^{-1}(T'_\ell, 1, 1)$
$(\ell,1,\ell)$	$\boldsymbol{\kappa}^*_{fgh}(\ell)\boldsymbol{\kappa}_g(\ell)^{-1}(1, T'_\ell, 1)$
$(\ell,\ell,1)$	$\boldsymbol{\kappa}^*_{fgh}(\ell)\boldsymbol{\kappa}_h(\ell)^{-1}(1, 1, T'_\ell)$

and $\boldsymbol{\kappa}_f, \boldsymbol{\kappa}_g, \boldsymbol{\kappa}_h : G_{\mathbf{Q}} \to \Lambda^\times$ denote the composition of the characters (2.18) with ε_{cyc}. If we also have that $(\ell, m) = 1$ then

(i,j,k)	$\bigstar$
$(1,1,1)$	$(\ell + 1)$
(ℓ,ℓ,ℓ)	$(\ell+1)\boldsymbol{\kappa}^*_{fgh}(\ell)$

Proof. With π_i replaced by pr_i and the classes $\boldsymbol{\kappa}^{(1)}_m$ replaced by $\tilde{\boldsymbol{\kappa}}^{(1)}_m$, the stated relations with an extra factor of $\ell - 1$ follow immediately from equations (174) and (176) in [BSV22] (adding the prime ℓ to the level, rather than p). The stated relations for $\boldsymbol{\kappa}^{(1)}_m$ then follow in the same way as in Proposition 2.2.4. $\square$

Assume that

$$
(\text{dist}) \qquad\qquad \xi_i(\mathfrak{p}) \not\equiv \xi_i(\bar{\mathfrak{p}}) \pmod{\mathfrak{P}},
$$

for $i = 1, 2$, so the Galois representation associated to the CM family $\boldsymbol{\theta}_{\xi_i}(Z_i)$ are p-distinguished. Note that the specialisation of $\boldsymbol{\theta}_{\xi_i}(Z_i)$ to $Z_i = 0$ gives the ordinary p-stabilisation (with U_p-eigenvalue $\xi_i\psi_0^{-1}(\bar{\mathfrak{p}})$) of

the weight 2 theta series associated to $\xi_i\psi_0^{-1}$, which may be seen as the p-adic avatar of a Hecke character of K of infinity type $(-1,0)$.

Let $\mathfrak{f}_i \subset \mathcal{O}_K$ with $(p,\mathfrak{f}_i) = 1$ be a modulus for ξ_i, and put $N_{\xi_i} = N_{K/\mathbf{Q}}(\mathfrak{f}_i)D_K$. Then for every $r \geq 1$ and every integer $m = \mathfrak{m}\overline{\mathfrak{m}}$ coprime to p and divisible only by primes split in K, we have Hecke algebra homomorphisms

$$(2.21) \qquad \phi_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r} : \mathbb{T}(1, N_{\xi_1}mp^r)' \to \mathcal{O}[H_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}], \quad \phi_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r} : \mathbb{T}(1, N_{\xi_2}mp^r)' \to \mathcal{O}[H_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}]$$

associated to $\xi_1\psi_0^{-1}, \xi_2\psi_0^{-1}$, respectively, and by Theorem 1.3.3 these induce isomorphisms

$$\nu_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r} : H^1_{\text{ét}}(Y_1(N_{\xi_1}mp^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}] \xrightarrow{\simeq} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r)}\mathcal{O}(\xi_1^{-1}\psi_0)$$

$$\nu_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r} : H^1_{\text{ét}}(Y_1(N_{\xi_2}mp^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}] \xrightarrow{\simeq} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r)}\mathcal{O}(\xi_2^{-1}\psi_0)$$

satisfying the natural compatibility as r varies. On the other hand, as explained in [ACR23, §5.6] we have $G_{\mathbf{Q}}$-module isomorphisms

$$(2.22) \qquad H^1(\Gamma(1, N_{\xi_i}m(p)), \mathcal{D}'_j(1)) \simeq e_j \varprojlim_r H^1_{\text{ét}}(Y_1(N_{\xi_i}mp^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)),$$

where $e_j = \frac{1}{p-1}\sum_{a \in (\mathbf{Z}/p\mathbf{Z})^\times} \omega^{-j}(a)[a]$ is the projector onto the ω^j-isotypic component of $\mathbf{Z}_p[\mathbf{Z}_p^\times]$. Therefore, combining (2.22) with the inverse limit $\varprojlim_r \nu_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}$ and using the decompositions $H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty} \simeq H^{(p)}_{\mathfrak{f}_1\mathfrak{m}} \times \Gamma_{\mathfrak{p}}$ and $H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty} \simeq H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}} \times \Gamma_{\mathfrak{p}}$, we obtain the $G_{\mathbf{Q}}$-equivariant isomorphisms

$$(2.23) \qquad \begin{aligned} \nu_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty} : H^1(\Gamma(1, N_{\xi_1}m(p)), \mathcal{D}'_{-1}(1))\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty}]\!] \xrightarrow{\simeq} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1\mathfrak{m})}\Lambda_{\mathfrak{p}}(\xi_1^{-1}\psi_0), \\ \nu_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty} : H^1(\Gamma(1, N_{\xi_2}m(p)), \mathcal{D}'_{-1}(1))\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty}]\!] \xrightarrow{\simeq} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2\overline{\mathfrak{m}})}\Lambda_{\mathfrak{p}}(\xi_2^{-1}\psi_0), \end{aligned}$$

where $\Lambda_{\mathfrak{p}} = \mathcal{O}[\![\Gamma_{\mathfrak{p}^\infty}]\!]$ with the G_K-action given by the tautological character $G_K \twoheadrightarrow \Gamma_{\mathfrak{p}^\infty} \hookrightarrow \Lambda_{\mathfrak{p}}^\times$.

Continuing with the construction in this section, as in §2.1, the maps used to arrive at $\boldsymbol{\kappa}^{(2)}_m$ in (2.20) are compatible under correspondences. Therefore, after tensoring with $\mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}]$ and $\mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}]$ via $\phi_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}$ and $\phi_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}$, respectively, and letting $r \to \infty$, the same construction gives rise to a class

$$\boldsymbol{\kappa}^{(3)}_m \in H^1\big(\mathbf{Q}, H^1(\Gamma(1, N(p)), D'_{2r-2})\hat{\otimes}_{\mathcal{O}}(H^1(\Gamma(m,p), \mathcal{D}'_{-1})\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty}]\!])$$
$$\hat{\otimes}_{\mathcal{O}[D_m]}(H^1(\Gamma(m,p), \mathcal{D}'_{-1})\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty}]\!])(2 - \boldsymbol{\kappa}^*_{f\boldsymbol{g}\boldsymbol{h}})\big).$$

Now let $(\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ be a triple of level-N test vectors for $(f, \boldsymbol{g}, \boldsymbol{h})$. Then we obtain a $G_{\mathbf{Q}}$-equivariant map

$$(2.24) \qquad \varpi_{\breve{f}} : H^1(\Gamma(1, N(p)), D'_{2r-2}(1))[f] \to T_f^\vee.$$

Composing with the isomorphisms (2.23) and the natural projections $H^{(p)}_{\mathfrak{f}_1\mathfrak{m}} \to H^{(p)}_{\mathfrak{m}}$, $H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}} \to H^{(p)}_{\overline{\mathfrak{m}}}$, we also obtain the $G_{\mathbf{Q}}$-equivariant maps

$$(2.25) \qquad \begin{aligned} \varpi_{\breve{\boldsymbol{g}}} : H^1(\Gamma(m,p), \mathcal{D}'_{-1}(1))\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty}]\!] \to \mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\xi_1^{-1}\psi_0}[H^{(p)}_{\mathfrak{m}}][\![\Gamma_{\mathfrak{p}}]\!], \\ \varpi_{\breve{\boldsymbol{h}}} : H^1(\Gamma(m,p), \mathcal{D}'_{-1}(1))\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty}]\!] \to \mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\xi_2^{-1}\psi_0}[H^{(p)}_{\overline{\mathfrak{m}}}][\![\Gamma_{\mathfrak{p}}]\!]. \end{aligned}$$

Taking the image of $\boldsymbol{\kappa}^{(3)}_m$ under the natural maps induced by (2.24) and (2.25) we thus obtain

$$\boldsymbol{\kappa}^{(4)}_{f,\xi_1,\xi_2,m} \in H^1\big(\mathbf{Q}, T_f^\vee\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\xi_1^{-1}\psi_0}[H^{(p)}_{\mathfrak{m}}][\![\Gamma_{\mathfrak{p}}]\!])\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\xi_2^{-1}\psi_0}[H^{(p)}_{\overline{\mathfrak{m}}}][\![\Gamma_{\mathfrak{p}}]\!])(-1 - \boldsymbol{\kappa}^*_{f\boldsymbol{g}\boldsymbol{h}})\big),$$

using that by (sd) the above $\otimes_{\mathbf{Z}_p[D_m]}$ can be replaced by $\hat{\otimes}_{\mathbf{Z}_p}$.

Next, we note that, with the identifications $\Lambda_{\mathfrak{p}} \cong \mathcal{O}[\![Z_1]\!], \Lambda_{\mathfrak{p}} \cong \mathcal{O}[\![Z_2]\!]$ defined by our choice of topological generator $\gamma_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}$, we have the following equalities as G_K-representations

$$\mathcal{O}_{\xi_1^{-1}\psi_0}[\![\Gamma_{\mathfrak{p}}]\!] = \xi_1^{-1}\psi_0\Psi_{Z_1}, \qquad \mathcal{O}_{\xi_2^{-1}\psi_0}[\![\Gamma_{\mathfrak{p}}]\!] = \xi_2^{-1}\psi_0\Psi_{Z_2},$$

where the terms on the right-hand side of these isomorphisms denote the free $\mathcal{O}[\![Z_i]\!]$-module of rank one with G_K-action via $\xi_i^{-1}\psi_0\Psi_{Z_i}$; and the character $-1 - \boldsymbol{\kappa}_{fgh}^*$ (written additively as in [BSV22, §8.1]) is the same as $\varepsilon_{\mathrm{cyc}}^{1-r}(\psi_0^{-1}\Psi_{Z_1}^{-1/2}\Psi_{Z_2}^{-1/2} \circ \mathcal{V})$. Thus, we may equivalently write

(2.26)
$$\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(4)} \in H^1\big(\mathbf{Q}, T_f^\vee(1-r)\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\psi_0\Psi_{Z_1}[H_{\mathfrak{m}}^{(p)}])\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\psi_0\Psi_{Z_2}[H_{\overline{\mathfrak{m}}}^{(p)}])\otimes(\psi_0^{-1}\Psi_{Z_1}^{-1/2}\Psi_{Z_2}^{-1/2}\circ\mathcal{V})\big),$$

which applying the diagonal map ξ_Δ in (2.7) gives rise to the class

(2.27) $$\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(5)} \in H^1\big(\mathbf{Q}, T_f^\vee(1-r)\hat{\otimes}_{\mathcal{O}}\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{Z_1}^{(1-\mathbf{c})/2}\Psi_{Z_2}^{(1-\mathbf{c})/2}[H[m]^{(p)}]\big),$$

where $\Psi_{Z_i}^{\mathbf{c}} : \Gamma_\infty \to \mathcal{O}[\![Z_i]\!]^\times$ denotes the character given by

$$\Psi_{Z_i}^{\mathbf{c}}(\sigma) = (1+Z_i)^{l(\sigma)},$$

with $l(\sigma) \in \mathbf{Z}_p$ determined by $\sigma|_{K_{\overline{\mathfrak{p}}^\infty}^\circ} = \gamma_{\overline{\mathfrak{p}}}^{l(\sigma)}$.

2.4.3. *Anticyclotomic Iwasawa cohomology classes.* The action of the complex conjugation $\mathbf{c} \in \mathrm{Gal}(K/\mathbf{Q})$ (or more precisely, any lift of $\mathbf{c}$ to $\mathrm{Gal}(K_\infty/K)$) on Γ_∞ yields the subgroup decomposition

$$\Gamma_\infty \simeq \Gamma^+ \times \Gamma^-,$$

with Γ^- representing the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension K_∞^-/K. We can then identify $\Gamma_{\mathfrak{p}^\infty} \cong \Gamma^-$ by mapping the topological generator $\gamma_{\mathfrak{p}}$ to $\gamma_{\mathfrak{p}}^{1/2}\gamma_{\overline{\mathfrak{p}}}^{-1/2} =: \gamma_- \in \Gamma^-$. Puting $V_1 := (1+Z_1)^{1/2}-1$, we have $\mathcal{O}[\![\Gamma^-]\!] \cong \mathcal{O}[\![V_1]\!]$. The character

$$\Psi_{V_1}^{1-\mathbf{c}} = \Psi_{Z_1}^{(1-\mathbf{c})/2} : \Gamma_\infty \to \mathcal{O}[\![V_1]\!]^\times$$

factors through Γ^- and is identified with the tautological character $\Gamma^- \hookrightarrow \mathcal{O}[\![\Gamma^-]\!]^\times$. Thus setting $Z_2 = 0$ in (2.27), by Shapiro's lemma the class $\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(5)}$ finally gives rise to

(2.28) $$\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(6)} \in H_{\mathrm{Iw}}^1(K[mp^\infty], T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}),$$

where $H_{\mathrm{Iw}}^1(K[mp^\infty], T)$ denotes the limit $\varprojlim_s H^1(K[mp^s], T)$ with respect to corestriction. Thus, we arrive at the following key result.

Theorem 2.4.2. *Let* $f \in S_{2r}(\Gamma_0(N_f))$ *be a p-ordinary newform of weight $2r \geq 2$, let K be an imaginary quadratic field satisfying* (spl) *and* (cn), *let ξ_1, ξ_2 be ray class characters of K satisfying* (sd) *and* (dist) *with moduli $\mathfrak{f}_1, \mathfrak{f}_2 \subset \mathcal{O}_K$, and suppose $(p, N_f\mathfrak{f}_1\mathfrak{f}_2) = 1$. Let $\phi : \Gamma^- \to \mathcal{O}^\times$ be the p-adic avatar of an anticyclotomic Hecke character of K of infinity type $(-j, j)$ with $j \in \mathbf{Z}$, and consider the G_K-representations*

$$T_{f,\xi_1\xi_2\phi} = T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\phi^{-1}, \qquad T_{f,\xi_1\xi_2^{\mathbf{c}}\phi} = T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\phi^{-1}.$$

Let $m = \mathfrak{m}\overline{\mathfrak{m}}$ run over the squarefree integers divisible only by primes split in K and coprime to pN, where $N = \mathrm{lcm}(N_f, N_{\xi_1}, N_{\xi_2})$. Then there exists collections of Iwasawa cohomology classes

$$\mathbf{z}_{f,\xi_1,\xi_2,\phi,m} \in H_{\mathrm{Iw}}^1\big(K[mp^\infty], T_{f,\xi_1\xi_2\phi}\big), \qquad {}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m} \in H_{\mathrm{Iw}}^1\big(K[mp^\infty], T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}\big)$$

such that for every prime $\ell = \mathfrak{l}\overline{\mathfrak{l}}$ split in K with $(\ell, mpN) = 1$ we have the norm relations

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(\mathbf{z}_{f,\xi_1,\xi_2,\phi,m\ell}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})(\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}),$$

where $P_{\mathfrak{l}}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} \mid T_{f,\xi_1\xi_2\phi}^\vee(1))$, and

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}({}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m\ell}) = P_{\mathfrak{l}}^{\mathbf{c}}(\mathrm{Frob}_{\mathfrak{l}})({}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}),$$

where $P_{\mathfrak{l}}^{\mathbf{c}}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} \mid T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}^\vee(1))$.

Proof. For $\phi = \psi_0^{\mathbf{c}-1}$ (which factor through Γ^- and corresponds to the p-adic avatar of a Hecke character of K of infinity type $(-1,1)$), the construction of the classes $\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ satisfying the stated norm relations follows from a direct adaptation of the proof of Theorem 2.2.6 applied to the classes $\boldsymbol{\kappa}^{(6)}_{f,\xi_1,\xi_2,m}$ of (2.28) using Proposition 2.4.1; the construction of $\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ for general ϕ then following by twisting by $\phi^{-1}\psi_0^{\mathbf{c}-1}$ using [Rub00, Thm. 6.3.5].

The construction of the 'conjugate' variant classes $^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ follows from an adaptation of the construction described in Section 2.3. Indeed, replacing the homomorphism $\phi_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}$ in (2.21) by

$$\phi_{\mathfrak{f}_2\mathfrak{m}\mathfrak{p}^r} : \mathbb{T}(1, N_{\xi_2}mp^r)' \to \mathcal{O}[H_{\mathfrak{f}_2\mathfrak{m}\mathfrak{p}^r}],$$

we arrive at the $G_{\mathbf{Q}}$-equivariant isomorphism

$$\nu_{\mathfrak{f}_2\mathfrak{m}\mathfrak{p}^\infty} : H^1(\Gamma(1, N_{\xi_2}m(p)), \mathcal{D}'_{-1}(1))\hat{\otimes}_{\mathbf{Z}_p}\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\mathfrak{m}\mathfrak{p}^\infty}]\!] \xrightarrow{\simeq} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2\mathfrak{m})}\Lambda_{\mathfrak{p}}(\xi_2^{-1}\psi_0),$$

and in the same manner as above from the classes $\boldsymbol{\kappa}^{(2)}_m$ in (2.20) we obtain classes
(2.29)
$$^{\mathbf{c}}\boldsymbol{\kappa}^{(4)}_{f,\xi_1,\xi_2,m} \in H^1\big(\mathbf{Q}, T_f^\vee(1-r)\otimes(\mathrm{Ind}^{\mathbf{Q}}_K\xi_1^{-1}\psi_0\Psi_{Z_1}[H^{(p)}_{\mathfrak{m}}])\hat{\otimes}(\mathrm{Ind}^{\mathbf{Q}}_K\xi_2^{-1}\psi_0\Psi_{Z_2}[H^{(p)}_{\mathfrak{m}}])\otimes(\psi_0^{-1}\Psi_{Z_1}^{-1/2}\Psi_{Z_2}^{-1/2}\circ\mathscr{V})\big).$$

Applying to these the 'conjugate' map $\xi_\Delta^{\mathbf{c}}$ in (2.14) we obtain

$$^{\mathbf{c}}\boldsymbol{\kappa}^{(5)}_{f,\xi_1,\xi_2,m} \in H^1\big(\mathbf{Q}, T_f^\vee(1-r)\otimes_{\mathcal{O}}\mathrm{Ind}^{\mathbf{Q}}_K\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{Z_1}^{(1-\mathbf{c})/2}\Psi_{Z_2}^{(\mathbf{c}-1)/2}[H[m]^{(p)}]\big),$$

which after setting $Z_2 = 0$ result in classes

$$^{\mathbf{c}}\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m} \in H^1_{\mathrm{Iw}}(K[mp^\infty], T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}).$$

Applying the argument in the proof of Theorem 2.2.6 to these classes yields the construction of $^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ for $\phi = 1$, and the construction for general ϕ then follows again by twisting (by ϕ^{-1}, in this case). $\qquad\square$

Remark 2.4.3. Denote by h_2 the weight 2 specialisation of $\boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(Z_2)$ obtained by setting $Z_2 = 0$, put

(2.30)
$$\begin{aligned}\mathbb{V}^\dagger &= T_f^\vee(1-r)\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}^{\mathbf{Q}}_K\xi_1^{-1}\psi_0\Psi_{Z_1})\hat{\otimes}_{\mathcal{O}}(\mathrm{Ind}^{\mathbf{Q}}_K\xi_2^{-1}\psi_0)\otimes(\psi_0^{-1}\Psi_{Z_1}^{-1/2}\circ\mathscr{V}) \\ &\cong \big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\big)\end{aligned}$$

and let $\kappa(\breve{f},\breve{g},\breve{h}_2) \in H^1(\mathbf{Q},\mathbb{V}^\dagger)$ be the projection associated to the level-N test vectors $(\breve{f},\breve{h},\breve{h}_2)$ of the corresponding specialisation of the $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$-isotypic component of the class $\boldsymbol{\kappa}^{(1)}_m$ in (2.17) for $m = 1$. Then, writing

(2.31)
$$\kappa(\breve{f},\breve{g},\breve{h}_2) = (\kappa_1(\breve{f},\breve{g},\breve{h}_2), \kappa_2(\breve{f},\breve{g},\breve{h}_2))$$

according to the decomposition

$$\begin{aligned}H^1(\mathbf{Q},\mathbb{V}^\dagger) &\cong H^1(\mathbf{Q}, T_f^\vee(1-r)\hat{\otimes}_{\mathcal{O}}\xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}) \oplus H^1(\mathbf{Q}, T_f^\vee(1-r)\hat{\otimes}_{\mathcal{O}}\mathrm{Ind}^{\mathbf{Q}}_K\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}) \\ &\cong H^1_{\mathrm{Iw}}(K[p^\infty], T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}) \oplus H^1_{\mathrm{Iw}}(K[p^\infty], T_{f,\xi_1\xi_2^{\mathbf{c}}})\end{aligned}$$

from (2.30) and Shapiro's lemma, we see directly from the proof of Theorem 2.4.2 that

(2.32)
$$(\kappa_1(\breve{f},\breve{g},\breve{h}_2), \kappa_2(\breve{f},\breve{g},\breve{h}_2)) = (\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},1}, {}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,1,1}).$$

3. Anticyclotomic Euler systems

In this section we show that the systems of Iwasawa cohomology classes constructed in Theorem 2.4.2, which form an anticyclotomic Euler systems in the sense of Jetchev–Nekovář–Skinner [JNS], land in certain Selmer groups defined in the style of Greenberg. We then record the bounds on these Selmer groups that follow by applying their machinery to our construction.

3.1. **Selmer groups.** Let $f \in S_{2r}(\Gamma_0(N_f))$ be a p-ordinary newform of weight $2r \geq 2$ with $p \nmid N_f$, and K be an imaginary quadratic field in which $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits. Let χ be an anticyclotomic Hecke character of K of infinity type $(-j, j)$, and consider the conjugate self-dual G_K-representation

$$V_{f,\chi} := V_f^\vee(1 - r) \otimes \chi^{-1}.$$

Given a prime $v \mid p$ of K and a G_{K_v}-stable subspace $\mathscr{F}_v^+(V_{f,\chi}) \subset V_{f,\chi}$, we put $\mathscr{F}_v^-(V_{f,\chi}) = V_{f,\chi}/\mathscr{F}_v^+(V_{f,\chi})$.

Definition 3.1.1. Let L be a finite extension of K, and fix $\mathscr{F} = \{\mathscr{F}_v^+(V_{f,\chi})\}_{v|p}$. The associated *Greenberg Selmer group* $\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi})$ is defined by

$$\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_{\mathscr{F}}(L_w, V_{f,\chi})} \right\},$$

where w runs over the finite primes of L, and the local conditions are given by

$$H^1_{\mathscr{F}}(L_w, V_{f,\chi}) = \begin{cases} \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\} & \text{if } w \nmid p, \\ \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, \mathscr{F}_v^-(V_{f,\chi}))\} & \text{if } w \mid v \mid p. \end{cases}$$

Given any lattice $T_{f,\chi} \subset V_{f,\chi}$, we let $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ be the inverse image of $H^1_{\mathscr{F}}(L_w, V_{f,\chi})$ under the natural map $H^1(L_w, T_{f,\chi}) \to H^1(L_w, V_{f,\chi})$, and define $\mathrm{Sel}_{\mathscr{F}}(L, T_{f,\chi})$ in the same manner; and given any $\mathbf{Z}_p$-extension $L_\infty = \bigcup_n L_n$ of L, we put

$$\mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) := \varprojlim_n \mathrm{Sel}_{\mathscr{F}}(L_n, T_{f,\chi}),$$

with limit with respect to corestriction, and also put $\mathrm{Sel}_{\mathscr{F}}(L_\infty, V_{f,\chi}) := \mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (which is independent of the chosen $T_{f,\chi}$).

We shall be particularly interested in the following two instances of these definitions:

- The *relaxed-strict Selmer group* $\mathrm{Sel}_{\mathrm{rel,str}}(L, V_{f,\chi})$ obtained by taking

$$\mathscr{F}_v^+(V_{f,\chi}) = \begin{cases} V_{f,\chi} & \text{if } v = \mathfrak{p}, \\ 0 & \text{if } v = \bar{\mathfrak{p}}. \end{cases}$$

- The *ordinary Selmer group* $\mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi})$. Since f is p-ordinary, upon restriction to $G_{\mathbf{Q}_p} \subset G_{\mathbf{Q}}$ the Galois representation $V_f^\vee$ fits into a short exact sequence

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0$$

  with $V_f^{\vee,\pm}$ one-dimensional, and with the $G_{\mathbf{Q}_p}$-action on $V_f^{\vee,-}$ being unramified (see §1.2.2). Then $\mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi})$ is the Greenberg Selmer group defined by

(3.1) $$\mathscr{F}_v^+(V_{f,\chi}) = V_{f,\chi}^+ := V_f^{\vee,+}(1 - k/2) \otimes \chi^{-1}$$

  for all $v \mid p$.

Following [BK90], we also define the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi})$ by

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) := \ker\left\{ H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_f(L_w, V_{f,\chi})} \right\},$$

where as before w runs over the finite primes of L, and the local conditions are given by

$$H^1_f(L_w, V_{f,\chi}) = \begin{cases} \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\} & \text{if } w \nmid p, \\ \ker\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, V_{f,\chi} \otimes \mathbf{B}_{\mathrm{cris}})\} & \text{if } w \mid p, \end{cases}$$

with $\mathbf{B}_{\mathrm{cris}}$ being Fontaine's crystalline period ring. The local conditions $H^1_f(L_w, T_{f,\chi}) \subset H^1(L_w, T_{f,\chi})$ are then defined by propagation.

For our later convenience, we now recall the well-known relation between these different Selmer groups. Here we shall adopt the convention that the p-adic cyclotomic character has Hodge–Tate weight -1. Thus,

since χ has infinity type $(-j, j)$ (see §1.3.1 for our convention regarding infinity types), the p-adic avatar of χ has Hodge–Tate weight j at $\mathfrak{p}$ and $-j$ at $\bar{\mathfrak{p}}$. Since it will suffice for our applications, we suppose $j \geq 0$.

Lemma 3.1.2. *Suppose f has weight $2r \geq 2$. Then for any finite extension L of K we have*

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) = \begin{cases} \mathrm{Sel}_{\mathrm{rel,str}}(L, V_{f,\chi}) & \text{if } j \geq r, \\ \mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi}) & \text{if } 0 \leq j < r. \end{cases}$$

Proof. Combining the results of [Nek00, (3.1)-(3.2)] and [Fla90, Lem. 2, p. 125], for every prime $w \mid v \mid p$ of $L/K/\mathbf{Q}$ we have

$$H^1_f(L_w, V_{f,\chi}) = \mathrm{im}\{H^1(L_w, \mathrm{Fil}^1_v(V_{f,\chi})) \to H^1(L_w, V_{f,\chi})\},$$

where $\mathrm{Fil}^1_v(V_{f,\chi}) \subset V_{f,\chi}$ is a G_{K_v}-stable subspace (assuming it exists) such that the Hodge–Tate weights of $\mathrm{Fil}^1_v(V_{f,\chi})$ (resp. $V_{f,\chi}/\mathrm{Fil}^1_v(V_{f,\chi})$) are all < 0 (resp. ≥ 0).

Now, the Hodge–Tate weights of $V^+_{f,\chi}$ and $V^-_{f,\chi} := V_{f,\chi}/V^+_{f,\chi}$ at the primes of K above p are given by:

	$V^+_{f,\chi}$	$V^-_{f,\chi}$
HT weight at $\mathfrak{p}$	$-j - r$	$-j - 1 + r$
HT weight at $\bar{\mathfrak{p}}$	$j - r$	$j - 1 + r$

and so we find $\mathrm{Fil}^1_{\mathfrak{p}}(V_{f,\chi}) = V_{f,\chi}$ and $\mathrm{Fil}^1_{\bar{\mathfrak{p}}}(V_{f,\chi}) = 0$ when $j \geq r$, and $\mathrm{Fil}^1_{\mathfrak{p}}(V_{f,\chi}) = \mathrm{Fil}^1_{\bar{\mathfrak{p}}}(V_{f,\chi}) = V^+_{f,\chi}$ when $0 \leq j < r$, yielding the equalities in the lemma. $\square$

For $A_{f,\chi} := \mathrm{Hom}_{\mathbf{Z}_p}(T_{f,\chi}, \mu_{p^\infty})$, and a choice of Galois stable subspaces $\mathscr{F} = \{\mathscr{F}^+_v(V_{f,\chi})\}_{v|p}$, we define the associated *dual Selmer group* $\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi})$ by

$$\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi}) := \ker\left\{H^1(L, A_{f,\chi}) \to \prod_w \frac{H^1(L_w, A_{f,\chi})}{H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})}\right\},$$

where $H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})$ is the orthogonal complement of $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ under local Tate duality

$$H^1(L_w, T_{f,\chi}) \times H^1(L_w, A_{f,\chi}) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

In particular, we find that:

- The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict (resp. relaxed) condition at the primes $w|\mathfrak{p}$ (resp. $w|\bar{\mathfrak{p}}$); we shall denote this by $\mathrm{Sel}_{\mathrm{str,rel}}(L, A_{f,\chi})$.
- The dual Selmer group of $\mathrm{Sel}_{\mathrm{ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p, and land in the image of the natural map

$$H^1(L_w, \mathscr{F}^+_v(A_{f,\chi})) \to H^1(L_w, A_{f,\chi}), \quad \mathscr{F}^+_v(A_{f,\chi}) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{F}^-_v(T_{f,\chi}), \mu_{p^\infty}),$$

  for $w \mid v \mid p$; we shall denote this by $\mathrm{Sel}_{\mathrm{ord,ord}}(L, A_{f,\chi})$.

3.2. **Local conditions.** We now determine the Selmer groups where the classes $\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ and $^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ of Theorem 2.4.2 live.

Theorem 3.2.1. *For any ξ_1, ξ_2, ϕ, and m as in Theorem 2.4.2, we have the inclusion*

$$\mathbf{z}_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{rel,str}}(K[mp^\infty], T_{f,\xi_1\xi_2\phi}),$$

$$^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{ord,ord}}(K[mp^\infty], T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}).$$

Proof. With notations as in the proof of Theorem 2.4.2, it suffices to check the result for $\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},m}$ and $^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,1,m}$; the result for arbitrary ϕ then follows by twisting.

We begin by explaining the case $m = 1$. Letting $\mathbb{V}^\dagger$ and $\kappa(\check{f}, \check{g}, \check{h}_2) \in H^1(\mathbf{Q}, \mathbb{V}^\dagger)$ be as in Remark 2.4.3, by (2.32) we need to show the inclusions

$$(3.2) \qquad \kappa_1(\check{f}, \check{g}, \check{h}_2) \in \mathrm{Sel}_{\mathrm{rel,str}}(K[p^\infty], T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}), \qquad \kappa_2(\check{f}, \check{g}, \check{h}_2) \in \mathrm{Sel}_{\mathrm{ord,ord}}(K[p^\infty], T_{f,\xi_1\xi_2}).$$

It follows from [BSV22, Cor. 8.2] that the class $\kappa(\breve{f}, \breve{g}, \breve{h}_2)$ lands in the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger)$ of Definition 4.2.2 below. Restricted to G_K, the $G_\mathbf{Q}$-representation $\mathbb{V}^\dagger$ decomposes as

$$(3.3) \qquad \mathbb{V}^\dagger|_{G_K} = \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right),$$

where $V_1 = (1+Z_1)^{1/2} - 1$, and from we readily find that the local condition $\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger)$ at p cutting out the balanced Selmer group corresponds to

$$(3.4) \quad \begin{aligned} \mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\mathbb{V}^\dagger|_{G_K}) &= \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right), \\ \mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(\mathbb{V}^\dagger|_{G_K}) &= \{0\} \qquad\qquad\qquad\qquad\qquad\; \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right), \end{aligned}$$

showing that the classes in (3.2) satisfy the right local conditions at the primes above p.

For the finite primes $w \nmid p$, we can give an argument that applies to all m. Since $V_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}$ is conjugate self-dual and pure of weight -1, we see that

$$H^0(K[mp^s]_w, V_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}) = H^2(K[mp^s]_w, V_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}) = 0$$

for all $s \geq 0$, and therefore $H^1(K[mp^s]_w, V_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}) = 0$ by Tate's local Euler characteristic formula. This shows that $H^1(K[mp^s]_w, T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}})$ is torsion, and as a result the inclusion

$$\mathrm{res}_w(\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},m}) \in \varprojlim_s H_f^1(K[mp^s]_w, T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}})$$

follows automatically. A similar argument shows that the classes ${}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,m}$ are unramified at the primes outside p, thereby concluding the proof of the inclusions (3.2) and hence the result for $m = 1$.

It remains to show that for general m, the classes in the statement satisfy the claimed local condition at the primes above p. Specialising the class $\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(4)}$ in (2.26) to $Z_2 = 0$, it suffices to show the result for $m = 1$ with ξ_1 replaced by $\xi_1\eta$ and ξ_2 replaced by $\xi_2\eta'$, where characters $\eta: H_{\mathfrak{m}}^{(p)} \to \mu_{p^\infty}, \eta': H_{\bar{\mathfrak{m}}}^{(p)} \to \mu_{p^\infty}$ have inverse central characters. We obtain classes $\kappa_{f,\xi_1\eta,\xi_2\eta',m} \in H^1(\mathbf{Q}, \mathbb{V}_{\eta,\eta'}^\dagger)$, where

$$\mathbb{V}_{\eta,\eta'}^\dagger := T_f^\vee(1-r) \otimes (\mathrm{Ind}_K^\mathbf{Q} \xi_1^{-1}\eta\psi_0\Psi_{Z_1}) \otimes (\mathrm{Ind}_K^\mathbf{Q} \xi_2^{-1}\eta'\psi_0) \otimes (\psi_0^{-1}\Psi_{Z_1}^{-1/2} \circ \mathscr{V}),$$

landing in $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}_{\eta,\eta'}^\dagger)$ as a consequence of [BSV22, Cor. 8.2]. Since the map ξ_Δ in (2.7) has the effect of projecting onto the first direct summand in the decomposition

$$(3.5) \qquad \mathbb{V}_{\eta,\eta'}^\dagger|_{G_K} = \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\eta\eta'\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\eta(\eta')^{\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}\right),$$

from the description of $\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\mathbb{V}_{\eta,\eta'}^\dagger|_{G_K})$ and $\mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(\mathbb{V}_{\eta,\eta'}^\dagger|_{G_K})$ analogous to (3.4), and letting η and η' vary, the inclusions

$$\mathrm{res}_w(\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},m}) \in \begin{cases} H_{\mathrm{Iw}}^1(K[mp^\infty]_w, T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}) & \text{if } w \mid \mathfrak{p}, \\ \{0\} & \text{if } w \mid \bar{\mathfrak{p}} \end{cases}$$

follow, concluding the proof of the first inclusion in the theorem. Finally, the inclusions

$$\mathrm{res}_w({}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,m}) \in H_{\mathrm{Iw}}^1(K[mp^\infty], \mathscr{F}_w^+(T_{f,\xi_1\xi_2}))$$

for all $w \mid p$ can be shown in the same manner, now specialising the class ${}^{\mathbf{c}}\boldsymbol{\kappa}_{f,\xi_1,\xi_2,m}^{(4)}$ in (2.29) to $Z_2 = 0$ and characters $\nu, \nu': H_{\mathfrak{m}}^{(p)} \to \mu_{p^\infty}$ with inverse central characters, and using the fact that the map $\xi_\Delta^{\mathbf{c}}$ in (2.14) has the effect of projecting onto the second direct summand in the decomposition (3.5). $\qquad\square$

3.3. **Applying the general machinery.** In this section we give some direct arithmetic applications that follow by applying to the classes of Theorem 2.4.2 the general Euler system machinery of Jetchev–Nekovář–Skinner [JNS]. Later in the paper, by exploiting the relation between the bottom class of our Euler systems and special values of complex and p-adic L-functions, we shall deduce from these results applications to the Bloch–Kato conjecture and the anticyclotomic Iwasawa main conjecture.

For ξ_1, ξ_2, ϕ and m as in Theorem 2.4.2, denote by

$$z_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[m], T_{f,\xi_1\xi_2\phi}),$$
$$^{\mathbf{c}}z_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K[m], T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}),$$

the image of $\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}, {}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}$ under the projections

$$\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[mp^\infty], T_{f,\xi_1\xi_2\phi}) \to \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[m], T_{f,\xi_1\xi_2\phi}),$$
$$\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K[mp^\infty], T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}) \to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K[m], T_{f,\xi_1\xi_2^{\mathbf{c}}\phi}),$$

respectively, and put

$$z_{f,\xi_1,\xi_2,\phi} := z_{f,\xi_1,\xi_2,\phi,1} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, T_{f,\xi_1\xi_2\phi}),$$
$$^{\mathbf{c}}z_{f,\xi_1,\xi_2,\phi} := {}^{\mathbf{c}}z_{f,\xi_1,\xi_2,\phi,1} \in \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$$

(recall that we assume (cn), so $K[1] = K$).

3.3.1. *Rank one results.*

Theorem 3.3.1. *Let the hypotheses be as in Theorem 2.4.2. Assume also that f is not of CM-type. Then the following hold:*

(I) *If $z_{f,\xi_1,\xi_2,\phi}$ is non-torsion, then $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\xi_1\xi_2\phi})$ is one-dimensional.*

(II) *If $^{\mathbf{c}}z_{f,\xi_1,\xi_2,\phi}$ is non-torsion, then $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$ is one-dimensional.*

Proof. By Theorem 2.4.2 and Theorem 3.2.1, the system of classes

$$(3.6) \qquad \left\{ z_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[m], T_{f,\xi_1\xi_2\phi}) \right\}_m$$

forms an anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner [JNS] for the relaxed-strict Greenberg Selmer group. Hence from their general results[1] the one-dimensionality of $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\xi_1\xi_2\phi})$ is implied by the nonvanishing of $z_{f,\xi_1,\xi_2,\phi} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\xi_1\xi_2\phi})$ provided the G_K-representation $V := V_{f,\xi_1\xi_2\phi}$ satisfies the following hypotheses:

(i) V is absolutely irreducible;

(ii) There is an element $\sigma \in G_K$ fixing $K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $V/(\sigma - 1)V$ is one-dimensional;

(iii) There is an element $\gamma \in G_K$ fixing $K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $V^{\gamma=1} = 0$.

Since f is not of CM-type, hypotheses (i)–(iii) follow easily from Momose's big image results [Mom81] as in [LLZ15, Prop. 7.1.4], whence part (I) of the theorem holds; the proof of part (II) is the same. $\qquad \square$

3.3.2. *Iwasawa-theoretic results.*

Recall that K_∞^- denotes the anticyclotomic $\mathbf{Z}_p$-extension of K, and put $\Lambda_K^- = \mathcal{O}[\![\mathrm{Gal}(K_\infty^-/K)]\!]$. Let

$$\mathbf{z}_{f,\xi_1,\xi_2,\phi} := \mathbf{z}_{f,\xi_1,\xi_2,\phi,1} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K_\infty^-, T_{f,\xi_1\xi_2\phi}),$$
$$^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi} := {}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,1} \in \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, T_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$$

be the bottom classes of the systems $\{\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}\}_m$ and $\{^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi,m}\}_m$ from Theorem 2.4.2, where the inclusions follow from Theorem 3.2.1.

Notation 3.3.2. As in [LLZ15, §7.1], we shall say that f *has big image* if the image of $G_{\mathbf{Q}}$ in $\mathrm{Aut}_\mathcal{O}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbf{Z}_p)$.

We also note that, by a theorem of Ribet [Rib85], if f is not of CM-type, then it has big image at $\mathfrak{P}$ all but finitely many primes $\mathfrak{P}$ of L.

Put

$$X_{\mathrm{str},\mathrm{rel}}(K_\infty^-, A_{f,\xi_1\xi_2\phi}) = \mathrm{Hom}_{\mathbf{Z}_p}\left(\varinjlim_n \mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K_n^-, A_{f,\xi_1\xi_2\phi}), \mathbf{Q}_p/\mathbf{Z}_p \right),$$

where K_n^- denotes the subextension of K_∞^- of with $[K_n^- : K] = p^n$, and likewise for $X_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$.

---

[1]See also [ACR23, §8.1] for an exposition of the relevant results from [JNS], which at the time of writing is not publicly available yet.

The next result can be seen as a divisibility towards an anticyclotomic Iwasawa main conjecture 'without L-functions'.

Theorem 3.3.3. *Let the hypotheses be as in Theorem 2.4.2, and assume in addition that f has big image. Then the following hold:*

(I) *If $\mathbf{z}_{f,\xi_1,\xi_2,\phi}$ is non-torsion, then $X_{\mathrm{str,rel}}(K_\infty^-, A_{f,\xi_1\xi_2\phi})$ and $\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\xi_1\xi_2\phi})$ both have Λ_K^--rank one, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}(X_{\mathrm{str,rel}}(K_\infty^-, A_{f,\xi_1\xi_2\phi})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\xi_1\xi_2\phi})}{\Lambda_K^- \cdot \mathbf{z}_{f,\xi_1,\xi_2,\phi}}\right)^2$$

in Λ_K^-.

(II) *If $^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi}$ is non-torsion, then $X_{\mathrm{ord,ord}}(K_\infty^-, A_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$ and $\mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\xi_1\xi_2^{\mathbf{c}}\phi})$ both have Λ_K^--rank one, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}(X_{\mathrm{ord,ord}}(K_\infty^-, A_{f,\xi_1\xi_2^{\mathbf{c}}\phi})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\xi_1\xi_2^{\mathbf{c}}\phi})}{\Lambda_K^- \cdot {}^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,\phi}}\right)^2$$

in Λ_K^-.

Here, in both (I) *and* (II)*, the subscript* tors *denotes the Λ_K^--torsion submodule.*

Proof. By Theorem 2.4.2 and Theorem 3.2.1, the system of classes

$$(3.7) \qquad \left\{\mathbf{z}_{f,\xi_1,\xi_2,\phi,m} \in \mathrm{Sel}_{\mathrm{rel,str}}(K[mp^\infty], T_{f,\xi_1\xi_2\phi})\right\}_m$$

forms a Λ_K^--adic anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner for the relaxed-strict Selmer group, and so by the general results of [JNS] (see [ACR23, §8.1] for a summary) the non-torsionness of $\mathbf{z}_{f,\xi_1,\xi_2,\phi}$ implies the conclusions in part (I) of the theorem provided the G_K-module $T = T_{f,\xi_1\xi_2\phi}$ satisfies the following hypotheses:

(i) $\bar{T} := T/\mathfrak{P}T$ is absolutely irreducible;

(ii) There is an element $\sigma \in G_K$ fixing $K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $T/(\sigma - 1)T$ is free of rank 1 over $\mathcal{O}$;

(iii) There is an element $\gamma \in G_K$ fixing $K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and acting as multiplication by a scalar $a_\gamma \neq 1$ on $\bar{T}$;

but these are easily checked under our assumption that f has big image (see [LLZ15, Prop. 7.1.6]). This shows part (I) of the theorem, and part (II) follows in the same manner. $\square$

Part 2. Applications

4. PRELIMINARIES

In this section, we briefly review the unbalanced triple product p-adic L-function constructed in [Hsi21] and their associated Selmer groups. We also recall from [BSV22] the explicit reciprocity law for diagonal classes.

4.1. Triple product p-adic L-function.

4.1.1. *Hida families.* Let $\mathbb{I}$ be a normal domain, finite flat over

$$\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!],$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $L_{\mathfrak{P}}$ of $\mathbf{Q}_p$. (Here, as in §2, $L_{\mathfrak{P}}$ denotes the completion of a number field L at a prime $\mathfrak{P}$ above p induced by our fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.) For an integer $N > 0$ with $p \nmid N$, and a Dirichlet character $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$, we denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ as defined in [Hsi21, §3.1].

Denote by $\mathfrak{X}_{\mathbb{I}}^+ \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ the set of *arithmetic points* of $\mathbb{I}$, consisting of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that $Q|_{1+p\mathbf{Z}_p}$ is given by $z \mapsto z^{k_Q-2}\epsilon_Q(z)$ for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the *weight of Q* and $\epsilon_Q(z) \in \mu_{p^\infty}$. (Note that here we center the weight map at weight 2, rather than weight 0 as done in *loc. cit.*). As in [Hsi21, §3.1], we say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $Q \in \mathfrak{X}_{\mathbb{I}}^+$ the specialisation $\boldsymbol{f}_Q$ gives the q-expansion of a $\mathfrak{P}$-ordinary p-stabilised newform of weight k_Q and tame conductor N. Attached to such $\boldsymbol{f}$ we let $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}}$ be the set of ring homomorphisms Q as above with $k_Q \in \mathbf{Z}_{\geq 1}$ such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form (thus $\mathfrak{X}_{\mathbb{I}}^{\mathrm{cls}}$ contains $\mathfrak{X}_{\mathbb{I}}^+$).

For $\boldsymbol{f}$ a primitive Hida family of tame level N, we let

$$\text{(4.1)} \qquad \rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \operatorname{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \operatorname{GL}_2(\mathbb{I})$$

denote the associated Galois representation, which here we take to be the *dual* of that in [Hsi21, §3.2]; in particular, the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_{\mathbb{I}} \cdot \varepsilon_{\mathrm{cyc}}$ in the notations of *loc. cit.*, where $\varepsilon_{\mathrm{cyc}}$ is the p-adic cyclotomic character. A priori, $\rho_{\boldsymbol{f}}$ is just realised over in the fraction field $\operatorname{Frac}(\mathbb{I})$, but we shall always assume that its associated residual representation $\bar{\rho}_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \operatorname{GL}_2(\kappa_{\mathbb{I}})$, where $\kappa_{\mathbb{I}}$ denotes the residue field of $\mathbb{I}$, is absolutely irreducible, in which case an integral model as in (4.1) can always be found.

Restricting to $G_{\mathbf{Q}_p}$, the Galois representation $V_{\boldsymbol{f}}$ fits into a short exact sequence

$$0 \to V_{\boldsymbol{f}}^+ \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^- \to 0,$$

where the quotient $V_{\boldsymbol{f}}^-$ is free of rank one over $\mathbb{I}$, with the $G_{\mathbf{Q}_p}$-action given by the unramified character sending an arithmetic Frobenius $\operatorname{Frob}_p^{-1}$ to $a_p(\boldsymbol{f})$, see [Wil88, Thm. 2.2.2].

Denote by $\mathbb{T}(N, \mathbb{I})$ the Hecke algebra acting on $\oplus_\chi S^o(N, \chi, \mathbb{I})$, with χ running over the Dirichlet characters modulo Np. Associated with $\boldsymbol{f}$ there is a $\mathbb{I}$-algebra homomorphism

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \mathbb{I}) \to \mathbb{I}$$

factoring through a local component $\mathbb{T}_{\mathfrak{m}}$. Following [Hid88a], we define the *congruence ideal $C(\boldsymbol{f})$ of $\boldsymbol{f}$* by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\operatorname{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

If $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished, it follows from the results of [Wil95] and [Hid88a] that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}}^{cong} \in \mathbb{I}$.

4.1.2. *Triple products of Hida families.* Let

$$(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_f, \chi_f, \mathbb{I}_f) \times S^o(N_g, \chi_g, \mathbb{I}_g) \times S^o(N_h, \chi_h, \mathbb{I}_h)$$

be a triple of primitive Hida families with

(sd-triple) $\qquad\qquad\qquad\qquad \chi_f \chi_g \chi_h = \omega^{2a}$ for some $a \in \mathbf{Z}$,

where ω is the Teichmüller character. Put

$$\mathcal{R} = \mathbb{I}_f \hat{\otimes}_{\mathcal{O}} \mathbb{I}_g \hat{\otimes}_{\mathcal{O}} \mathbb{I}_h,$$

which is a finite extension of the three-variable Iwasawa algebra $\Lambda \hat{\otimes}_{\mathcal{O}} \Lambda \hat{\otimes}_{\mathcal{O}} \Lambda$.

Let $\mathfrak{X}_{\mathcal{R}}^+ \subset \operatorname{Spec} \mathcal{R}(\overline{\mathbf{Q}}_p)$ be the weight space of $\mathcal{R}$ given by

$$\mathfrak{X}_{\mathcal{R}}^+ := \left\{ \underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathbb{I}_f}^+ \times \mathfrak{X}_{\mathbb{I}_g}^{\mathrm{cls}} \times \mathfrak{X}_{\mathbb{I}_h}^{\mathrm{cls}} \ : \ k_{Q_0} + k_{Q_1} + k_{Q_2} \equiv 0 \ (\mathrm{mod}\ 2) \right\}.$$

This can be written as the disjoint union $\mathfrak{X}_{\mathcal{R}}^+ = \mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}} \sqcup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} \sqcup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{g}} \sqcup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{h}}$, where

$$\mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}} := \left\{ \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+ \ : \ k_{Q_0} + k_{Q_1} + k_{Q_2} > 2k_{Q_i} \text{ for all } i = 0, 1, 2 \right\}$$

is the set of *balanced* weights, i.e. where each weight k_{Q_i} is smaller than the sum of the other two, and

$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} := \left\{ \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+ \ : \ k_{Q_0} \geq k_{Q_1} + k_{Q_2} \right\},$$
$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{g}} := \left\{ \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+ \ : \ k_{Q_1} \geq k_{Q_0} + k_{Q_2} \right\},$$
$$\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{h}} := \left\{ \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^+ \ : \ k_{Q_2} \geq k_{Q_0} + k_{Q_1} \right\},$$

are the sets of $\boldsymbol{f}$-, $\boldsymbol{g}$-, and $\boldsymbol{h}$-*unbalanced* weights, respectively.

Let $\mathbf{V} = V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. Write the determinant of $\mathbf{V}$ in the form $\det \mathbf{V} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ (note that this is possible by (sd-triple) and $p > 2$), and put

$$(4.2) \qquad\qquad\qquad\qquad \mathbf{V}^\dagger := \mathbf{V} \otimes \mathcal{X}^{-1},$$

which is a self-dual twist of $\mathbf{V}$.

4.1.3. *Unbalanced triple product p-adic L-function.* Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace $\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger)$ of $\mathbf{V}^\dagger$ by

$$(4.3) \qquad\qquad\qquad \mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^\dagger) := V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}} \otimes \mathcal{X}^{-1},$$

and for any $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ denote by $\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger) \subset \mathbf{V}_{\underline{Q}}^\dagger$ the corresponding specialisations.

For a rational prime ℓ, let $\varepsilon_\ell(\mathbf{V}_{\underline{Q}}^\dagger)$ be the epsilon factor attached to the local representation $\mathbf{V}_{\underline{Q}}^\dagger|_{G_{\mathbf{Q}_\ell}}$ (see [Tat79, p. 21]), and assume that

(sgn) $\qquad\qquad$ for some $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$, we have $\varepsilon_\ell(\mathbf{V}_{\underline{Q}}^\dagger) = +1$ for all prime factors ℓ of $N_f N_g N_h$.

As explained in [Hsi21, §1.2], it is known that condition (sgn) is independent of $\underline{Q}$, and it implies that the sign in the functional equation for the triple product L-function (with center at $s = 0$)

$$L(\mathbf{V}_{\underline{Q}}^\dagger, s)$$

is $+1$ (resp. -1) for all $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}} \cup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{g}} \cup \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{h}}$ (resp. $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}}$).

Theorem 4.1.1. *Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families satisfying conditions (sd-triple) and (sgn). Assume in addition that*

- $\gcd(N_f, N_g, N_h)$ *is square-free;*
- $\bar{\rho}_{\boldsymbol{f}}$ *is absolutely irreducible and p-distinguished;*

and fix a generator $\eta_{\boldsymbol{f}}^{cong}$ of the congruence ideal of $\boldsymbol{f}$. Then there exists a unique element

$$\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}$$

such that for all $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight (k_0, k_1, k_2) with $\epsilon_{Q_0} = 1$ we have

$$\left(\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(\underline{Q}) \right)^2 = \Gamma_{\mathbf{V}_{\underline{Q}}^\dagger}(0) \cdot \frac{L(\mathbf{V}_{\underline{Q}}^\dagger, 0)}{(\sqrt{-1})^{2k_0} \cdot \Omega_{\boldsymbol{f}_{Q_0}}^2} \cdot \mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^\dagger)) \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} (1 + q^{-1})^2,$$

where:

- $\Gamma_{\mathbf{V}_{\underline{Q}}^{\dagger}}(0) = \Gamma_{\mathbf{C}}(c_{\underline{Q}})\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 2 - k_1 - k_2)\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_1)\Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_2)$, *with*

$$c_{\underline{Q}} = (k_0 + k_1 + k_2 - 2)/2$$

 and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$;
- $\Omega_{\boldsymbol{f}_{Q_0}}$ *is the canonical period*

$$\Omega_{\boldsymbol{f}_{Q_0}} := (-2\sqrt{-1})^{k_0+1} \cdot \frac{\|\boldsymbol{f}_{Q_0}^{\circ}\|_{\Gamma_0(N_f)}^2}{\eta_{\boldsymbol{f}_{Q_0}}^{cong}} \cdot \left(1 - \frac{\chi_f'(p)p^{k_0-1}}{\alpha_{Q_0}^2}\right)\left(1 - \frac{\chi_f'(p)p^{k_0-2}}{\alpha_{Q_0}^2}\right),$$

 with $\boldsymbol{f}_{Q_0}^{\circ} \in S_{k_0}(\Gamma_0(N_f))$ *the newform of conductor* N_f *associated with* $\boldsymbol{f}_{Q_0}$, χ_f' *the prime-to-p part of* χ_f, *and* α_{Q_0} *the specialisation of* $a_p(\boldsymbol{f}) \in \mathbb{I}_f^{\times}$ *at* Q_0;
- $\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^{\dagger}))$ *is the modified p-Euler factor*

$$\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^{\dagger})) := \frac{L_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^{\dagger}), 0)}{\varepsilon_p(\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^{\dagger})) \cdot L_p(\mathbf{V}_{\underline{Q}}^{\dagger}/\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}_{\underline{Q}}^{\dagger}), 0)} \cdot \frac{1}{L_p(\mathbf{V}_{\underline{Q}}^{\dagger}, 0)},$$

and Σ_{exc} *is an explicitly defined subset of the prime factors of* $N_f N_g N_h$, [Hsi21, p. 416].

Proof. This is Theorem A in [Hsi21], which in fact proves a more general interpolation formula. $\square$

Remark 4.1.2. The construction of the p-adic L-function $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is based on Hida's p-adic Rankin–Selberg method [Hid88b], and the proof of the above exact interpolation formula relies on a suitable choice of test vectors $(\breve{\boldsymbol{f}}^{\star}, \breve{\boldsymbol{g}}^{\star}, \breve{\boldsymbol{h}}^{\star})$ for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of level $N = \mathrm{lcm}(N_f, N_g, N_h)$. In general (without any additional hypotheses on $\bar{\rho}_{\boldsymbol{f}}$), for any choice $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ of level-N test vectors, Hida's method produces an element

$$(4.4) \qquad\qquad \mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}}) \in \mathrm{Frac}(\mathbb{I}_f)\hat{\otimes}_{\mathcal{O}}\mathbb{I}_g\hat{\otimes}_{\mathcal{O}}\mathbb{I}_h,$$

and by virtue of the proof of Jacquet's conjecture by Harris–Kudla [HK91] (see also [DR14, Rem. 4.8]), for any fixed $\underline{Q}_0 \in \mathcal{X}_{\mathcal{R}}^{\breve{\boldsymbol{f}}}$, if the central L-value $L(\mathbf{V}_{\underline{Q}_0}^{\dagger}, 0)$ is nonzero, one can find $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ such that

$$\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})(\underline{Q}_0) \neq 0.$$

In terms of (4.4), under the hypotheses on $\bar{\rho}_f$ in Theorem 4.1.1, the p-adic L-function $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is given by $\eta_{\boldsymbol{f}}^{cong} \cdot \mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}^{\star}, \breve{\boldsymbol{g}}^{\star}, \breve{\boldsymbol{h}}^{\star})$.

4.2. **Triple product Selmer groups.** Let $\mathbf{V}^{\dagger} = \mathbf{V} \otimes \mathcal{X}^{-1}$ be the self-dual twist of the Galois representation associated to a triple of primitive Hida families $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfying (sd-triple).

Definition 4.2.1. Put

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^{\dagger}) := \left(V_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}^+ + V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}}^+ \otimes V_{\boldsymbol{h}}\right) \otimes \mathcal{X}^{-1},$$

and define the *balanced local condition* $\mathrm{H}_{\mathrm{bal}}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger})$ by

$$\mathrm{H}_{\mathrm{bal}}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger}) := \mathrm{im}\left(\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^{\dagger})) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger})\right).$$

As in (4.3), put $\mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^{\dagger}) := \left(V_{\boldsymbol{f}}^+ \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}\right) \otimes \mathcal{X}^{-1}$, and define the $\boldsymbol{f}$-*unbalanced local condition* $\mathrm{H}_{\boldsymbol{f}}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger})$ by

$$\mathrm{H}_{\boldsymbol{f}}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger}) := \mathrm{im}\left(\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\boldsymbol{f}}(\mathbf{V}^{\dagger})) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^{\dagger})\right).$$

It is easy to see that the maps appearing in these definitions are injective, and in the following we shall use this to identify $\mathrm{H}_?^1(\mathbf{Q}_p, \mathbf{V}^{\dagger})$ with $\mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^?(\mathbf{V}^{\dagger}))$ for $? \in \{\mathrm{bal}, \boldsymbol{f}\}$.

Definition 4.2.2. Let $? \in \{\mathrm{bal}, \boldsymbol{f}\}$, and define the Selmer group $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger)$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{V}^\dagger) := \ker\left\{ \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{V}^\dagger) \right\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$ (resp. $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}^\dagger)$) the *balanced* (resp. $\boldsymbol{f}$-*unbalanced*) Selmer group.

Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ and for $? \in \{\mathrm{bal}, \boldsymbol{f}\}$ define $\mathrm{H}^1_{?*}(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate duality

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

We then define the balanced and $\boldsymbol{f}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\left\{ \mathrm{H}^1(\mathbf{Q}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_{?*}(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{A}^\dagger) \right\},$$

and let $X^?(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ denote the Pontryagin dual of $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger)$.

4.3. Explicit reciprocity law.

We continue to denote by $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ a triple of primitive Hida families as in §4.1.2 satisfying (sd-triple), and put $N = \mathrm{lcm}(N_f, N_g, N_h)$. Let

$$(4.5) \qquad\qquad \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger(\mathcal{N}))$$

be the big diagonal class constructed in [BSV22, §8.1], where $\mathbf{V}^\dagger(\mathcal{N})$ denotes a free $\mathcal{R}$-module isomorphic to finitely many copies of $\mathbf{V}^\dagger$.

Remark 4.3.1. By construction, $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is the same as the $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$-isotypic projection of the class $\boldsymbol{\kappa}_m^{(1)}$ in (2.19) with $m = 1$.

The definition of the Selmer groups in §4.2 extends immediately to $\mathbf{V}^\dagger(\mathcal{N})$, and by Corollary 8.2 in *loc. cit.* one knows that $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger(\mathcal{N}))$.

Put

$$\mathscr{F}_p^3(\mathbf{V}^\dagger) = V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1} \subset \mathbf{V}^\dagger.$$

Then clearly $\mathscr{F}_p^3(\mathbf{V}^\dagger) \subset \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)$, with quotient given by

$$(4.6) \qquad\qquad \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger) \cong \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} \oplus \mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} \oplus \mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}},$$

where

$$(4.7) \qquad \begin{aligned} \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} &= V_{\boldsymbol{f}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\ \mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\ \mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^- \otimes \mathcal{X}^{-1}. \end{aligned}$$

We similarly define the level-N version $\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)(\mathcal{N})$, $\mathbb{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}(\mathcal{N})$, etc..

4.3.1. *Three-variable reciprocity law.*

As explained in [BSV22, §7.3] (see also [DR22, §5.1]), for every choice of level-N test vectors $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ one can deduce from results in [KLZ17] the construction of an injective three-variable p-adic regulator map with pseudo-null cokernel

$$(4.8) \qquad\qquad \mathrm{Log}_{(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})}^{\boldsymbol{f}} : \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}(\mathcal{N})) \to C(\boldsymbol{f})^{-1} \mathbb{I}_f \hat{\otimes}_{\mathcal{O}} \mathbb{I}_g \hat{\otimes}_{\mathcal{O}} \mathbb{I}_h,$$

where $C(\boldsymbol{f}) \subset \mathbb{I}_f$ is the congruence ideal of $\boldsymbol{f}$, characterised by the property that for all $\mathfrak{Z} \in \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}(\mathscr{N}))$ and all points $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^+$ of weight (k_0, k_1, k_2) with $\epsilon_{Q_i} = 1$ $(i = 0, 1, 2)$ we have

$$\mathrm{Log}_{(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})}^{\boldsymbol{f}}(\mathfrak{Z})(\underline{Q}) = (p-1)\alpha_{Q_0}\left(1 - \frac{\beta_{Q_0}\alpha_{Q_1}\alpha_{Q_2}}{p^{c_{\underline{Q}}}}\right)\left(1 - \frac{\alpha_{Q_0}\beta_{Q_1}\beta_{Q_2}}{p^{c_{\underline{Q}}}}\right)^{-1}$$

$$\times \begin{cases} \frac{(-1)^{c_{\underline{Q}}-k_0}}{(c_{\underline{Q}}-k_0)!} \cdot \left\langle \log_p(\mathfrak{Z}_{\underline{Q}}), \eta_{\breve{\boldsymbol{f}}_{Q_0}} \otimes \omega_{\breve{\boldsymbol{g}}_{Q_1}} \otimes \omega_{\breve{\boldsymbol{h}}_{Q_2}} \right\rangle_{\mathrm{dR}}, & \text{if } \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}}, \\ (k_0 - c_{\underline{Q}} - 1)! \cdot \left\langle \exp_p^*(\mathfrak{Z}_{\underline{Q}}), \eta_{\breve{\boldsymbol{f}}_{Q_0}} \otimes \omega_{\breve{\boldsymbol{g}}_{Q_1}} \otimes \omega_{\breve{\boldsymbol{h}}_{Q_2}} \right\rangle_{\mathrm{dR}}, & \text{if } \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}. \end{cases}$$

Here,

- $c_Q = (k_0 + k_1 + k_2 - 2)/2$ is as in Theorem 4.1.1;
- α_{Q_0} denotes the specialisation of $a_p(\boldsymbol{f})$ at Q_0 and we put $\beta_{Q_0} = \chi'_f(p)p^{k_0-1}\alpha_{Q_0}^{-1}$; and $(\alpha_{Q_1}, \beta_{Q_1})$ (resp. $(\alpha_{Q_2}, \beta_{Q_2})$) are defined likewise with $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) in place of $\boldsymbol{f}$;
- $\log_p$ and $\exp_p^*$ are the Bloch–Kato logarithm and dual exponential maps as reviewed in [BSV22, pp. 51-52];
- $\eta_{\breve{\boldsymbol{f}}_{Q_0}}$ (resp. $\omega_{\breve{\boldsymbol{g}}_{Q_1}}, \omega_{\breve{\boldsymbol{h}}_{Q_2}}$) is the differential attached to $\breve{\boldsymbol{f}}_{Q_0}$ (resp. $\breve{\boldsymbol{g}}_{Q_1}, \breve{\boldsymbol{h}}_{Q_2}$) as in [BSV22, Eq. (30)] (resp. [BSV22, Eq. (34)]); and
- $\langle -, - \rangle_{\mathrm{dR}}$ denotes the de Rham pairing of [BSV22, Eq. (32)].

Denote by $\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}$ the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under natural maps

$$(4.9) \quad \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger(\mathscr{N})) \xrightarrow{\mathrm{res}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger(\mathscr{N}))) \to \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger(\mathscr{N}))/\mathscr{F}_p^3(\mathbf{V}^\dagger(\mathscr{N})))$$
$$\to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}(\mathscr{N}))$$

arising from the restriction at p and the projection onto the first direct summand in (4.6).

Theorem 4.3.2. *Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families satisfying* (sd-triple). *Then for every triple $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ of level-N test vectors for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ we have*

$$\mathrm{Log}_{(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})}^{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}) = \mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}}),$$

where $\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ is as in (4.4).

Proof. This is Theorem A in [BSV22] (see also [DR22, Thm. 5.1]). $\qquad\square$

Remark 4.3.3. In particular, if $\bar{\rho}_f$ is absolutely irreducible and p-distinguished, then Theorem 4.3.2 gives

$$(4.10) \qquad \eta_{\boldsymbol{f}}^{cong} \cdot \mathrm{Log}_{(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)}^{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}) = \mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}),$$

where $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is as in Theorem 4.1.1. For the proof of the arithmetic applications later in this paper, in addition to (4.10) we shall use its counterpart in the $\boldsymbol{g}$-unbalanced case.

5. Definite case

In this section we deduce our applications to the Bloch–Kato conjecture and the Iwasawa main conjecture for anticyclotomic twists of f/K in the case where $\epsilon(f/K) = +1$.

Throughout this section, we let $f = \sum_{n=1}^\infty a_n q^n \in S_{2r}(\Gamma_0(N_f))$, with $p \nmid N_f$, be a p-ordinary newform of weight $2r \geq 2$ defined over $\mathcal{O}$, and K be an imaginary quadratic field satisfying (spl) and (cn).

5.1. **Anticyclotomic p-adic L-functions.** Recall that $\Gamma^- = \mathrm{Gal}(K_\infty^-/K)$ denotes the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K, and $\gamma_- \in \Gamma^-$ is a topological generator. Write

$$N_f = N^+ N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in K, and fix an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Theorem 5.1.1. *Let χ_0 be an $\mathcal{O}$-valued ring class character of K of conductor $c\mathcal{O}_K$, and suppose:*

(i) $(pN_f, cD_K) = 1$,

(ii) N^- *is the squarefree product of an odd number of primes.*

Then there exists a unique element $\Theta_p^{\mathrm{BD}}(f/K, \chi_0)(W) \in \mathcal{O}[\![W]\!]$ such that for every character ϕ of Γ^- of infinity type $(-j, j)$ with $0 \leq j < r$ and conductor p^n, we have

$$\Theta_p^{\mathrm{BD}}(f/K, \chi_0)^2(\phi(\gamma_-) - 1) = \frac{p^{(2r-1)n}}{\alpha_p^{2n}} \cdot \Gamma(r+j)\Gamma(r-j) \cdot \mathcal{E}_p(f, \chi_0\phi)^2 \cdot \frac{L(f/K, \chi_0\phi, r)}{(2\pi)^{2r} \cdot \Omega_{f, N^-}} \cdot u_K^2 \sqrt{D_K} \chi_0\phi(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

where:

- $\alpha_p \in \mathcal{O}^\times$ *is the p-adic unit root of $x^2 - a_p x + p^{2r-1}$,*
- $\mathcal{E}_p(f, \chi_0\phi) = \begin{cases} (1 - \alpha_p^{-1} p^{r-1}\chi_0\phi(\mathfrak{p}))(1 - \alpha_p p^{r-1}\chi_0\phi(\bar{\mathfrak{p}})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- $\Omega_{f, N^-} = 2^{2r} \cdot \|f\|_{\Gamma_0(N_f)}^2 \cdot \eta_{f, N^-}^{-1}$ *is the* Gross period *of f (see [Hsi21, p. 524]),*
- $u_K = |\mathcal{O}_K^\times|/2$,
- $\sigma_{\mathfrak{N}^+} \in \Gamma_\infty^-$ *is Artin symbol of $\mathfrak{N}^+$,*
- $\varepsilon_p \in \{\pm 1\}$ *is the local root number of f at p.*

Proof. This is Theorem A in [CH18b] (as extended in [Hun17, Thm. A] for $c > 1$), extending and refining a construction in [BD96] in weight 2. $\square$

5.2. Factorisation of triple product p-adic L-functions.

Let $\boldsymbol{f} \in S^o(N_f, \omega^{2r-2}, \mathbb{I})$ be the primitive Hida family specialising to the ordinary p-stabilisation of f at an arithmetic point $Q_0 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight $2r$. Let ξ_1, ξ_2 be ray class characters of K of conductors dividing the ideals $\mathfrak{f}_1, \mathfrak{f}_2 \subset \mathcal{O}_K$ coprime to p satisfying (sd), and let

$$(5.1) \qquad (\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2)) \in \mathcal{O}[\![Z_1]\!][\![q]\!] \times \mathcal{O}[\![Z_2]\!][\![q]\!]$$

be the CM Hida families attached to ξ_1, ξ_2 as in (2.16). Then $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfies conditions (sd-triple) and (sgn). Assume also that

(irr-dist) $\qquad\qquad\qquad \bar{\rho}_f$ is absolutely irreducible and p-distinguished,

so the hypotheses in Theorem 4.1.1 are satisfied. The ensuing $\boldsymbol{f}$-unbalanced triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is an element in $\mathcal{R} = \mathbb{I}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![Z_1]\!]\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![Z_2]\!] \simeq \mathbb{I}[\![Z_1, Z_2]\!]$, and in the following we let

$$(5.2) \qquad \mathscr{L}_p^{\boldsymbol{f}}(f, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{O}[\![Z_1, Z_2]\!]$$

denote its image under the natural map $\mathbb{I}[\![Z_1, Z_2]\!] \to \mathcal{O}[\![Z_1, Z_2]\!]$ defined by Q_0. More generally (in particular, without assuming (irr-dist)), for any choice of level-N test vectors, we let

$$(5.3) \qquad \mathscr{L}_p^{\boldsymbol{f}}(\breve{f}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}}) \in L_{\mathfrak{P}} \otimes_{\mathcal{O}} \mathcal{O}[\![Z_1, Z_2]\!]$$

be the image of (4.4) under the map induced by Q_0.

Proposition 5.2.1. *Assume that N^- is the square-free product of an odd number of primes. Set*

$$S_i = \mathbf{u}^2(1 + Z_i) - 1$$

for $i = 1, 2$, and

$$W_1 = \mathbf{u}^{-1}(1 + S_1)^{1/2}(1 + S_2)^{1/2} - 1, \qquad W_2 = (1 + S_1)^{1/2}(1 + S_2)^{-1/2} - 1.$$

Then

$$\mathscr{L}_p^{\boldsymbol{f}}(f, \boldsymbol{g}, \boldsymbol{h})(S_1, S_2) = \pm\mathbf{w} \cdot \Theta_p^{\mathrm{BD}}(f/K, \xi_1\xi_2)(W_1) \cdot \Theta_p^{\mathrm{BD}}(f/K, \xi_1\xi_2^{\mathbf{c}})(W_2) \cdot \frac{\eta_f^{cong}}{\eta_{f, N^-}},$$

where $\mathbf{w}$ is a unit in $\mathcal{O}[\![Z_1, Z_2]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. This is an immediate extension of Proposition 8.1 in [Hsi21], where the case $\xi_2 = \xi_1^{-1}$ is treated (and where the weight map is centered at 0 rather than 2, accounting for the change of variables from Z_i to S_i). The need to invert p in the above equality arises from the term $\prod_{q \in \Sigma_{\mathrm{exc}}} (1 + q^{-1})^2$ in Theorem 4.1.1. $\square$

5.3. Selmer group decompositions.

Assume further that the characters ξ_1 and ξ_2 satisfy (dist), so the associated big Galois representations $V_{\boldsymbol{g}}$ and $V_{\boldsymbol{h}}$ are such that

$$(5.4) \qquad V_{\boldsymbol{g}} \cong \mathrm{Ind}_K^{\mathbf{Q}}(\xi_1^{-1}\psi_0\Psi_{Z_1}), \qquad V_{\boldsymbol{h}} \cong \mathrm{Ind}_K^{\mathbf{Q}}(\xi_2^{-1}\psi_0\Psi_{Z_2}),$$

where Ψ_{Z_i} and $\psi_0 = \Psi_{\mathbf{u}-1}$ are as in §2.4.1.

Recall from §1.2 that the p-adic Galois representation $V_f^\vee$ associated to f satisfies $\det(V_f^\vee) = \varepsilon_{\mathrm{cyc}}^{2r-1}$. On the other hand, $\det(V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}) = \psi_0^2 \Psi_{Z_1}\Psi_{Z_2} \circ \mathscr{V}$. Thus, writing $\mathbf{V}_{Q_0}^\dagger$ for the specialization of $\mathbf{V}^\dagger$ to Q_0 we find

$$(5.5) \qquad \begin{aligned} \mathbf{V}_{Q_0}^\dagger &\simeq T_f^\vee \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\psi_0\Psi_{Z_1}) \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\psi_0\Psi_{Z_2}) \otimes \varepsilon_{\mathrm{cyc}}^{1-r}(\psi_0^{-1}\Psi_{Z_1}^{-1/2}\Psi_{Z_2}^{-1/2} \circ \mathscr{V}) \\ &\simeq \left(T_f^\vee(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}})\right) \oplus \left(T_f^\vee(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}})\right), \end{aligned}$$

where we put

$$(5.6) \qquad V_1 = (1+Z_1)^{1/2}(1+Z_2)^{1/2} - 1 = \mathbf{u}^{-1}(1+W_1) - 1,$$

(previously, in §2.4.3, we let $Z_2 = 0$ so $V_1 = (1+Z_1)^{1/2} - 1$) and W_1, W_2 are as in Proposition 5.2.1. In particular, since (5.6) gives

$$(5.7) \qquad \Psi_{W_1}^{1-\mathbf{c}} = \Psi_{\mathbf{u}-1}^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}} = \psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}$$

(see §2.4.1 for the second equality), we get

$$(5.8) \qquad \mathrm{H}^1(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{H}^1(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}})$$

by Shapiro's lemma.

Proposition 5.3.1. *Under (5.8), the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}),$$

and the $\boldsymbol{f}$-unbalanced Selmer group $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}).$$

Proof. From (5.5) we see that the balanced local condition is given by

$$(5.9) \qquad \begin{aligned} \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}_{Q_0}^\dagger) &\simeq \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right) \\ &\qquad\qquad\qquad \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-\mathbf{c}}\xi_2^{-1}\Psi_{W_2}^{\mathbf{c}-1}\right). \end{aligned}$$

Put

$$\widetilde{\mathbf{V}}_{Q_0}^\dagger = \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right),$$

so by (5.5) we have

$$(5.10) \qquad \mathrm{H}^1(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{H}^1(K, \widetilde{\mathbf{V}}_{Q_0}^\dagger).$$

Then from (5.9) we obtain

$$\begin{aligned} \mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) &\simeq \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right), \\ \mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) &\simeq \{0\} \qquad\qquad\qquad\quad \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right), \end{aligned}$$

which yields the claimed description of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$. Similarly, we find that the $\boldsymbol{f}$-unbalanced local condition is given by

$$\begin{aligned} \mathscr{F}_{\mathfrak{p}}^{\boldsymbol{f}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) &\simeq \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right), \\ \mathscr{F}_{\bar{\mathfrak{p}}}^{\boldsymbol{f}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) &\simeq \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right), \end{aligned}$$

from where the claimed description of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger})$ follows. $\square$

As a consequence, we also obtain the following decomposition for the Selmer groups with coefficients in $\mathbf{A}_{Q_0}^{\dagger} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}_{Q_0}^{\dagger}, \mu_{p^\infty})$, mirroring in the case of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{A}_{Q_0}^{\dagger})$ the factorisation of p-adic L-functions in Proposition 5.2.1.

Corollary 5.3.2. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{str,rel}}(K, A_f(r) \otimes \xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\mathrm{ord,ord}}(K, A_f(r) \otimes \xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}),$$

where $A_f(r) = \mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r), \mu_{p^\infty})$*; and the* $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{A}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbf{A}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{ord,ord}}(K, A_f(r) \otimes \xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\mathrm{ord,ord}}(K, A_f(r) \otimes \xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}).$$

Proof. This is immediate from Proposition 5.3.1 and local Tate duality. $\square$

5.4. **Explicit reciprocity law.** Now, we put

$$\mathbb{V}^{\dagger} = \mathbf{V}_{Q_0}^{\dagger} \otimes_{\mathcal{O}[\![Z_1, Z_2]\!]} \mathcal{O}[\![Z_1, Z_2]\!]/(Z_2),$$

where we let h_2 be the weight 2 CM form obtained by specialising $\boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(Z_2)$ to $Z_2 = 0$, and let

(5.11) $\kappa(f, \boldsymbol{g}, h_2) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}(\mathscr{N}))$

be the resulting 1-variable specialisation of the big diagonal class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in (4.5). Likewise, we denote by $\mathscr{L}_p^{\boldsymbol{f}}(f, \boldsymbol{g}, h_2)$ the image of (5.2) in $\mathcal{O}[\![Z_1, Z_2]\!]/(Z_2) \simeq \mathcal{O}[\![Z_1]\!]$, and similarly for any choice of level-N test vectors we let

$$\mathscr{L}_p^{\boldsymbol{f}}(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2) \in L_{\mathfrak{P}} \otimes_{\mathcal{O}} \mathcal{O}[\![Z_1]\!]$$

be the natural image of (5.3). Since we have the inclusion $\kappa(f, \boldsymbol{g}, h_2) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger}(\mathscr{N}))$ as a consequence of [BSV22, Cor. 8.2], we can write

(5.12) $\kappa(f, \boldsymbol{g}, h_2) = (\kappa_1(f, \boldsymbol{g}, h_2), \kappa_2(f, \boldsymbol{g}, h_2))$

according to the decomposition from Proposition 5.3.1; in particular, we have

(5.13) $\kappa_1(f, \boldsymbol{g}, h_2) \in \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}(\mathscr{N})).$

Let $\mathfrak{X}_{\mathcal{O}[\![W_1]\!]}^+$ be the set of ring homomorphisms $Q \in \mathrm{Spec}(\mathcal{O}[\![W_1]\!])(\overline{\mathbf{Q}}_p)$ with $Q(1+W_1) = \zeta_Q \mathbf{u}^{j_Q}$ for some $\zeta_Q \in \mu_{p^\infty}$ and $j_Q \in \mathbf{Z}_{\geq 0}$, and for any $\mathcal{O}[\![W_1]\!]$-module M we let M_Q denote the corresponding specialisation. Write

$$T_f^{\vee,-} := T_f^\vee / T_f^{\vee,+},$$

and denote by $p_f^- : T_f^\vee(1-r) \to T_f^{\vee,-}(1-r)$ the natural projection. Let $c_f = \eta_f^{cong} \in \mathcal{O}$ be the congruence number of f.

With a slight abuse, we shall refer to as a triple of 'level-N' test vectors for $(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)$ the triple obtained by specialising level-N test vectors $(\breve{\boldsymbol{f}}, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ for $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$.

Theorem 5.4.1. *For every triple* $(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)$ *of level-N test vectors for* $(f, \boldsymbol{g}, h_2)$ *there is an injective* $\mathcal{O}[\![W_1]\!]$-*module homomorphism with pseudo-null cokernel*

$$\mathrm{Log}_{\mathfrak{p},(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)}^f : \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}(\mathscr{N})) \to c_f^{-1}\mathcal{O}[\![W_1]\!]$$

such that for all $\mathfrak{Z} \in \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}(\mathscr{N}))$ *and* $Q \in \mathfrak{X}_{\mathcal{O}[\![W_1]\!]}^+$ *with* $0 \leq j_Q < r$ *we have*

$$\mathrm{Log}_{\mathfrak{p},(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)}^f(\mathfrak{Z})_Q = c_Q \cdot \left\langle \mathrm{exp}_p^*(\mathfrak{Z}_Q), \eta_{\breve{f}} \otimes \omega_{\breve{\boldsymbol{g}}_{Q'}} \otimes \omega_{h_2} \right\rangle_{\mathrm{dR}},$$

where c_Q *is an explicit nonzero constant, and* $Q' \in \mathrm{Spec}(\mathcal{O}[\![Z_1]\!])(\overline{\mathbf{Q}}_p)$ *is the weight $2j_Q$ specialisation given by* $Q'(1+Z_1) = \zeta_Q^2 \mathbf{u}^{2j_Q-2}$. *Moreover, we have the* explicit reciprocity law

$$\mathrm{Log}_{\mathfrak{p},(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)}^f\big(p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f, \boldsymbol{g}, h_2)))\big)(W_1) = \mathscr{L}_p^{\boldsymbol{f}}(\breve{f}, \breve{\boldsymbol{g}}, \breve{h}_2)(S_1),$$

where $S_1 = \mathbf{u}^2(1 + Z_1) - 1 = (1 + W_1)^2 - 1$.

Proof. In terms of (5.5), we find that

$$\mathscr{F}_p^3(\mathbb{V}^\dagger) = T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}} = T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}.$$

Together with (5.9), this gives the decomposition

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger)/\mathscr{F}_p^3(\mathbb{V}^\dagger) \cong \left(T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right)$$
$$\oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-\mathbf{c}}\xi_2^{-1}\Psi_{W_2}^{\mathbf{c}-1}\right),$$

with the terms in the direct sum corresponding to $\mathbb{V}_f^{\boldsymbol{g}h_2}$, $\mathbb{V}_{h_2}^{f\boldsymbol{g}}$, and $\mathbb{V}_{\boldsymbol{g}}^{fh_2}$ from (4.6), respectively, and where $W_2 = (1 + Z_1)^{1/2} - 1$ is as in Proposition 5.2.1 (with $Z_2 = 0$).

Thus we find that under the first isomorphism of Proposition 5.3.1, the composite map in (4.9) corresponds to the projection to $\mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})$ (the first factor in that decomposition) composed with the natural map

$$\mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \xrightarrow{\mathrm{res}_{\mathfrak{p}}} \mathrm{H}^1(K_{\mathfrak{p}}, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})$$

$$\xrightarrow{p_f^-} \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}),$$

and so under the corresponding isomorphisms we have $\mathrm{res}_p(\kappa(f, \boldsymbol{g}, h_2))_f = p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f, \boldsymbol{g}, h_2)))$ in

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}_f^{\boldsymbol{g}h_2}(\mathscr{N})) \cong \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}(\mathscr{N})).$$

Finally, the construction of $\mathrm{Log}_{\mathfrak{p},(\breve{f},\breve{g},\breve{h}_2)}^f$ is deduced from a specialisation of the 3-variable p-adic regulator map $\mathrm{Log}_{(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})}^{\boldsymbol{f}}$ in (4.8) by the same argument as in [ACR23, Prop. 7.3], and the stated explicit reciprocity law then follows from Theorem 4.3.2. $\qquad\square$

5.5. On the Bloch–Kato conjecture in rank 0.

In this section we deduce our first applications to the Bloch–Kato conjecture in analytic rank zero for the twisted G_K-representation

$$V_{f,\chi} := V_f^\vee(1-r) \otimes \chi^{-1}.$$

Denote by $K[c]$ the ring class field of K of conductor c. If χ is a Hecke character of conductor $c\mathcal{O}_K$, then its p-adic avatar is a locally algebraic character of $\mathrm{Gal}(K[cp^\infty]/K)$. The Galois group $\Gamma^- = \mathrm{Gal}(K_\infty^-/K)$ of the anticyclotomic $\mathbf{Z}_p$-extension of K arises as the maximal $\mathbf{Z}_p$-free quotient of $\mathrm{Gal}(K[cp^\infty]/K)$. Fix a (non-canonical) splitting

$$(5.14) \qquad\qquad \mathrm{Gal}(K[cp^\infty]/K) \simeq \Delta_c \times \Gamma^-,$$

where Δ_c is the torsion subgroup of $\mathrm{Gal}(K[cp^\infty]/K)$. Then every character of Δ_c can be viewed as the p-adic avatar of a ring class character of K of conductor dividing $cp^s\mathcal{O}_K$ for sufficiently large s. If χ is as above, we then write $\chi = \chi_t \cdot \chi_w$ according to the decomposition (5.14).

Theorem 5.5.1. *Let $f \in S_{2r}(\Gamma_0(N_f))$, with $p \nmid N_f$, be a p-ordinary newform of weight $2r \geq 2$, let K be an imaginary quadratic field satisfying* (spl) *and* (cn)*, and let χ be an anticyclotomic Hecke character of conductor $c\mathcal{O}_K$ and infinity type $(-j, j)$, $j \geq 0$. Assume that:*

- *N^- is a square-free product of an odd number of primes;*
- *$(pN_f, cD_K) = 1$;*
- *χ_t has conductor prime-to-p;*
- *$\bar{\rho}_f$ is absolutely irreducible and p-distinguihed;*
- *$p > 2r - 2$;*
- *f is not of CM-type.*

Then

$$L(f/K, \chi, r) \neq 0 \quad \implies \quad \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ holds in analytic rank zero.

Proof. We begin by noting that for $j \geq r$ the sign in the functional equation of $L(f/K, \chi, s)$ is -1 (and so $L(f/K, \chi, r) = 0$, in which case there is nothing to show), so without loss of generality below we assume that $0 \leq j < r$.

Write $\chi_t = \alpha/\alpha^{\mathbf{c}}$ with α a ray class character of K of conductor $\mathfrak{f} \subset \mathcal{O}_K$ prime-to-p (as is possible by e.g. [DR17, Lem. 6.9] or [Hid06b, Lem. 5.31] and our assumption on χ_t). For a prime $q \neq p$ split in K and an auxiliary ring class character β of q-power conductor (both to be further specified below), we consider the setting of §5.2 with the CM Hida families $(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2))$ for the ray class characters

$$\xi_1 := \beta\alpha, \qquad \xi_2 := \beta^{-1}\alpha^{-\mathbf{c}}.$$

Using $\xi_1\xi_2 = \chi_t$ and $\xi_1\xi_2^{\mathbf{c}} = \beta^2$, when specialised to $Z_2 = 0$, the factorisation in Proposition 5.2.1 becomes

$$\mathscr{L}_p^{\boldsymbol{f}}(f, \boldsymbol{g}, h_2)(S_1) = \pm\mathbf{w} \cdot \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1) \cdot \Theta_p^{\mathrm{BD}}(f/K, \beta^2)(W_2) \cdot \frac{\eta_f^{cong}}{\eta_{f,N^-}},$$

where $S_1 = \mathbf{u}^2(1 + Z_1) - 1$, $W_1 = \mathbf{u}(1 + Z_1)^{1/2} - 1$, $W_2 = (1 + Z_1)^{1/2} - 1$, and $\mathbf{w}$ is a unit in $\mathcal{O}[\![Z_1]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. By [CH18b, Thm. D] we may take q and β so that $\Theta_p^{\mathrm{BD}}(f/K, \beta^2)(W_2)$ is a unit in $\mathcal{O}[\![W_2]\!]$, and with such a choice the explicit reciprocity law of Theorem 5.4.1 can be rewritten as

$$\mathrm{Log}_{\mathfrak{p},(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)}^f \left(p_f^-(\mathrm{res}_\mathfrak{p}(\kappa_1(f, \boldsymbol{g}, h_2))) \right)(W_1) = \pm\mathbf{w}' \cdot \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1)$$

with $\mathbf{w}'$ a unit in $\mathcal{O}[\![W_1]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $(\check{f}^\star, \check{g}^\star, \check{h}_2)$ the triple of level-N test vectors from Theorem 4.1.1.

Denote by $Q \in \mathfrak{X}_{\mathcal{O}[\![W_1]\!]}^+$ the specialisation $W_1 \mapsto \zeta_Q\mathbf{u}^j - 1$ ($\zeta_Q \in \mu_{p^\infty}$) corresponding to χ_w, in the sense that

$$\chi_w = \Psi_{W_1}^{\mathbf{c}-1}|_{W_1 = \zeta_Q\mathbf{u}^j - 1} = \psi_0^{\mathbf{c}-1}\Psi_{V_1}^{\mathbf{c}-1}|_{V_1 = \zeta_Q\mathbf{u}^{j-1} - 1}.$$

Then from the above together with Theorem 5.1.1 and Theorem 5.4.1 we find

$$(5.15) \qquad \begin{aligned} L(f/K, \chi, r) \neq 0 &\implies \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(\chi_w(\gamma_-) - 1) \neq 0 \\ &\implies p_f^-(\mathrm{res}_\mathfrak{p}(\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)_Q)) \neq 0, \end{aligned}$$

where $\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)$ denotes the image of the class $\kappa_1(f, \boldsymbol{g}, h_2)$ in (5.13) under the projection

$$\mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi_t^{-1}\Psi_{W_1}^{1-\mathbf{c}}(\mathscr{N})) \to \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi_t^{-1}\Psi_{W_1}^{1-\mathbf{c}})$$
$$= \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi_t^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}})$$

associated to $(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)$.

As noted in Remark 2.4.3, the class $\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)$ is the bottom class of the anticyclotomic Euler system $\{\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},m}\}_m$ of Theorem 2.4.2 for $T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}}$ (and the given choice of level-N test vectors). Therefore, by construction, letting $\mathrm{tw}_{V_1,\psi_0^{\mathbf{c}-1}\chi_w^{-1}}(\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star))$ denote the image of $\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)$ under the 'twisting' map

$$\mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi_t^{-1}\psi_0^{1-\mathbf{c}}\Psi_{V_1}^{1-\mathbf{c}}) \to \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi^{-1}\Psi_{V_1}^{1-\mathbf{c}})$$

induced by the change of variables $V_1 \mapsto \zeta_Q^{-1}\mathbf{u}^{1-j}(1 + V_1) - 1$, it follows that $\mathrm{tw}_{V_1,\psi_0^{\mathbf{c}-1}\chi_w^{-1}}(\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star))$ is the bottom class of the twisted Euler system of Theorem 2.4.2

$$(5.16) \qquad \{\mathbf{z}_{f,\chi,m}\}_m := \{\mathbf{z}_{f,\xi_1,\xi_2,\psi_0^{\mathbf{c}-1},m} \otimes \psi_0^{\mathbf{c}-1}\chi_w^{-1}\}_m$$

for $T_{f,\xi_1\xi_2\psi_0^{\mathbf{c}-1}} \otimes \psi_0^{\mathbf{c}-1}\chi_w^{-1} = T_{f,\chi}$.

Since the class $\kappa_1(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)_Q$ in (5.15) is the same as the image of the bottom class $\mathbf{z}_{f,\chi,1}$ of the system (5.16) under natural map

$$\mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^\vee(1 - r) \otimes \chi^{-1}\Psi_{V_1}^{1-\mathbf{c}}) \cong \mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi}) \to \mathrm{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi}),$$

from Theorem 3.3.1 we deduce that $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi})$ is one-dimensional, spanned by $\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q$. Since we have in fact shown that $p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q)) \neq 0$, from the global duality exact sequence

$$0 \to \mathrm{Sel}_{\mathrm{ord},\mathrm{str}}(K, V_{f,\chi}) \to \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi}) \xrightarrow{\mathrm{res}_{\mathfrak{p}}} \frac{H^1(K_{\mathfrak{p}}, V_{f,\chi})}{H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, V_{f,\chi})}$$
$$\to \mathrm{Sel}_{\mathrm{rel},\mathrm{ord}}(K, V_{f,\chi})^\vee \to \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi})^\vee \to 0,$$

we deduce that $\mathrm{Sel}_{\mathrm{rel},\mathrm{ord}}(K, V_{f,\chi})$ is also one-dimensional and spanned by $\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q$ (hence equal to $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi})$). Finally, from another global duality exact sequence

$$0 \to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\chi}) \to \mathrm{Sel}_{\mathrm{rel},\mathrm{ord}}(K, V_{f,\chi}) \xrightarrow{\mathrm{res}_{\mathfrak{p}}} \frac{H^1(K_{\mathfrak{p}}, V_{f,\chi})}{H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, V_{f,\chi})}$$
$$\to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\chi})^\vee \to \mathrm{Sel}_{\mathrm{ord},\mathrm{str}}(K, V_{f,\chi})^\vee \to 0,$$

we deduce the vanishing of $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\chi})$; since by Lemma 3.1.2, for $0 \leq j < r$ the latter group agrees with $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$, this yields the result. $\qquad\square$

5.6. On the Iwasawa main conjecture.
Our next application is to a divisibility in the anticyclotomic Iwasawa main conjecture for modular forms in the definite setting.

For any eigenform f of weight $2r \geq 2$ and an anticyclotomic Hecke character χ, put

$$A_{f,\chi} = \mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r) \otimes \chi^{-1}, \mu_{p^\infty}),$$

and writing $\chi = \chi_t \cdot \chi_w$ as in Theorem 5.5.1, let $\Theta_p^{\mathrm{BD}}(f/K, \chi)$ denote the image of the p-adic L-function $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)$ of Theorem 5.1.1 attached to the ring class character χ_t under the twisting homomorphism $\mathrm{tw}_{\chi_w} : \mathcal{O}[\![W_1]\!] \to \mathcal{O}[\![W_1]\!]$ given by $W_1 \mapsto \chi_w(\gamma_-)(1 + W_1) - 1$.

Theorem 5.6.1. *Let the hypotheses be as in Theorem 5.5.1, and assume in addition that f has big image. Then $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_{f,\chi})$ is cotorsion over Λ_K^-, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}\big(\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_{f,\chi})^\vee\big) \supset \big(\Theta_p^{\mathrm{BD}}(f/K, \chi)^2\big)$$

in $\Lambda_K^- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. Repeating the argument in the proof of Theorem 5.5.1, we arrive at the equality

$$(5.17) \qquad \mathrm{Log}^f_{\mathfrak{p}, (\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)}\big(p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f, g, h_2)))\big)(W_1) = \pm\mathbf{w}' \cdot \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1)$$

with $\mathbf{w}'$ a unit in $\mathcal{O}[\![W_1]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. It follows from Vatsal's result [Vat03, Thm. 1.1] (as extended in [CH18b, Thm. C] and [Hun17, Thm. B] to higher weights) that the p-adic L-function $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1)$ is nonzero. Thus, letting

$$\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star) \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, T_f^\vee(1-r) \otimes \chi_t^{-1} \psi_0^{1-\mathbf{c}} \Psi_{V_1}^{1-\mathbf{c}})$$

be as in the proof of Theorem 5.5.1, we conclude that

$$\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2) \text{ is non-torsion}$$

from (5.17).

Now, as noted in the proof of Theorem 5.5.1, the twisted class $\mathrm{tw}_{V_1, \psi_0^{\mathbf{c}-1} \chi_w^{-1}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star))$ is the bottom class of the Euler system $\{\mathbf{z}_{f,\chi,m}\}_m$ for $T_{f,\chi}$ constructed in Theorem 2.4.2. Hence from Theorem 3.3.3 we deduce that $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, T_{f,\chi})$ and $X_{\mathrm{str},\mathrm{rel}}(K, A_{f,\chi})$ have both Λ_K^--rank one, and we have the divisibility

$$(5.18) \qquad \mathrm{char}_{\Lambda_K^-}\big(X_{\mathrm{str},\mathrm{rel}}(K, A_{f,\chi})_{\mathrm{tors}}\big) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, T_{f,\chi})}{\Lambda_K^- \cdot \mathrm{tw}_{V_1, \psi_0^{\mathbf{c}-1} \chi_w^{-1}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star))}\right)^2$$

in Λ_K^-. Since from (5.17) we deduce an explicit reciprocity law relating

$$\mathrm{res}_{\mathfrak{p}}(\mathrm{tw}_{V_1, \psi_0^{\mathbf{c}-1} \chi_w^{-1}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star))) = \mathrm{res}_{\mathfrak{p}}(\mathrm{tw}_{W_1, \chi_w^{-1}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)))$$

to $\operatorname{tw}_{\chi_w}(\Theta_p^{\mathrm{BD}}(f/K, \chi_t)) = \Theta_p^{\mathrm{BD}}(f/K, \chi)$, the result now follows from (5.18) and global duality by the same argument as in [BCK21, Thm. 5.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.6.2. An upper bound divisibility in the anticyclotomic Iwasawa main conjecture for $V_{f,\chi}$ as in Theorem 5.6.1 was first obtained by Bertolini–Darmon [BD05] for finite order character χ in weight 2 and by Chida–Hsieh [CH15] in higher weights $2 \leq k < p-1$ using Heegner cycles and level-raising congruences. Our proof of Theorem 5.6.1 is completely different from theirs (instead, it is more in line with Kolyvagin's original arguments), and allows us to dispense with their ramification hypotheses on $\bar{\rho}_f$.

5.7. **On the Bloch–Kato conjecture in rank** 1. The arguments in the proof of Theorem 5.6.1 give the following result towards the Bloch–Kato conjecture in rank 1.

Theorem 5.7.1. *Let the hypotheses be as in Theorem 5.5.1. If $j \geq r$ (which implies $L(f/K, \chi, r) = 0$), then*
$$\dim_{L_{\mathfrak{P}}} \operatorname{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \geq 1.$$
Moreover, there exists a class $z_{f,\chi} \in \operatorname{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ such that
$$z_{f,\chi} \neq 0 \quad \Longrightarrow \quad \dim_{L_{\mathfrak{P}}} \operatorname{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 1.$$

Proof. The proof of Theorem 5.6.1 showed that the class
$$\mathbf{z}_{f,\chi} := \operatorname{tw}_{V_1, \psi_0^{\mathbf{c}-1}\chi_w^{-1}}(\kappa_1(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)) \in \operatorname{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi})$$
is non-torsion over Λ_K^- (note that f is not required to have big image for this). On the other hand, one readily checks that the natural map
$$(5.19) \qquad \operatorname{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi})/(\gamma_- - 1)\operatorname{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi}) \to \operatorname{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi})$$
is injective. Thus we conclude that $\operatorname{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi})$ has positive $\mathcal{O}$-rank, which together with Lemma 3.1.2 yields the first part of the theorem. Letting $z_{f,\chi} \in \operatorname{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi})$ be the image of $\mathbf{z}_{f,\chi}$ under (5.19), the second claim follows from Theorem 3.3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.7.2. From the Euler system of Beilinson–Flach elements constructed by Lei–Loeffler–Zerbes and Kings–Loeffler–Zerbes [LLZ14, KLZ17] attached to the Rankin–Selberg convolution of f and a suitable CM form, one can produce a class $\mathrm{BF}_{f,\chi} \in \mathrm{H}^1(K, V_{f,\chi})$. As shown in [LLZ15] and [BL18], this class extends to a full Euler system for the G_K-representation $V_{f,\chi}$, but not for the correct local conditions at p. Indeed, with notations as in the proof of Theorem 5.5.1, it follows from the explicit reciprocity law of [KLZ17] that, for $j \geq r$, the class $\mathrm{BF}_{f,\chi}$ lands in $\operatorname{Sel}_{\mathrm{rel,str}}(K, V_{f,\chi}) = \operatorname{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ precisely when
$$(5.20) \qquad\qquad\qquad \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(\chi_w(\gamma_-) - 1) = 0$$
(see [Cas17, Thm. 2.4] and [BL18, Thm. 3.11] for a specialisation of the results of [KLZ17] to this case). However, for $j \geq r$ the character χ_w is *outside* the range of interpolation of $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)$, and so the vanishing (5.20) is *not* a consequence of $L(f/K, \chi, r) = 0$. As a result, Theorem 5.7.1 seems to fall outside the scope of methods building on these classes. (On the other hand, Heegner cycles also seem to not be enough, since N^- is assumed to have an odd number of prime factors, rending Heegner cycles not directly accessible, and in this definite setting the level-raising techniques of Bertolini–Darmon [BD05] are only known to yield results towards the Bloch–Kato conjecture in rank 0, see e.g. [LV10].)

6. INDEFINITE CASE

In this section we deduce our applications to the Bloch–Kato conjecture (in ranks 0 and 1) for anticyclotomic twists of f/K when $\epsilon(f/K) = -1$. Since the nonvanishing results we shall need from [Hsi14] are currently only available in the literature under the classical Heegner hypothesis, in the following we shall restrict to this case, but we note that with the required extension of [Hsi14] at hand (see [Bur17, Mag22] for progress in this direction), our results directly extend to the general indefinite case.

6.1. **Anticyclotomic p-adic L-functions.** Keeping the setting introduced in Section 5, we assume now that K satisfies the classical *Heegner hypothesis*:

(Heeg)   every prime $\ell \mid N_f$ splits in K,

and fix an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N_f\mathbf{Z}$.

   Let Ω_p and Ω_K be the CM periods attached to K as in [CH18a, §2.5], and put

$$\Lambda_K^{-,\mathrm{ur}} = \Lambda_K^- \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p^{\mathrm{ur}},$$

where $\mathbf{Z}_p^{\mathrm{ur}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$.

Theorem 6.1.1. *Let χ_0 be an $\mathcal{O}$-valued ring class character of K of conductor $c\mathcal{O}_K$ with $(pN_f, cD_K) = 1$. Then there exists a unique element $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_0) \in \mathbf{Z}_p^{\mathrm{ur}}[\![W]\!] \otimes_{\mathbf{Z}_p} \mathcal{O}$ such that every character ϕ of Γ^- of infinity type $(-j, j)$ with $j \geq r$ and conductor p^n, we have*

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_0)^2(\phi(\gamma_-) - 1) = \frac{\Omega_p^{4j}}{\Omega_K^{4j}} \cdot \frac{\Gamma(r+j)\Gamma(j+1-r)\phi(\mathfrak{N}^{-1})}{4(2\pi)^{2j+1}\sqrt{D_K}^{2j-1}} \cdot e_{\mathfrak{p}}(f, \chi_0\phi) \cdot L(f/K, \chi_0\phi, r),$$

where

$$e_{\mathfrak{p}}(f, \chi_0\phi) = \begin{cases} \left(1 - a_p\chi_0\phi(\bar{\mathfrak{p}})p^{-r} + \chi_0\phi(\bar{\mathfrak{p}})^2 p^{-1}\right)^2 & \textit{if } n = 0, \\ \varepsilon(\tfrac{1}{2}, (\chi_0\phi)_{\mathfrak{p}})^{-2} & \textit{else}, \end{cases}$$

with $\varepsilon(\tfrac{1}{2}, (\chi_0\phi)_{\mathfrak{p}})$ the epsilon-factor in [CH18a, p. 570] attached to the component of $\chi_0\phi$ at $\mathfrak{p}$. Moreover, $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_0)$ is a nonzero element of $\Lambda_K^{-,\mathrm{ur}}$.

Proof. This is a reformulation of results contained in [CH18a, §3]. In particular, since $(N_f, D_K) = 1$ as a consequence of (Heeg), the nonvanishing of $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_0)$ follows from [CH18a, Thm. 3.9]. $\square$

Remark 6.1.2. The CM period $\Omega_K \in \mathbf{C}^\times$ in Theorem 6.1.1 agrees with that in [BDP13, (5.1.16)], but is *different* from the period Ω_∞ defined in [dS87, p. 66] and [HT93, (4.4b)]. In fact, one has

$$\Omega_\infty = 2\pi i \cdot \Omega_K.$$

In terms of Ω_∞, the interpolation formula in Theorem 6.1.1 reads

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_0)^2(\phi(\gamma_-) - 1) = \frac{\Omega_p^{4j}}{\Omega_\infty^{4j}} \cdot \frac{\Gamma(r+j)\Gamma(j+1-r)\phi(\mathfrak{N}^{-1})}{4(2\pi)^{1-2j}\sqrt{D_K}^{2j-1}} \cdot e_{\mathfrak{p}}(f, \chi_0\phi) \cdot L(f/K, \chi_0\phi, r).$$

This is the form of the interpolation that we shall use later.

6.2. **Factorisation of triple product p-adic L-function.** As in §5.2, we consider the triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, with $\boldsymbol{f} \in S^o(N_f, \omega^{2r-2}, \mathbb{I})$ the Hida family specialising to f at an arithmetic point $Q_0 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight $2r$, and

$$(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2)) \in \mathcal{O}[\![Z_1]\!][\![q]\!] \times \mathcal{O}[\![Z_2]\!][\![q]\!]$$

the CM Hida families of §2.4.1 attached to the ray class characters ξ_1, ξ_2 satisfying (sd), and also (dist).

   The triple product p-adic L-function of interest in this section is the $\boldsymbol{g}$-unbalanced p-adic L-function

(6.1) $$\mathscr{L}_p^{\boldsymbol{g}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R} = \mathbb{I}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![Z_1]\!]\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![Z_2]\!] \simeq \mathbb{I}[\![Z_1, Z_2]\!]$$

obtained from Theorem 4.1.1 with the roles of $\boldsymbol{f}$ and $\boldsymbol{g}$ reversed (note that the conditions in Theorem 4.1.1 in this setting are ensured by (dist) and our hypothesis on the conductor of ξ_1). In the following we let

$$\mathscr{L}_p^{\boldsymbol{g}}(f, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{O}[\![Z_1, Z_2]\!]$$

be the image of $\mathscr{L}_p^{\boldsymbol{g}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under the map $\mathbb{I}[\![Z_1, Z_2]\!] \to \mathcal{O}[\![Z_1, Z_2]\!]$ given by $Q_0 : \mathbb{I} \to \mathcal{O}$.

6.2.1. *Anticyclotomic Katz p-adic L-function.* Before we can state and prove the main result of this section, we need to recall the interpolation property of the Katz p-adic L-functions [Kat78], following the exposition in [dS87]. For any ideal $\mathfrak{c} \subset \mathcal{O}_K$ coprime to p, we let $Z(\mathfrak{c})$ denote the ray class group of K of conductor $\mathfrak{c}p^\infty$ (so $Z(\mathfrak{c}) \simeq \mathrm{Gal}(K_{\mathfrak{c}p^\infty}/K)$).

Theorem 6.2.1. *There exists an element $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}} \in \mathcal{O}[\![Z(\mathfrak{c})]\!]\hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}^{\mathrm{ur}}_p$ such that for every character ξ of $Z(\mathfrak{c})$ that is crystalline at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, corresponding to a Hecke character of infinity type (k, j) with $k > 0$ and $j \leq 0$, then ξ satisfies*

$$\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}(\xi) = \frac{\Omega^{k-j}_p}{\Omega^{k-j}_\infty} \cdot \Gamma(k) \cdot \left(\frac{\sqrt{D_K}}{2\pi} \right)^j \cdot (1 - \xi^{-1}(\mathfrak{p})p^{-1})(1 - \xi(\bar{\mathfrak{p}})) \cdot L_{\mathfrak{c}}(\xi, 0),$$

where $L_{\mathfrak{c}}(\xi, s)$ denotes the Hecke L-function of ξ with the Euler factors at the primes dividing $\mathfrak{c}$ removed. Moreover, we have the functional equation

$$\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}(\xi) = \mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\bar{\mathfrak{c}}}(\xi^{-\mathbf{c}} \mathbf{N}^{-1}),$$

where the equality is up to a p-adic unit.

Proof. Our $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}$ corresponds to the measure denoted $\mu(\mathfrak{c}\bar{\mathfrak{p}}^\infty)$ in [dS87, Thm. II.4.14], and the functional equation is given in [dS87, Thm. II.6.4] (which allows to extend the interpolation property from $k > -j \geq 0$ to the entire range in the statement). □

Let $\Gamma_{\mathfrak{c}}$ be the maximal torsion-free subgroup of $Z(\mathfrak{c})$, and fix a (non-canonical) splitting

$$Z(\mathfrak{c}) \simeq \Delta_{\mathfrak{c}} \times \Gamma_{\mathfrak{c}}$$

with $\Delta_{\mathfrak{c}}$ a finite group and $\Gamma_{\mathfrak{c}} \simeq \mathbf{Z}^2_p$. For $\mathfrak{c}' \supset \mathfrak{c}$ the natural projection $Z(\mathfrak{c}) \twoheadrightarrow Z(\mathfrak{c}')$ takes $\Delta_{\mathfrak{c}}$ to $\Delta_{\mathfrak{c}'}$, inducing an isomorphism $\Gamma_{\mathfrak{c}} \xrightarrow{\sim} \Gamma_{\mathfrak{c}'}$. Thus in the following we shall identify $\Gamma_{\mathfrak{c}}$ with $\Gamma_\infty := \Gamma_{(1)}$, the Galois group of the $\mathbf{Z}^2_p$-extension of K as introduced in §2.4.1.

Suppose η is a Hecke character of K of conductor dividing $\mathfrak{c}p^\infty$. Viewing η as a character on $Z(\mathfrak{c}) \simeq \Delta_{\mathfrak{c}} \times \Gamma_K$, we put $\bar{\eta} := \eta|_{\Delta_{\mathfrak{c}}}$, and denote by $\mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\eta}}$ the image of $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}$ under the composite map

$$\mathcal{O}[\![Z(\mathfrak{c})]\!]\hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}^{\mathrm{ur}}_p \to \mathcal{O}[\![\Gamma_K]\!]\hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}^{\mathrm{ur}}_p \to \Lambda^{-,\mathrm{ur}}_K,$$

where the first arrow is the natural projection defined by $\bar{\eta}$, and the second arrow is defined by $\gamma \mapsto \gamma^{1-\mathbf{c}}$ for $\gamma \in \Gamma_\infty$. Put also $\bar{\eta}^- := \bar{\eta}^{\mathbf{c}-1}$.

Lemma 6.2.2. *Let ξ be a ray class character of K such that $\bar{\xi}^-$ has conductor $\mathfrak{c}$ prime-to-p. Assume that:*
  (i) *$\mathfrak{c}$ is only divisible by primes that are split in K;*
  (ii) *$\Delta_{\mathfrak{c}}$ has order prime-to-p;*
  (iii) *$\bar{\xi}^-|_{G_{K_v}} \neq 1$ for all primes $v \mid p$ in K;*
  (iv) *$\bar{\xi}^-$ has order at least 3.*
Then the congruence ideal of the CM Hida family $\boldsymbol{\theta}_\xi(Z)$ in (2.16) is principal, generated by $\mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\xi}^-}$.

Proof. As explained in [ACR25, Prop. 4.6], this is a consequence of the proof of the anticyclotomic Iwasawa main conjecture for Hecke characters by Hida–Tilouine [HT93, HT94] and Hida [Hid06a] (recall that here we assume (cn), so the omitted term h_K is a p-adic unit). □

6.2.2. *The factorisation result.* We now fix our choice of generator of the congruence ideal of $\boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}(Z_1)$.

Definition 6.2.3. For ξ_1 satisfying the conditions of Lemma 6.2.2 (in particular, note that (iii) is equivalent to (dist)), put

$$\mathscr{L}^{\boldsymbol{g}}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) := \mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\xi}^-_1} \cdot \mathscr{L}^{\boldsymbol{g}}_p(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star),$$

where $(\breve{\boldsymbol{f}}^\star, \breve{\boldsymbol{g}}^\star, \breve{\boldsymbol{h}}^\star)$ is the triple of level-N test vectors from Theorem 4.1.1 (see also Remark 4.1.2), and let $\mathscr{L}^{\boldsymbol{g}}_p(f, \boldsymbol{g}, \boldsymbol{h})$ denote its image under the map induced by $Q_0 : \mathbb{I} \to \mathcal{O}$.

Note that ξ_1 can be replaced by a twist $\xi_1 \cdot \phi \circ \mathbf{N}$ for a Dirichlet character ϕ without changing $\bar{\xi}_1^-$, and thus in the following we may assume that ξ_1 satisfies the following minimality hypotheses:

(6.2) $\qquad\qquad\qquad$ the conductor of ξ_1 is minimal among Dirichlet twists.

The following is an analogue of Proposition 5.2.1 in the indefinite setting. Note that a variant of this result first appeared in the work of Darmon–Lauder–Rotger (see [DLR15, Thm. 3.9]), but the formulation of their result is not well-suited for our Iwasawa-theoretic purposes in this paper.

Proposition 6.2.4. *Assume that ξ_1 satisfies the conditions in Lemma 6.2.2. Set*
$$S_i = \mathbf{u}^2(1 + Z_i) - 1$$
for $i = 1, 2$, and
$$W_1 = \mathbf{u}^{-1}(1 + S_1)^{1/2}(1 + S_2)^{1/2} - 1, \qquad W_2 = (1 + S_1)^{1/2}(1 + S_2)^{-1/2} - 1.$$
Then
$$\mathscr{L}_p^{\boldsymbol{g}}(f, \boldsymbol{g}, \boldsymbol{h})(S_1, S_2) = \pm \mathbf{w} \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2)(W_1) \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2^{\mathbf{c}})(W_2),$$
where $\mathbf{w}$ is a unit in $\mathcal{O}[\![Z_1, Z_2]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. Let k_1, k_2 be integers with $k_1 \equiv k_2 \pmod 2$ and $k_1 \geq k_2 + 2r$. Set $S_i = \mathbf{u}^{k_i} - 1$ for $i = 1, 2$, so the corresponding specialisations of W_i are given by
$$W_1 = \mathbf{u}^{(k_1+k_2-2)/2} - 1, \qquad W_2 = \mathbf{u}^{(k_1-k_2)/2} - 1,$$
and denote by $\mathbf{V}_{\underline{Q}}^{\dagger}$ the specialisation of $\mathbf{V}^{\dagger}$ at $\underline{Q} = (Q_0, S_1, S_2)$. Putting
$$T_i = \mathbf{u}^{-1}(1 + S_i) - 1 = \mathbf{u}(1 + Z_i) - 1$$
for the ease of notation, we have
$$\det(T_f^{\vee} \otimes V_{\boldsymbol{g}_{T_1}} \otimes V_{\boldsymbol{h}_{T_2}}) = \varepsilon_{\mathrm{cyc}}^{2r-1} \cdot (\xi_1\xi_2\Psi_{T_1}\Psi_{T_2} \circ \mathscr{V}) = \varepsilon_{\mathrm{cyc}}^{2r-1} \cdot (\Psi_{T_1}\Psi_{T_2} \circ \mathscr{V}),$$
using that the central characters of ξ_1 and ξ_2 are inverses of each other for the second equality, and so

(6.3)
$$\begin{aligned}
\mathbf{V}_{\underline{Q}}^{\dagger} &= T_f^{\vee} \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\Psi_{T_1}) \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\Psi_{T_2}) \otimes \varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2} \circ \mathscr{V}) \\
&\simeq \left(T_f^{\vee}(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee}(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right).
\end{aligned}$$

Thus we find that the completed L-value appearing in the interpolation formula of Theorem 4.1.1 is given by

(6.4)
$$\begin{aligned}
\Gamma_{\mathbf{V}_{\underline{Q}}^{\dagger}}(0) \cdot L(\mathbf{V}_{\underline{Q}}^{\dagger}, 0) = {}& \frac{\Gamma\left(\frac{k_1+k_2}{2} + r - 1\right)\Gamma\left(\frac{k_1-k_2}{2} - r + 1\right)\Gamma\left(\frac{k_1+k_2}{2} - r\right)\Gamma\left(\frac{k_1-k_2}{2} + r\right)}{2^4 \cdot (2\pi)^{2k_1}} \\
&\times L(f/K, \xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}, r) \cdot L(f/K, \xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}, r),
\end{aligned}$$

and similarly the modified Euler factor decomposes as

(6.5)
$$\begin{aligned}
\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{g}}(\mathbf{V}_{\underline{Q}}^{\dagger})) = {}& \left(1 - a_p(\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})(\bar{\mathfrak{p}})p^{-r} + (\xi_1\xi_2\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})^2 p^{-1}\right)^2 \\
&\times \left(1 - a_p(\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})p^{-r} + (\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})^2 p^{-1}\right)^2.
\end{aligned}$$

Moreover, letting χ_g' be the prime-to-p part of the nebentypus character of $\boldsymbol{g}_{T_1}$, we have
$$\left(1 - \frac{\chi_g'(p)p^{k_1-1}}{\xi_1\Psi_{T_1}^{-1}(\bar{\mathfrak{p}})^2}\right)\left(1 - \frac{\chi_g'(p)p^{k_1-2}}{\xi_1\Psi_{T_1}^{-1}(\bar{\mathfrak{p}})^2}\right) = \left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})\right)\left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})p^{-1}\right),$$

and therefore the canonical period $\Omega_{\boldsymbol{g}_{T_1}}$ in Theorem 4.1.1 (associated with the generator $\eta_{\boldsymbol{g}}^{\star} = \mathcal{L}_{\mathfrak{p}, \bar{\xi}_1^-}^{\mathrm{Katz},-}$ of $C(\boldsymbol{g})$ from Lemma 6.2.2) is given by

(6.6)
$$\Omega_{\boldsymbol{g}_{T_1}} = (-2\sqrt{-1})^{k_1+1} \cdot \frac{\|\boldsymbol{g}_{T_1}^{\circ}\|_{\Gamma_0(C)}^2}{\eta_{\boldsymbol{g}_{T_1}}^{\star}} \cdot \left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})\right)\left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})p^{-1}\right),$$

where $C = N_{K/\mathbf{Q}}(\mathfrak{f}_1)D_K$, and we note that we may ignore the term $\prod_{q \in \Sigma_{\mathrm{exc}}}(1+q^{-1})^2$ from Theorem 4.1.1, since (up to a p-adic unit) it only contributes a fixed power of p (in particular, independent of k_1, k_2).

On the other hand, since g_{T_1} has weight k_1, from Hida's formula for the adjoint L-value [HT93, Thm. 7.1] (using that ξ_1 satisfies the minimality condition (6.2)) and Dirichlet's class number formula we obtain

$$\|\boldsymbol{g}_{T_1}^{\circ}\|_{\Gamma_0(C)}^2 = \Gamma(k_1) \cdot \frac{D_K^2}{2^{2k_1}\pi^{k_1+1}} \cdot \frac{2\pi h_K}{w_K\sqrt{D_K}} \cdot L(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}, 1),$$

where $w_K = |\mathcal{O}_K^{\times}|$.

Note that $L(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}, 1) = L(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}\mathbf{N}^{-1}, 0)$, and $\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}\mathbf{N}^{-1}$ has infinity type $(k_1, 2-k_1)$. Hence for $k_1 \geq 2$ this character lies in the range of interpolation of $\mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}$, where $\mathfrak{f}$ denotes the conductor of $\xi_1^{1-\mathbf{c}}$, and from the right above formula for $L(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}, 1)$ and Theorem 6.2.1 we obtain

$$
(6.7) \quad \begin{aligned}
\mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}\mathbf{N}^{-1}) &= \left(\frac{\Omega_p}{\Omega_\infty}\right)^{2k_1-2} \cdot \frac{\pi^{2k_1-2} \cdot 2^{3k_1-3}}{\sqrt{D_K}^{k_1+1}} \\
&\times \left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})\right)\left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})p^{-1}\right) \cdot \|\boldsymbol{g}_{T_1}^{\circ}\|_{\Gamma_0(C)}^2 \cdot \frac{w_K}{h_K}.
\end{aligned}
$$

Moreover, by the functional equation of Theorem 6.2.1 and the definition of $\eta_{\boldsymbol{g}}^{\star}$ we have the relation

$$(6.8) \qquad \frac{h_K}{w_K} \cdot \mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}(\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}\mathbf{N}^{-1}) \sim_p \eta_{\boldsymbol{g}_{T_1}}^{\star},$$

where $\sim_p$ denotes equality up to a p-adic unit. Therefore, equations (6.7) and (6.8) imply that

$$(6.9) \qquad \frac{\|\boldsymbol{g}_{T_1}^{\circ}\|_{\Gamma_0(C)}^2}{\eta_{\boldsymbol{g}_{T_1}}^{\star}} \cdot \left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})\right)\left(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\bar{\mathfrak{p}})p^{-1}\right) \sim_p \left(\frac{\Omega_\infty}{\Omega_p}\right)^{2k_1-2} \cdot \frac{\sqrt{D_K}^{k_1+1}}{(2\pi)^{2k_1-2}}$$

Hence, from (6.6) and (6.9) we arrive at

$$(6.10) \qquad \frac{1}{\Omega_{\boldsymbol{g}_{T_1}}} \sim_p \left(\frac{\Omega_p}{\Omega_\infty}\right)^{2k_1-2} \cdot \frac{(2\pi)^{2k_1-2}}{\sqrt{-D_K}^{k_1+1}}.$$

Finally, note that the characters $\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}$ and $\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}$ in the right-hand side of (6.4) are both anticyclotomic, and of infinity type $(-(k_1+k_2-2)/2, (k_1+k_2-2)/2)$ and $(-(k_1-k_2)/2, (k_1-k_2)/2)$, respectively, and so for $k_1 \geq k_2 + 2r$ they are in the range of interpolation for $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2)$ and $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2^{\mathbf{c}})$, respectively. Thus substituting (6.4), (6.5), and (6.10) into the interpolation formula for $\mathscr{L}_p^{\boldsymbol{g}}(f, \boldsymbol{g}, \boldsymbol{h})$ in Theorem 4.1.1 and comparing with Theorem 6.1.1 we finally arrive at

$$\mathscr{L}_p^{\boldsymbol{g}}(f, \boldsymbol{g}, \boldsymbol{h})^2(S_1, S_2) \sim_p \frac{-1}{D_K^{k_1+1}} \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2)^2(W_1) \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2^{\mathbf{c}})^2(W_2),$$

and this yields the proof of the result.   □

6.3. Selmer group decomposition.

Proposition 6.3.1. *Under the direct sum decomposition*

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{H}^1(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}})$$

of (5.8), *the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}});$$

and the $\boldsymbol{g}$-unbalanced Selmer group $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\mathrm{rel,str}}(K, T_f^{\vee}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}).$$

Proof. The result for $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$ is given in Proposition 5.3.1, so we focus on $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$. Put

$$\widetilde{\mathbf{V}}_{Q_0}^\dagger = \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right),$$

so by Shapiro's lemma we have

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{H}^1(K, \widetilde{\mathbf{V}}_{Q_0}^\dagger).$$

Putting $T_i = \mathbf{u}(1+Z_i) - 1$ as in the proof of Proposition 6.2.4, so (5.4) can be rewritten as

$$V_{\boldsymbol{g}} \cong \mathrm{Ind}_K^{\mathbf{Q}}(\xi_1^{-1}\Psi_{T_1}), \qquad V_{\boldsymbol{h}} \cong \mathrm{Ind}_K^{\mathbf{Q}}(\xi_2^{-1}\Psi_{T_2}),$$

a direct computation shows that the $\boldsymbol{g}$-unbalanced local condition is given by

$$\mathscr{F}_p^{\boldsymbol{g}}(\mathbf{V}_{Q_0}^\dagger) = T_f^\vee \otimes \xi_1^{-1}\Psi_{T_1} \otimes \left(\xi_2^{-1}\Psi_{T_2} \oplus \xi_2^{-\mathbf{c}}\Psi_{T_2}^{\mathbf{c}}\right) \otimes \varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2} \circ \mathscr{V})$$
$$= \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right) \oplus \left(T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right).$$

Therefore, we have

$$\mathscr{F}_{\mathfrak{p}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) = \widetilde{\mathbf{V}}_{Q_0}^\dagger, \quad \mathscr{F}_{\bar{\mathfrak{p}}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger) = 0,$$

and this yields the stated decomposition for $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger)$. $\qquad\square$

Corollary 6.3.2. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}_{Q_0}^\dagger) \simeq \mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K, A_f(r) \otimes \xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, A_f(r) \otimes \xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1});$$

and the $\boldsymbol{g}$-*unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{A}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q}, \mathbf{A}_{Q_0}^\dagger) \simeq \mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K, A_f(r) \otimes \xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K, A_f(r) \otimes \xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}).$$

Proof. As in Corollary 5.3.2, this is immediate from Proposition 6.3.1 and local Tate duality. $\qquad\square$

6.4. **Explicit reciprocity law.** As in §5.4, we put

$$\mathbb{V}^\dagger = \mathbf{V}_{Q_0}^\dagger \otimes_{\mathcal{O}\llbracket Z_1, Z_2\rrbracket} \mathcal{O}\llbracket Z_1, Z_2\rrbracket/(Z_2),$$

let h_2 be specialisation of $\boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(Z_2)$ of weight 2 given by $Z_2 = 0$, but now consider the second component $\kappa_2(f, \boldsymbol{g}, h_2)$ of the specialised big diagonal class

$$\kappa(f, \boldsymbol{g}, h_2) = (\kappa_1(f, \boldsymbol{g}, h_2), \kappa_2(f, \boldsymbol{g}, h_2))$$

according to the decomposition of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger(\mathscr{N}))$ from Proposition 6.3.1; in particular, we have

$$(6.11) \qquad \kappa_2(f, \boldsymbol{g}, h_2) \in \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N})),$$

where $W_2 = (1+Z_1)^{1/2} - 1$.

Let $\mathfrak{X}_{\mathcal{O}\llbracket W_2\rrbracket}^+$ be the set of ring homomorphisms $Q \in \mathrm{Spec}(\mathcal{O}\llbracket W_2\rrbracket)(\overline{\mathbf{Q}}_p)$ with $Q(1+W_2) = \zeta_Q \mathbf{u}^{j_Q}$ for some $\zeta_Q \in \mu_{p^\infty}$ and $j_Q \in \mathbf{Z}_{\geq 0}$, and for any $\mathcal{O}\llbracket W_2\rrbracket$-module M we let M_Q denote the corresponding specialisation.

Theorem 6.4.1. *For every triple* $(\check{f}, \check{\boldsymbol{g}}, \check{h}_2)$ *of level-N test vectors for* $(f, \boldsymbol{g}, h_2)$ *there is an injective* $\mathcal{O}\llbracket W_2\rrbracket$-*module homomorphism with pseudo-null cokernel*

$$\mathrm{Log}_{\bar{\mathfrak{p}}, (\check{f}, \check{\boldsymbol{g}}, \check{h}_2)}^{\boldsymbol{g}} : \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N})) \to C(\boldsymbol{g})^{-1}\mathcal{O}\llbracket W_2\rrbracket$$

such that for any $\mathfrak{Z} \in \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N}))$ *and* $Q \in \mathfrak{X}_{\mathcal{O}\llbracket W_2\rrbracket}^+$ *of weight* $j_Q \geq r$ *we have*

$$\mathrm{Log}_{\bar{\mathfrak{p}}, (\check{f}, \check{\boldsymbol{g}}, \check{h}_2)}^{\boldsymbol{g}}(\mathfrak{Z})_Q = c_Q \cdot \left\langle \exp_p^*(\mathfrak{Z}_Q), \omega_{\check{f}} \otimes \eta_{\check{\boldsymbol{g}}_{Q'}} \otimes \omega_{\check{h}_2} \right\rangle_{\mathrm{dR}},$$

where c_Q *is an explicit nonzero constant, and* $Q' \in \mathrm{Spec}(\mathcal{O}\llbracket Z_1\rrbracket)(\overline{\mathbf{Q}}_p)$ *is given by* $Q'(1+Z_1) = \zeta_Q^2 \mathbf{u}^{2j_Q}$. *Moreover, we have the* explicit reciprocity law

$$\mathrm{Log}_{\bar{\mathfrak{p}}, (\check{f}, \check{\boldsymbol{g}}, \check{h}_2)}^{\boldsymbol{g}}\left(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f, \boldsymbol{g}, h_2))\right)(W_2) = \mathscr{L}_p^{\boldsymbol{g}}(\check{f}, \check{\boldsymbol{g}}, \check{h}_2)(S_1),$$

where $S_1 = \mathbf{u}^2(1+Z_1) - 1 = \mathbf{u}^2(1+W_2)^2 - 1$.

Proof. As shown in the proof of Theorem 5.4.1, we have

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger)/\mathscr{F}_p^3(\mathbb{V}^\dagger) \cong \left(T_f^{\vee,-}(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right)$$
$$\oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right) \oplus \left(T_f^{\vee,+}(1-r) \otimes \xi_1^{-\mathbf{c}}\xi_2^{-1}\Psi_{W_2}^{\mathbf{c}-1}\right),$$

with the direct summands corresponding to $\mathbb{V}_f^{\boldsymbol{g}h_2}$, $\mathbb{V}_{h_2}^{f\boldsymbol{g}}$, and $\mathbb{V}_{\boldsymbol{g}}^{fh_2}$ from (4.6), respectively. As a result, the analogue of the composite map (4.9) in the present $\boldsymbol{g}$-unbalanced case:

$$(6.12) \qquad \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger(\mathscr{N})) \xrightarrow{\mathrm{res}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger(\mathscr{N}))) \to \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger(\mathscr{N}))/\mathscr{F}_p^3(\mathbb{V}^\dagger(\mathscr{N})))$$
$$\to \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}_{\boldsymbol{g}}^{fh_2}(\mathscr{N}))$$

corresponds, under the isomorphism of Proposition 6.3.1, to the projection onto $\mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1-r) \otimes$
$\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N}))$ (the second factor in that decomposition) composed with the restriction map

$$\mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N})) \xrightarrow{\mathrm{res}_{\bar{\mathfrak{p}}}} \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N})).$$

In particular, under the corresponding identifications the image $\mathrm{res}_p(\kappa(f,\boldsymbol{g},h_2))_{\boldsymbol{g}}$ of $\kappa(f,\boldsymbol{g},h_2)$ under (6.12) is such that

$$\mathrm{res}_p(\kappa(f,\boldsymbol{g},h_2))_{\boldsymbol{g}} = \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f,\boldsymbol{g},h_2))$$

in

$$\mathrm{H}^1(\mathbf{Q}_p, V_{\boldsymbol{g}}^{fh_2}(\mathscr{N})) \simeq \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_f^{\vee,+}(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}(\mathscr{N})).$$

On the other hand, the construction of $\mathrm{Log}_{\bar{\mathfrak{p}},(\breve{f},\breve{g},\breve{h}_2)}^{\boldsymbol{g}}$ is deduced from a specialisation of the 3-variable p-adic regulator map $\mathrm{Log}_{(\breve{f},\breve{g},\breve{h})}^{\boldsymbol{g}}$ in §4.3 by the same argument as in [ACR23, Prop. 7.3]), and the associated explicit reciprocity law then follows from Theorem 4.3.2. $\square$

In particular, for the choice of level-N test vectors from Theorem 4.1.1 we deduce the following.

Corollary 6.4.2. *Assume that ξ_1 satisfies the conditions in Lemma 6.2.2, and put*

$$S_1 = \mathbf{u}^2(1+Z_1) - 1, \quad W_1 = \mathbf{u}(1+Z_1)^{1/2} - 1, \quad W_2 = (1+Z_1)^{1/2} - 1.$$

Then

$$\mathcal{L}_{\mathfrak{p},\bar{\xi}_1^-}^{\mathrm{Katz},-} \cdot \mathrm{Log}_{\bar{\mathfrak{p}},(\breve{f}^\star,\breve{g}^\star,\breve{h}_2^\star)}^{\boldsymbol{g}}\left(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f,\boldsymbol{g},h_2))\right)(W_2) = \mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},h_2)(S_1)$$
$$= \pm\mathbf{w} \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2)(W_1) \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \xi_1\xi_2^{\mathbf{c}})(W_2),$$

where $\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},h_2)$ is the specialisation of $\mathscr{L}_p^{\boldsymbol{g}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in Definition 6.2.3 and $\mathbf{w}$ is a unit in $\mathcal{O}[\![Z_1]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. The first equality is immediate from Lemma 6.2.2 and Theorem 6.4.1, and the second follows from Proposition 6.2.4. $\square$

6.5. **On the Bloch–Kato conjecture in rank** 0. As another application of the Euler system construction in this paper, we now deduce a result towards the Bloch–Kato conjecture for

$$V_{f,\chi} = V_f^\vee(1-r) \otimes \chi^{-1}$$

analogous to Theorem 5.5.1 but in the indefinite setting.

Theorem 6.5.1. *Let $f \in S_{2r}(\Gamma_0(N_f))$, with $p \nmid N_f$, be a p-ordinary newform of weight $2r \geq 2$, let K be an imaginary quadratic field satisfying (spl) and (cn), and let χ be an anticyclotomic Hecke character of conductor $c\mathcal{O}_K$ and infinity type $(-j,j)$, $j \geq 0$. Assume that:*

- *every prime $\ell \mid N_f$ splits in K;*
- *$(pN_f, cD_K) = 1$;*
- *χ_t has conductor prime-to-p;*
- *$\bar{\rho}_f$ is absolutely irreducible;*
- *f is not of CM-type.*

Then

$$L(f/K, \chi, r) \neq 0 \quad \implies \quad \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ holds in analytic rank zero.

Proof. We argue similarly as in the proof of Theorem 5.5.1, with some modifications. By our assumption on N_f, the sign in the functional equation of $L(f/K, \chi, s)$ is -1 for $0 \leq j < r$, so without loss of generality we assume that $j \geq r$.

Write $\chi_t = \alpha/\alpha^{\mathbf{c}}$ for a ray class character α as in the proof of Theorem 5.5.1, but now put

$$(6.13) \qquad \xi_1 := \beta\alpha, \qquad \xi_2 := (\beta^{-1}\alpha^{-\mathbf{c}})^{\mathbf{c}} = \beta\alpha^{-1},$$

with β an auxiliary ring class character of K of q-power conductor for a suitable prime $q \neq p$ split in K. Consider the setting of §6.2 with the CM Hida families

$$(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}(Z_1), \boldsymbol{\theta}_{\xi_2}(Z_2)).$$

By [CH18b, Thm. C] we may take q and β so that $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \beta^2)(W_1)$ is a unit in $\mathbf{Z}_p^{\mathrm{ur}}[\![W_1]\!] \otimes_{\mathbf{Z}_p} \mathcal{O}$ and ξ_1 satisfies the hypotheses of Lemma 6.2.2. With such a choice, using the equalities $\xi_1\xi_2 = \beta^2$ and $\xi_1\xi_2^{\mathbf{c}} = \chi_t$ the explicit reciprocity law of Corollary 6.4.2 becomes

$$(6.14) \qquad \mathscr{L}_{\mathfrak{p}, \bar{\chi}_1^-}^{\mathrm{Katz},-} \cdot \mathrm{Log}_{\bar{\mathfrak{p}}, (\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)}^{\boldsymbol{g}} \big(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f, \boldsymbol{g}, h_2)) \big)(W_2) = \pm\mathbf{w}' \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)(W_2),$$

where $W_2 = (1 + Z_1)^{1/2} - 1 = V_1$, with $\mathbf{w}'$ is a unit in $\mathbf{Z}_p^{\mathrm{ur}}[\![W_2]\!] \otimes_{\mathbf{Z}_p} L_{\mathfrak{P}}$.

Denoting by $Q \in \mathfrak{X}_{\mathcal{O}[\![W_2]\!]}^+$ the specialisation $W_2 \mapsto \zeta_Q \mathbf{u}^j - 1$ ($\zeta_Q \in \mu_{p^\infty}$) such that

$$\chi_w = \Psi_{W_2}^{\mathbf{c}-1}|_{W_2 = \zeta_Q \mathbf{u}^j - 1},$$

from (6.14), Theorem 6.1.1, and Theorem 6.4.1 we find

$$(6.15) \qquad \begin{aligned} L(f/K, \chi, r) \neq 0 \quad &\implies \quad \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)(\chi_w(\gamma_-) - 1) \neq 0 \\ &\implies \quad \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q) \neq 0, \end{aligned}$$

where $\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)$ denotes the image of the class $\kappa_2(f, \boldsymbol{g}, h_2)$ in (6.11) under the projection

$$\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \chi_t^{-1} \Psi_{W_2}^{1-\mathbf{c}}(\mathcal{N})) \to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \chi_t^{-1} \Psi_{W_2}^{1-\mathbf{c}})$$

associated to $(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)$.

As noted in Remark 2.4.3, the class $\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)$ is the bottom class of the anticyclotomic Euler system $\{^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,m}\}_m$ of Theorem 2.4.2 for $T_{f,\xi_1\xi_2^{\mathbf{c}}} = T_{f,\chi_t}$ (and the given choice of test vectors). Therefore, letting $\mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star))$ denote the image of $\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)$ under the 'twisting' map

$$\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \chi_t^{-1} \Psi_{W_2}^{1-\mathbf{c}}) \to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \chi^{-1} \Psi_{W_2}^{1-\mathbf{c}})$$

given by the change of variables $W_2 \mapsto \zeta_Q^{-1}\mathbf{u}^{-j}(1 + W_2) - 1$, it follows that $\mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star))$ is the bottom class of the Euler system

$$(6.16) \qquad \{^{\mathbf{c}}\mathbf{z}_{f,\chi,m}\}_m := \{^{\mathbf{c}}\mathbf{z}_{f,\xi_1,\xi_2,,m} \otimes \chi_w^{-1}\}_m$$

of Theorem 2.4.2 for $T_{f,\xi_1\xi_2^{\mathbf{c}}} \otimes \chi_w^{-1} = T_{f,\chi}$.

Since by construction the class $\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q$ in (6.15) agrees with the image of the bottom class $^{\mathbf{c}}\mathbf{z}_{f,\chi,1}$ of the system (6.16) under natural map

$$\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_f^\vee(1-r) \otimes \chi^{-1} \Psi_{W_2}^{1-\mathbf{c}}) \cong \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, T_{f,\chi}) \to \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, T_{f,\chi}),$$

from Theorem 3.3.1 we deduce that $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\chi})$ is one-dimensional, spanned by $\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q$. Since we have in fact shown that $\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(\breve{f}^\star, \breve{g}^\star, \breve{h}_2^\star)_Q) \neq 0$, the vanishing of $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi})$ then follows by global duality similarly as in the proof of Theorem 5.5.1; and since by Lemma 3.1.2, for $j \geq r$ the latter group agrees with $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$, this yields the result. $\qquad \square$

6.6. **On the Iwasawa main conjecture.** Writing any anticyclotomic Hecke character χ of K as $\chi = \chi_t \cdot \chi_w$ as in §5.5, we let $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi)$ denote the image of $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)$ under the twisting homomorphism $\mathrm{tw}_{\chi_w} : \mathbf{Z}_p^{\mathrm{ur}}[\![W_2]\!] \otimes_{\mathbf{Z}_p} \mathcal{O} \to \mathbf{Z}_p^{\mathrm{ur}}[\![W_2]\!] \otimes_{\mathbf{Z}_p} \mathcal{O}$ given by $W_2 \mapsto \chi_w(\gamma_-)(1 + W_2) - 1$.

Our next application is to the Iwasawa–Greenberg main conjecture for $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi)$.

Theorem 6.6.1. *Let the hypotheses be an in Theorem 6.5.1, and assume in addition that f has big image. Then $\mathrm{Sel}_{\mathrm{str,rel}}(K, A_{f,\chi})$ is cotorsion over Λ_K^-, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}\big(\mathrm{Sel}_{\mathrm{str,rel}}(K, A_{f,\chi})^\vee\big) \supset \big(\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi)^2\big)$$

in $\Lambda_K^{-,\mathrm{ur}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. Repeating the argument in the proof of Theorem 5.5.1, we arrive at the equality

$$(6.17) \qquad \mathcal{L}_{\mathfrak{p}, \bar{\xi}_1^-}^{\mathrm{Katz}, -} \cdot \mathrm{Log}_{\bar{\mathfrak{p}}, (\check{f}^\star, \check{g}^\star, \check{h}_2^\star)}^{\boldsymbol{g}}\big(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f, \boldsymbol{g}, h_2))\big)(W_2) = \pm \mathbf{w}' \cdot \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)(W_2),$$

with $\mathbf{w}'$ is a unit in $\mathbf{Z}_p^{\mathrm{ur}}[\![W_2]\!] \otimes_{\mathbf{Z}_p} L_{\mathfrak{P}}$. Since $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)(W_2)$ is nonzero by Theorem 6.1.1, letting

$$\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2^\star) \in \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_f^\vee(1 - r) \otimes \chi_t^{-1}\Psi_{W_2}^{1 - \mathbf{c}})$$

be as in the proof of Theorem 6.5.1, from (6.17) we conclude that $\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2)$ is non-torsion.

Since $\mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2^\star))$ is the bottom class of the Euler system $\{^{\mathbf{c}}\mathbf{z}_{f,\chi,m}\}_m$ for $T_{f,\chi}$ constructed in Theorem 2.4.2, from Theorem 3.3.3 we deduce that $\mathrm{Sel}_{\mathrm{ord,ord}}(K, T_{f,\chi})$ and $X_{\mathrm{ord,ord}}(K, A_{f,\chi})$ have both Λ_K^--rank one, and we have the divisibility

$$(6.18) \qquad \mathrm{char}_{\Lambda_K^-}\big(X_{\mathrm{ord,ord}}(K, A_{f,\chi})_{\mathrm{tors}}\big) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{ord,ord}}(K, T_{f,\chi})}{\Lambda_K^- \cdot \mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2^\star))}\right)^2$$

in Λ_K^-. Since from (6.17) we deduce an explicit reciprocity law relating

$$\mathrm{res}_{\bar{\mathfrak{p}}}(\mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)))$$

to $\mathrm{tw}_{\chi_w}(\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi_t)) = \mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K, \chi)$, the result now follows from (6.18) and global duality by the same argument as in [BCK21, Thm. 5.1]. $\qquad\square$

Remark 6.6.2. Note that Theorem 6.6.1 also yields a proof of a divisibility towards the Perrin-Riou main conjecture for generalised Heegner cycles formulated in [LV19] (see [BCK21, Thm. 5.2] for the argument), removing some of the hypotheses in the main result of [LV19].

6.7. **On the Bloch–Kato conjecture in rank** 1. We can also give an analogue of Theorem 5.7.1 in the indefinite case.

Theorem 6.7.1. *Let the hypotheses be as in Theorem 6.5.1. If $0 \le j < r$ (which implies $L(f/K, \chi, r) = 0$), then*

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \ge 1.$$

Moreover, there exists a class $^{\mathbf{c}}z_{f,\chi} \in \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ such that

$$^{\mathbf{c}}z_{f,\chi} \ne 0 \quad \Longrightarrow \quad \dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 1.$$

Proof. The proof of Theorem 6.6.1 showed that the class

$$^{\mathbf{c}}\mathbf{z}_{f,\chi} := \mathrm{tw}_{\chi_w^{-1}}(\kappa_2(\check{f}^\star, \check{g}^\star, \check{h}_2^\star)) \in \mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\chi})$$

is non-torsion over Λ_K^-. On the other hand, one readily checks that the natural map

$$(6.19) \qquad \mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\chi})/(\gamma_- - 1)\mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\chi}) \to \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_{f,\chi})$$

is injective. Thus we conclude that $\mathrm{Sel}_{\mathrm{ord,ord}}(K, T_{f,\chi})$ has positive $\mathcal{O}$-rank, which together with Lemma 3.1.2 yields the first part of the theorem. Letting $^{\mathbf{c}}z_{f,\chi} \in \mathrm{Sel}_{\mathrm{ord,ord}}(K, T_{f,\chi})$ be the image of $^{\mathbf{c}}\mathbf{z}_{f,\chi}$ under (6.19), the last claim follows from Theorem 3.3.1. $\qquad\square$

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