MATRICES:

Dfn:

A Matrix is an ordered rectangular arrangement of numbers usually enclosed in Parenthesis or square brackets.

⇒ Upper case letters are used to denote Matrices;

Order of a Matrix

The size of a Matrix is specified by the number of rows and the number of columns.

e.g $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 0 & 1 \end{pmatrix}$ is a 3×2 Matrix.

A general Matrix of order $m \times n$ is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

i - i^{th} row

j - j^{th} column

A square Matrix is one with same number of rows and columns i.e $n \times n$ matrix. Two matrices are of the same size if they have the same order.

A vector is a matrix with a one row $(1 \times n)$ or one column $(m \times 1)$. A row vector is of the form $1 \times n$ and a column vector is of the form $m \times 1$.

A zero / null matrix is an $m \times n$ matrix in which every entry is a zero i.e $a_{ij} = 0$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots$

Similarly we talk of zero row or column vector;

$\underset{\sim}{0} = (0, 0, \ldots, 0)$ or $\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

A diagonal matrix is a square Matrix whose every entry is a zero except entries in the main diagonal, or principal or leading diagonal. The main diagonal runs from upper left to the main bottom right of the Matrix.

e.g $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ is a diagonal matrix.

In Particular an Identity matrix is a diagonal matrix whose every diagonal entry is a unit

e.g $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity Matrix

An upper triangular matrix is a square matrix whose every entry below the main diagonal is a zero

e.g $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}$

is an upper 4×4 Matrix.

Similarly, a lower triangular matrix is a square matrix whose every entry above the main diagonal is a zero.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 4 & 5 & 0 \\ 0 & 7 & 6 & 2 \end{pmatrix}$$

Matrix Algebra.

(i) Equality

Two Matrices A and B are said to be equal if they have the same order $m \times n$ and $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots m$ and all

$j = 1, 2, \ldots n$.

Example:

Given that $\begin{pmatrix} x+y & 2s+r \\ x-y & s-r \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ Solve for x, y and s

Soln.

Since $\begin{pmatrix} x+y & 2s+r \\ x-y & s-r \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$, Then

$x + y = 4$ --- (i)
$x - y = 2$ --- (ii)
$2s + r = 6$ --- (iii)
$s - r = 3$ --- (iv)

From i and ii, $x = 3$, $y = 1$
Similarly
From (iii) and (iv), $s = 3$ $r = 0$

(ii) Addition and Subtraction.

Performed on Matrices of the same order (size). Let A and B be $m \times n$ Matrices, then.

$A \pm B = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$

$= C_{m \times n}$

Example

$\begin{pmatrix} 0 & k & 4 \\ r & 7 & 2 \end{pmatrix} + \begin{pmatrix} d & 5 & 6 \\ 3 & 2 & e \end{pmatrix} = \begin{matrix} 7 & 8 & 10 \\ 5 & 9 & 3 \end{matrix}$

Values of d, k and r are:
$d = 7$
$k = 3$
$r = 2$
$e = 1$

(iii) Scalar Multiplication.

Let a be any Matrix, then $kA = k(a_{ij}) = (k a_{ij})$, where k

Example

Determine $7A + 6B$ given that $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$

Soln.

$7A + 6B = 7\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} + 6\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$

$= \begin{pmatrix} 7 & 21 \\ 28 & 14 \end{pmatrix} + \begin{pmatrix} 18 & -12 \\ 42 & 24 \end{pmatrix}$

$= \begin{pmatrix} 25 & 9 \\ 70 & 38 \end{pmatrix}$

(iv) Matrix Multiplication.

Dot Product.

let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be any two vectors of size n (matrix with a single row or column) i.e $\underset{\sim}{a} = (a_1, a_2 \ldots a_n)$ and $\underset{\sim}{b} = (b_1, b_2 \ldots, b_n)$. Then the dot product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, denoted $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is given by,

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1, a_2 \ldots a_n) \cdot (b_1, b_2 \ldots b_n)$

$= a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$= \sum_{i=1}^{n} a_i b_i$ (scalar)

The dot product is also called Scalar Product.

Example:

Let $\underset{\sim}{a} = (1, 2, 3)$ and $\underset{\sim}{b} = (2, 0, 1)$.
Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$

Soln

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1\ 2\ 3) \cdot (2, 0, 1)$

$= (1 \times 2) + (2 \times 0) + (3 \times 1)$

$= 5$

Dot product of a row and a column vector of order n is given by

$$(a_1, a_2, \dots a_n) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Dfn (Matrix Multiplication)

Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times s$ matrix.
The matrix product AB is the $m \times s$ matrix (C_{ij}) where C_{ij} is the dot product of the ith row of A and j^{th} column of B

$$AB = (a_{ik})(b_{kj}) = (C_{ij}) \text{ where}$$

$$C_{ij} = A_i \cdot B_j = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Remark

If the number of columns in A is not equal to the number of rows in B; then AB is undefined

Example

Let $A = \begin{pmatrix} 6 & 4 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 5 & 7 \\ 3 & 1 & 1 \end{pmatrix}$

Compute (a) A^2
(b) AB and BA
is $AB = BA$?

Soln

(a) $A^2 = \begin{pmatrix} 6 & 4 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 6 & 4 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$36 - 8 + 3$ ✓

$$= \begin{pmatrix} 31 & 33 & 20 \\ -19 & 2 & -11 \\ 10 & -1 & 8 \end{pmatrix}$$

(b) AB $\begin{pmatrix} 6 & 4 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 5 & 7 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 29 & 29 & 37 \\ -1 & 12 & 18 \\ 6 & -2 & -4 \end{pmatrix}$

BA $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 5 & 7 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 4 & 3 \\ -2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 10 & 7 \\ 9 & 16 & 15 \\ 17 & 14 & 10 \end{pmatrix}$

Hence $AB \neq BA$

Conclusion:
~~Contribution~~. Matrix Multiplication is **not** commutative. ✓✓

Transposition

The transpose of a matrix A is the matrix $B = A^T$, such that $b_{ij} = a_{ji}$ i.e the rows are exchanged for columns and columns for rows if $A = A^T$, we say that A is symmetric and skew-symmetric if $A^T = -A$

Example:

Find A^T and $(A^T)^T$ given that

$A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 4 & -2 & 1 & 2 \end{pmatrix}$

Soln:

(a) $A^T = \begin{pmatrix} 2 & 4 \\ 1 & -2 \\ 3 & 1 \\ 5 & 2 \end{pmatrix}$

$(A^T)^T = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 4 & -2 & 1 & 2 \end{pmatrix} = A$. ✓

$A \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 3 & 6 \end{pmatrix}$

$A^T \begin{pmatrix} 1 & 5 & 4 \\ 2 & 6 & 3 \\ 3 & 7 & 6 \end{pmatrix}$ ✓✓

$A = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ is an example of a <u>symmetry</u> matrix.

$A^T = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$

The transpose operation on matrices satisfies the following properties

(i) $(A + B)^t = A^t + B^t$

$(A^t)^t = A$

$(kA)^t = kA^t$, for k a scalar

$(AB)^t = B^t A^t$

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Example 2.

For what values of x, y and z $A = \begin{pmatrix} x & y & z \\ 2 & 0 & 3 \\ 4 & 3 & 3 \end{pmatrix}$ symmetric?

Soln.

$A^T = \begin{pmatrix} x & 2 & 4 \\ y & 0 & 3 \\ z & 3 & 3 \end{pmatrix}$

For A to be symmetric $A^T = A$

$\begin{pmatrix} x & 2 & 4 \\ y & 0 & 3 \\ z & 3 & 3 \end{pmatrix} = \begin{pmatrix} x & y & z \\ 2 & 0 & 3 \\ 4 & 3 & 3 \end{pmatrix}$

Thus x is any real number

$y = 2$ ✓

$z = 4$ ✓

Properties of Matrix Operations

1. $A + B = B + A$, addition is <u>commutative</u>.

2. $(A + B) + C = A + (B + C)$, addition is <u>associative</u>

3. $A + 0 = 0 + A = A$, 0 is identity for addition.

4. $\alpha(A + B) = \alpha A + \alpha B$, left distributive law.

5. $(\alpha + \beta)A = \alpha A + \beta A$, right distributive law.

6. $(\alpha\beta)A = \alpha(\beta A)$, associativity of scalar multiplicative law.

7. $(\alpha A)\beta = A(\alpha\beta)$, scalar pull through.

8. $(AB)C = A(BC)$, associativity of matrix multiplication.

9. $I_n A = A$, $A I_m = A$, $A_{n \times m}$, identity for matrix multiplication.

10. $A(B + C) = AB + AC$, left distributive law.

11. $(A + B)C = AC + BC$, right distributive law.

12. $(A^T)^T = A$

13. $(A + B)^T = A^T + B^T$ transpose of a sum.

14. $(AB)^T = B^T A^T$ transpose of a product.

NB. Proof of most of these properties involve just routine computations with components and entries.

Proof of law 9.

9. A of order $n \times m$

$A = \begin{matrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \end{matrix}$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$I_n A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = A.$$

Example.

Show that $(A-B)(A+B) = A^2 - B^2$ if and only if $AB = BA$ i.e A and B Commutative

Soln

$\Rightarrow$ Suppose that $(A-B)(A+B) = A^2 - B^2$ and show that A and B Commute.

Now

$$A^2 - B^2 = (A-B)(A+B)$$
$$= A^2 + AB - BA - B^2$$

Hence $AB - BA = 0 \Rightarrow AB = BA$

Hence A and B Commute.

$\Leftarrow$ Assume that A and B Commute and show that $(A-B)(A+B) = A^2 - B^2$

Since Matrix A and B commute, then $(A-B)(A+B) = A^2 + AB - BA - B^2$

$$= A^2 + AB - AB - B^2$$
$$= A^2 - B^2$$

Proof (Property 14)

$$(AB)^T = B^T A^T$$

$A_{m \times n}$; $B_{n \times s}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1s} \\ b_{21} & b_{22} & \cdots & b_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{ns} \end{pmatrix}$$

$(AB)^T$ is $s \times m$

$$AB = a_{11}b_{11} + a_{12}b_{21} \cdots + a_{1n}b_{n}$$

Example:

The statement

$$\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$

$x + y = 3$ $\qquad$ $2z + w = 5$

$x - y = 1$ $\qquad$ $z - w = 4$

$2x = 4$ $\qquad$ $3z = 9$

$x = 2$ ✓ $\qquad$ $z = 3$ ✓

$y = 1$ ✓ $\qquad$ $w = -1$ ✓

$$A = \begin{pmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & -1 \end{pmatrix}.$$ Find $3A + 4B - 2C$

$$\begin{pmatrix} 6 & -15 & 3 \\ 9 & 0 & -12 \end{pmatrix} + \begin{pmatrix} 4 & -8 & -12 \\ 0 & -4 & 20 \end{pmatrix} - \begin{pmatrix} 0 & 2 & -4 \\ 2 & -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -25 & -5 \\ 7 & -2 & 10 \end{pmatrix} ✓$$

Find x, y, z and w if

$$3\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3x & 3y \\ 3z & 3w \end{pmatrix} = \begin{pmatrix} 4+x & 6+x+y \\ -1+z+w & 2w+3 \end{pmatrix}$$

$3x = 4 + x$

$3y = 6 + x + y$

$3z = -1 + z + w$

$3w = 2w + 3$

$w = 3$ ✓

$2z = 2 \qquad z = 1$ ✓

$2y = x + 6$

$x = 2$ ✓

$y = 4$ ✓

Reduction of matrix to echelon form.

A matrix $A = (a_{ij})$ is an echelon matrix or is said to be in echelon form, if the number of zeros preceeding the 1st non zero entry of a row increases row by row until only zero rows remain (if possible).

The left most non zero entry in a non-zero row of an echelon matrix is called a leading entry of that row.

Example

1 The following matrices are in echelon form where the leading entries have been circled.

$$\begin{pmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 7 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2 The following matrix is $\boxed{\text{not}}$ in echelon form

$$\begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix}$$

In particular, an echelon matrix is called a row reduced echelon matrix if

(i) The leading entry in each non-zero row is 1

(ii) Each leading 1 is the only non-zero entry in its column.

e.g $\begin{pmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

We perform the following operations on the rows of a matrix when reducing it to echelon form:

1. Interchange the ith row with j^{th} row

$$R_i \longleftrightarrow R_j$$

2. Multiply the ith row by a non-zero scalar, say k.

$$R_i \longleftrightarrow kR_i \quad , \quad k \neq 0$$

3. Add a scalar multiple of row j to row i,

$$R_i \longrightarrow R_i + kR_j$$

In actual practice we apply operations 2 and 3 in one step i.e

$$R_i \longrightarrow kR_i + k'R_j \quad , \quad k \neq 0$$

The operations are called elementary row operations.

Example.

Reduce $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{pmatrix}$ to echelon form.

soln.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 + R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 2R_1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & -3 & 0 & 2 \end{pmatrix} \quad R_4 \to R_4 - R_2$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix} \quad R_4 \to R_4 - 2R_3$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

2. Reduce $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$ to echelon form.

Soln.

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix} \quad R_3 \to 3R_3 - 7R_2$$

$$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix} \checkmark$$

3. Reduce $\begin{pmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{pmatrix}$

swap. $R_2 \longleftrightarrow R_1$

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ 3 & 4 & -1 & 1 \\ 4 & -3 & 11 & 2 \end{pmatrix}$$

$R_2 \longrightarrow R_2 - 3R_1$

$R_3 \longrightarrow R_3 - 4R_1$

$$= \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 7 & -10 & -2 \\ 0 & 1 & -1 & -2 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_3}$$

$$= \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 7 & -10 & -2 \end{pmatrix} \quad R_3 - 7R_2 \quad \begin{matrix} 7-7 \\ -10+7 \quad -2+14 \end{matrix}$$

$$= \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -3 & +12 \end{pmatrix} \checkmark$$

4. Find the reduced echelon form of the matrix.

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{pmatrix} \quad R_2 \Rightarrow R_2 - 3R_1, \; R_3 - 2R_1$$

Soln.

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{pmatrix} \longrightarrow R_2 \to \frac{1}{2} R_2$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{pmatrix} \to R_3 \to R_3 - 3R_2$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \checkmark \quad \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

5. Find the row reduced echelon form of the matrix:

$$\begin{pmatrix} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Soln.

$$\begin{pmatrix} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 \to 4R_3 - 3R_1]{R_2 \to 2R_2 - R_1}$$

$$\begin{pmatrix} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 4 & 1 & -3 & 0 & 4 \end{pmatrix}$$

$R_3 \to R_3 - R_2$ $\quad \begin{pmatrix} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \end{pmatrix}$

$R_1 \to 2R_1 - R_3$

$\to$

$R_2 \to 2R_2 + R_3$

$$\begin{pmatrix} 8 & 0 & 0 & 4 & 2 & -4 \\ 0 & 8 & 0 & -4 & 2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{pmatrix} \quad \begin{matrix} R_1 \to \frac{1}{8}R_1 \\ R_2 \to \frac{1}{8}R_2 \\ R_3 \to \frac{1}{2}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 2 \end{pmatrix}$$

Determinant of 2x2 and 3x3 Matrices.

Determinants are defined for square matrices 2x2 matrices.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then,

$$\det A = |A| = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Which is a scalar.

Example:

1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ Compute det A

Soln

$$\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 3) = -2$$

2. If $\begin{vmatrix} x & x \\ 3 & x-2 \end{vmatrix} = 0$ i.e a singular matrix find x

Soln.

$$x(x-2) - 3x = 0$$

$$x^2 - 5x = 0$$

$$x(x-5) = 0$$

$$x = 0, \text{ or } x = 5$$

3 x 3 matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example:

Find $\begin{vmatrix} 5 & -2 & -3 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

Soln.

$$\begin{vmatrix} 5 & -2 & -3 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix} = 5(1) + 2(5) - 3(-3)$$

$$= 5 + 10 + 12 \qquad 5 + 10 + 12$$

$$= 27 \qquad = 27$$

Exercise

Compute.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 7 \\ 9 & 1 \end{vmatrix} - 3\begin{vmatrix} 5 & 7 \\ 8 & 1 \end{vmatrix} + 4\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

$$= 2(-57) - 3(-51) + 4(-3)$$

$$-114 + 153 - 12$$

$$= +27 = 27$$

Minors and Co-Factors

Df. Let A be a matrix of order $n \times n$ and let a_{ij} be a general element of A.

The minor of element a_{ij} which is denoted by M_{ij} is the det of the $(n-1) \times (n-1)$ submatrix

Which is obtained after deleting the ith row and j^{th} column of A. The number $A_{ij} = (-1)^{i+j} M_{ij}$ is called the **Cofactor** of a_{ij}

Example.

Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$, find the minors and cofactors. A

Soln

Minors,

$M_{11} = \begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -20+2 = -18$

$M_{12} = \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 0-2 = -2$

$M_{13} = \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$

$M_{21} = \begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = 15-4 = 11$

$M_{22} = \begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 10+4 = 14$

$M_{23} = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2-3 = -5$

$M_{31} = \begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = 6-16 = -10$

$M_{32} = \begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = 4-0 = 4$

$M_{33} = \begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$

Cofactors

$A_{11} = (-1)^2(-18) = -18$

$A_{12} = (-1)^3(-2) = 2$

$A_{13} = (-1)^4(4) = 4$

$A_{21} = (-1)^3(11) = -11$

$A_{22} = (-1)^4(14) = 14$

$A_{23} = (-1)^5(-5) = 5$

$A_{31} = (-1)^4(-10) = -10$

$A_{32} = (-1)^5(4) = -4$

$A_{33} = (-1)^6(-8) = -8$

Theorem [Laplace's Expansion Theorem].

The determinant of the matrix $A = (a_{ij})$ is equal to the sum of products obtained by multiplying elements of any row (for any column) by their respective cofactors

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

or

$$|A| = a_{ij}A_{ij} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Example.

Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$. Find $|A|$ by expansion of elements in:

(a) 1st row
(b) 3rd row
(c) 3rd column

Soln:

1st row **formula method**.

Note that A is the same as the matrix in the previous example

$|A| = 2(-18) - 3(-2) - 4(4)$

$-36 + 6 - 16$

$= -46$ ✓

Using cofactor expansion method.

$|A| = 2(-18) + 3(2) + (-4)(4)$

$= -46$

(ii) 3rd row

$|A| = 1(-10) + (-1)(-4) + 5(-8)$

$= -10 + 4 - 40$

$= -46$

(iii) 3rd column.

$|A| = -4(4) + 2(5) + 5(-8) = -46$

Remark:

Determinant of a 3×3 matrix can also be obtained by using the following elementary scheme.

$$\begin{array}{cccccc} + & + & + & - & - & - \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$-$$

Then.

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example:

Find the determinant of A where

$$A = \begin{pmatrix} 2 & -3 & -4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{pmatrix}$$

Soln

$$\begin{array}{cccccc} + & + & + & - & - & - \end{array}$$

$$\begin{pmatrix} 2 & -3 & -4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{pmatrix} \begin{matrix} 2 & -3 \\ -1 & 4 \\ 3 & 10 \end{matrix}$$

$$(2\times4\times1) + (-3\times2\times3) + (-4\times-1\times10) - (-4\times4\times3) - (2\times2\times10) - (-3\times-1\times1)$$

$$8 + (-18) + (40) + 48 - 40 - (3)$$

$$= 35 \checkmark \qquad = 35 \checkmark$$

Inverses of 2×2 and 3×3 Matrices

A square matrix A is said to be invertible if there exists a matrix B with the property that

$AB = BA = I$ (I is the identity matrix)

Such a matrix B is unique and its called the inverse of A, denoted A^{-1}.

Defn (Classical adjoint)

Classical adjoint of a square matrix A which is denoted by adjoint A is the transpose of matrix of co-factors of A.

Example

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$

Compute

(a) adj A.

(b) (adj A) A and A (adj A).

(c) $|A|$

Soln.

Soln

$A_{11} = +\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$ $\quad$ -9 to 0

$A_{12} = -\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} = -6$

$A_{13} = +\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1$

$A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$

$A_{22} = +\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$ $\quad$ 1-9

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3$

$A_{31} = +\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 3$ $\quad$ 1-6

$A_{33} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$

Therefore the matrix of cofactors of A is:

$$\begin{pmatrix} 2 & -6 & 1 \\ -3 & 0 & 3 \\ -1 & 3 & -5 \end{pmatrix}$$

and $\text{adj } A = \begin{pmatrix} 2 & -3 & -1 \\ -6 & 0 & 3 \\ 1 & 3 & -5 \end{pmatrix}$

(b)

$$(\text{adj } A)\, A = \begin{pmatrix} 2 & -3 & -1 \\ -6 & 0 & 3 \\ 1 & 3 & -5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

$$A(A\,adj) = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 & -1 \\ -6 & 9 & 3 \\ 1 & 3 & -5 \end{pmatrix} = \begin{matrix} -9 \\ (2-12+1) \\ -9+0+0 \end{matrix}$$

$$= \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

(c) Cofactors expansion along column 2

$|A| = 2(-6) + 1(0) + 1(3) = -9$ ✓

From the above example we see that $(adj\,A)\,A = A\,(adj\,A) = |A|\,I$

In general this is always $\boxed{\text{true}}$ ✓

Theorem

For any $n \times n$ matrix A, $A(adj A) = (adj A)A = |A|\,I$, where I is the Identity Matrix. This if $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}(adj\,A)$$

In particular when $n = 2$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then Cofactors of A are

$A_{11} = +d \qquad A_{12} = -c$

$A_{21} = -b \qquad A_{22} = +a$

Matrix Cofactor is

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

and $A\,adj = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det A = ad - bc$

$$A^{-1} = \frac{1}{|A|}\,adj\,A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Properties of Inverses:

1. $AA^{-1} = A^{-1}A = I$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^{-1})^T = (A^T)^{-1}$
4. $(A^{-1})^{-1} = A$
5. $|A| = \dfrac{1}{|A^{-1}|}$

Example:

1. Let $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, find A^{-1}.

Soln

$$\frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$ ✓

2. Find A^{-1} given that

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix}$$

Cofactors are

$A_{11} = +\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7$

$A_{12} = -\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = 2$

$A_{13} = +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

$A_{21} = -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = 6$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$

$A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2$

$A_{31} = +\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} = -3$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2$

$A_{33} = +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$

Matrix of Co-factors of A: $= \begin{pmatrix} -7 & 2 & 3 \\ 6 & -2 & -2 \\ -3 & 2 & 1 \end{pmatrix}$ $\quad A\,adj = \begin{pmatrix} -7 & 6 & -3 \\ 2 & -2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$

Cofactor expansion along row 1.

$|A| = 1(-7) + 0(2) + 3(3) = -7 + 9 = 2$

Thus.

$$A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{2}\begin{pmatrix} -7 & 6 & -3 \\ 2 & -2 & 2 \\ 3 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{7}{2} & 3 & -\frac{3}{2} \\ 1 & -1 & 1 \\ \frac{3}{2} & -1 & \frac{1}{2} \end{pmatrix}$$ ✓

Remarks

(i) Every matrix is row-equivalent to one and only one reduced echelon matrix.

(ii) The inverse of a matrix A can be obtained using row reduction to echelon form of the augumented matrix (A/I) to give (I/A^{-1})

Example:

Find A^{-1} given that $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

Soln.

augumented matrix

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to 4R_3 - 3R_1]{R_2 \to 2R_2 - R_1}$$

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 4 & 1 & -3 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right) \xrightarrow[R_2 \to 2R_2 + R_3]{R_1 \to 2R_1 - R_3}$$

echelon form.

$$= \left(\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & 2 & -4 \\ 0 & 8 & 0 & -4 & 2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right) \xrightarrow[R_2 \to \frac{1}{8}R_2,\ R_3 = \frac{1}{2}R_3]{R_1 \to \frac{1}{8}R_1}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right)$$

$$(A/I) \to (I/B)$$

Thus $A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{pmatrix}$

Systems of linear equations

A linear equation is an equation of the form:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots a_n x_n = b$$

where $a_1, a_2, a_3 \cdots a_n$ are constants while $x_1, x_2, x_3 \cdots x_n$ are variables to be determined.

A solution is an ordered n-tuple $(c_1, c_2, \cdots, c_n)$ such that:

$$a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots a_n c_n = b$$

Example

$(3, 5, 3, 2)$ is a soln of:

$5x_1 - 3x_2 + 6x_3 - 4x_4 = 10$, since

$5(3) - 3(5) + 6(3) - 4(2) = \underline{\underline{10}}$ ✓

A system of linear equations is a collection of several linear equations. A solution to a system in n variables is an n-tuple $(c_1, c_2 \cdots c_n)$ that satisfies each equation in the system.

The set of solution to a particular system is called the soln set.

A linear system is said to be consistent if it has at least one soln and inconsistent if it has none.

ELIMINATION METHOD:

The strategy is to eliminate one variable at a time by performing one of the following elementary operations:

(a) Multiply one equation by a non zero constant.

(b) Interchange two equations

(c) Add a constant multiple of one equation to another

Example:

Solve

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20 \qquad R_2 - 3R_1$$
$$2x + 7y + 9z = 23$$

Soln

First we add -3 times the 1st eqt to the 2nd eqt to obtain

$$-3x - 6y - 3z = -12$$
$$3x + 8y + 7z = 20 \quad +$$
$$2y + 4z = 8$$
$$y + 2z = 4$$

Then add -2 times the 1st eqt to the 3rd equation

$$x + 2y + z = 4$$
$$y + 2z = 4$$
$$3y + 7z = 15$$

Having eliminated x from the 2nd and 3rd equation. We simplify the process of eliminating y from the 3rd eqt by multiplying the 2nd by $\frac{1}{2}$ to obtain

$$x + 2y + z = 4$$
$$y + 2z = 4$$
$$3y + 7y = 15$$

Finally additive of -3 times 2nd to the 3rd equation yields

$$x + 2y + z = 4$$
$$y + 2z = 4$$
$$z = 3$$

By back substitution in ② and ③ we get $y = -2$ and $x = 5$

Soln $x = 5$, $y = -2$, $z = 3$

The elimination method is very tedious where there are many variables. We use the more organized elimination method called Gauss - Jordan elimination method

Gauss elimination Method:

Given the linear system;

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$

The system can be represented in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

or simply $Ax = b$, where A is the $n \times n$ coefficient matrix, $x = (x_1, x_2, \ldots, x_n)$ and $b = (b_1, b_2, \ldots, b_n)$

An augumented matrix is a matrix of the form $(A \mid B)$

The Gauss elimination method involves reducing the augumented matrix $(A \mid b)$ to echelon form.

Example

1. Solve the system

$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 + 8x_2 + 7x_3 = 20$$
$$2x_1 + 7x_2 + 9x_3 = 23$$

Soln

The matrix form is

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \\ 23 \end{pmatrix}$$

Reducing the augumented matrix to echelon form,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right) \xrightarrow{\frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The corresponding system is

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

$$x_1 + 2x_2 + x_3 = 4$$
$$x_2 + 2x_3 = 4$$
$$x_3 = 3$$

Soln: $x_1 = 5$, $x_2 = -2$ $x_3 = 3$

2. Solve the system.

$$2x_1 - 4x_2 + 6x_3 = 20$$
$$3x_1 - 6x_2 + x_3 = 22$$
$$-2x_1 + 5x_2 - 2x_3 = -18$$

The matrix form of the system.

$$\begin{pmatrix} 2 & -4 & 6 \\ 3 & -6 & 1 \\ -2 & 5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 22 \\ -18 \end{pmatrix}$$

Reduce the augumented matrix to echelon form:

$$\left(\begin{array}{ccc|c} 2 & -4 & 6 & 20 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-3R_1}$$

$$= \left(\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 0 & -8 & -8 \\ 0 & 1 & 4 & 2 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

The Corresponding system is

$$x_1 - 2x_2 + 3x_3 = 10$$
$$x_2 + 4x_3 = 2$$
$$-8x_3 = -8$$

$x_1 + 4 + 3 = 10$
$x_1 = 3$

$$x_3 = 1$$
$$x_2 = -2$$
$$x_1 = 3$$

Solution: $x_1 = 3$, $x_2 = -2$, $x_3 = 1$

Remark:

1. Suppose a linear system is in echelon form i.e the augumented matrix is an echelon matrix. Then the variables that correspond to columns containing leading entries are called leading / dependent variables; all other variables are called free / independent variables

2. A system of linear equations may have a unique, many soln or no soln.

Unknowns and equations are equal in the echelon form of the equation. Many solution: Equations are less than the unknowns in the echelon form of the system.

No solution: Last row of the echelon form of the system is of the form

$$(0 \ 0 \ \cdots \ 0 \mid a \neq 0)$$

Example

Solve the system

(a) $x_1 + x_2 - x_3 + 2x_4 = 4$
$-2x_1 + x_2 + 3x_3 + x_4 = 5$
$-x_1 + 2x_2 + 2x_3 + 3x_4 = 6$

Soln.

Matrix form is:

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ -1 & 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Reducing the augumented matrix to echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 4 \\ -2 & 1 & 3 & 1 & 5 \\ -1 & 2 & 2 & 3 & 6 \end{array}\right) \xrightarrow[R_3+R_1]{R_2+2R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 & 13 \\ 0 & 3 & 1 & 5 & 10 \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 & 13 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

Therefore

$$\left.\begin{array}{l} x_1 + x_2 - x_3 + 2x_4 = 4 \\ 3x_2 - x_3 + 5x_4 = 13 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = -3 \end{array}\right\} \text{meaning(s)}$$

Thus the system has no solution.

(b) $x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$
$2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7$
$3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27$

Soln

We reduce the augumented matrix to echelon form

$$\left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right)$$

$$R_2 \leftrightarrow R_3 \quad \left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{array}\right) \xrightarrow{R_3 - 2R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right) \xrightarrow{-R_3}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right)$$

The Corresponding System is

$$\left.\begin{array}{l} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ x_3 + 2x_5 = -3 \\ x_4 - 4x_5 = 7 \end{array}\right\} \Rightarrow \text{Infinitely many solns.}$$

The leading entries are in the 1st ~~in the~~ 3rd and 4th Columns. Hence x_1, x_3 and x_4 are the leading variables. and x_2 and x_5 are the free variables

Let $x_2 = s$, $x_5 = t$, where s and t are any real no.s

(a) By back substitution;

$x_4 = 7 + 4x_5 = 7 + 4t$

$x_3 = -3 - 2x_5 = -3 - 2t$

$x_1 = 10 + 2x_2 - 3x_3 - 2x_4 - x_5 =$

$10 + 2s - 3(-3 - 2t) - 2(7 + 4t) - t$

$= 5 + 2s - 3t$

Soln

$x_1 = 5 + 2s - 3t$

$x_2 = s$

$x_3 = -3 - 2t$

$x_4 = 7 + 4t$

$x_5 = t$

or $\cancel{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 10 \end{pmatrix}}$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -3 \\ 7 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -2 \\ 4 \\ 1 \end{pmatrix} \quad s, t \in \mathbb{R}$$

Exercise

Solve by Gauss elimination

(a)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 2x_2 - x_3 - x_4 &= 7 \\ 2x_1 - x_2 - x_3 - x_4 &= 8 \\ x_1 - x_2 + 2x_3 - 2x_4 &= -7 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 5x_2 + 2x_3 &= -1 \\ 7x_1 + 17x_2 + 5x_3 &= -1 \end{aligned}$$

Gauss Jordan elimination

This method involves reducing the augumented matrix (A/b) to row reduced echelon form.

Example

Solve the system by Gauss Jordan elimination method.

$$x_1 + 2x_2 + x_3 = 4$$
$$3x_1 + 8x_2 + 7x_3 = 20$$
$$2x_1 + 7x_2 + 9x_3 = 23$$

We transform the augmented matrix to row reduced echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{array}\right) \xrightarrow{\frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{array}\right) \xrightarrow{R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1-2R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[R_2-2R_3]{R_1+3R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The corresponding system is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$$

Soln $x_1 = 5$
$x_2 = -2$
$x_3 = 3$

Homogenious System

A system of linear equations is said to be homogenious if it can be written in the form $Ax = 0$, where A is the coefficient matrix and 0 is the zero column vector. Such a system always has a ~~constant~~ solution namely $x = 0$ (zero vector) this is usually called a trivial soln. A non trivial soln exists if and only if the echelon form of the system has at least one free variable.

Example

Solve the homogenious system:

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Soln.

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$$R_3 + 3R_2 \quad \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding system:

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$3x_2 = 0$$

x_1 and x_2 are the leading variables and so x_3 is a free variable, Thus the system has a nontrivial solution.

Let $x_3 = s$, where s is a real number. Thus

$x_1 = \frac{4}{3}s$

$3x_1 + 0 - 4s = 0$
$3x_1 = 4s$
$x_1 = \frac{4}{3}s$ ✓

Soln. $x_1 = \frac{4}{3}s$, $x_2 = 0$, $x_3 = s$

2. Solve the homogenious system

$$x_1 + x_2 - x_3 = 0$$
$$2x_1 + 4x_2 - x_3 = 0$$
$$3x_1 + 2x_2 + 2x_3 = 0$$

Soln

Reduce the augumented matrix to echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & 4 & -1 & 0 \\ 3 & 2 & 2 & 0 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 5 & 0 \end{array}\right) \xrightarrow{2R_3 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 11 & 0 \end{array}\right).$$

leading variables
x_1
x_2 } hence no free variable
x_3

$\therefore$ $x_1 + x_2 - x_3 = 0$

$\therefore$ $2x_2 + x_3 = 0$

$11x_3 = 0$

There is no free variable, hence the system has no non-trivial soln. Infact $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

Exercise

Solve the system

(a) $x_1 - 5x_2 + 9x_3 = 0$
$-x_1 + 4x_2 - 3x_3 = 0$
$2x_1 - 8x_2 + 9x_3 = 0$

(b) $3x_1 + 6x_2 - 4x_3 - x_4 = 0$
$-5x_1 + 8x_3 + 3x_4 = 0$
$8x_1 - x_2 + 7x_4 = 0$

CRAMERS RULES

Theorem (Cramers Rule)

Consider a system of n equation and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Let Δ denote the determinant of matrix coefficients;

$A = (a_{ij})$. Also let the determinant of the matrix obtained by replacing the ith column of A by the matrix of T constant terms i.e $b = (b_1, b_2, \cdots, b_n)$.

The above system has a unique solution if and only if $\Delta \neq 0$. In this case the unique solution is given by

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \cdots, \quad x_n = \frac{\Delta_n}{\Delta}$$

Example:

Use Cramer's rule to solve

$$2x + 3y - z = 1$$
$$3x + 5y + 2z = 8$$
$$x - 2y - 3z = -1$$

Soln

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= 2(-11) - 3(-11) + 11$$

$$= \underline{\underline{22}} \checkmark$$

$$\Delta_1 = \begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = -1\begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 8 & 2 \\ -1 & -3 \end{vmatrix} - 1\begin{vmatrix} 8 & 5 \\ -1 & -2 \end{vmatrix}$$

$$= -11 - 3(-22) - 1(-11)$$

$$= 66$$

$$\Delta_2 = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{pmatrix} = 2\begin{vmatrix} 8 & 2 \\ -1 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix}$$

$$= 2(-22) - 1(-11) - 1(-11)$$

$$= \underline{\underline{-22}}$$

$$\Delta_3 = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{pmatrix} = 2\begin{vmatrix} 5 & 8 \\ -2 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & 8 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix}$$

$$= 2|11| - 3|-11| + 1|-11|$$

$$= 22 + 33 - 11$$

$$= \underline{\underline{44}}$$

By Cramer's Rule

$$x = \frac{\Delta_1}{\Delta} = \frac{66}{22} = 3 \checkmark \qquad y = \frac{\Delta_2}{\Delta} = \frac{-22}{22} = -1 \checkmark$$

$$z = \frac{\Delta_3}{\Delta} = \frac{44}{22} = \underline{\underline{2}} \checkmark$$

INVERSE MATRIX METHOD:

Consider the system of linear equations represented in matrix form by $Ax = b$, where A is the matrix coefficients.

If $\det A \neq 0$, then A^{-1} exists and $A^{-1}(Ax) = A^{-1}b \Longrightarrow$

$$Ix = A^{-1}b$$

This method works only if $\det A \neq 0$ and there is a unique solution

$$\begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} = A^{-1}b = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$$

$$x_1 = C_1, \quad x_2 = C_2, \quad \cdots, \quad x_n = C_n$$

Example:

1. Solve the following systems by Inverse matrix method.

$$x + 2y = 6$$
$$4x + 3y = 3$$

Soln

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad \det A = (1\times 3) - (4\times 2) = -5$$

$$b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\text{Inverse } (A^{-1}) = \frac{1}{-5}\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3/5 & 2/5 \\ 4/5 & -1/5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \times b$$

$$= \begin{pmatrix} -3/5 & 2/5 \\ 4/5 & -1/5 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$x = -18/5 + 6/5 = \frac{-12}{5}$$

$$y = \frac{24}{5} - 3/5 = +\frac{21}{5}$$

2. Solve by inverse matrix method

$$x_1 + 3x_2 + 2x_3 = 3$$
$$2x_1 + 4x_2 + 2x_3 = 8$$
$$x_1 + 2x_2 - x_3 = 10$$

Soln

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 8 \\ 10 \end{pmatrix}$$

Co-factors

$A_{11} = +\begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix} = -8$ $\qquad A_{21} = +\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = 7$

$A_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = 4$ $\qquad A_{22} = -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = 3$

$A_{13} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 0$ $\qquad A_{23} = +\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 1$

$A_{31} = +\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = -2$ $\quad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2$ $\quad A_{33} = +\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$

Thus the matrix of co-factors.

$$= \begin{bmatrix} -8 & 4 & 0 \\ 7 & -3 & 1 \\ -2 & 2 & -2 \end{bmatrix}.$$

$$\text{and adj } A = \begin{pmatrix} -8 & 7 & -2 \\ 4 & -3 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

Expanding the ~~third~~ 1st column to obtain (det).

$$\det = 1(-8) + 4(3) + 2(0) = 4$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{4}\begin{pmatrix} -8 & 7 & -2 \\ 4 & -3 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

Soln

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}b = \frac{1}{4}\begin{pmatrix} -8 & 7 & -2 \\ 4 & -3 & 2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 3 \\ 8 \\ 10 \end{pmatrix}$$

$$= \begin{matrix} -24 + 56 - 20 & = 12 \\ 12 - 24 + 20 & = 8 \\ 0 + 8 - 20 & = -12 \end{matrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 12 \\ 8 \\ -12 \end{pmatrix} = \begin{matrix} 3 \\ 2 \\ -3 \end{matrix}$$

$$x_1 = 3 \quad x_2 = 2 \quad x_3 = -3$$

Exercise

Solve by inverse matrix method

$$2x_1 + 3x_2 - x_3 = 1 \qquad x_1 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 8 \qquad x_2 = -1$$
$$x_1 - 2x_2 - 3x_3 = -1 \qquad x_3 = 2$$

Soln

$$\begin{pmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{pmatrix}$$

Its inverse is given by co-factors.

$A_{11} \begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix} = -15 + 4 = -11$

$A_{12} \begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = -9 - 2 = +12 = 11$

$$A_{13} = \begin{vmatrix} 3 & 5 \\ 1 & -2 \end{vmatrix} = -6-5 = -11$$

$$A_{21} = \begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} = -9-2 = 11$$

$$A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -6+1 = -5$$

$$A_{23} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4-3 = 7$$

$$A_{31} = \begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 6+5 = 11$$

$$A_{32} = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4+3 = -7$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10-9 = 1$$

$\therefore$ Its Inverse is given by

$$\begin{matrix} \cancel{-11} & \cancel{-5} & \cancel{-7} \\ \cancel{12} & \cancel{7} & \\ \cancel{-11} & \cancel{11} & \end{matrix} \quad \begin{pmatrix} -11 & 11 & 11 \\ 11 & -5 & -7 \\ -11 & 7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 8 \\ -1 \end{pmatrix}$$

$$x_1 = -11 + 88 - 11 = 66$$
$$x_2 = 11 - 40 + 7 = -22 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$
$$x_3 = -11 + 56 - 1 = \underline{\underline{44}}$$

Proof:

$$2x_1 + 3x_2 - x_3 = 1$$
$$\cancel{12} + (\cancel{-15}) - (\cancel{44}) =$$
$$6 - 3 - 1 = 1$$

$1 = 1$ hence proven

VECTORS:

Coordinate System:

Any point on the Cartesian plane can be represented as pair (x_0, y_0) where x_0 is the x - coordinate and y^0 is the y - co-ordinate. The Cartesian plane is simply $\mathbb{R}^2$. The x axis, y axis and z axis meet at right angles at ~~called~~ the origin $(0\ 0\ 0)$

$\mathbb{R}^3$ consists of three planes, xy-plane, yz-plane, & xz - plane

The x-axis consist of points $(x_0, 0, 0)$ y axis $(0\ y_0\ 0)$ and z axis $(0\ 0\ z_0)$

In general, $\mathbb{R}^n$ consists of all <u>n-tuples</u> of real entries.

$$\mathbb{R}^n = \{a_1, a_2, \ldots, a_n \mid a_i \in \mathbb{R}\}$$

such That

Vectors

These are quantities defined by <u>Magnitude</u> and <u>direction</u>. ✓✓

A <u>Column vector</u> is an <u>n-tuple</u> of numbers written vertically

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ or } n\times 1 \text{ Matrix}$$

If the numbers $a_1, a_2, \ldots a_n$ are real we have a real column vector; if they are complex we have complex column vector.

The number a_i in the ith slot is the ith component of the vector.

A <u>row</u> <u>n-vector</u> is an n-tuple $(a_1, a_2, \ldots, a_n)$ of numbers written horizontally.

The Zero Vector

The zero vector in $\mathbb{R}^n$ is a vector of the form

$$\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underbrace{(0, 0, \ldots, 0)}_{n\text{-tuples}}$$

Unit vector

It is a vector of unit length i.e length 1

In $\mathbb{R}^2$, the unit vector along the x-axis is denoted by $\underset{\sim}{i}$ and along the y-axis by $\underset{\sim}{j}$

$$\underset{\sim}{i} = (1, 0), \quad \underset{\sim}{j} = (0, 1)$$

Similarly in $\mathbb{R}^3$ the unit vector along the x axis is denoted by $\underset{\sim}{i}$ along the y-axis by $\underset{\sim}{j}$ and along the z-axis by $\underset{\sim}{k}$.

$$\underset{\sim}{i} = (1,0,0) \quad \underset{\sim}{j} = (0,1,0) \text{ and}$$

$$k = (0,0,1)$$

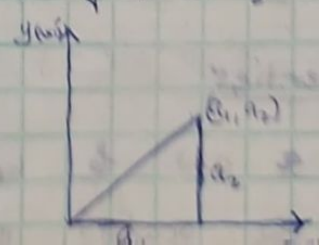
if $(x, y, z) \in \mathbb{R}^3$ then

$$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$
$$= x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$

Magnitude of a Vector

Let $\underset{\sim}{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$. Then by Pythagoras theorem,

$$|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2}$$



In general if $\underset{\sim}{a} = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n$

then

$$|\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Example:

If $\underset{\sim}{a}\ (1, 2, 5) \in \mathbb{R}^3$, then

$$|\underset{\sim}{a}| = \sqrt{1^2 + 2^2 + 5^2} = \sqrt{30}$$

$$= \underline{\underline{5.48}}$$

Vector algebra

(a) Vector addition and Subtraction

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots b_n) \in \mathbb{R}^n$ then

$$\underset{\sim}{a} + \underset{\sim}{b} = (a_1, a_2, \ldots, a_n) \pm (b_1, b_2, \ldots, b_n)$$

$$= (a_1 \pm b_1, a_2 \pm b_2, \ldots a_n \pm b_n)$$

(b) Multiplication by Scalar

$k\underset{\sim}{a} = k(a_1, a_2, \ldots a_n)$, $\in \mathbb{R}$, then

$$k\underset{\sim}{a} = k(a_1, a_2, \ldots, a_n)$$

$$= (ka_1, ka_2, \ldots ka_n)$$ where

k is a scalar

Eg $\underset{\sim}{u} = (4, -3)$ $\underset{\sim}{v} = (-2, 3)$ then

$$\underset{\sim}{u} + \underset{\sim}{v} = (4, -3) + (-2, 3) = (2, 0)$$

$$\underset{\sim}{u} - \underset{\sim}{v} = (4, -3) - (-2, 3) = (6, -6)$$

$$-2\underset{\sim}{v} = -2(-2, 3) = (4\ -6)$$

Properties

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be vectors, λ and μ be scalars, then

1. $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$ (Commutativity)

2. $\underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c}) = (\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c}$ associativity.

3. $\underset{\sim}{a} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{a} = \underset{\sim}{a}$ Zero element.

4. $\underset{\sim}{a} + (-\underset{\sim}{a}) = (-\underset{\sim}{a}) + \underset{\sim}{a} = 0$ Additive Inverse.

5. $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$ (Distributivity) left

6. $(\lambda + \mu)\underset{\sim}{a} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{a}$ (Distributivity) right

7. $(\lambda\mu)\underset{\sim}{a} = \lambda(\mu\underset{\sim}{a})$ Associativity with to multiplication by scalar.

8. $(\lambda\mu)\underset{\sim}{a} = \mu(\lambda\underset{\sim}{a})$ scalar Pull Through

9. $I\underset{\sim}{u} = \underset{\sim}{u}$ multiplicative identity.

Vector equality

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots b_n)$

if $a_i = b_i$ for all $i = 1, 2, \ldots, n$

i.e $a_1 = b_1$, $a_2 = b_2$, $\ldots$, $a_n = b_n$

Example.

Solve for λ and μ if

$$\lambda(3, 4) + \mu(1, -1) = (1, 2)$$

Soln.

$$(3\lambda, 4\lambda) + (\mu, -\mu) = (1, 2)$$

$3\lambda + \mu = 1$
$4\lambda - \mu = 2$ +

$7\lambda = 3$

$\lambda = \frac{3}{7}$ ✓✓

$\frac{9}{7} + \mu = 1$

$\mu = -\frac{2}{7}$

$\mu = -\frac{2}{7}$ ✓✓

2. Solve for x, y and z if
$x(5i + j) + y(j + k) + zk = 5i + 3j - k$

Soln

Simplifying

$5x\underset{\sim}{i} + x\underset{\sim}{j} + y\underset{\sim}{j} + y\underset{\sim}{k} + z\underset{\sim}{k} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

$= 5x\underset{\sim}{i} + (x+y)\underset{\sim}{j} + (y+z)\underset{\sim}{k} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

$5x = 5$ $\quad 1 + y = 3$ $\quad 2 + z = -1$

$x = 1$ ✓✓ $\quad y = 2$ ✓✓ $\quad z = -3$ ✓✓

so $(5\ 3\ -1) = 1(5, 1, 0) + 2(0, 1, 1) - 3(0\ 0\ 1)$

✓✓ **The dot Product (inner Product) (or scalar Product)**

Let $\underset{\sim}{a} = (a_1, a_2)$ and $\underset{\sim}{b} = (b_1, b_2)$

Then

$$\boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2}$$

is the algebraic definition of dot product.

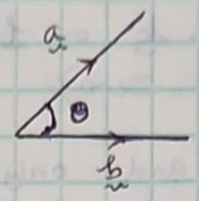
Example

1. $(1, 2) \cdot (2, 3) = (1 \times 2) + (2 \times 3) = 8$ ✓

2. $(1, 2, 3) \cdot (-1, 2, 1) = (1 \times -1) + (2 \times 2) + (3 \times 1)$
$= 6$ ✓

3. $(2\underset{\sim}{i} + \underset{\sim}{j}) \cdot (3\underset{\sim}{i} - \underset{\sim}{j}) = (2 \times 3) + 1(-1)$
$6 - 1 = 5$ ✓

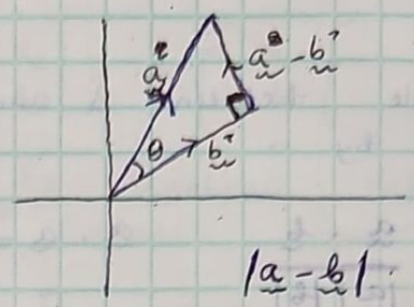
GEOMETRIC DEFINITION



$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

where θ is the ∠ bwt a and b.

Proof



$|\underset{\sim}{a} - \underset{\sim}{b}|$

$|\underset{\sim}{b}||\underset{\sim}{a}|$

By Cosine rule.

$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

Therefore

$2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} - \underset{\sim}{b}|^2$

Let $\underset{\sim}{a} = (a_1, a_2)$ $\quad \underset{\sim}{b} = (b_1, b_2)$ then

$\underset{\sim}{a} - \underset{\sim}{b} = (a_1 - b_1, a_2 - b_2)$

RHS $\quad a_1^2 + a_2^2 + b_1^2 + b_2^2 - (a_1^2 - 2a_1b_1 + b_1^2 + a_2^2$

$= 2a_1b_1 + 2a_2b_2$

$2(\cancel{a_1b_1 + a_2b_2})$

$\therefore |a||b| \cos\theta = \underbrace{a_1b_1 + a_2b_2}_{\text{algebraic definition}}$

PROPERTIES OF DOT PRODUCT

1. $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$ Commutativity

2. $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$ Distributive

3. $\underset{\sim}{a} \cdot \underset{\sim}{0} = 0$

are $\perp$, therefore the angle bwt them is $90°$.)

5. $\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$ (they make an angle of zero)

6. $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular (orthogonal, at right angles or normal to each other.)

7. The angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} \qquad \underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$$

8. $|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

Proof

For if $\underset{\sim}{a} = (a_1, a_2)$ then

$$(\underset{\sim}{a} \cdot \underset{\sim}{a}) = (a_1, a_2) \cdot (a_1, a_2)$$

$$= a_1^2 + a_2^2 \qquad |\underset{\sim}{a}| = \sqrt{a_1^2 + a_2^2}$$

$$= |\underset{\sim}{a}|^2 \qquad |\underset{\sim}{a}|^2 = a_1^2 + a_2^2$$

$$\Rightarrow |\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$$

* **Example:**

Find the angle between $\underset{\sim}{a} = i + 4j + 8k$ $\underset{\sim}{b} = 5\underset{\sim}{i} + 4\underset{\sim}{j} + 3k$.

Soln

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} = \frac{(1, 4, 8) \cdot (5, 4, 3)}{|1, 4, 8|\,|5, 4, 3|}$$

$$= \frac{5 + 16 + 24}{\sqrt{1^2 + 4^2 + 8^2}\,\sqrt{5^2 + 4^2 + 3^2}} = \frac{45}{\sqrt{81}\,\sqrt{50}}$$

$$= \frac{45}{9 \times 5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \qquad 45° \text{ or } \frac{\pi}{4}$$

2. Determine C so that $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = C\underset{\sim}{i} + 5\underset{\sim}{j} - 3k$ are orthogonal.

Soln

Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are orthogonal,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$$

Now,

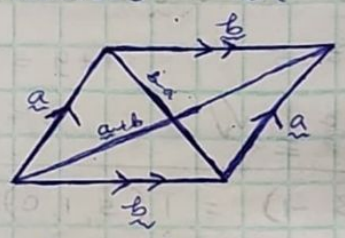
$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3, 4, -2) \cdot (C, 5, -3) = 0$$

$$3C + 20 + 6 = 0$$

$$3C = -26$$

$$C = -\frac{26}{3}$$

3. Prove that the diagonals of a rhombus are perpendicular.



The diagonals are $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$

and $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

To show that the diagonals are perpendicular, we need to prove that

$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$$

Now,

$$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b} \quad \text{Distributivity}$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b} \quad \text{Commutativity}$$

$$|\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2 = 0$$

Since $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

Exercise

The Cauchy - Schwarz inequality states that for any vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}|\,|\underset{\sim}{b}|$. Prove this inequality

[Hint $|\cos\theta| \leq 1$ for all θ]

Solution

(*) $\underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}|\,|\underset{\sim}{b}|$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a|\,|b| \cos\theta$

$|\underset{\sim}{a} \cdot \underset{\sim}{b}| = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\,|\cos\theta|$

$|\underset{\sim}{a} \cdot \underset{\sim}{b}| \leq |\underset{\sim}{a}|\,|\underset{\sim}{b}|$

Hence a prove

The Cross Product | Vector Products

These are defined in three dimension

Let $\underset{\sim}{a} = (a_1, a_2, a_3)$ and $\underset{\sim}{b} = (b_1, b_2, b_3)$

Then

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

is the algebraic definition of cross Product.

* Example

1. Find $(1,2,1) \times (0,1,3)$

Soln

$$(1\ 2\ 1) \times (0\ 1\ 3) = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 5\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$$

2. Find $\underset{\sim}{i} \times \underset{\sim}{j}$

Soln

$$\underset{\sim}{i} \times \underset{\sim}{j} = (1,0,0) \times (0,1,0) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$i \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \underset{\sim}{k}$$

3. Find $\underset{\sim}{j} \times \underset{\sim}{i}$

Soln

$$\underset{\sim}{j} \times \underset{\sim}{i} = (0,1,0) \times (1,0,0) = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$i \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - j \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -k$$

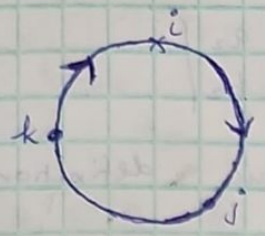
From examples ② and ③ we see that Cross products are **not** Commutative.

$\underset{\sim}{i}\ \underset{\sim}{j}\ \underset{\sim}{k}\ \underset{\sim}{i}\ \underset{\sim}{j}\ \underset{\sim}{k}$

$\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$ $\quad$ $\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}$

$\underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}$ $\quad$ $\underset{\sim}{k} \times \underset{\sim}{j} = -\underset{\sim}{i}$

$\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j}$ $\quad$ $\underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j}$



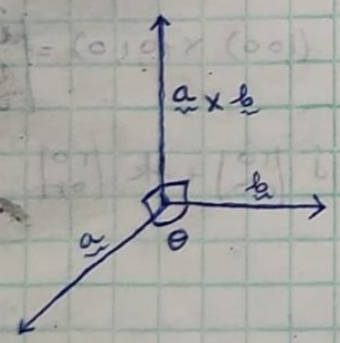
Geometric definition of vector Products:

$\underset{\sim}{a} \times \underset{\sim}{b}$ is a vector whose magnitude is equal to $|\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Thus,

$\underset{\sim}{a} \times \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta\ \hat{u}$, where $\hat{u}$ is a unit vector $\perp$ to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$

Thus $\underset{\sim}{a} \times \underset{\sim}{b}$ is a vector that is Orthogonal to both vector $\underset{\sim}{a}$ and vector $\underset{\sim}{b}$.



$\underset{\sim}{a} \times \underset{\sim}{b}$ is Orthogonal to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$

PROPERTIES OF VECTOR PRODUCT:

1. $\underset{\sim}{a} \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) = 0$

i.e $\underset{\sim}{a} \times \underset{\sim}{b}$ is Orthogonal to $\underset{\sim}{a}$ and $\underset{\sim}{b}$

2. $\underset{\sim}{a} \times \underset{\sim}{b} = -(\underset{\sim}{b} \times \underset{\sim}{a})$

3. $\underset{\sim}{a} \times \underset{\sim}{a} = 0$

$\left.\begin{array}{l} \underset{\sim}{i} \times (\underset{\sim}{i} \times \underset{\sim}{j}) = \underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j} \\ (\underset{\sim}{i} \times \underset{\sim}{i}) \times \underset{\sim}{j} = \underset{\sim}{0} \times \underset{\sim}{j} = 0 \end{array}\right\}$ NO associativity.

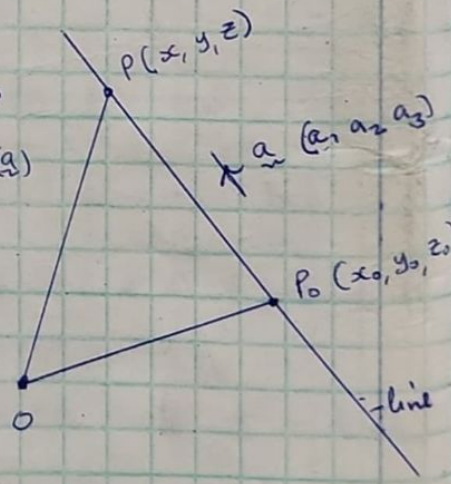
Lines and planes in Space:

Lines:

A line in the plane is determined by a single point P_0 on it and a vector $\underset{\sim}{a}$ that determines the direction of the line. Let $P_0(x_0, y_0, z_0)$ be any point in the line L which parallel to the vector $\underset{\sim}{a} = (a_1, a_2, a_3)$. If $P(x, y, z)$ is any point on the line L.

$OP = OP_0 + P_0P$

$= OP_0 + \lambda(\underset{\sim}{a})$



there exists a scalar λ such that

$$(x, y, z) = (x_0, y_0, z_0) + \lambda(a_1, a_2, a_3)$$

① is called the vector equation of the line

Equating the components of the vector in ①

$$\left.\begin{aligned} x &= x_0 + t a_1 \\ y &= y_0 + t a_2 \\ z &= z^0 + t a_3 \end{aligned}\right\} \quad ②$$

These are the <u>parametric equations</u> of the line L through $P_0(x_0, y_0, z_0)$ and parallel to the vector

$\underset{\sim}{a} = (a_1, a_2, a_3)$

Eliminating t from (2), we have $\dfrac{x - x_0}{a_1} = \dfrac{y - y_0}{a_2} = \dfrac{z - z_0}{a_3} \rightarrow ③$

③ is called <u>symmetric equation</u> of the line L.

NOTE:

(a_1, a_2, a_3) is a vector parallel to the line L.

Example 1,

Find the <u>Parametric</u> and <u>Symmetric</u> equation of the line passing through the point $(2, -3, 4)$ and parallel to the vector $(3, 5, -6)$

Soln

Let $P(x, y, z)$ be any point on the line

$(x\ y\ z) = (2, -3, 4) + t(3, 5, -6)$

is the vector equation of the line

Now,

$$\left.\begin{aligned} x &= 2 + 3t \\ y &= -3 + 5t \\ z &= 4 - 6t \end{aligned}\right\} \text{ Parametric equation of the line.}$$

Eliminating t,

$\rightarrow ④ \quad \dfrac{x-2}{3} = \dfrac{y+3}{5} = \dfrac{z-4}{-6}$ is the <u>symmetric</u> equation of the line

line L

Example 2:

Find the vector equation of the line passing through the points $A(1, 2, 3)$ and $B(4, 4, 4)$ and the coordinates where this line meets the plane $z = 0$

Soln

A vector parallel to the line is $\vec{AB} = (4, 4, 4) - (1, 2, 3)$

$= (3, 2, 1)$

Therefore the vector equation of the line is

$(x, y, z) = (1, 2, 3) + t(3, 2, 1)$

The line meets $z = 0$ where

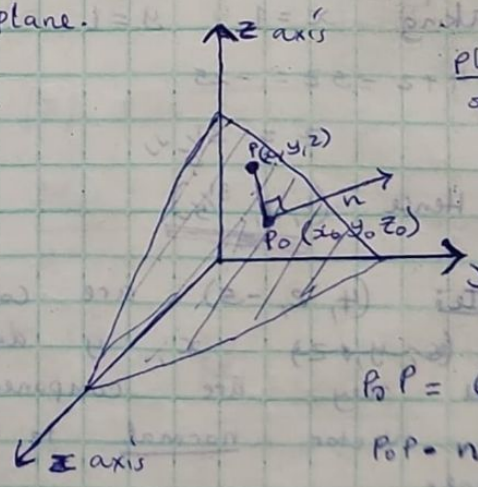
$3 + t = 0$

$\Rightarrow t = -3$, substituting this in the equation of the line, we get

$(x, y, z) = (1, 2, 3) - 3(3, 2, 1) = (-8, -4, 0)$

PLANES IN 3-DIMENSIONS

A plane in 3-dimension can be specified by giving a <u>non-zero</u> vector that is perpendicular to the <u>plane</u> (called normal) and one point on the plane.



<u>Plane</u> – Portion of space bounded by a line.

$P_0P = (x, y, z) - (x_0, y_0, z_0)$

$P_0P \cdot n = 0$

Then $\vec{P_0P}$ is orthogonal to n, that is $n \cdot \vec{P_0P} = 0$ --- (i)

Since $P_0P = (x - x_0, y - y_0, z - z_0)$

equation ① becomes,

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{--- ②}$$

We call eqt ② the <u>Point normal form</u> of the equation of the plane.

<u>Example</u>

Find the equation of the plane passing through $(3, -1, 7)$ and perpendicular to the vector $(4, 2, -5)$

<u>Soln</u>

Let (x, y, z) be a general point on the plane, then

$(x, y, z) - (3, -1, 7) = (x-3, y+1, z-7)$ is a general vector on the plane and must be normal to $(4, 2, -5)$. Therefore

$$(4, 2, -5) \cdot (x-3, y+1, z-7) = 0$$

$$4(x-3) + 2(y+1) - 5(z-7) = 0$$

$$4x - 12 + 2y + 2 - 5z + 35 = 0$$

$$4x + 2y - 5z = -25$$

taking $x = 1$, $y = 1$.

$$4 + 2 - 5z = -25$$

$$z = 31/5$$

Hence $(1, 1, 31/5)$

<u>Note:</u> $(4, 2, -5)$ are coefficient of ~~(x, y, z)~~ x, y and z and they are components of the vector <u>normal</u> to the plane.

THEOREM

If a, b, c and d are constants and a, b, c are not zero, then the graph of the equation

$$ax + by + cz = d$$

is a plane having vector $n = (a, b, c)$ as normal.

Find the equation of the plane passing through the points $P_1(1, 2, -1)$, $P_2 = (2, 3, 1)$ and $P_3(3, -1, 2)$

<u>Soln</u>

Two vectors on the plane are:

$$\overrightarrow{P_1P_2} = (2, 3, 1) - (1, 2, -1) = (1, 1, 2)$$

and

$$\overrightarrow{P_1P_3} = (3, -1, 2) - (1, 2, -1) = (2, -3, 3)$$

A vector normal to the plane is

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix}$$

$$= i\begin{vmatrix} 1 & 2 \\ -3 & 3 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= 9i + j - 5k$$

Let (x, y, z) be a general point on the plane, then $(x, y, z) - (1, 2, -1) = (x-1, y-2, z+1)$ is a general vector in the plane.

Thus the equation of the plane is

$$(9, 1, -5) \cdot (x-1, y-2, z+1) = 0$$

$$9(x-1) + 1(y-2) - 5(z+1) = 0$$

$$9x - 9 + y - 2 - 5z - 5 = 0$$

$$9x + y - 5z = 16$$

4. Find the equation of the plane through $(-1, 2, 3)$ and perpendicular to the

α β

$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$

$2x - 3y + 4z = 1$ and $3x - 5y + 2z = 3$

Soln

A vector Perpendicular to $2x - 3y + 4z = 1$ is $(2, -3, 4)$

A vector Perpendicular to $3x - 5y + 2z = 3$ is $(3, -5, 2)$

A vector Parallel to the two planes is

$$(2, -3, 4) \times (3, -5, 2) = \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix} -3 & 4 \\ -5 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix}$$

$$= 14\underset{\sim}{i} + 8\underset{\sim}{j} - \underset{\sim}{k}$$

And this vector is Perpendicular to the plane in question. Let (x, y, z) be any point on the plane, then $(x, y, z) - (1, 2, 3)$

$= (x+1)\ y-2,\ z-3)$ is a general vector on the plane so that

$(14, 8, -1) \cdot (x+1\quad y-2\quad z-3) = 0$

$14(x+1) + 8(y-2) - 1(z-3) = 0$

$14x + 14 + 8y - 16 - z + 3 = 0$

$14x + 8y - z = -1$

Find the angle between the following planes $2x + y - 2z = 1$ and $x - 2y - 2z = 2$

Soln

A vector perpendicular to $2x+y-2z=1$ is $(2, 1, -2)$

A vector perpendicular to $x - 2y - 2z = 2$ $(1, -2, -2)$

Angle bwt the two planes is the same as the angle bwt their normals.

$$\cos\theta = \frac{(2, 1, -2) \cdot (1, -2, -2)}{|2, 1, -2|\,|1, -2, -2|} = \frac{2 - 2 + 4}{\sqrt{9}\sqrt{9}}$$

$$= 4/9$$

$$\theta = \cos^{-1} 4/9 = \cos^{-1}(0.4444) = 63.61^\circ$$

Distance from the Origin to the Plane $ax + by + cz = d$

We know that (a, b, c) is a vector perpendicular to the plane. Place the tail of (a, b, c) at the Origin and multiply (a, b, c) with an appropriate scalar π such that

$\pi(a, b, c) = (\pi a, \pi b, \pi c)$ lies on the plane.

Since $(\pi a, \pi b, \pi c)$ lies on the plane, it must satisfy the equation of the plane. Thus,

$a(\pi a) + b(\pi b) + c(\pi c) = d$

$a^2\pi + b^2\pi + c^2\pi = d$

$$\pi = \frac{|d|}{a^2 + b^2 + c^2}$$

Now the distance from $(0, 0, 0)$ to the plane is

$$|\pi a, \pi b, \pi c| = \sqrt{(\pi a)^2 + (\pi b)^2 + (\pi c)^2}$$

$$\sqrt{\pi^2(a^2 + b^2 + c^2)}$$

$$= |\pi|\sqrt{a^2 + b^2 + c^2}$$

$$\frac{|d|}{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2}$$

$$= \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance of the plane from the Origin

$$\boxed{D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}}$$

Example.

Find the distance from the origin to the plane $2x + 3y - z = 2$

$a=2 \quad b=3 \quad c=-1 \quad d=2$

Soln

$a = 2, \quad b = 3, \quad c = -1, \quad d = 2$

$$D = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|2|}{\sqrt{2^2+3^2+(-1)^2}}$$

$$= \frac{2}{\sqrt{4+9+1}} = \frac{2}{\sqrt{14}}$$

$$= \frac{2\sqrt{14}}{14} = \frac{\sqrt{14}}{7} \text{ units}$$

Distance from a point $P_0(x_0 y_0 z_0)$ and the plane $ax + by + cz = d$

Place the tail of (a, b, c) at the point $P_0(x_0, y_0, z_0)$ and multiply (a, b, c) with appropriate scalar such that

$(x_0 \; y_0 \; z_0) + r(a, b, c) = (x_0 + ra, \; y_0 + rb, \; z_0 + rc)$ lies on the plane.

Since $(x_0 + ra, \; y_0 + rb, \; z_0 + rc)$ lies on the plane.

$$a(x_0 + ra) + b(y_0 + rb) + c(z_0 + rc) = d$$

$$= ax_0 + ra^2 + by_0 + rb^2 + cz_0 + rc^2 = d$$

$$r a^2 + r b^2 + r c^2 = d - ax_0 - by_0 - cz_0$$

$$r = \frac{d - ax_0 - by_0 - cz_0}{a^2+b^2+c^2}$$

Now the distance from $P_0(x_0, y_0, z_0)$ to the plane is

$$D = |ra, rb, rc| = |r|\sqrt{a^2+b^2+c^2}$$

$$= \frac{|d - ax_0 - by_0 - cz_0|}{a^2+b^2+c^2} \cdot \sqrt{a^2+b^2+c^2}$$

$$= \frac{|d - ax_0 - by_0 + cz_0|}{\sqrt{a^2+b^2+c^2}}$$

or

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

Example.

Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$

$a=2 \quad b=-3 \quad c=6 \quad d=-1$

Soln. $2(1) + (-3)(-4) + 6(-3) = -1$

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|2(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2+(-3)^2+6^2}}$$

$$= \frac{|-3|}{\sqrt{49}} = \frac{3}{7}$$

$\frac{2+12-18+1}{\sqrt{4+9+36}} = \frac{|-3|}{7} = \frac{3}{7}$

Intersecting and Parallel Planes

Given two planes, either they intersect, in which case we can seek for their line of intersection or they are parallel, in which case we can ask for distance between them.

Two planes are parallel if their normals are parallel vectors.

Example:

1 Find the parametric equations for the line of intersection of the planes $3x + 2y - 4z - 6 = 0$ and $x - 3y - 2z - 4 = 0$.

Soln

The line of intersection consist of all points that satisfy the two equations in the system.

$$3x + 2y - 4z = 6$$
$$x - 3y - 2z = 4$$

$$3x + 2y - 4z = 6$$
$$3x - 9y - 6z = 12$$
$$11y + 2z = -6$$

Let $z = t$, then $y = \frac{-6}{11} - \frac{2t}{11}$,

$$x = \frac{26}{11} + \frac{16}{11}t$$

The parametric equations of the line of intersection are

$$x = \frac{26}{11} + \frac{16}{11}t$$

$$y = -\frac{6}{11} - \frac{2}{11}t \qquad (-\infty < t < \infty)$$

$$z = t$$

* 2 The planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$ are parallel. Find the distance between them.

Soln

To find the distance D bwt the two planes we may select an arbitrary point in one of the planes and compute its distance to the plane. By setting $y = z = 0$ in the equation $x + 2y - 3z = 3$, we obtain $P(3,0,0)$ in the plane. The distance between P_0 and the plane $2x + 4y - 4z = 7$ is

$$D = \frac{|2(3) + 4(0) + -4(0) - 7|}{\sqrt{2^2 + 4^2 + (-4)^2}} = \frac{1}{\sqrt{36}}$$

$$= \frac{1}{6} \text{ units}$$

① Find the equation of the line of intersection of the planes $4x + 3y + z = 10$ and $x + y + z = 6$

② Find the distance between the given parallel planes

(a) $3x - 4y + z = 1$

(b) $6x - 8y + 2z = 3$

Solutions.

② $3x - 4y + z = 1$. **Soln**

By setting $y = z = 0$ in equation $6x - 8y + 2z = 3$ $(\frac{1}{2}, 0, 0)$

So the distance between $3x - 4y + z = 1$ and $P(\frac{1}{2}, 0, 0)$

$$D = \frac{|\frac{1}{2}(3) + -4(0) + 1(0) - 1|}{\sqrt{9 + 16 + 1}}$$

$$= \frac{0.5}{\sqrt{26}} = 0.098$$

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$\frac{2(3) - 7}{\sqrt{4+3}} = \frac{-11}{6} = \frac{1}{6}$$

$$4x + 3y + z = 0$$
$$x + y + z = 6$$
$$2 = -6$$

LINEAR DEPENDENCE AND INDEPENDENCE

Dfn: (Linear Combination)

A vector w is called a <u>linear Combination</u> of vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}$ provided there <u>exists</u> <u>scalars</u> $c_1, c_2, \ldots c_k$ such that

$$\underset{\sim}{w} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \cdots + c_k \underset{\sim}{v_k}$$

Example

Let

1. Determine wheter vector $\underset{\sim}{w} = (2, -6, 3) \in R^3$ is a linear Combination of the vectors

$\underset{\sim}{v_1} = (1, -2, -1)$ and $\underset{\sim}{v_2} = (3, -5, 4)$

Soln

Set $\underset{\sim}{w} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2}$, that is

$$c_1(1, -2, -1) + c_2(3, -5, 4) = (2, -6, 3)$$

$$c_1 + 3c_2 = 2$$
$$-2c_1 - 5c_2 = -6$$
$$c_1 + 4c_2 = 3$$

Reducing the augumented matrix to echelon form.

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & -5 & -6 \\ 1 & 4 & 3 \end{array}\right) \xrightarrow[R_3 + R_1]{R_2 + 2R_1}$$

$$\begin{array}{rrr} -2 & -5 & -6 \\ 2 & 6 & 4 \\ \hline 0 & -1 & -2 \end{array} \qquad \begin{array}{rrr} -1 & 4 & 3 \\ 1 & 3 & 2 \\ \hline 0 & 7 & 5 \end{array}$$

$$R_3 - 7R_2 \quad \begin{array}{rrr} 0 & 7 & -14 \\ \hline 0 & 0 & 19 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 7 & 5 \end{array}\right) \longrightarrow R_3 - 7R_2 \longrightarrow$$

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{array}\right)$$

We see from the last row that the system has no <u>solution</u> (<u>inconsistent</u>), so desired scalars c_1 and c_2 do not <u>exist</u>. Hence w is not a <u>linear Combination</u> of $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$

2. Express $\underset{\sim}{w} = (-7, 7, 11)$ as a linear Combination of the vectors $\underset{\sim}{v_1} = (1, 2, 1)$ $\underset{\sim}{v_2} = (-4\ -1\ 2)$ and $\underset{\sim}{v_3} = (-3, 1, 3)$

Soln.

We write

$\underset{\sim}{w} = c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3}$, that is

$$c_1(1, 2, 1) + c_2(-4, -1, 2) + c_3(-3, 1, 3) =$$

$$c_1 - 4c_2 - 3c_3 = -7$$
$$2c_2 - c_2 + c_3 = 7$$
$$c_1 + 2c_2 + 3c_3 = 11$$

$$\left(\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} = \left(\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 6 & 6 & 18 \end{array}\right) \xrightarrow{R_3 - 6R_2} \left(\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{rrr} 0 & 6 & 18 \\ 0 & 6 & 18 \\ \hline 0 & 0 & 0 \end{array} \checkmark$$

Therefore

$$c_1 - 4c_2 - 3c_3 = -7$$
$$c_2 + c_3 = 3$$

Thus c_3 is a <u>free variable</u>. set $c_3 = t$, then $c_2 = 3 - t$ and $c_1 = 5 - t$.

When $t = 1$

$c_1 = 4$ $\quad c_2 = 2$ $\quad c_3 = 1$

$$\underset{\sim}{w} = 4\underset{\sim}{v_1} + 2\underset{\sim}{v_2} + \underset{\sim}{v_3}$$

Set $t = -2$ $\quad c_1 = 7$ $\quad c_2 = 5$ $\quad c_3 =$

$$\underset{\sim}{w} = 7\underset{\sim}{v_1} + 5\underset{\sim}{v_2} - 2\underset{\sim}{v_3}$$

Dfn (linear dependence)

If we have a vector $v_1, v_2, \dots v_k \in \mathbb{R}^n$, we say that $v_1, v_2, \dots, v_k$ are linearly dependent if one of the vectors can be written as a linear combination of the others.

e.g

$\{(-7,7,11), (-4,-1,2), (1,2,1), (-3,1,3)\}$

In the previous example are linearly dependent since $(-7,7,11)$ is a linear combination of the others.

Theorem

$\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_k} \in \mathbb{R}^n$ are linearly dependent if and only if there exists scalars $c_1, c_2, \dots c_k$ not all zero such that

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \dots + c_k \underset{\sim}{v_k} = \underset{\sim}{0}$$

Proof.

$\Rightarrow$ Assume $\underset{\sim}{v_1}, v_2, \dots, v_k$ are linearly dependent so one of them say v_k can be written as a linear combination of the others;

$$v_k = c_1 v_1 + c_2 v_2 + \dots + c_{k-1} v_{k-1}$$

Therefore

$$c_1 v_1 + c_2 v_2 + \dots + c_{k-1} v_{k-1} + c_k v_k = 0$$

where $c_k = -1 \neq 0$

$\Leftarrow$ Assume that $c_1 v_1 + c_2 v_2 + \dots + c_k v_k = \underset{\sim}{0}$ where not all c_i's are zero, say $c_k \neq 0$

Then

$$c_k v_k = -c_1 v_1 - c_2 v_2 - \dots - c_{k-1} v_{k-1}$$

$$v_k = -\frac{c_1}{c_k} \underset{\sim}{v_1} - \frac{c_2}{c_k} \underset{\sim}{v_2} - \dots - \frac{c_{k-1}}{c_k} v_{k-1}$$

Thus $\underset{\sim}{v_k}$ is a scalar combination of the vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2} \dots v_{k+1}$, and so the vectors are linearly dependent.

Example

1. Are the vectors $\underset{\sim}{v_1} = (1,2,2,1)$, $v_2 = (2,3,4,1)$ and $v_3 = (3,8,7,5)$ in $\mathbb{R}^4$ linearly dependent?

Soln.

set $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ That is

$$c_1(1\,2\,2\,1) + c_2(2,3,4,1) + c_3(3,8,7,5) = (0\,0\,0\,0)$$

$$\Rightarrow \begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 3c_2 + 8c_3 &= 0 \\ 2c_1 + 4c_2 + 7c_3 &= 0 \\ c_1 + c_2 + 5c_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 8 & 0 \\ 2 & 4 & 7 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right) \xrightarrow[\substack{R_3 - 2R_1 \\ R_4 - R_1}]{R_2 - 2R_1} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{R_4 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ -c_2 + 2c_3 &= 0 \\ c_3 &= 0 \end{aligned} \Rightarrow$

$$c_1 = c_2 = c_3$$

Thus the vectors are not linearly dependent. They are linearly independent.

Dfn.

The vectors $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots \underset{\sim}{v_k}$ are <u>linearly independent</u> if they are not <u>linearly dependent</u>. Hence $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_k}$ are linearly independent if

$$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + \cdots + c_k \underset{\sim}{v_k} = 0 \Rightarrow$$

$$c_1 = c_2 = \cdots = c_k = 0$$

Example:

1. Determine wheter the vectors $\underset{\sim}{v_1} = (1, -2, 1)$ $v_2 = (2, 1, -1)$ $\underset{\sim}{v_3} = (7, -4, 1)$ are linearly independent.

Soln

Set $c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} = \underset{\sim}{0}$

$$c(1, -2, 1) + c_2(2, 1, -1) + c_3(7, -4, 1)$$

$$= \underset{\sim}{0} = (0, 0, 0)$$

$$c_1 + 2c_2 + 7c_3 = 0$$
$$-2c_1 + c_2 - 4c_3 = 0$$
$$c_1 - c_2 + c_3 = 0$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 5 & 10 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{5}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right) \xrightarrow{R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $c_1 + 2c_2 + 7c_3 = 0$
$c_2 + 2c_3 = 0$

Since c_3 is a <u>free variable</u>, the <u>homogeneous system</u> has non-trivial soln. Thus the 3 vectors are not <u>linearly independent</u>.

2. Show that the vectors $\underset{\sim}{v_1} = [6, 2, 3, 4]$, $v_2 = [0, 5, -3, 1]$ and $v_3 = [0, 0, 7, -2]$ are linearly independent.

Set $c_1 \underset{\sim}{v_1} + c_2 v_2 + c_3 \underset{\sim}{v_3} = \underset{\sim}{0}$

$$c_1(6, 2, 3, 4) + c_2(0, 5, -3, 1) + c_3(0, 0, 7, -2) = \underset{\sim}{0} = (0, 0, 0, 0)$$

$$6c_1 = 0 \qquad c_1 = 0$$
$$2c_1 + 5c_2 = 0 \qquad c_2 = 0$$
$$3c_1 - 3c_2 + 7c_3 = 0 \qquad c_3 = 0$$
$$4c_1 + c_2 - 2c_3 = 0$$

Hence three vectors are <u>linearl Independent</u>.

Observe that the vectors in the preceeding example for a matrix in echelon form

$$\begin{bmatrix} 6 & 2 & 3 & 4 \\ 0 & 5 & -3 & +1 \\ 0 & 0 & 7 & -2 \end{bmatrix}$$

Thus we have shown that the non-zero rows of the above echelon matrix are <u>linearly Independent</u>. The results hold in general.

Theorem

The non-zero rows of a matrix in echelon form are <u>linearly Independent</u>.

Example

Determine wheter the following vectors in $\mathbb{R}^4$ are linearly dependent or Independent.

(i) $\{(1, 3, -1, 4)\ (13, 8, -5, 7)\ (2, 7, 4, 23)\}$

(ii) $\{(1,-2,4,1)\ (2,1,0,-3)\ (3,-6,1,4)\}$

(i) Soln.

Form a matrix whose row are the given vectors and row reduce to echelon form.

$$\begin{pmatrix} 1 & 3 & -1 & 4 \\ 3 & 8 & -5 & 7 \\ 2 & 9 & 4 & 23 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & 3 & 6 & 15 \end{pmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{pmatrix} 1 & 3 & -1 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the echelon matrix has a zero row, the vectors are <u>linearly dependent</u>.

(ii) $$\begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 1 & 0 & -3 \\ 3 & -6 & 1 & 4 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 5 & -8 & -5 \\ 0 & 0 & -11 & 1 \end{pmatrix}$$

Since the echelon matrix has no zero rows, the vectors are linearly Independent.

Exercise

Suppose that $\underset{\sim}{v_1}, \underset{\sim}{v_2} \ldots v_k$ are non zero vectors such that $v_i \cdot v_j = 0$ whenever $i \neq j$, show that $\underset{\sim}{v_1} + \underset{\sim}{v_2} + \cdots + \underset{\sim}{v_k}$ are linearly Independent.

$$C_1 v_1 + C_2 v_2 + \cdots + C_k v_k = \underset{\sim}{0}$$

Vector Spaces:

A vector space is a non-empty set V, of objects called vectors, on which are defined two operations, called addition and multiplication by scalar (real numbers), subject to the following axioms (rules) below. The axioms must hold for all vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ in V and for all scalars c and d.

1. The sum of $\underset{\sim}{u}$ and $\underset{\sim}{v}$, denoted $\underset{\sim}{u} + \underset{\sim}{v}$ is in V.

2. $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$

3. $\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w}$

4. There is a zero vector 0 in V, such that $\underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{u}$

5. For each vector $\underset{\sim}{u} \in V$, there is a vector in V. such that $\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0}$

6. The scalar multiple of $\underset{\sim}{u}$ by c, denoted $c\underset{\sim}{u} \in V$

7. $c \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = c\underset{\sim}{u} + c\underset{\sim}{v}$

8. $(c + d)u = c\underset{\sim}{u} + d\underset{\sim}{u}$

9. $c(d\underset{\sim}{u}) = (cd)\underset{\sim}{u}$

10. $1\underset{\sim}{u} = u$

Examples

1. The set $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R}\}$ is a vector space.

2. $M_{n,m}(\mathbb{R})$, the set of all $n \times m$ matrices is a vector with respect to the usual matrix addition and multiplication by scalar.

3. $P_n(x)$ - The set of all polynomials, $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ with coefficients from $\mathbb{R}$ is a vector space with respect to addition of polynomials and multiplication by a constant.

4. $F(\mathbb{R})$ be set of all functions $f: \mathbb{R} \to \mathbb{R}$, $F(\mathbb{R})$ is a vector space, Given f and g in $F(\mathbb{R})$ and a real constant c, the function $f+g$ and cf are defined as follows:

$$(f+g)x = f(x) + g(x) \quad \text{and}$$

$$(cf)(x) = c\,f(x)$$

Example:

Write the polynomial $\underset{\sim}{v} = t^2 + 4t - 3$ as a linear combination of the polynomials $e_1 = t^2 - 2t + 5$, $e_2 = 2t^2 - 3t$, $e_3 = t + 3$

Soln

Set $\underset{\sim}{v} = c_1 e_1 + c_2 e_2 + c_3 e_3$

$$t^2 + 4t - 3 = c_1(t^2 - 2t + 5) + c_2(2t^2 - 3t) + c_3(t+3)$$

$$= c_1 t^2 - 2c_1 t + 5c_1 + 2c_2 t^2 - 3c_2 t + c_3 t + 3c_3$$

$$= (c_1 + 2c_2)t^2 + (-2c_1 - 3c_2 + c_3)t + 5c_1 + 3c_2$$

Equating the coefficients

$$c_1 + 2c_2 = 1$$
$$-2c_1 - 3c_2 + c_3 = 4$$
$$5c_1 + 3c_3 = -3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 4 \\ 5 & 0 & 3 & -3 \end{array}\right) \xrightarrow[R_3 - 5R_1]{R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & -10 & 3 & -8 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 13 & 52 \end{array}\right)$$

$$\left.\begin{aligned} c_1 + 2c_2 &= 1 \\ c_2 + c_3 &= 6 \\ 13c_3 &= 52 \end{aligned}\right\} \begin{aligned} c_3 &= 4 \\ c_2 &= 2 \\ c_1 &= -3 \end{aligned}$$

$$\underset{\sim}{x} = -3e_1 + 2e_2 + 4e_3$$

2. Determine wheter the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

are linearly Independent.

Soln.

set

$$a_1 A_1 + a_2 A_2 + a_3 A_3 = 0$$

$$a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_1 + \quad + a_3 &= 0 \\ a_1 \quad &= 0 \\ a_1 + a_2 \quad &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 = 0 \quad a_2 = 0 \\ a_3 = 0 \end{aligned}$$

Hence the matrixes are linearly independent.

Subspace

Let W be a non-empty subset of a vector space V. Then W is a subspace of V provided that W itself is a vector space with respect to the operations of addition and multiplication by scalar defined in V.

Theorem.

The non-empty subset W of a vector space V is a subspace of V if and only if it satisfies the following axioms.

(i) $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors in W, the $\underset{\sim}{u} + \underset{\sim}{v}$ is also in W.

(ii) If $\underset{\sim}{u}$ is a vector in W and k is a scalar, then the vector $k\underset{\sim}{u}$ is in W.

Corollary

W is a subspace of V if and only if

1. $\underset{\sim}{0} \in W$ (or $W \neq \phi$), and (not an empty set)
2. $\underset{\sim}{u}, \underset{\sim}{v} \in W$ implies $k\underset{\sim}{u} + k'\underset{\sim}{v} \in W$ for every scalars k and k'.

Examples

1. Show that $W = \{(x, y) \mid x = 0\}$ is a subspace $\mathbb{R}^2$

Soln.

W is the y-axis

(i) Let $\underset{\sim}{u} = (x_1, y_1)$, $\underset{\sim}{v} = (x_2, y_2) \in W$

then $x_1 = x_2 = 0$

Now

$$\underset{\sim}{u} + \underset{\sim}{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $x_1 + x_2 = 0 + 0 = 0$

Hence $\underset{\sim}{u} + \underset{\sim}{v} \in W$

(ii) If k is a scalar, then

$$k\underset{\sim}{u} = k(x_1, y_1) = (kx_1, ky_1)$$

and

$$kx_1 = k(0) = 0$$

Thus $k\underset{\sim}{u} \in W$

Since the two axioms are satisfied then W is a subspace of $V = \mathbb{R}^2$

2. Show that $W = \{(x, y) \mid x = 2y\}$ is a subspace of $\mathbb{R}^2$

Soln.

W is a line in $\mathbb{R}^2$ its passes through the origin

(i) Let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in W$,

then $x_1 = 2y_1$ and $x_2 = 2y_2$

Now;

$\underset{\sim}{u} + \underset{\sim}{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

and

$x_1 + x_2 = 2y_1 + 2y_2 = 2(y_1 + y_2)$

Hence $u + v \in W$

(ii) If k is a scalar, then

$k\underset{\sim}{u} = k(x_1, y_1) = (kx_1, ky_1)$

and

$kx_1 = k(2y_1) = 2(ky_1)$

Thus $k\underset{\sim}{u} \in W$, Hence W is a subspace of $\mathbb{R}^2$.

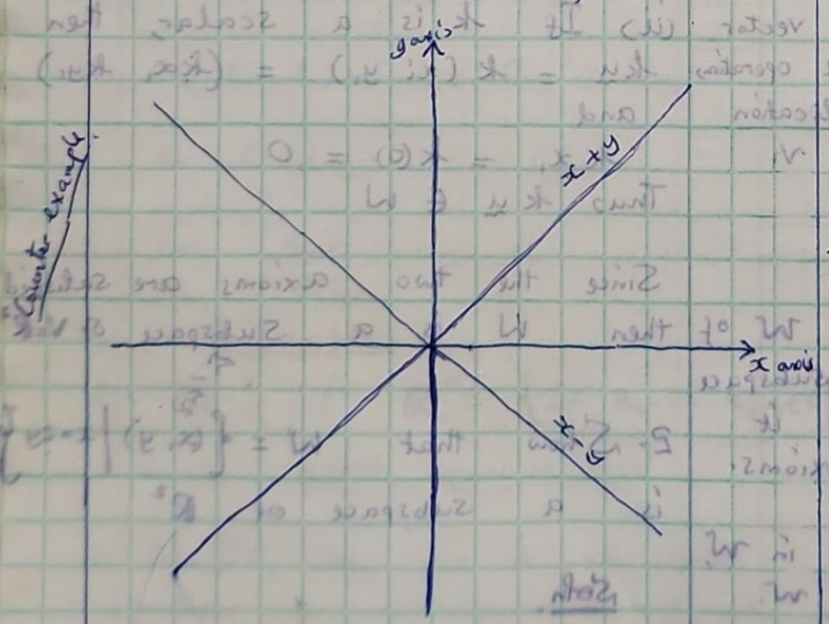
3. Show that $W = \{(x, y) \mid x^2 = y^2\}$ is <u>not</u> a subspace of $\mathbb{R}^2$

<u>Soln</u>

$x^2 = y^2 \implies x = \pm y$

i.e $x = y$ or $x = -y$

W is a union of 2 lines



$\underset{\sim}{u} = (1, 1)$, $\underset{\sim}{v} = (-1, 1) \in W$

But

$\underset{\sim}{u} + \underset{\sim}{v} = (1, 1) + (-1, 1) = (0, 2) \notin W$

Since $0^2 \neq 2^2$. Hence W is not a subspace of $\mathbb{R}^2$.

<u>Remark</u>

(i) Subspaces of $\mathbb{R}^2$: $\mathbb{R}^3$, $\{0\}$, are lines through the Origin.

(ii) To show that W is a subspace, Prove in general and to show that W is not a <u>subspace</u> you give a <u>Counter example</u>.

4. Determine wheter $W = \{(x, y) \mid y = 0\}$ is a subspace of $\mathbb{R}$?

<u>Soln</u>

$\underset{\sim}{u} = (1, 1)$ and $\underset{\sim}{v} = (-1, 1)$

$\underset{\sim}{u} + \underset{\sim}{v} = (1, 1) + (-1, 1) = (0, 2) \in W$

Since $2 \neq 0^2$

Hence W is <u>not</u> a subspace of $\mathbb{R}^2$.

5. Show that $W = \{(x, y, z) \mid x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$.

<u>Soln</u>

W is a plane through the origin

(i) Let $\underset{\sim}{u} = (x, y, z)$, $v_2 = (x_2, y_2, z_2) \in W$,

then

$x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$

Now;

$\underset{\sim}{u} + \underset{\sim}{v} = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

and

$\{(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2)\} = 0$

$(x_1+y_1+z_1)+(x_2+y_2+z_2) = 0+0=0$

$\Rightarrow \underset{\sim}{u}+\underset{\sim}{v} \in W$

(ii) If k is a scalar, then $k\underset{\sim}{u}$ $= k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$ and

$kx_1 + ky_1 + kz_1 = k(x_1+y_1+z_1) = k(0)$
$= 0$

$\Rightarrow k\underset{\sim}{u} \in W$

Hence W is a subspace of $\mathbb{R}^3$

Remarks

Subspaces of $\mathbb{R}^3$: $\{0\}$, $\mathbb{R}^3$, Planes through the Origin, lines through the Origin.

6. Show that $W = \{(x,y,z) \mid x+y+z=0$ and $x-y=1\}$ is not subspace of $\mathbb{R}^3$

Soln

let $\underset{\sim}{u} = (1,0,-1)$ & $\underset{\sim}{v} = (0,-1,1)$

But $\underset{\sim}{u}+\underset{\sim}{v} = (1,0,-1)+(0,-1,1) = (1,-1,0) \notin W$

$1-1+0=0$

Since $x-y=1$ $= 1-(-1) = 2 \neq 1$

Hence W is not a subspace of $\mathbb{R}^3$.

7. Determine wheter $W = \{(a,b,c) \mid ab=0\}$ is a subspace of $\mathbb{R}^3$.

Soln

We use a Counter example

$\underset{\sim}{u} = (0,1,2)$ $\underset{\sim}{v} = (1,0,6) \in W$

But

$\underset{\sim}{u}+\underset{\sim}{v} = (0,1,2)+(1,0,6) = (1,1,8) \notin W$

Since $|x| = 1 \neq 0$

Hence W is not a subspace of $\mathbb{R}^3$

Exercise

1. Is $W = \{(a,b,c) \mid a \geq 1\}$ a subspace of $\mathbb{R}^3$

2. Let $\underset{\sim}{a} \in \mathbb{R}^n$ and let $U = \{\underset{\sim}{u} \in \mathbb{R}^n \mid \underset{\sim}{a}\cdot\underset{\sim}{u} = 0\}$ show that U is a subspace of $\mathbb{R}^n$

3. Show wheter or not the following subspaces of $\mathbb{R}^3$.

(a) $U = \{(x,y,z) \mid x+y+z=1\}$

(b) $W = \{(x,y,z) \mid x-2y+z=3\}$

*4. Suppose U and V are subspaces of $\mathbb{R}^n$ show that $U \cap V$ is also a subspace of $\mathbb{R}^n$.

$U \cap V = \{\underset{\sim}{a} \in \mathbb{R}^n \mid \underset{\sim}{a} \in U$ and $\underset{\sim}{a} \in V\}$

Theorem:

Let $u_1, u_2, \ldots, u_m \in R_n$ and W be set of all linear combination of $u_1, u_2, \ldots, u_m$. Then W is a subspace of $\mathbb{R}^n$

Proof

(I) Let $x, y \in W$, then

$$x = k_1 u_1 + k_2 u_2 + \cdots + k_m u_m$$
$$y = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$$

Now,

$$x + y = (k_1 u_1 + k_2 u_2 + \cdots + k_m u_m) + (c_1 u_1 + c_2 u_2 + \cdots + c_m u_m)$$
$$= (k_1 + c_1) u_1 + (k_2 + c_2) u_2 + \cdots + (k_m + c_m) u_m \in W$$

(II) Let k' be a scalar, then

$$k' x = k' (k_1 u_1 + k_2 u_2 + \cdots + k_m u_m)$$
$$= k' k_1 u_1 + k' k_2 u_2 + \cdots + k' k_m u_m \in W$$

Hence W is a subspace of $\mathbb{R}$ $\boxtimes$ end of the proof.

Def: (Spanning Sets)

If $u_1, u_2, \ldots, u_m$ are vectors in a vector space V, then the set of all linear combinations of $u_1, u_2, \ldots, u_m$ form a subspace of V. This subspace is said to be spanned by $u_1, u_2, \ldots, u_m$. Now, the set $\{u_1, u_2, \ldots, u_m\}$ is called the spanning set of the subspace

Example

1. $\mathbb{R}^3$ is spanned by $\{(1,0,0), (0,1,0), (0,0,1)\}$. Since $(x, y, z) \in \mathbb{R}^3$ then $(x, y) \in \mathbb{R}^2$

Since $(x, y, z) \in \mathbb{R}^3$, then $(x, y) \in \mathbb{R}^2$

$$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

2. $\{(1,1), (2,1)\}$ span $\mathbb{R}^2$, since

Let $(x, y) \in \mathbb{R}^2$

$$(x, y) = a(1,1) + b(2,1)$$
$$\Rightarrow \begin{cases} a + 2b = x \\ a + b = y \end{cases} \quad \begin{cases} b = x - y \\ a = y - (x-y) = 2y - x \end{cases}$$

$$(x, y) = (2y - x)(1,1) + (x - y)(2,1)$$

Hence $\{(1,1), 2, 1\}$ spans $\mathbb{R}^2$

*3. Show that $\{(1,1,1), (0,1,1), (0,1,-1)\}$ span $\mathbb{R}^3$

soln.

Let $(x, y, z) \in \mathbb{R}^3$, set

$$(x, y, z) = a(1,1,1) + b(0,1,1) + c(0,1,-1)$$
$$\Rightarrow \begin{aligned} a &= x \\ a + b + c &= y \\ a + b - c &= z \end{aligned} \quad \text{or} \quad \begin{aligned} x + b + c &= y \\ x + b - c &= z \\ a &= x \end{aligned}$$

$$\begin{aligned} a + b + c &= y \\ -2c &= z - y \\ a &= x \end{aligned} \quad \begin{cases} a = x \\ c = \frac{y-z}{2} \\ b = \frac{-2x+y+z}{2} \end{cases}$$

$$(x, y, z) = x(1,1,1) + \frac{-2x+y+z}{2}(0,1,1) + \frac{y-z}{2}(0,1,-1)$$

Hence $\{(1,1,1), (0,1,1), (0,1,-1)\}$ span $\mathbb{R}^3$

Is $\{(1,1)\}$ a spanning set of $\mathbb{R}^2$?

$\{(1,1)\}$ does not span $\mathbb{R}^2$ since for example $(1,3) \in \mathbb{R}^2$ but $(1,3) = a(1,1)$ has no soln

Dfn (Basis)

*Let B be a finite subset of a vector space V. Then B is said to be a basis of V if

(i) B spans V

(ii) B is a linearly independent

Examples

1. $\{(1,0), (0,1)\}$ is a basis for $\mathbb{R}^2$ since if $(x, y) \in \mathbb{R}^2$ then $(x, y) = x(1,0) + y(0,1)$, thus $\{(1,0),(0,1)\}$ Spans $\mathbb{R}^2$

Also

$a(1,0) + b(0,1) = \underline{0} = (0,0)$

$\Rightarrow a = 0, \quad b = 0$

Hence $\{(1,0), (0,1)\}$ is linearly independent

Hence $\{(1,0), (0,1)\}$ is a basis for $\mathbb{R}^2$

2. Already we have shown that $\{(1,1), (2,1)\}$ span $\mathbb{R}^2$. It can be checked that $\{(1,1), (2,1)\}$ is a linearly independent set. Hence $\{(1,1), (2,1)\}$ is a basis for $\mathbb{R}^2$.

3. $\{(1,0,0) (0,1,0) (0,0,1)\}$ is a basis for $\mathbb{R}^3$.

$\{(1,0,0) (0,1,0) (0,0,1), (1,1,1)\}$ Spans $\mathbb{R}^3$ Since if $(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1) \boxed{+ 0(1,1,1)}$

But $\{(1,0,0) (0,1,0), (0,0,1), (1,1,1)\}$ is <u>not</u> linearly independent since

$(1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

and hence $\{(1,0,0), (0,1,0) (0,0,1)(1,1,1)\}$ is not a basis of $\mathbb{R}^3$.

Remark.

A basis for $\mathbb{R}^n$ contains exactly n vectors.

Example

* Determine whether the vectors $\{(1,1,1) (1,2,3) (2,-1,1)\}$ form a basis for $\mathbb{R}^3$.

Soln

Since the vectors are in $\mathbb{R}^3$ check for linearly independence

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow{R_3 + 3R_2}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \checkmark\checkmark$$

The echelon matrix has no zero rows hence the three vectors are linearly independent and hence form a basis for $\mathbb{R}^3$

5. $\{1, 1\}$ is not a basis for $\mathbb{R}^2$ since $\{1, 1\}$ does not span $\mathbb{R}^2$.

DIMENSION:

Let V be a vector space and B be a basis for V, then dimension of V which is denoted by $\dim V$ is the number of <u>vectors</u> in B.

e.g $\dim(\mathbb{R}^2) = 2$, $\dim \mathbb{R}^3 = 3$

Theorem

Suppose B_1 and B_2 are both basis for a vector space V. Then B_1 and B_2 have the same number of vectors.

Therefore the dimension of a vector space is non-ambiguous. It does not depend on the basis that you choose.

Corollary

Dimension of $\mathbb{R}^n$ is n

Let $e_1 = (1, 0, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

$e_3 = (0, 0, 1, \ldots, 0)$

$e_n = (0, 0, 0, \ldots, 1)$

Where e_i has 1 in the ith position and zero elsewhere, then $\{e_i\}$ forms a basis of $\mathbb{R}^n$. This is called the standard basis for $\mathbb{R}^n$

Example

1. $\{(1,0), (0,1)\}$ standard basis for $\mathbb{R}^2$

2. $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a standard basis for $\mathbb{R}^4$

Example

Let W be a subspace of $\mathbb{R}^4$ generated by the vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$

(i) Find a basis and the dimension of W.

(ii) Extend this basis to a basis of the whole vector space $\mathbb{R}^4$.

Soln

We check for L.I vectors out of the given 3.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \xrightarrow{R_3-2R_2}$$

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows $(1,-2,5,-3)$ $(0,7,-9,2)$ of the echelon matrix form a basis for W

The $\dim W = 2$.

(ii) We seek four independent vectors which include the above two vectors. The vectors $(1,-2,5,-3)$, $(0,7,-9,2)$, $(0,0,1,0)$ and $(0,0,0,1)$ are linearly independent (since they form an echelon matrix) and so they form a basis of $\mathbb{R}^4$ which is an extension of the basis of W.

Alternative definition

The dimension of a vector space can be defined as the number of degrees of freedom when choosing a vector in a vector space. It is also the number of free or independent variables when choosing a vector in a vector space.

Example

1. $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

We have 3 independent in choosing a vector in $\mathbb{R}^3$ hence $\dim(\mathbb{R}^3) = 3$. A basis for $\mathbb{R}^3$ $\{(1,1,2), (0,2,7), (0,0,1)\}$

2. $V = \{(x, y, z) \mid z = x+y\}$

$\dim V = 2$, since there are 2 independent variables.

A basis for V is $\{(1,1,2), (0,1,1)\}$

3. $U = \{x, y, z \mid x = 2y, z = 3y\}$

$\dim U = 1$ since there is one independent solution.

A basis of U is $\{(2,1,3)\}$

4. Find the dimension & a basis for $W = \{x, y, z, w \mid x+y+z+w = 0\}$

Soln

y, z and w are the free variables

hence $\dim W = 3$

Set (i) $y=1$, $z=0$, $w=0$

(ii) $y=0$, $z=1$, $w=0$

(iii) $y=0$, $z=0$, $w=1$

To obtain the respective basis

$\{(-1,1,0,0), (-1,0,1,0) (-1,0,0,1)\}$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{r} -1\ 0\ 1\ 0 \\ -1\ 1\ 0\ 0 \\ \hline 0\ -1\ 1\ 0 \end{array} \qquad \begin{array}{r} -1\ 0\ 0\ 1 \\ -1\ 1\ 0\ 0 \\ \hline 0\ -1\ 0\ 1 \\ 0\ -1\ 1\ 0 \\ \hline 0\ 0\ -1\ 1 \end{array}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Hence the vectors are L.I

5. Find the dimension and a basis for.

$W = \{a, b, c, d \mid b - 2c + d = 0\}$

Soln

$b - 2c + d = 0$ or $a \cdot 0 + b - 2c + d = 0$

We have 3 independent variables

hence $\dim W = 3$

Set (i) $a=1$, $c=0$, $d=0$

(ii) $a=0$, $c=1$, $d=0$

(iii) $a=0$, $c=0$, $d=1$

To obtain the respective basis

$\{(1,0,0,0) (0,2,1,0) (0,-1,0,1)\}$

6. Find the dimension and a basis for,

$U = \{(x,y,z,w,t) \mid x+y+z+w+t=0$

$x-y+z-w+t=0\}$

Soln

We first solve the two equations simultaneously

$$\begin{array}{l} x+y+z+w+t=0 \\ x-y+z-w+t=0 \end{array} \quad R_2-R_1$$

$$\begin{array}{l} x+y+z+w+t=0 \\ -2y \quad -w \quad = 0 \end{array}$$

There are 3 independent variables z, w and t the $\dim U = 3$

Set (i) $z=1$, $w=0$, $t=0$

(ii) $z=0$, $w=1$, $t=0$

(iii) $z=0$, $w=0$, $t=1$

To obtain the respective vectors

$\{(-1,0,1,0,0), (0,-1,0,1,0), (-1,0,0,0,1)\}$

DFN

Let U and V be subspaces of $\mathbb{R}^n$

Then $U+V = \{\underset{\sim}{u} + \underset{\sim}{v} \mid \underset{\sim}{u} \in U, \underset{\sim}{v} \in V\}$

this is subspace of $\mathbb{R}^n$ and is called the direct sum of $\underset{\sim}{U}$ and $\underset{\sim}{V}$.

Theorem $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$.

Example

Let $U = \{(x,y,z,w) \mid x+y+z+w=0$

$x-y+z-w=0\}$ and

$V = \{(x,y,z,w) \mid x-y-z-w=0\}$

Find the dimension and a basis of (i) U (ii) V (iii) $U+V$

Soln

$$\begin{array}{l} x+y+z+w=0 \\ x-y+z-w=0 \end{array} \quad R_2-R_1 \quad \text{or} \quad \begin{array}{l} x+y+z+w=0 \\ -2y \quad -2w=0 \end{array}$$

or $x+y+z+w=0$

$y \quad +w=0$

There are two independent variables z and w hence $\dim U = 2$

Set (i) $z=1$ $w=0$

(ii) $z=0$ $w=1$

$y=0, \quad w=0, \quad x=-1$
$w=1, \quad y=1 \quad x=$

To obtain the respective basis

$\{(-1,0,1,0), (0,-1,0,1)\}$

(ii) $V = \{(x,y,z,w) \mid x-y-w=0\}$

There are 3 free variables, y, z and w. The dim $V=3$

Set (i) $y=1, \quad z=0, \quad w=0 \quad \Big| \quad x=1$

(ii) $y=0, \quad z=1, \quad w=0$

(iii) $y=0, \quad z=0, \quad w=1$

We obtain the respective basis

$\{(1,1,0,0) \ (1,0,1,0), (1,0,0,1)\}$

(iii) $U \cap V$ Comprise of all vector (x,y,z,w) satisfy the conditions defining V and U. We solve

$$\begin{aligned} x+y+z+w &= 0 \\ x-y+z-w &= 0 \\ x-y-z-w &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x+y+z+w &= 0 \\ -2y \quad -2w &= 0 \\ -2y-2z-2w &= 0 \end{aligned}$$

$$\text{or} \quad \begin{aligned} x+y+z+w &= 0 \\ -2y \quad -2w &= 0 \\ -2z \quad &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x+y+z+w &= 0 \\ y \quad +w &= 0 \\ z \quad &= 0 \end{aligned}$$

The system has Independent variable, hence $\dim(U \cap V) = 1$. A basis for this subspace is:

$\{(0,-1,0,1)\}$

(iv) $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

$= 2+3-1 = 4$

A basis for $U+V$ is $\{(-1,0,1,0)$ $(0,-1,0,1)$ $(1,1,0,0)$, $(1,0,1,0)\}$ ✓

Exercise:

Let U and W be following subspaces of $\mathbb{R}^4$

$U = \{(a,b,c,d) \mid b+c+d=0\}$

$W = \{(a,b,c,d) \mid a+b=0, \ c=2d\}$

Find the dimension and a basis for

(a) U
(b) W
(c) $U \cap W$
(d) $U+W$