CSC420 : Assignment #1

Won Jin Kim
(1003611424)

Part I : Theoretical Problems

Q1 LTI system.

$T[x(n)] = h(n) * x(n)$

Any DT signal can be written as $\sum_k a_k \delta[n-k]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$\delta[n] \xrightarrow{T} h(n)$$

$$\delta[n-k] \xrightarrow{T} h(n-k) \quad \text{time invariant}$$

$$\alpha\delta[n-k] \xrightarrow{T} \alpha h(n-k) \quad \text{linear property}$$

Let $\alpha = x(k)$

$$x(k)\,\delta[n-k] \xrightarrow{T} x(k)\,h(n-k)$$

$$\underbrace{\sum_{k=-\infty}^{\infty} x(k)\,\delta[n-k]} \xrightarrow{T} \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$x[n] \xrightarrow{T} \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

in other word $T[x[n]] = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

$= x[n] * h[n]$

$= h[n] * x[n]$ (law of Commutative)

Q2 Polynomial Multiplication & Convolution

Consider 3rd-degree polynomial $(a_3x^3 + a_2x^2 + a_1x + a_0)$

this in vector form $\rightarrow [a_3\ a_2\ a_1\ a_0]$

let's create one other polynomial $(b_3x^3 + b_2x^2 + b_1x + b_0)$

w/ its vector form $[b_3, b_2\ b_1\ b_0]$

Using table we can show multiplication of polynomial

	a_0	a_1x	a_2x^2	a_3x^3
b_0	a_0b_0	a_1b_0x	$a_2b_0x^2$	$a_3b_0x^3$
b_1x	a_0b_1x	$a_1b_1x^2$	$a_2b_1x^3$	$a_3b_1x^4$
b_2x^2	$a_0b_2x^2$	$a_1b_2x^3$	$a_2b_2x^4$	$a_3b_2x^5$
b_3x^3	$a_0b_3x^3$	$a_1b_3x^4$	$a_2b_3x^5$	$a_3b_3x^6$

the result of multiplication between two polynomials could be obtained by adding all the terms in the table.

the color above is grouped by the power of argument x.

if we assume c to be a resulting coefficient then

$$
\begin{aligned}
&c_0 && a_0b_0 \\
&+ c_1x && + (a_0b_1 + a_1b_0)x \\
&+ c_2x^2 && + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \\
&+ c_3x^3 \quad = && + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 \\
&+ c_4x^4 && + (a_1b_3 + a_2b_2 + a_3b_1)x^4 \\
&+ c_5x^5 && + (a_2b_3 + a_3b_2)x^5 \\
&+ c_6x^6 && + (a_3b_3)x^6
\end{aligned}
$$

$\longrightarrow$

writing this in matrix form you get

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_0 & 0 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ b_2 & b_1 & b_0 & 0 \\ b_3 & b_2 & b_1 & b_0 \\ 0 & b_3 & b_2 & b_1 \\ 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We could generalize the observation above to polynomial multiplication w/ other degrees

$$C_i = \sum_j a(j)\, b(i-j+1) = a_i * b_i$$

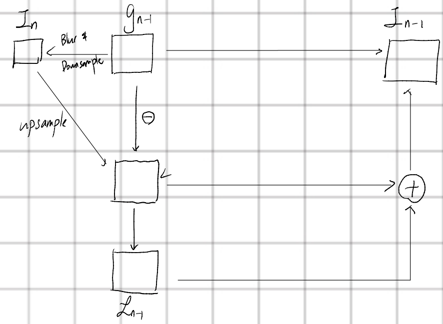
$\therefore$ convolving two vectors of polynomial coefficient is equal to multiplying the two polynomials they each represents.

$\boxed{Q3}$ Image Pyramids

We should be able to perform at least one gaussian pyramid operation in order to reconstruct image

thus Base case: for I_0 w/ size $2^n \times 2^n$, g_{n-1} is base case, where image has size of 2×2

let $F(\cdot)$ be upsampling function



Above diagram could be expressed as

$I_{n-1} = L_{n-1} + F(I_n)$ (*)

Given all the L_i, I_0 is minimum information to get I_0

Continuing w/ equation (*)

$$\begin{aligned} I_{n-2} &= L_{n-2} + F(I_{n-1}) \\ &= L_{n-2} + F(L_{n-1} + F(I_n)) \\ &= L_{n-2} + F(L_{n-1}) + F^2(I_n) \end{aligned}$$

$$\begin{aligned} I_{n-3} &= L_{n-3} + F(I_{n-2}) \\ &= L_{n-3} + F(L_{n-2} + F(L_{n-1}) + F^2(I_n)) \\ &= L_{n-3} + F(L_{n-2}) + F^2(L_{n-1}) + F^3(I_n) \end{aligned}$$

$\therefore I_{n-k} = L_{n-k} + \sum_{i=1}^{k} F^i(L_{n-k+i})$ where base case: $k=1$

at $k=0$ $\rightarrow$ $I_{n-1} = L_{n-1} + F(L_n)$ ✓

assume $k=k$ is true $I_{n-k} = L_{n-k} + \sum_{i=1}^{k} F^i(L_{n-k+i})$ (**)

when $k=k+1$

$$I_{n-k-1} = L_{n-k-1} + \sum_{i=1}^{k+1} F^i(L_{n-k-1+i})$$

$$I_{n-k-1} = L_{n-k-1} + F(L_{n-k}) + \sum_{i=2}^{k+1} F^i(L_{n-k-1+i})$$

from equ (**) $\rightarrow$ $L_{n-k} = I_{n-k} - \sum_{i=1}^{k} F^i(L_{n-k+i})$

$$I_{n-k-1} = L_{n-k-1} + F(I_{n-k}) - F\left(\sum_{i=1}^{k} F^i(L_{n-k+i})\right) \underbrace{+ \sum_{i=2}^{k+1} F^i(L_{n-k-1+i})}_{= \sum_{i=1}^{k} F^{i+1}(L_{n-k+1-1+i})}$$

$I_{n-k-1} = L_{n-k-1} + F(I_{n-k})$ ✓

$\longrightarrow$

Thus by mathmatical Induction we have shown that

$I_{n-k} = \mathcal{L}_{n-k} + \sum_{i=1}^{k} F^i(\mathcal{L}_{n-k+i})$ is true for $1 \leq k \leq n$

when $k=n \rightarrow I_0 = \mathcal{L}_0 + \sum_{i=1}^{n} F^i(\mathcal{L}_i)$ where $\mathcal{L}_n = I_n$

$\blacksquare$

$\boxed{Q4}$ Laplacian Operator

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta I = I_{xx} + I_{yy}$$

In $\mathbb{R}^2$, matrix rotation that rotates vector v_0 in the counter clock-wise angle θ is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{so } v' = R_\theta v_0$$

setting $v_0 = \begin{bmatrix} x \\ y \end{bmatrix}$ we could obtain two expressions for coordinates in polar notations

$$\begin{cases} r = x\cos\phi - y\sin\phi \\ r' = x\sin\phi + y\cos\phi \end{cases} \qquad I(r, r')$$

Now using chain rule,

$$f = \frac{\partial I(x,y)}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial I}{\partial r} + \frac{\partial r'}{\partial x}\frac{\partial I}{\partial r'}$$

$$= \cos\phi \frac{\partial I}{\partial r} + \sin\phi \frac{\partial I}{\partial r'}$$

$$\frac{\partial^2 I}{\partial x^2} = \frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial r'}{\partial x}\frac{\partial f}{\partial r}$$

$$= \cos\phi \frac{\partial f}{\partial r} + \sin\phi \frac{\partial f}{\partial r'}$$

$$= \cos\phi \frac{\partial}{\partial r}\left(\cos\phi \frac{\partial I}{\partial r} + \sin\phi \frac{\partial I}{\partial r'}\right)$$

$$+ \sin\phi \frac{\partial}{\partial r'}\left(\cos\phi \frac{\partial I}{\partial r} + \sin\phi \frac{\partial I}{\partial r'}\right)$$

$$= \cos^2\phi \frac{\partial^2 I}{\partial r^2} + 2\sin\phi\cos\phi \frac{\partial^2 I}{\partial r \partial r'} + \sin^2\phi \frac{\partial^2 I}{\partial r'^2}$$

$$g = \frac{\partial I(x,y)}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial I}{\partial r} + \frac{\partial r'}{\partial y}\frac{\partial I}{\partial r'}$$

$$= -\sin\phi \frac{\partial I}{\partial r} + \cos\phi \frac{\partial I}{\partial r'}$$

$$\frac{\partial^2 I}{\partial y^2} = \frac{\partial g}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial g}{\partial r} + \frac{\partial r'}{\partial y}\frac{\partial g}{\partial r'}$$

$$= -\sin\phi \frac{\partial g}{\partial r} + \cos\phi \frac{\partial g}{\partial r'}$$

$$= -\sin\phi \frac{\partial}{\partial r}\left(-\sin\phi \frac{\partial I}{\partial r} + \cos\phi \frac{\partial I}{\partial r'}\right)$$

$$+ \cos\phi \frac{\partial}{\partial r'}\left(-\sin\phi \frac{\partial I}{\partial r} + \cos\phi \frac{\partial I}{\partial r'}\right)$$

$$= \sin^2\phi \frac{\partial^2 I}{\partial r^2} - 2\sin\phi\cos\phi \frac{\partial^2 I}{\partial r \partial r'} + \cos^2\phi \frac{\partial^2 I}{\partial r'^2}$$

$$\Delta I = I_{xx} + I_{yy} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

$$= \cos^2\phi \frac{\partial^2 I}{\partial r^2} + \cancelto{0}{2 \sin\phi \cos\phi \frac{\partial^2 I}{\partial r \partial r'}} + \sin^2\phi \frac{\partial^2 I}{\partial r'^2}$$

$$+ \sin^2\phi \frac{\partial^2 I}{\partial r^2} \cancelto{0}{- 2 \sin\phi \cos\phi \frac{\partial^2 I}{\partial r \partial r'}} + \cos^2\phi \frac{\partial^2 I}{\partial r'^2}$$

$$= \underbrace{\left(\sin^2\phi + \cos^2\phi\right)}_{=1} \frac{\partial^2 I}{\partial r^2} + \underbrace{\left(\sin^2\phi + \cos^2\phi\right)}_{=1} \frac{\partial^2 I}{\partial r'^2}$$

$$= \frac{\partial^2 I}{\partial r^2} + \frac{\partial^2 I}{\partial r'^2} = I_{rr} + I_{r'r'}$$

$\therefore$ Laplacian is rotation invariant.