CSC420: Assignment #3

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"In solving the questions in this assignment, I worked alone. I confirm that I have written the solutions / code / report in my own words.

Part 1: Theoretical Problem

Q1:

[Question 1] Laplacian of Gaussian (25 marks)

The Laplacian of Gaussian operator is defined as:

$$\nabla^2 G(x,y,\sigma) = \frac{\partial^2 G(x,y,\sigma)}{\partial x^2} + \frac{\partial^2 G(x,y,\sigma)}{\partial y^2} = \frac{1}{\pi\sigma^4}\left(\frac{x^2+y^2}{2\sigma^2} - 1\right)e^{-\frac{x^2+y^2}{2\sigma^2}},$$

where the Gaussian filter G is:

$$G(x,y,\sigma) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$$

The characteristic scale is defined as the scale that produces the peak value (minimum or maximum) of the Laplacian response.

1. **(10 marks)** What scale (i.e. what value of σ) maximises the magnitude of the response of the Laplacian filter to an image of a black circle with diameter D on a white background? Justify your answer.

[1] To maximize magnitude of response of the Laplace filter to a binary circle w/ diameter D, the zeros of the Laplacian have to be aligned with the circle

$$\left(\frac{x^2+y^2}{2\sigma^2} - 1\right) = 0$$

$$\Leftrightarrow \left(\frac{r^2}{2\sigma^2} - 1\right) = 0$$

$$\Leftrightarrow \frac{r^2}{2\sigma^2} = 1 \Leftrightarrow \sigma^2 = \frac{r^2}{2} \Leftrightarrow \sigma = \frac{r}{\sqrt{2}}$$

$$\Leftrightarrow \sigma = \frac{D}{2\sqrt{2}}$$

2. **(5 marks)** What scale should we use if we want to instead detect a white circle of the same size on a black background?

$$\left(\frac{x^2+y^2}{2\sigma^2} - 1\right) = 1$$

$$\frac{r^2}{2\sigma^2} = 2 \Leftrightarrow 4\sigma^2 = r^2 \Leftrightarrow \sigma = \frac{r}{2}$$

$$\overset{r=\frac{D}{2}}{\Longleftrightarrow} \sigma = \frac{D}{4}$$

[3] on Implementation section

Q2:

[Question 2] **Corner Detection (25 marks)**

For corner detection, we defined the Second Moment Matrix as follows:

$$M = \sum_x \sum_y w(x,y) \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

Let's denote the 2×2 matrix used in the equation by N; i.e.:

$$N = \begin{pmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{pmatrix}$$

1. **(10 marks)** Compute the eigenvalues of N denoted by λ_1 and λ_2?

I $N = \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}$ $\quad Nx = \lambda x$

$$\det(N - I\lambda) = 0$$

$$\det\begin{pmatrix} I_x^2-\lambda & I_xI_y \\ I_xI_y & I_y^2-\lambda \end{pmatrix} = 0$$

$$(I_x^2-\lambda)(I_y^2-\lambda) - (I_xI_y)^2 = 0$$

$$I_x^2I_y^2 - I_x^2\lambda - I_y^2\lambda + \lambda^2 - (I_xI_y)^2 = 0$$

$$\lambda^2 - (I_x^2+I_y^2)\lambda + I_x^2I_y^2 - (I_xI_y)^2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda_\pm = \frac{(I_x^2+I_y^2) \pm \sqrt{(I_x^2+I_y^2)^2 - 4(I_x^2I_y^2 - (I_xI_y)^2)}}{2}$$

$$= \frac{(I_x^2+I_y^2) \pm \sqrt{I_x^4+I_y^4+2I_x^2I_y^2 - 4I_x^2I_y^2 - 4(I_xI_y)^2}}{2}$$

$$\boxed{\lambda_\pm = \frac{(I_x^2+I_y^2) \pm \sqrt{(I_x^2-I_y^2)^2 - 4(I_xI_y)^2}}{2}}$$

2. **(15 marks)** Prove that matrix M is positive semi-definite.

matrix $N = \begin{pmatrix} I_x^2 & I_{xy} \\ I_{xy} & I_y^2 \end{pmatrix}$ is symmetric matrix since $N = N^T$ then has only real eigenvalues

→ Considering (λ, v) as eigenpair of N

$$\langle Nv, Nv\rangle = v^TN^TNv = v^TN^2v$$
$$= \lambda^2 \|v\|_2^2$$

$\therefore \lambda^2 = \dfrac{\langle Nv, Nv\rangle}{\|v\|^2}$ is real non-negative #,

Hence λ must be real.

Given N symmetric matrix w/ non-negative eigenvaluse, we can diagonalize matrix as $Q^T\Lambda Q$ and matrix is PSD (positive-semi-definite) by definition.

$$M = \underbrace{\sum_x\sum_y w(x,y)}\cdot \underbrace{\sum_x\sum_y \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}}_{\text{matrix } A}$$

window fn is scaling of matrix A

and sum of PSD considering we have 2 PSDs with A & B where $x^TAx \geq 0$ $x^TBx \geq 0$ for all non-zero x. then notice that

$x^T(A+B)x = x^TAx + x^TBx$ &

$x^TAx + x^TBx \geq 0$

implies $x^T(A+B)x \geq 0$ thus our matrix $A = \sum_x\sum_y \begin{pmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{pmatrix}$ is also PSD

thus matrix M is PSD.