CSC420: Assignment #4

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"In solving the questions in this assignment, I worked alone. I confirm that I have written the solutions / code / report in my own words."

Part 1: Theoretical Problems

Part I: Theoretical Problems (75 marks)

[Question 1] Camera Models (25 marks)

Assume a plane passing through point $\vec{P}_0 = [X_0,\ Y_0,\ Z_0]^T$ with normal $\vec{n}$. The corresponding vanishing points for all the lines lying on this plane form a line, called the horizon. In this question, you are asked to prove the existence of the horizon line by following the steps below:

1. **(15 marks)** Find the pixel coordinates of the vanishing point corresponding to a line L, passing point $\vec{P}_0$ and going along direction $\vec{d}$.

   Hint: $\vec{P} = \vec{P}_0 + t\,\vec{d}$ are the points on line L, and $\vec{p} = \begin{pmatrix} \omega x \\ \omega y \\ \omega \end{pmatrix} = K\,\vec{P} = K \begin{pmatrix} X_0 + t\,d_x \\ Y_0 + t\,d_y \\ Z_0 + t\,d_z \end{pmatrix}$ are pixel coordinates of the same line in the image, and $K = \begin{pmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{pmatrix}$, where f is the camera focal length and (p_x, p_y) is the principal point.

$\square$

$$\vec{p} = \begin{pmatrix} \omega x \\ \omega y \\ \omega \end{pmatrix} = K\vec{P}$$

$$\vec{p} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 + td_x \\ Y_0 + td_y \\ Z_0 + td_z \end{bmatrix}$$

$$\begin{bmatrix} \omega x \\ \omega y \\ \omega \end{bmatrix} = \begin{bmatrix} fX_0 + ftd_x + p_x Z_0 + t p_x d_z \\ fY_0 + ftd_y + p_y Z_0 + t p_y d_z \\ Z_0 + t d_z \end{bmatrix}$$

$$\frac{\omega x}{\omega} = x = \frac{fX_0 + ftd_x + p_x Z_0 + t p_x d_z}{Z_0 + td_z}$$

$$\frac{\omega y}{\omega} = y = \frac{fY_0 + ftd_y + p_y Z_0 + t p_y d_z}{Z_0 + td_z}$$

moving infinitely far from camera

$$x' = \lim_{t \to \infty} x = \frac{fd_x + p_x d_z}{d_z}$$

$$y' = \lim_{t \to \infty} y = \frac{fd_y + p_y d_z}{d_z}$$

2. **(10 marks)** Prove the vanishing points of all the lines lying on the plane form a line.

   Hint: all the lines on the plane are perpendicular to the plane's normal $\vec{n}$; that is, $\vec{n}\,.\,\vec{d} = 0$, or $n_x\,d_x + n_y\,d_y + n_z\,d_z = 0$

Since vanishing points of all the lines lying on the plane we can simplify $d_z = 0$, $Z_0 = 1$

$$x = \frac{fX_0 + ftd_x + p_x Z_0 + t p_x d_z}{Z_0 + td_z} = fX_0 + ftd_x + p_x$$

$$y = \frac{fY_0 + ftd_y + p_y Z_0 + t p_y d_z}{Z_0 + td_z} = fY_0 + ftd_y + p_y$$

We can then find the value of $D = (d_x\ d_y\ d_z)^T$

$$D = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} \frac{x - fX_0 - p_x}{ft} \\ \frac{y - fY_0 - p_y}{ft} \\ 0 \end{pmatrix}$$

notice x is vanishing point, fX_0 is scaled version of X_0 and p_x is principal point that are all on the same plane normal to $\vec{n}$

thus $d_x \cdot \vec{n} = 0$ similarly $d_y \cdot \vec{n} = 0$

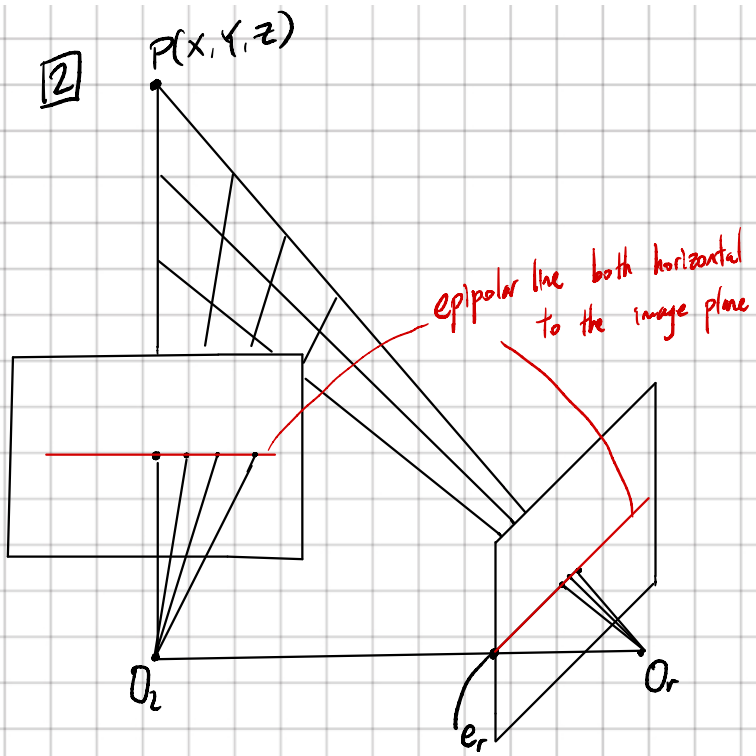
$\therefore D \cdot \vec{n} = 0$

therefore vanishing points of all the line lying on the plane form a line.

[Question 2] Epipolar Geometry (20 marks)

For a pair of rectified stereo cameras (i.e. two identical parallel cameras, with a fixed displacement perpendicular to its optical axis) the epipolar lines on each image plane form a set of parallel lines, with the epipole at infinity. Now, let's rotate the right camera for 45 degrees toward the left camera, as you see in the diagram below. For this stereo camera setup, show the epipolar lines and the epipole for each of the images planes. Make sure to include your reasoning that justifies your answer.

$\boxed{2}$



Since epipoles are where $O_l O_r$ intersects the image plane only epipole of the image plane that is rotated 45° has epipole e_r at the edge. The image before being rotated has been rectified thus horizontal rotation of the right image plane won't effect the epipolar lines that was rectified. therefore epipolar line are shown as graph above.

[Question 3] Homogeneous Coordinates (30 marks)

Using the homogeneous coordinates:

1. **(15 marks)** (a) Show that the intersection of the 2D line l and l' is the 2D point $p = l \times l'$.
2. **(15 marks)** (b) Show that the line that goes through the 2D points p and p' is $l = p \times p'$.

$\boxed{3}$

1) line in 2D plane : $ax + by + c = 0$ (*)

if $l = (a, b, c)^T$ & $x = (x, y, 1)^T$

(*) can be written as $l^T x = 0$

where x is homogeneous representation of $(x, y)^T$

point must lie on both l & l'

Let this point be x.

then $lx = 0$

$l'x = 0$

Since $l' \times l$ is vector orthogonal to both l' & l

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$$l(l' \times l) = l(-l \times l')$$
$$= (l \times -l) \cdot l'$$
$$= 0 \cdot l'$$
$$= 0$$

$\downarrow \vec{a}(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

similarly $l'x = l(l' \times l) = 0$

thus $l \times l'$ gives p which is the point of intersection between l & l'

alternatively ①

assume $l = [a_1 \ b_1 \ c_1]^T$

$l' = [a_2 \ b_2 \ c_2]^T$

$lp = 0$ & $l'p = 0$

can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad y = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$p = l \times l'$$

alternatively ②

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0 \end{cases}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{b_1 a_2 - a_1 b_2} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$x = \frac{b_2 c_1 - b_1 c_2}{b_1 a_2 - a_1 b_2} \quad y = \frac{a_1 c_2 - a_2 c_1}{b_1 a_2 - a_1 b_2} \quad (*)$$

$$l \times l' = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1 c_2 - b_2 c_1) i + (a_2 c_1 - a_1 c_2) j + (a_1 b_2 - a_2 b_1) k$$

$$x = \begin{bmatrix} b_1 c_1 - b_2 c_1 \\ a_2 c_1 - a_1 c_2 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} w x \\ w y \\ w \end{bmatrix}$$

$$x = \frac{wx}{w} = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}$$

$$y = \frac{wy}{w} = \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \quad (**)$$

notice (*) & (**) are same.

2) Using similar logic as 3-1

the equation $p^T l = 0$ & $p'^T l = 0$

where $p = [x_1 \ y_1 \ 1]^T$ $p' = [x_2 \ y_2 \ 1]^T$

$l = [a, b, c]^T$

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

it can be seen

$$a = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} \quad b = -\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} \quad c = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

& therefore the cross-product defines the line l i.e.

$$l = p \times p'$$

alternatively ①

$$\left.\begin{aligned} &\text{Let } m = \frac{y_2 - y_1}{x_2 - x_1} \quad g = y_1 = m x_1 \\ &\text{then } g = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \end{aligned}\right\} (*)$$

$$x \times x' = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 x_1 & w_1 y_1 & w_1 \\ w_2 x_2 & w_2 y_2 & w_2 \end{bmatrix} = \begin{matrix} w_1 w_2 (y_1 - y_2)\hat{i} \\ w_1 w_2 (x_2 - x_1)\hat{j} \\ w_1 w_2 (x_1 y_2 - x_2 y_1)\hat{k} \end{matrix}$$

$$l = w_1 w_2 \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = \begin{bmatrix} wm \\ -w \\ wg \end{bmatrix}$$

$$\left.\begin{aligned} m &= \frac{-wm}{-w} = \frac{y_2 - y_1}{x_2 - x_1} \\ g &= \frac{-wg}{-w} = \frac{x_2 y_2 - x_1 y_2}{x_2 - x_1} \end{aligned}\right\} (**)$$

notice $(*) = (**)$