Urban Transportation Models

Problem Set #2

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Task 1 Apply the F-W algorithm to solve BMW formulation and find out the UE link flows

BMW Model (link-based)

First, let us present the general BMW model for user equilibrium analysis.

$$\begin{aligned} \min & \sum_a \int_0^{x_a} t_a(\varpi) d\varpi \\ \text{s.t.} \quad & \sum_k f_k^{rs} = q_{rs} && \forall r, s \\ & f_k^{rs} \geq 0 && \forall k, r, s \\ & x_a = \sum_{rs} \sum_k f_k^{rs} \delta_{a,k}^{rs} && \forall a \in A \end{aligned}$$

The meanings of the notations can be found below.

Network Equilibrium Analysis

$\delta_{a,k}^{rs}$	Link-path incidence, which equals 1 if link a is on path k connecting origin r to destination s
x_a	Flow on link a
c_k^{rs}	Travel time of path k for O–D pair r–s
c_{min}^{rs}	Minimum travel time for O–D pair r–s
$\tilde{c}_k^{rs}$	Marginal travel time of path k for O–D pair r–s
$t_a(x)$	Performance function of link a
$\tilde{t}_a(x_a)$	Marginal travel time of link a
q_{rs}	Travel demand from origin r to destination s
(r, s)	O-D pair
f_k^{rs}	Flow on path k for O–D pair r–s
x_{ij}^{rs}	Flow on link ij for O–D pair r–s
$\phi = \frac{T_{UE}}{T_{SO}}$	Price of anarchy

F-W Algorithm

The BMW model given above is a constrained optimization problem and we will use FW-algorithm to solve the problem. The general F-W algorithm starts with a feasible solution and move it towards a direction by certain step which keeps it feasible until meets certain stopping criterion. Here, let us describe the basic process of the modified F-W algorithm specially for determining the equilibrium flows for transportation networks.

F-W Algorithm

Input: Transportation network, Link performance function
Ouput: Equilibrium flows for the given transportation network

Step 0: Initialize. Perform all-or-nothing assignment based on $t_a = t_a(0), \forall a$. Get x^1, set $n = 1$.
Step 1: Update. Set $t_a^n = t_a(x_a^n), \forall a$.
Step 2: Direction finding. Perform all-or-nothing assignment based on $t_a^n, \forall a$. This yields flows y^n.
Step 3: Line searh. Find $0 \leq \alpha_n \leq 1$ that solves

$$\sum_a (y^n - x^n) t_a(x^n + \alpha_n(y^n - x^n)) = 0$$

Step 4: Move. Set $x^{n+1} = x^n + \alpha_n(y^n - x^n)$.
Step 5: Convergence test. If $\frac{\sqrt{\sum_a (x_a^{n+1} - x_a^n)^2}}{\sum_a x_a^n} < \varepsilon$, stop; otherwise $n = n + 1$, go to Step1.

All-or-nothing Assignment with Floyd-Warshall

Input: Transportation network
Output: Feasible flows for the given transportation network

Step 1: Apply Floyd-Warshall algorithm to find the shortest paths for all pairs.
Step 2: Loop over all O-D pairs, assign the flows to the corresponding shortest path.

Note that we use Floyd-Warshall algorithm to find shortest paths for all pairs and keep track of the paths used. Floyd-Warshall algorithm is easy to implement and debug, and we can obtain all shortest paths in one run.

Code Structure

We use Python for coding. Note that we use matrices to store variables concerning links. Later, when we present the output result, they are also in matrices form. For example, for link flow matrix X, $x[i,j]$ is the link flow on link (i,j) connecting node i and node j.

Core Code

First, we will present the core of the implementation codes. This is in accordance of the F-W algorithm structure and we will show the functions in detail later.

Note that we choose stricter stopping criterion for the main part of F-W algorithm than in Bisection for efficiency and accuracy. Since the former is vital for the quality of the solution while the latter is merely related to the size of a step, which is itself approximate.

```
import time
time_start=time.time()
# F-W algorithm
# step 0 initialize and get x_1
x0=allornothing_assignment(graph0) # Func 1
# step 1 Update
graph=update_graph(x0)             # Func 2
# step 2 Find direction
y=allornothing_assignment(graph)
# step 3 Line search
alpha=bisect(func, x0, y)          # Func 3
# step 4 Move
x1=x0+alpha*(y-x0)
count=1 # record number of iterations
```

```
# Loop until meeting stopping criterion
while sum(sum(x1-x0)**2)/sum(sum(x0))>epsilon: # stopping criterion
    x0=x1
    graph=update_graph(x0)
    y=allornothing_assignment(graph)
    alpha=bisect(func, x0, y)
    x1=x0+alpha*(y-x0)
    count=count+1
    print(count)

time_end=time.time()
print('Totally Time',time_end-time_start)
```

Func1: All-or-nothing Assignment

Recall that we use Floyd-Warshall algorithm to find shortest paths in the All-or-nothing Assignment.

```
def allornothing_assignment(graph): # perform all-or-nothing assignment, get feasible direction
    y=np.zeros((n,n)) # link flow
    path=floyd_w(graph) # keep track of the shortest path obatined by Floyd-Warshall
    for i in range(n): # iterate over all O-D pairs (demands)
        for j in range(n):
            if demand[i,j]!=0:
                y=y+path_link(path,i,j) # add y_ij^rs to y_ij to get the link flow
    return y


def floyd_w(g): # Floyd-Warshall algorithm: get the shortest paths for all pairs
    path=copy.deepcopy(path0) # path0 is a global variable that helps to record shortest path, here we use deepcopy to avoid overwritten danger.
    graph=copy.deepcopy(g)
    for k in range(0,n):
        for i in range(0,n):
            for j in range(0,n):
                if graph[i,j] > graph[i,k] + graph[k,j]:
                    graph[i,j] = graph[i,k] + graph[k,j]
                    path[i,j] = path[i,k] # keep track of the shortest path
    return path


def path_link(path,r,s): # assign demand of the O-D pair to the link used
    q=demand[r,s] # demand is a global variable containing all q^rs
    y=np.zeros((n,n))
    while path[r,s]!=s:
        r_k=int(path[r,s]) # r->k->...->s
        y[r,r_k]=q
        r=r_k # keep tracking
    y[r,s]=q # don't forget this link!
    return y
```

Func2: Update travel time of all links

After we get a new feasible solution, we want to update travel time of all links based on it. Here, we use the BRF function.

$$t_a = t_a^0[1 + 0.15(\frac{x_a}{c_a})^4]$$

```
def update_graph(x): # update travel time based on free flow travel time
    graph=copy.deepcopy(graph0) # graph0 is a global variable containing t_a^0
    for i in range(n):
        for j in range(n):
            if capacity[i,j]!=0 and x[i,j]!=0:
                graph[i,j]=graph0[i,j]*(1+0.15*(x[i,j]/capacity[i,j])**4)
    return graph
```

Func3: Use bisection to perform line search

After we get a feasible direction, we need to find the step by line search. In our problem, this step is reduced to find the solution to an equation. Therefore, we perform bisection.

```
def bisect(func, x, y ,low=0, high=1):
    'Find root of continuous function where f(low) and f(high) have opposite
signs'
    assert func(low,x,y)*func(high,x,y)<=0
    while(high-low>0.0001): #stopping criterion
        midpoint = (low + high) / 2.0
        if func(low,x,y)*func(midpoint,x,y)>0:
            low = midpoint
        else:
            high = midpoint

    return midpoint

def func(alpha,x,y):
    s=0
    for i in range(len(x)):
        for j in range(len(x)):
            if capacity[i,j]!=0:
                temp=(y[i,j]-x[i,j])*(graph0[i,j]*(1+0.15*((x[i,j]+alpha*
(y[i,j]-x[i,j]))/capacity[i,j])**4))
                s=s+temp
    return s
```

Algorithm Input and Initialization

In this part, we will show the codes for data input.

```
n=24 # number of nodes
epsilon=0.00001 # parameter of stopping criterion
```

```
dfs = pd.read_excel('link.xlsx')
# get demand matrix from the given string
with open('demand_rs.txt') as f:
    data0=f.read()
data=data0.split('Origin')
demand=np.zeros((n,n))
for i in range(1,n+1):
    for test in data[i].split(';'):
        try:
            j=int(test.split(':')[0].split()[-1])
            q=float(test.split(':')[1].split()[0])
            demand[i-1,j-1]=q
        except:
            pass

# Initialization for global variables (Transportation network parameters)
graph0=np.zeros((n,n))+3000 # Adjacency matrix for the network with weight link
travel time; 3000 represents infinity
capacity=np.zeros((n,n)) # capacity matrix
for k in range(len(dfs)):
    i=dfs['Starting node'][k]-1 # starting from 0, 0 represents node 1
    j=dfs['Ending node'][k]-1
    graph0[i,j]=dfs['Free flow travel time (min)'][k] # initialize graph0 with
t(0)
    capacity[i,j]=dfs['Capacity (veh/hr)'][k] # initialize capacity matrix

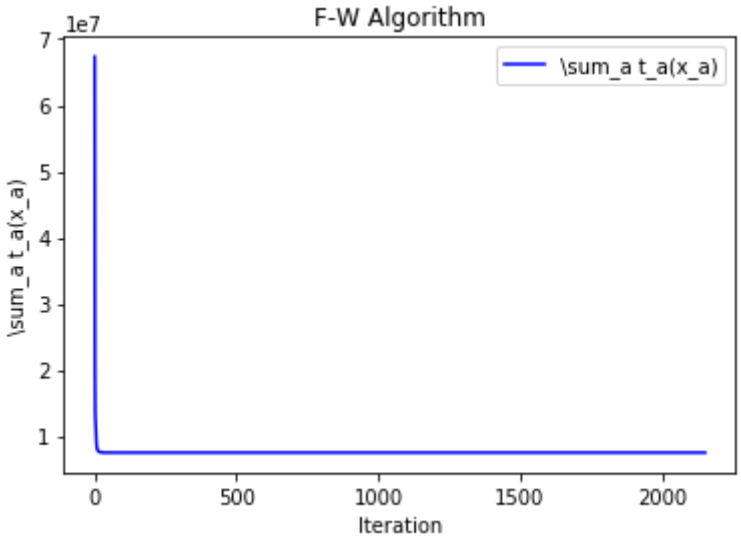
path0=np.zeros((n,n)) # auxiliary matrix used in Floyd-Warshall to track
shortest path
for j in range(n):
    path0[:,j]=j
```

Output analysis

Now we will show the algorithm output results we obtained.

Number of iterations	$\sum_a x_a t_a(x_a)$ **(converged)**	**Running time**
1639	7479960.4305	88.9293

Since it takes lots of time to calculate the integral, instead we show the value of $\sum_a t_a(x_a)$ as the algorithm proceed.



The converged link flow result is as follows.

```
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```

```
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```

```
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       10258.89356014,     0.        ,  7860.4787675 ,     0.        ]])
```

Task 2 Present the marginal-cost pricing toll pattern

Model Modification

By definition, the objective function is

$$\sum_a x_a t_a(x_a)$$

Therefore, in the F-W algorithm, we need to do all-or-nothing assignment based on

$$t_a(x_a^n) + x_a^n \frac{\partial t_a(x_a)}{\partial x_a}\bigg|_{x_a^n} = t_a^0[1 + 0.75(\frac{x_a^n}{c_a})^4].$$

Meanwhile, in Step 3, .we need to find $0 \leq \alpha_n \leq 1$ that solves

$$\sum_a (y^n - x^n)[t_a(x^n + \alpha_n(y^n - x^n)) + (x^n + \alpha_n(y^n - x^n)) * \frac{\partial t_a(x_a)}{\partial x_a}\bigg|_{x^n+\alpha_n(y^n-x^n)}] = 0$$

The modified codes are as follows.

```
def update_graph_SO(x):
    graph=copy.deepcopy(graph0)
    for i in range(n):
        for j in range(n):
            if capacity[i,j]!=0 and x[i,j]!=0:
                graph[i,j]=graph0[i,j]*(1+0.75*(x[i,j]/capacity[i,j])**4)
    return graph

def func_SO(alpha,x,y):
    s=0
    for i in range(len(x)):
        for j in range(len(x)):
            if capacity[i,j]!=0:
                temp=(y[i,j]-x[i,j])*(graph0[i,j]*(1+0.75*((x[i,j]+alpha*
(y[i,j]-x[i,j]))/capacity[i,j])**4))
                s=s+temp
    return s
```

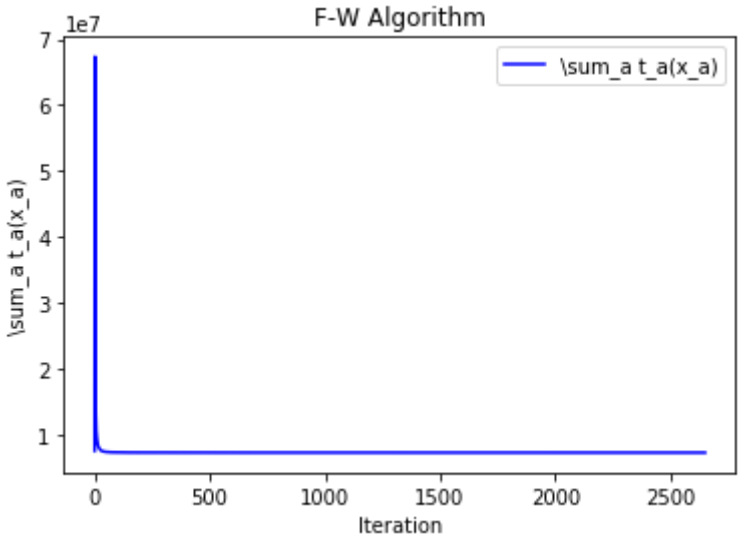
Output analysis

Now we will show the algorithm output results we obtained.

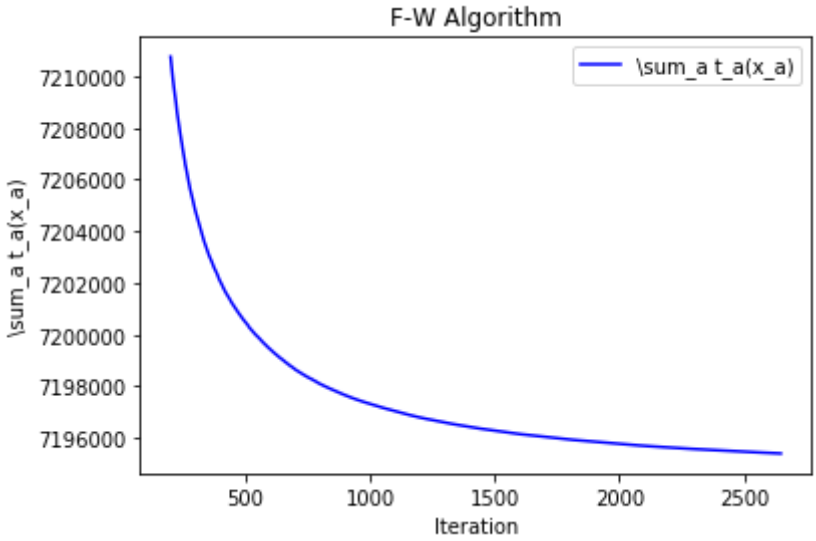
Number of iterations	$\sum_a x_a t_a(x_a)$ **(converged)**	**Running time**
2648	7195401.8372	109.9133

Compared to the 7479960.4305 obtained in UE, the objective function is 7195401.8372, indicating better system performance, which is in accordance to the idea of system optimum.

Let us show the value of $\sum_a t_a(x_a)$ as the algorithm proceed.



Let us take a closer look.



We can find that as we update the solution for several times, our solution tend to converge.

The converged link flow result is as follows.

```
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```

```
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        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        , 18805.3119893 ,
        0.        ,  6994.70845034,     0.        ,     0.        ,
    16898.1072643 ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,  6639.16506358,     0.        ,     0.        ,
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        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        , 13273.7910547 ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        , 16332.22147109,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
        0.        , 12595.16086466, 13226.19516692,     0.        ,
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        0.        ,     0.        ,     0.        ,     0.        ,
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       [    0.        ,     0.        ,     0.        ,     0.        ,
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        0.        , 21762.13887095,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
    21881.39835032,     0.        , 17469.16922751,     0.        ,
        0.        ,     0.        , 23361.72301608, 10743.72928381,
     8320.80053797,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,  6091.27751268,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        , 17389.62496314,     0.        ,  7326.59360312,
        0.        ,  9444.62950096,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        , 14325.66859691,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,  7325.33319084,     0.        ,
    15540.40028762,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        , 15647.14661226,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        ,     0.        ,
        0.        ,     0.        ,     0.        , 10538.78956411],
```

```
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,  9470.59885316,     0.        ,
            0.        ,     0.        ,  8917.97280543,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,  7616.27004408,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        , 23421.20044993,     0.        ,     0.        ,
            0.        ,  8955.60839743,     0.        ,     0.        ,
            0.        ,     0.        , 18551.3309181 ,     0.        ,
            0.        , 16177.53430444,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,  7987.71654475,
            0.        , 10766.1107548 ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
        10598.63653308, 18473.06718959,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
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            0.        ,     0.        ,     0.        , 10582.0680245 ,
            0.        ,     0.        ,  8256.45219591,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        , 16379.81735888,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        , 18532.90186081,
            0.        ,     0.        ,     0.        , 20704.31055449,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        , 18565.92339291,     0.        ,
         8256.82852622,     0.        ,     0.        ,  8697.60186795,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        , 20711.74111348,  8712.57067309,     0.        ,
         6362.15539493,  7119.19380637,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,  6317.85001726,
            0.        ,  7943.06357317,     0.        ,  9834.43664547],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        , 16160.05485548,     0.        ,
            0.        ,     0.        ,     0.        ,  7085.89854816,
         7952.10724045,     0.        ,  9299.44265877,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,  7604.60380428,     0.        ,     0.        ,
```

```
           0.        ,     0.        ,     0.        ,     0.        ,
           0.        ,  9257.71161889,     0.        ,  7738.80136576],
       [   0.        ,     0.        ,     0.        ,     0.        ,
           0.        ,     0.        ,     0.        ,     0.        ,
           0.        ,     0.        ,     0.        ,     0.        ,
       10545.53588874,     0.        ,     0.        ,     0.        ,
           0.        ,     0.        ,     0.        ,     0.        ,
        9781.08760052,     0.        ,  7685.40408607,     0.        ]])
```

Marginal-cost pricing tolls are as follows.

```
array([[ 0.        ,  0.05682504,  0.13274541,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.02885292,  0.        ,  0.        ,  0.        ,  0.        ,
        22.23123637,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.21344562,  0.        ,  0.        ,  1.76496944,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.20465057,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  1.69932199,  0.        ,  1.35666096,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         2.56768374,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  1.1591112 ,  0.        ,
         4.26065453,  0.        ,  0.        , 29.60875223,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        , 10.30291589,  0.        ,  0.        ,  6.29699813,
         0.        ,  0.        , 57.57811363,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        , 15.25227823,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.23480812,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        43.90397648, 11.03685597,  0.        , 39.72412445,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        28.8656225 ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        , 21.8672958 ,
         0.        ,  0.        , 25.25988659,  0.        , 14.53041656,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
```

```
        0.        ,  0.        ,  0.        , 10.05940017,  0.        ,
       36.88566137,  0.        ,  0.        ,  0.        , 31.42515988,
       80.2030837 , 36.63759127,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  4.42652155,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        , 22.74132949,
        0.        , 35.86165095,  0.        , 23.23458429,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.32776312,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
       17.75824931,  0.        ,  0.21728869,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.23337926,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        , 44.65267635],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
       23.56735066,  0.        ,  0.        ,  0.        , 30.24430016,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        , 14.60269368,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        , 31.18484724,
        0.        ,  0.        ,  0.        , 28.64640269,  0.        ,
        0.        ,  0.        ,  0.        ,  3.860232  ,  0.        ,
        0.        , 12.7926737 ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        , 17.74823818,  0.        , 80.66221482,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        , 30.39230677,  1.48863559,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        , 36.64039796,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
       18.00717261,  0.        ,  0.        , 14.78529899,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.30409354,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        1.63263659,  0.        ,  0.        ,  0.        ,  1.60669695,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  3.77757865,
        0.        ,  7.38902089,  0.        ,  0.        , 19.32697387,
        0.        ,  0.        ,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  2.06756999,  8.65197368,  0.        ,
       11.95865033, 10.24791975,  0.        ,  0.        ],
      [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        0.        ,  0.        ,  0.        ,  0.        , 10.25220266,
```

```
        0.        ,  4.93227154,  0.        , 31.42138975],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        , 12.23851447,
         0.        ,  0.        ,  0.        ,  0.        , 10.25750292,
         5.23073163,  0.        , 24.05330364,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        , 15.79911703,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        , 23.40280373,  0.        ,  6.90066124],
       [ 0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
         0.        ,  0.        , 47.6847745 ,  0.        ,  0.        ,
         0.        ,  0.        ,  0.        ,  0.        ,  0.        ,
        27.03807935,  0.        ,  7.23296189,  0.        ]])
```

Task 3 Solve the UE condition under marginal-cost pricing and compare it with the original UE without toll

Model Modification

By definition, the marginal-cost pricing for path a is

$$\tau_a = \tilde{t}_a(x_a^{SO}) - t_a(x_a^{SO}) = x_a^{SO} \frac{\partial t_a(x_a)}{\partial x_a}\bigg|_{x_a^{SO}}$$

Therefore, we need to modify the objective function

$$\min \sum_a \int_0^{x_a} (t_a(\varpi) + \tau_a) d\varpi$$

Therefore, in the F-W algorithm, we need to do all-or-nothing assignment based on

$$t_a(x_a^n) + x_a^{SO} \frac{\partial t_a(x_a)}{\partial x_a}\bigg|_{x_a^{SO}}$$

using the results obtained in task 2.

Meanwhile, in Step 3, .we need to find $0 \le \alpha_n \le 1$ that solves

$$\sum_a (y^n - x^n)[t_a(x^n + \alpha_n(y^n - x^n)) + \tau_a] = 0$$

The modified codes are as follows.

```
# marginal is a matrix containing all marginal-cost pricing
def update_graph_m(x):
    graph=copy.deepcopy(graph0)
    for i in range(n):
        for j in range(n):
            if capacity[i,j]!=0 and x[i,j]!=0:
```

```
                graph[i,j]=graph0[i,j]*(1+0.15*
(x[i,j]/capacity[i,j])**4)+marginal[i,j]
    return graph

def func_m(alpha,x,y):
    s=0
    for i in range(len(x)):
        for j in range(len(x)):
            if capacity[i,j]!=0:
                temp=(y[i,j]-x[i,j])*(graph0[i,j]*(1+0.15*((x[i,j]+alpha*
(y[i,j]-x[i,j]))/capacity[i,j])**4)+marginal[i,j])
                s=s+temp
    return s
```

Output analysis

Now we will show the algorithm output results we obtained.

Number of iterations	$\sum_a x_a t_a(x_a)$ (converged)	Running time
2208	7194533.3204	116.8562

The converged link flow result is as follows.

```
array([[    0.        ,  9180.41779904, 11349.64771705,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [ 7749.52226999,     0.        ,     0.        ,     0.        ,
            0.        ,  8180.55861644,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [12780.5432461 ,     0.        ,     0.        , 15845.09288863,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        , 12646.80270191,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        , 15695.65381534,     0.        ,
        18336.75662266,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,  4511.15199764,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        , 17629.33634855,
            0.        ,  5711.43926707,     0.        ,     0.        ,
        17724.52938357,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
```

```
        0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,  6749.66308739,     0.        ,     0.        ,
         6297.37961428,     0.        ,     0.        , 12892.58280681,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        , 13379.25532033,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        , 15565.50736202,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
            0.        , 12047.62762497, 12339.84218572,     0.        ,
         8100.90911605,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,  8886.83044947,
            0.        ,     0.        ,     0.        ,     0.        ,
            0.        ,     0.        ,     0.        ,     0.        ],
       [    0.        ,     0.        ,     0.        ,     0.        ,
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Discussion

In this part, we will discuss the results obtained above.

Task	Number of iterations	$\sum_a x_a t_a(x_a)$(converged)	Running time
1	1639	7479960.4305	88.9293
2	2648	7195401.8372	109.9133
3	2208	7194533.3204	116.8562

Denote the system travel time under SO by T_{SO} and the one under UE by T_{UE}.

$$\phi = \frac{T_{UE}}{T_{SO}} = 7479960.4305/7195401.8372 = 1.0395$$

For BPR, the price of anarchy is 2.151, which is bigger than ϕ we get here. So we may conclude that in our problem setting, the improvement potential is relatively small.

Now let us look at T_{SO} and the optimal system travel time under marginal-cost pricing. Compared to the 7479960.4305 obtained in UE, the objective function obtained under marginal-cost pricing is 7194533.3204, which is smaller and pretty close to the result we obtained under SO.

Since we have $\frac{7194533.3204-7194533.3204}{7194533.3204} = 0.012\%$, we may say the result can be treated as the same.