Chap.2 Getting Started

- Here we study two sorting algorithms
- **Insertion sort** sorts by inserting into a sorted list the elements of the input array one after the other (incremental approach)
- **Merge sort** sorts by recursively the input array into halves, sorting the halves separately, and then merging them into a full sorted list (divide & conquer approach)

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Example of insertion sort

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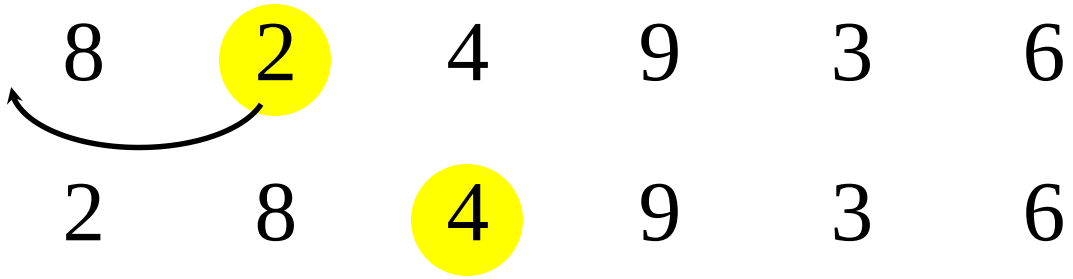
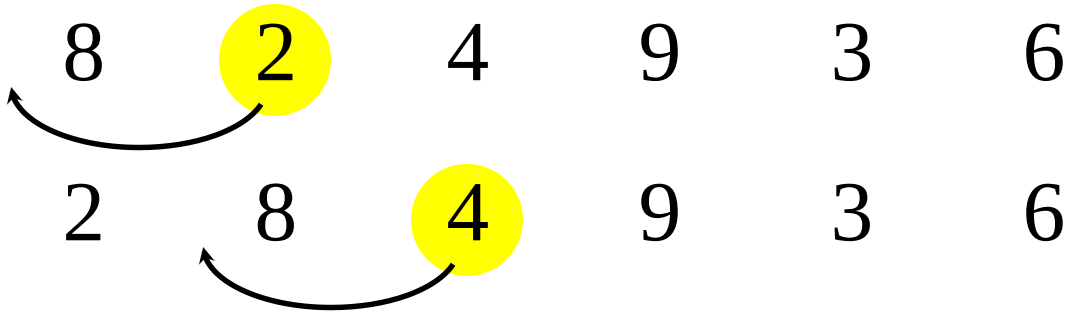


Example of insertion sort



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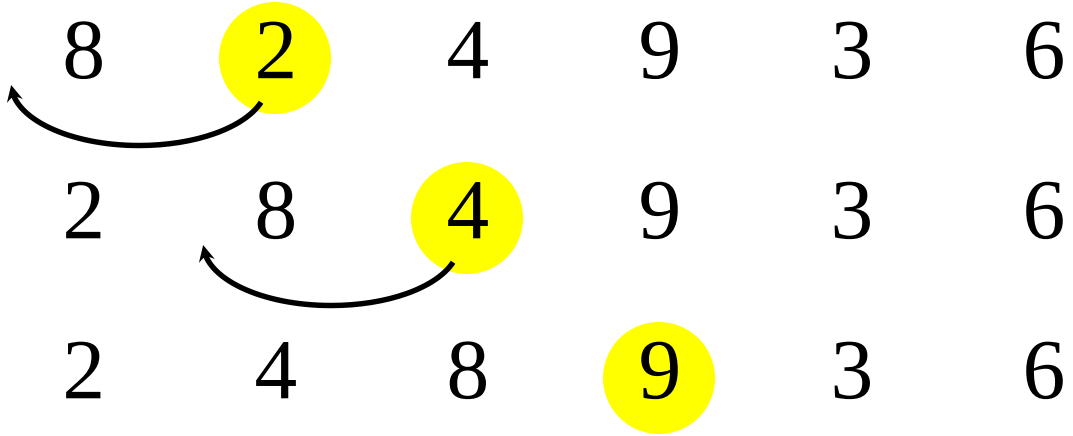

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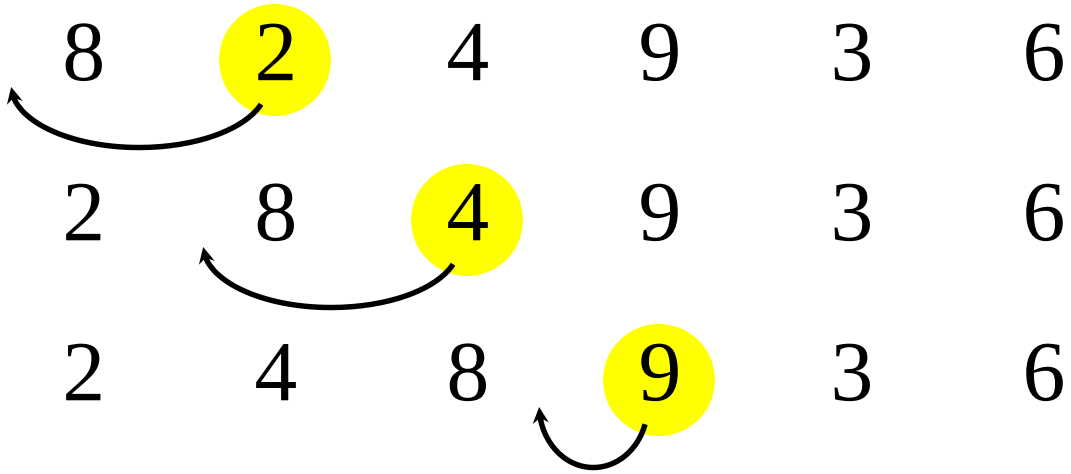

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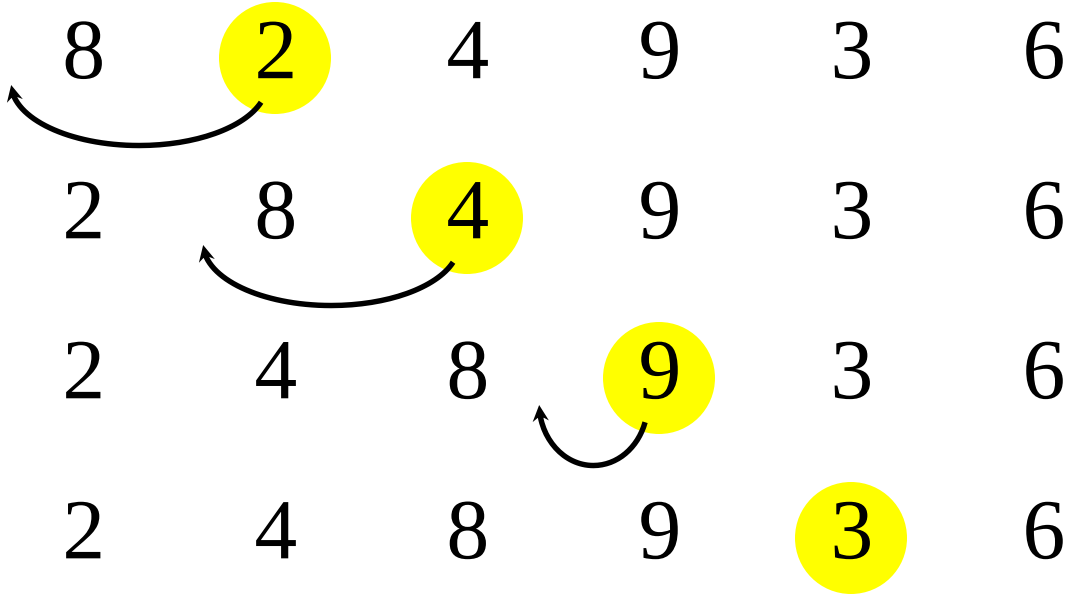

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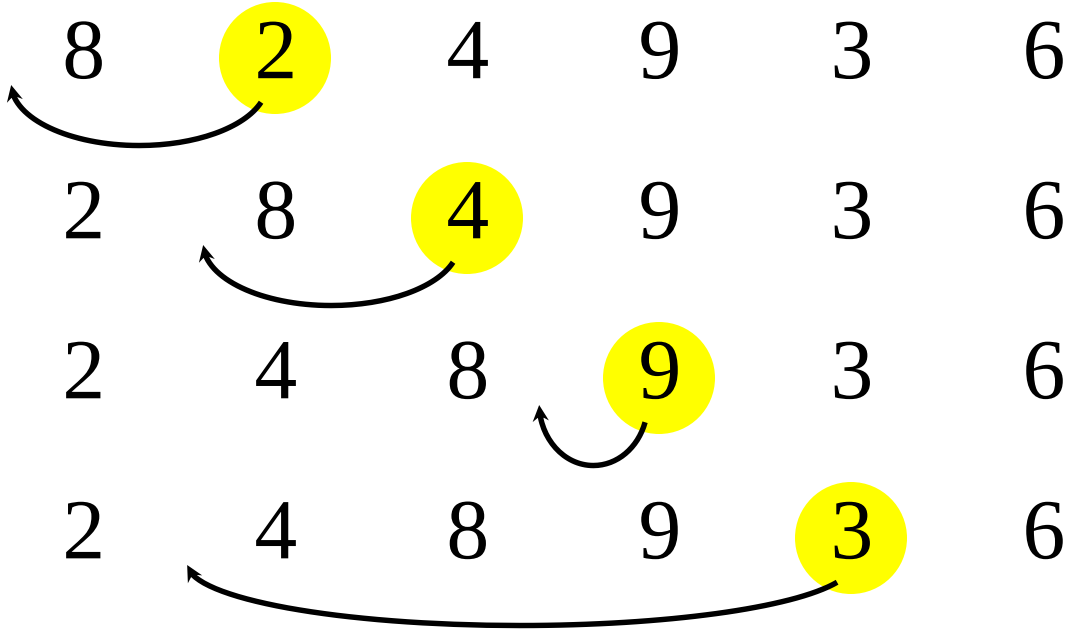

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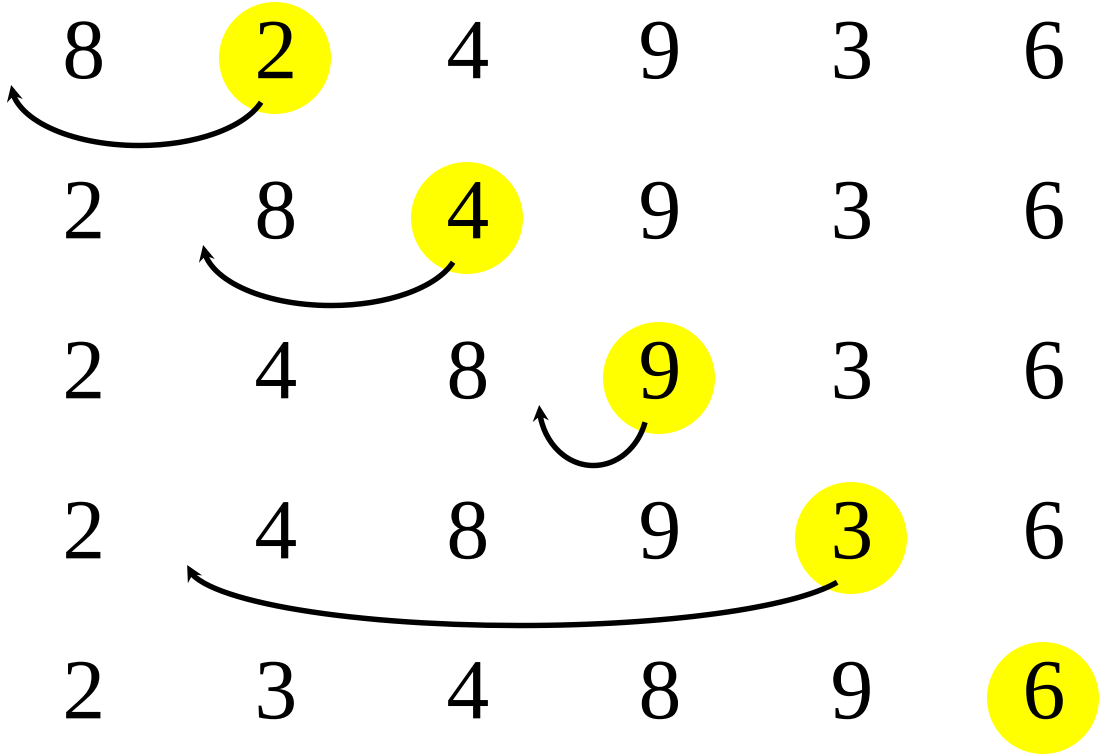

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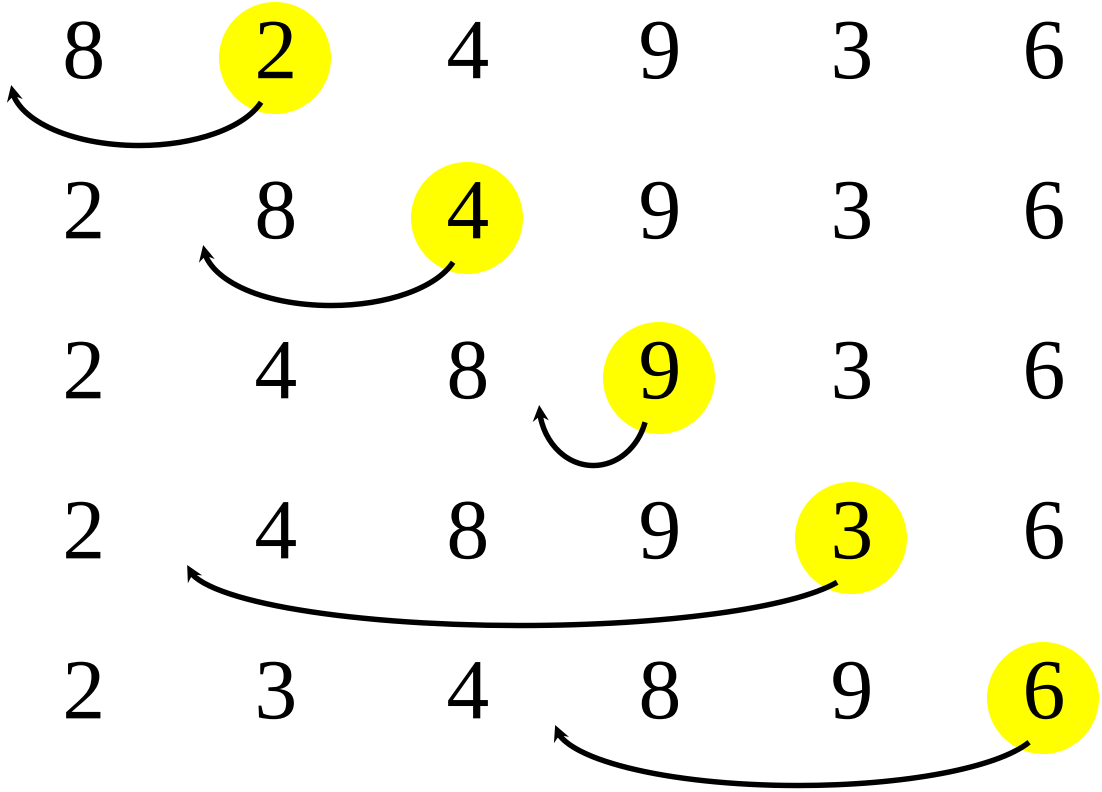
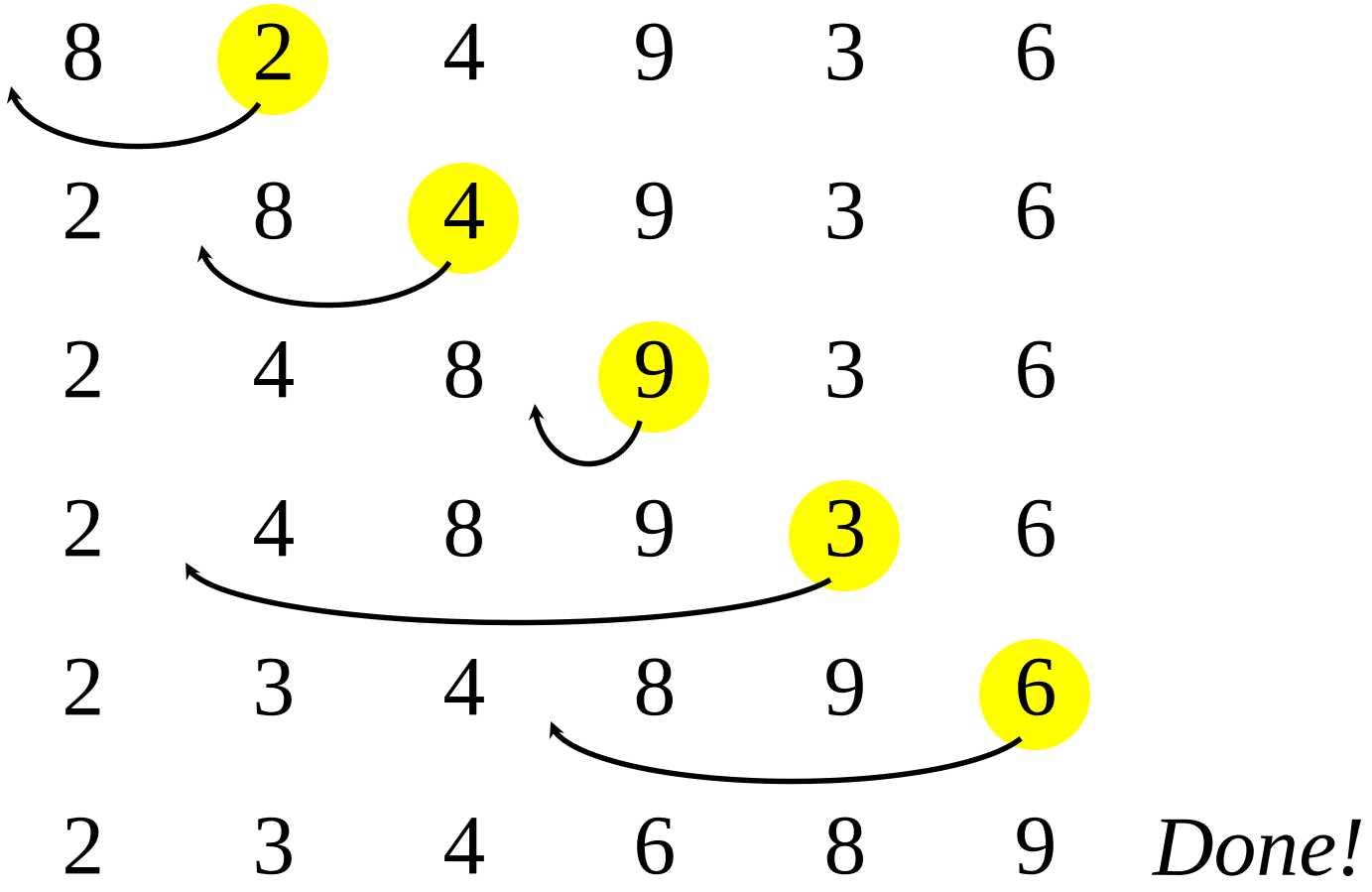


Example of insertion sort



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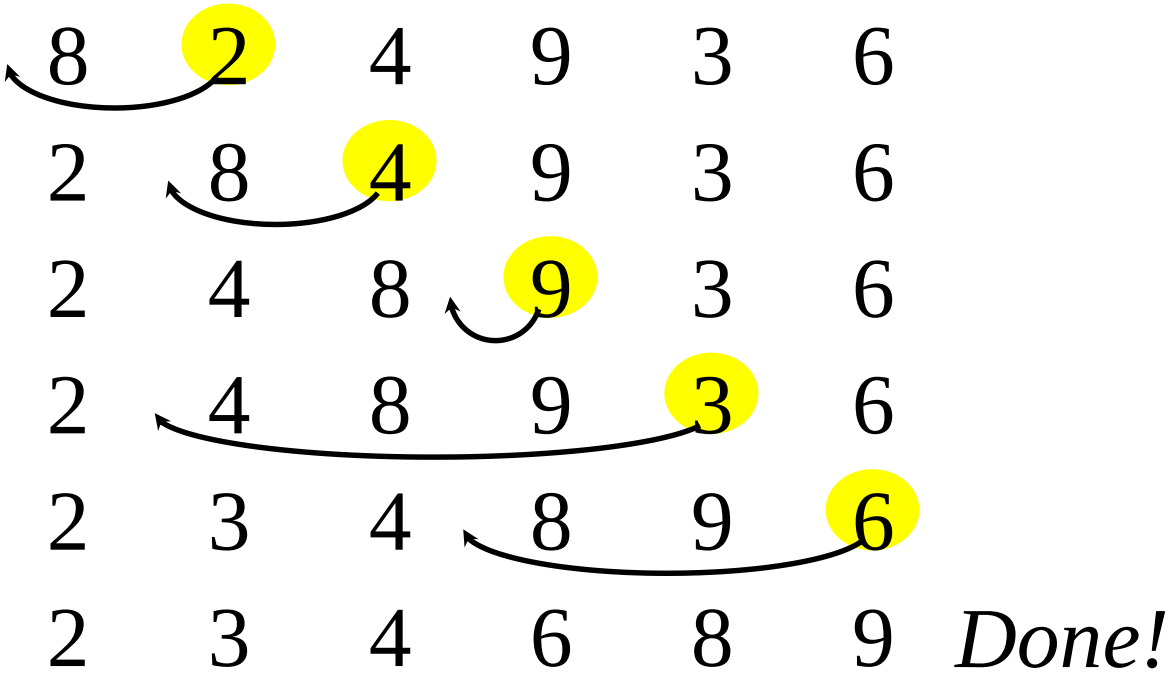
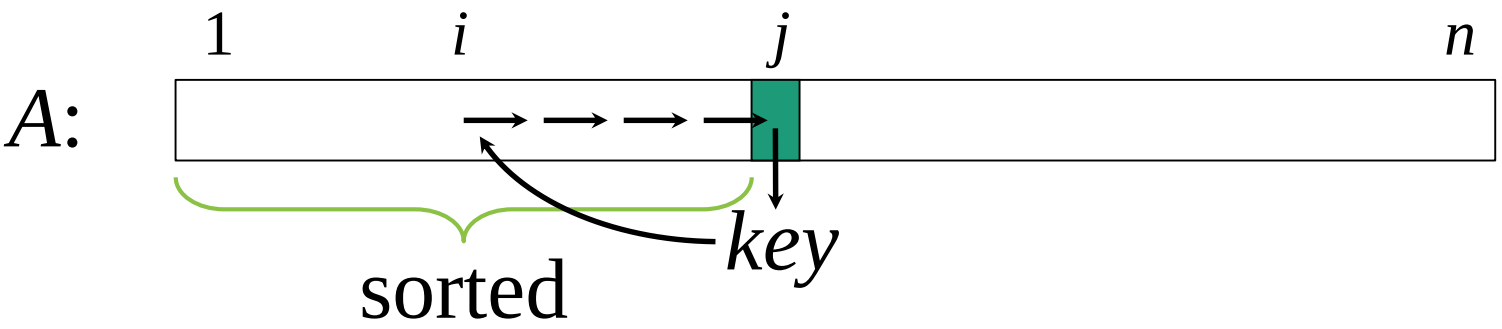




Insertion Sort

INSERTION-SORT(A)

1. **for** j = 2 to length[A]
2. **do** $key \leftarrow A[j]$
3. //insert A[j] to sorted sequence A[1..j-1]
4. $i \leftarrow j$-1
5. **while** i >0 and A[i]>key
6. **do** $A[i+1] \leftarrow A[i]$ //move A[i] one position right
7. $i \leftarrow i$-1
8. $A[i+1] \leftarrow key$

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.

- Major Simplifying Convention: Parameterize the running time by the size of the input,

  $T_A(n)$ = time of A on length n inputs

- Generally, we seek upper bounds on the running time, to have a guarantee of performance.

Analysis of Algorithm

Worst-case:

$T(n)$ = maximum time of algorithm on any input of size n.

Average-case:

$T(n)$ = expected time of algorithm over all inputs of size n.

Need assumption of statistical distribution of inputs.

Best-case:

works fast on *some* input.

Analysis of Algorithm

- When we analyze an algorithm, we are often primarily interested in its *worst-case* performance.
- Why?
  - The worst-case is an *upper bound on the running time of an algorithm*. (We know its performance can't be any worse than that.)
  - For some algorithms, the worse case occurs fairly often.

Machine-independent time

What is insertion sort's worst-case time?

Basic Idea:

Ignore machine dependent constants,
otherwise impossible to verify and to compare algorithms

Look at ***growth*** of $T(n)$ as $n \to \infty$.

Asymptotic Analysis

Insertion sort analysis

- Worst case
  - Reverse sorted list
  - $O(n^2)$
- Best Case
  - Sorted input
  - $O(n)$
- Is insertion sort a fast sorting algorithm?
  - Moderately so, for small *n*.
  - Not at all, for large *n*.

Merge Sort

- Insertion sort used an *incremental approach* to sorting: sort the smallest subarray (1 item), add one more item to the subarray, sort it, add one more item, sort it, etc.
- Merge sort uses a *divide-and-conquer approach,* based on the concept of *recursion.*

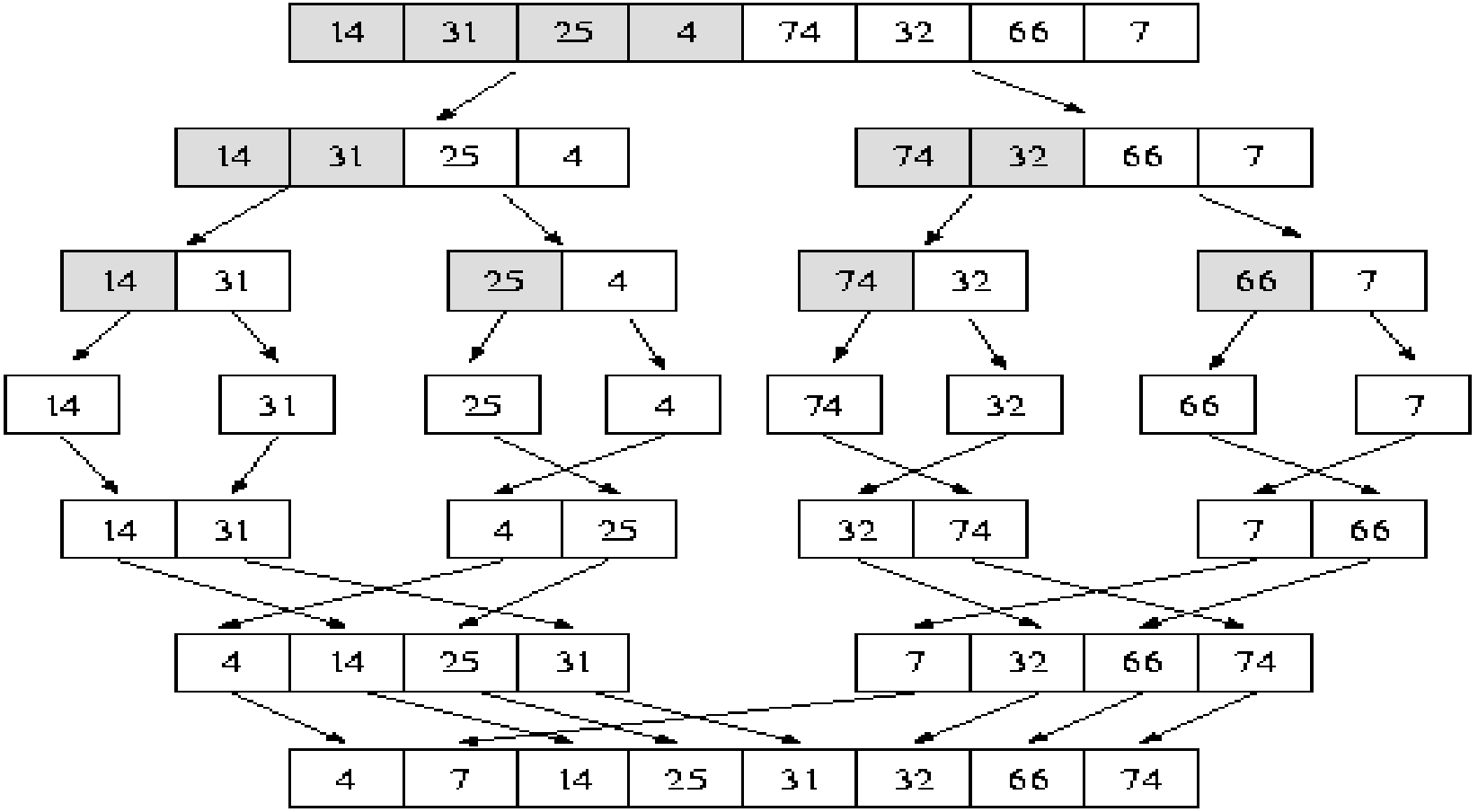
Merge Sort

- ***Divide-and-conquer:***
  - *Divide* the problem into several subproblems.
  - *Conquer* the subproblems by solving them recursively. If the subproblems are small enough, solve them directly.
  - *Combine* the solutions to the subproblems to get the solution for the original problem.

Merge Sort (divide & conquer)

- *Divide the n-element sequence to be sorted into* two subsequences of n/2 each.
- *Conquer by sorting the subsequences recursively* by calling merge sort again. If the subsequences are small enough (of length 1), solve them directly.(Arrays of length 1 are already sorted.)
- *Combine the two sorted subsequences by merging* them to get a sorted sequence.

Merge Sort Example



Merge sort

- Merge sort basically consists of recursive calls to itself.
- The base case (which stops the recursion) occurs when a subsequence has a size of 1.
- The combine step is accomplished by a call to an algorithm called *Merge*.

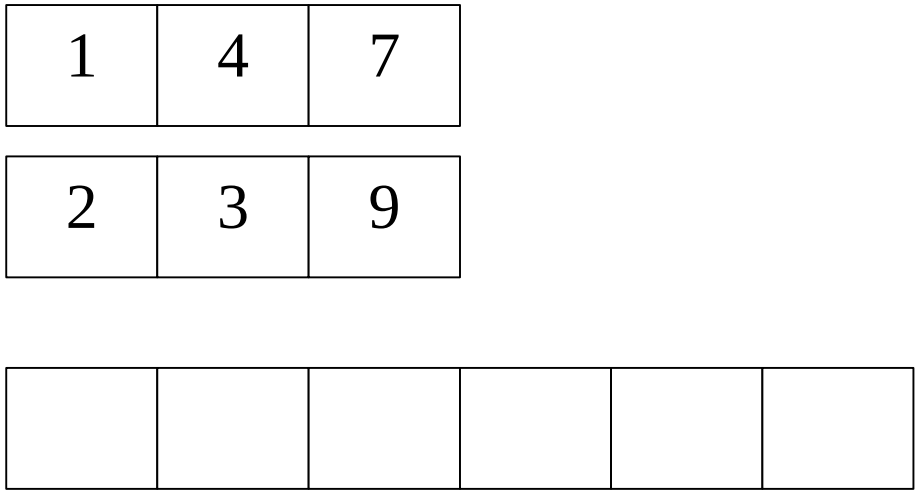
Merge-Sort(A,p,r)

1. if $p < r$
2. then $q \leftarrow \lfloor (p+r)/2 \rfloor$
3. Merge-Sort(A, p, q)
4. Merge-Sort(A, q+1, r)
5. MERGE(A, p, q, r)

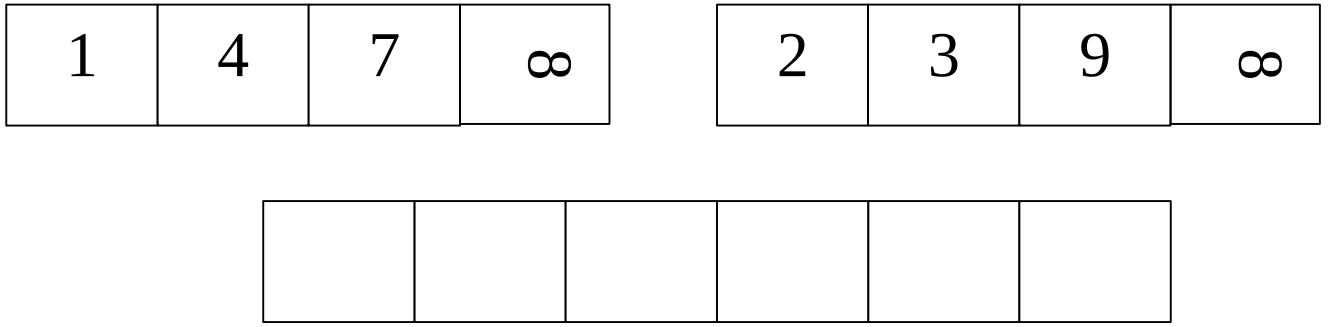
Call to Merge-Sort (A,1,n) (suppose n=length(A))

Merge

- Merge works by assuming you have two already sorted sublists and an empty array

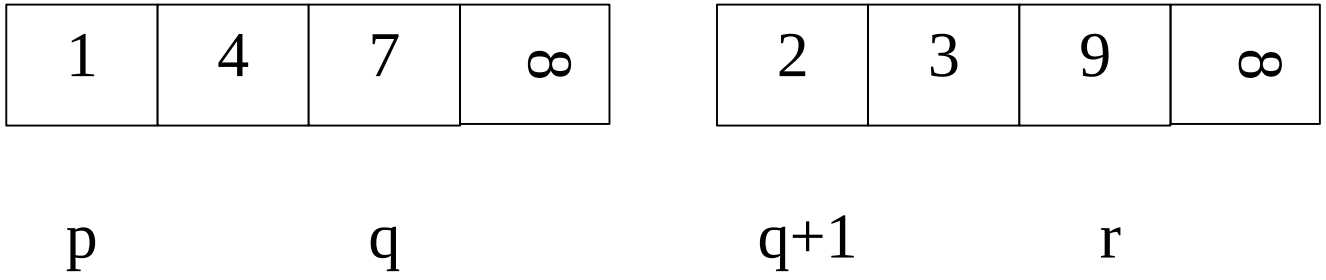


Merge



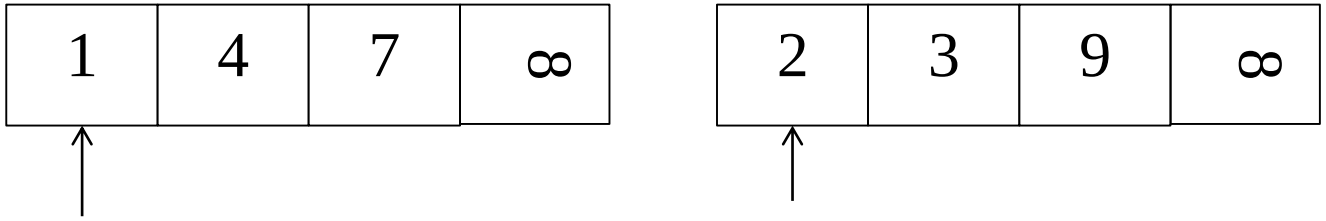
- Let's assume we have an *infinity,* which is guaranteed to be larger than the last item at the end of each sublist which lets us know when we have hit the end of the sublist.

Merge

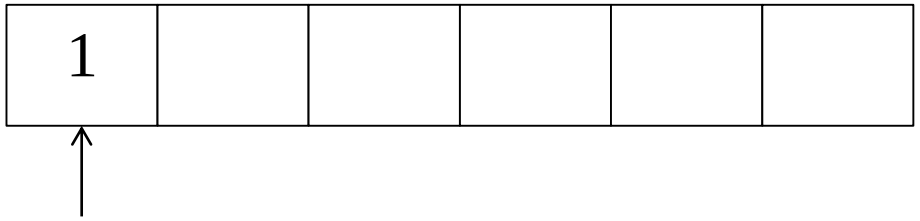


- The two sublists are indexed from p to q (for the first sublist) and from q+1 to r for the second sublist. There are (r – p) + 1 items in the two sublists combined, so we will need an output array of that size.

Merge



- Look at the first item in each subarray. Choose the smallest item.
- Move the chosen item to the output array



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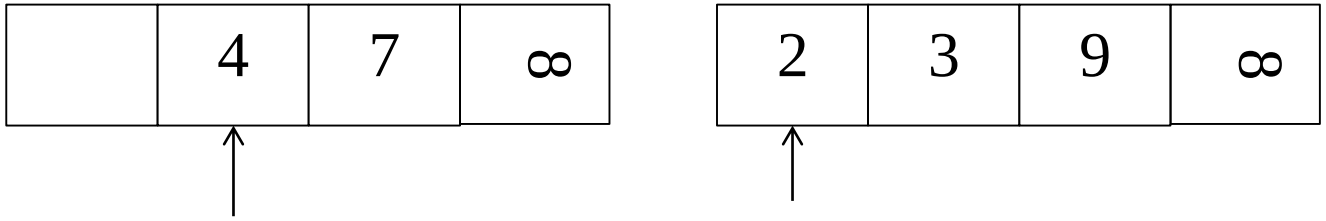

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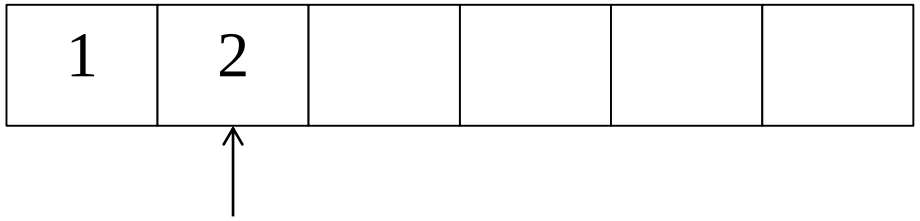

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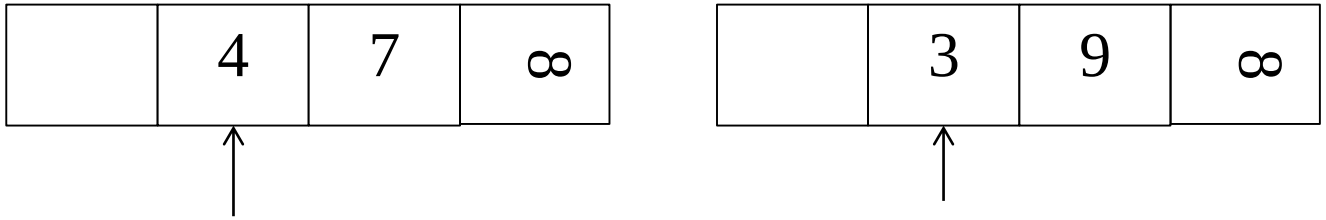

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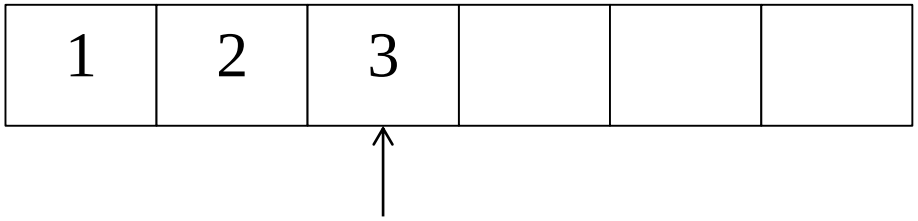

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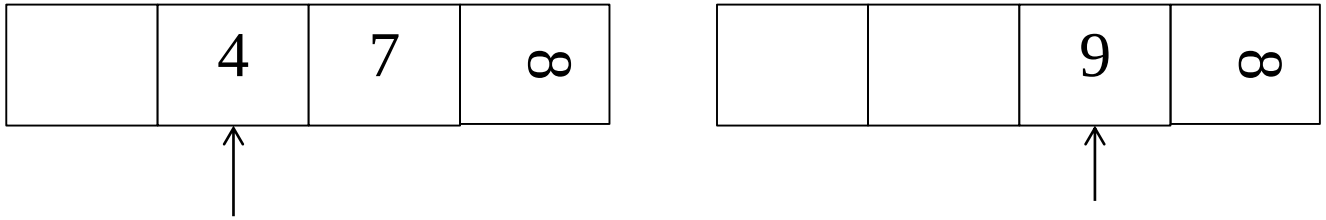

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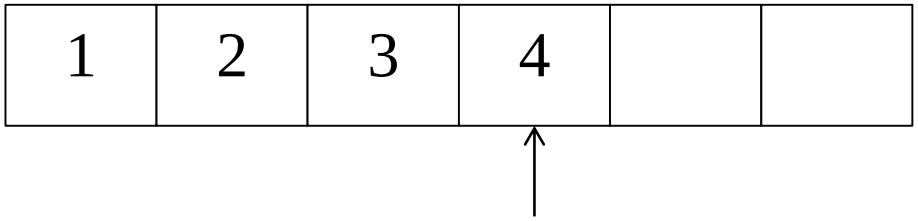

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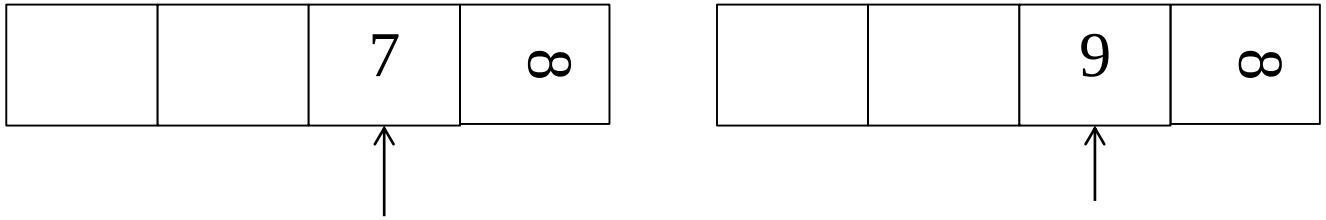

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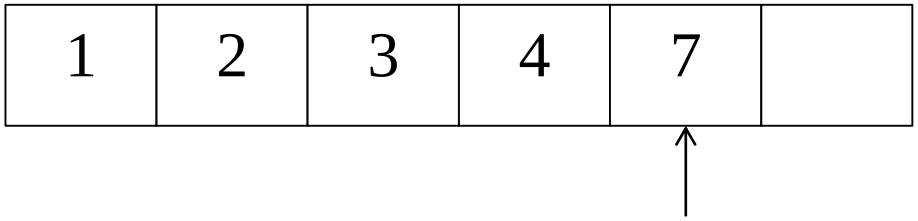

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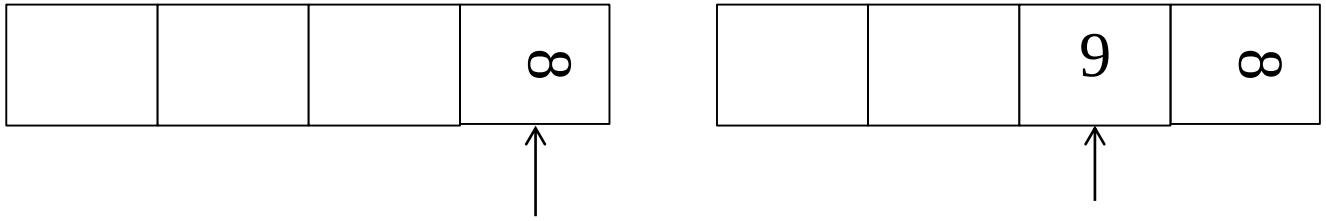

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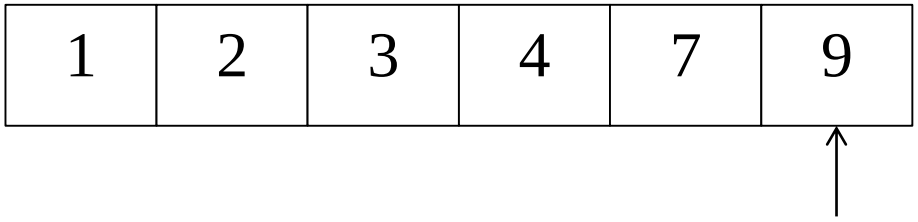

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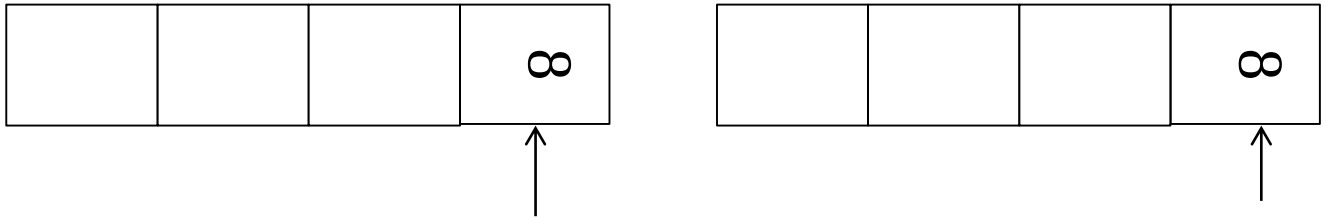
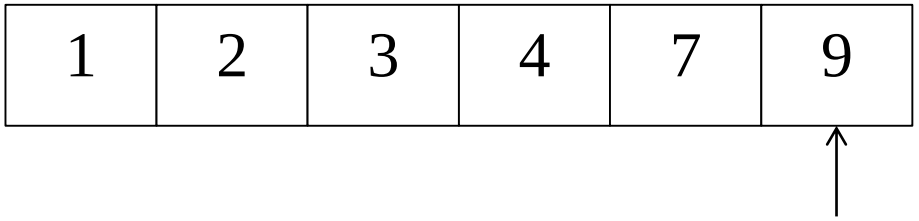


- We know that we have only n = (r – p) + 1 items. So, we will make only (r – p) + 1 moves.
- Here r = 1 and p = 6, and (6 – 1) + 1 = 6, so when we have made our 6^{th} move we're through.



Merge(A,p,q,r)

```
MERGE(A, p, q, r)
 1  n1 ← q − p + 1
 2  n2 ← r − q
 3  create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
 4  for i ← 1 to n1
 5        do L[i] ← A[p + i − 1]
 6  for j ← 1 to n2
 7        do R[j] ← A[q + j]
 8  L[n1 + 1] ← ∞
 9  R[n2 + 1] ← ∞
10  i ← 1
11  j ← 1
12  for k ← p to r
13        do if L[i] ≤ R[j]
14              then A[k] ← L[i]
15                   i ← i + 1
16              else A[k] ← R[j]
17                   j ← j + 1
```

Analysis of Divide-and-Conquer

- The Merge-Sort algorithm contains a recursive call to itself. When an algorithm contains a recursive call to itself, its running time often can be described by a *recurrence equation, or recurrence.*
- The recurrence equation describes the running time on a problem of size n in terms of the running time on smaller inputs.
- We can use mathematical tools to solve the recurrence and provide bounds on the performance of the algorithm.

Analysis of Divide-and-Conquer

- A recurrence of a divide-and-conquer algorithm is based on its 3 parts: divide, conquer, and combine.
- Let T(n) be the running time on a problem of size n.
- If the problem is small enough, say n <= c, we can solve it in a straightforward manner, which takes constant time, which we write as Θ(1).
- If the problem is bigger, we solve it by dividing the problem to get ***a*** *subproblems, each of which is* *1/**b*** *the size of the original.*
  For Merge-sort, both *a and b are 2*.

Analysis of Divide-and-Conquer

- Described by recursive equation
- Suppose $T(n)$ is the running time on a problem of size n.
- $T(n) = \begin{cases} \Theta(1) & \text{if } n \leq c \\ aT(n/b)+D(n)+C(n) & \text{otherwise} \end{cases}$

  Where a: number of subproblems

  n/b: size of each subproblem

  $D(n)$: cost of divide operation

  $C(n)$: cost of combine operation

Analysis of Merge-sort

- **Base case:** n = 1. Merge sort on an array of size 1 takes constant time, $\Theta(1)$.
- **Divide:** The Divide step of Merge-Sort just calculates the middle of the subarray. This takes constant time.
  So $D(n) = \Theta(1)$.
- **Conquer:** We make 2 calls to Merge-Sort. Each call handles 1/2 of the subarray that we pass as a parameter to the call. The total time required is $2T(n/2)$.
- **Combine:** Running Merge on an n-element subarray takes $\Theta(n)$, so $C(n) = \Theta(n)$.

Analysis of Merge-sort

- Here is what we get:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(1) + \Theta(n) & \text{if } n>1 \end{cases}$$

- By inspection, we can see that we can ignore the Θ(1) factor, as it is irrelevant compared to Θ(n). We can rewrite this recurrence as:

$$T(n) = \begin{cases} c & \text{if } n=1 \\ 2T(n/2) + c(n) & \text{if } n>1 \end{cases}$$

Analysis of Merge-sort

- How many Divide steps?
- Let's assume that n is some power of 2.
- Then for an array of size n, it will take us $\log_2 n$ steps to recursively subdivide the array into subarrays of size 1.

Analysis of MERGE-SORT

Example: $8 = 2^3$

Step 0

Step 1

Step 2

Step 3

Analysis of MERGE-SORT

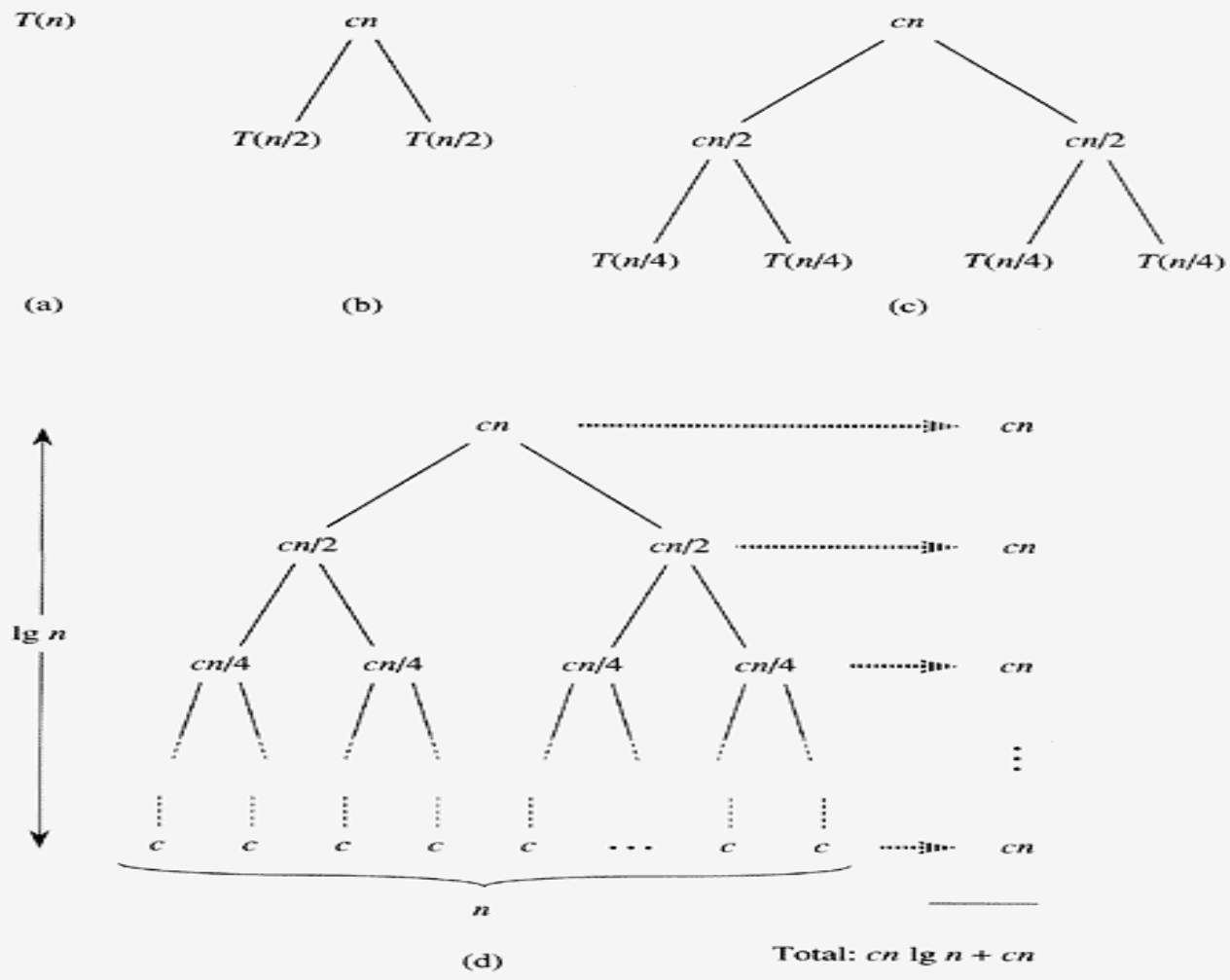


Figure 2.5 The construction of a recursion tree for the recurrence $T(n) = 2T(n/2) + cn$. Part **(a)** shows $T(n)$, which is progressively expanded in **(b)–(d)** to form the recursion tree. The fully expanded tree in part (d) has $\lg n + 1$ levels (i.e., it has height $\lg n$, as indicated), and each level contributes a total cost of cn. The total cost, therefore, is $cn \lg n + cn$, which is $\Theta(n \lg n)$.

Analysis of MERGE-SORT

- So, it took us $\log_2 n$ steps to divide the array all the way down into subarrays of size 1.
- As a result, we will have $\log_2 n + 1$ layers in the recurrence tree.
- Every layer of the recurrence tree it takes us n steps, since we have to put each array item into its proper position within each array.

Analysis of MERGE-SORT

- Consequently, the total cost can be expressed as:
  $cn(\log_2 n + 1)$
  $= cn(\log_2 n) + cn$
- Ignoring the low-order term and the constant c gives:

  $\Theta(n \cdot \log_2 n)$

Summary

- Insertion sort
  - *incremental approach*
  - Worst case : $O(n^2)$ (Reverse sorted list)
  - Best Case : O(n) (Sorted input)
- Merge sort
  - *divide-and-conquer approach (recursive)*
  - Worst case : O(nlogn)
  - Best Case : O(nlogn)