Morse Theory

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June 29, 2019

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0 Introduction

Morse theory reconstructs the topology of a manifold from the level sets of special functions on the manifold. Actually, these Morse functions are not that special, as there are enough of them to approximate any smooth function. The level sets of Morse functions turn out to change shape only at critical values. So, the index of the critical points determines the homology of the manifold. We justify these ideas and apply them to well-known manifolds.

1 Basic Definitions and the Morse Lemma

We briefly review the definition of a critical point. We then define the Hessian, which we use to distinguish between degenerate and non-degenerate critical points.

Definition (Critical point)**.** p is a *critical point* for $f : M \to \mathbb{R}$ if $df|_p : TM|_p \to T\mathbb{R}|_{f(p)}$ is not surjective.

Definition (Hessian)**.** We define the *Hessian* at a critical point p as a bilinear form $f_{**} : TM|_p \times TM|_p \to \mathbb{R}$ that takes $v, w \in TM|_p$ to

$$f_{**}(v,w) := \tilde{V}_p(\tilde{W}(f))$$

, where $\tilde{V}, \tilde{W} \in \mathfrak{X}(M)$ are extensions of v, w.

Now it seems at first that the definition of the Hessian is unnatural, as it does not agree with the Hessian defined in $\mathbb{R}^n$. We show that the Hessian is independent of the extensions $\tilde{V}, \tilde{W} \in \mathfrak{X}(M)$, and agrees with the usual Hessian of $\mathbb{R}^n$.

Proposition. *The Hessian $f_{**}(v,w)$ is symmetric, and does not depend on the extensions $\tilde{V}, \tilde{W}$.*

Proof. Since p is a critical point for f,

$$\tilde{V}_p(\tilde{W}(f)) - \tilde{W}_p(\tilde{V}(f)) = [V,W]_p(f) = 0$$

for fixed $\tilde{V}, \tilde{W}$.
Now we take two different vector field extensions of v,w: $\tilde{V}_1, \tilde{V}_2, \tilde{W}_1, \tilde{W}_2$. Then $\tilde{V}_{1,p} = \tilde{V}_{2,p} = v$, $\tilde{W}_{1,p} = \tilde{W}_{2,p} = w$, so

$$\begin{aligned} \tilde{V}_{1,p}(\tilde{W}_1(f)) &= \tilde{V}_{2,p}(\tilde{W}_1(f)) \\ &= \tilde{W}_{1,p}(\tilde{V}_2(f)) \\ &= \tilde{W}_{2,p}(\tilde{V}_2(f)) \\ &= \tilde{V}_{2,p}(\tilde{W}_2(f)) \end{aligned}$$

□

Now, take a local coordinate chart $(x^1, ..., x^n)$ around p. Then,

$$f_{**}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \frac{\partial^2}{\partial x^i \partial x^j}(f)$$

Therefore, with basis $\{\frac{\partial}{\partial x^i}\}$ for $TM|_p$, the Hessian takes the form

$$(\frac{\partial^2 f}{\partial x^i \partial x^j})_{1 \leq i,j \leq n}$$

We now see that this definition of the Hessian agrees with our usual definition of the Hessian in $\mathbb{R}^n$.

Now we use the Hessian to distinguish between degenerate and non-degenerate critical points.

Definition (Degenerate Critical Point)**.** For $f \in C^\infty(M)$, a critical point p of f is *non-degenerate* if its Hessian has nonzero determinant.

Definition (Morse Function)**.** **f** is a *Morse function* if all of its critical points are non-degenerate.

Now we introduce a lemma important to understanding the behavior of a Morse function near a critical point.

Theorem (Morse Lemma)**.** *For any critical point p of a Morse function f, we can find suitable coordinates $x^1, .., x^n$ such that locally f takes the form*

$$f = -x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2 + f(p)$$

Proof. We use that $\forall f \in C^\infty$(M), $\exists g_1, ..., g_n \in C^\infty$(M) s.t. $g_i(p) = \frac{\partial f}{\partial x^i}$ and

$$f = f(p) + \Sigma x^i g_i$$

locally near p.

Now, take any local coordinate $\phi, (x^1, .., x^n)$ near p, and assume $f(p) = 0$. Then, $\exists g_1, ..., g_n$ s.t.

$$f = \Sigma x^i g_i$$

Since $g_i(p) = \frac{\partial f}{\partial x^i} \mid_p = 0$, take again new functions g_{ij} such that

$$g_i = \Sigma x^j g_{ij}$$

Now we have for $h_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$,

$$f = \Sigma x^i x^j h_{ij}$$

Now we can assume $h_{11}(p) \neq 0$, since p is non-degenerate. Now,

$$\begin{aligned} f \circ \phi^{-1}(x^1, ..., x^n) &= h_{11} x_1^2 + 2(\Sigma x_i h_{1i}) x_1 + H(x_2, ..., x_n) \\ &= \pm X_1^2 + H(x_2, ..., x_n) \end{aligned}$$

where $X_1 = \sqrt{|h_1|}(x_1 + \Sigma x_i \frac{h_{11}}{h_{1i}})$, and H is a quadratic form for $x_2, ..., x_n$. We can do this because $h_{11} \neq 0$ locally near p.

Now, repeating this for $x_2, .., x_n$ gives new coordinates $X_1, ..., X_n$ such that locally near p,

$$f = \pm X_1^2 \pm X_2^2 ... \pm X_n^2$$

Now, we can conclude that the number of minuses in the above equation is equal to the index of the critical point by calculating the Hessian. □

Corollary. [Critical Points of Manifolds]

(i). Non-degenerate critical points are isolated.

(ii). For compact M, Morse functions have finite critical points.

2 Existence of Morse Functions

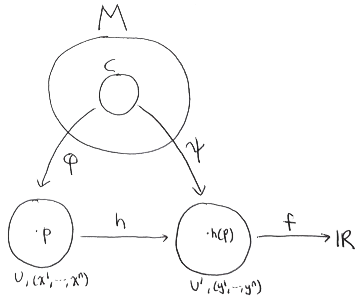
The level sets of Morse functions can be used to determine the topology of the manifold. But do Morse functions exist for any manifold? This section proves that actually, Morse functions form an open dense set of C^∞(M), given a topology called the C^2 topology.

We first introduce the C^2 topology on C^∞(M).

Definition (The C^2 topology on C^∞(M)**)**. Take compact coordinate neighborhoods of M: $(K_1, \phi_1), ..., (K_n, \phi_n)$. For $f \in C^\infty(M), \epsilon > 0$, define $N_f(\epsilon)$ as $\{g \in C^\infty(M) | (g - f) \circ \phi_i^{-1} : \phi(K_i) \to \mathbb{R}$ has $|0, 1, 2^{nd}$ partial derivative$| < \epsilon\}$. The C^2 *topology* is defined as the topology generated by basis $N_f(\epsilon)$.

Of course we have to check well-definedness.

Proposition (Well-definedness of the C^2 topology on C^∞(M))
Let $h : U \to U'$ be a diffeomorphism between open sets of $\mathbb{R}^n$. Then for any compact $K \subset U$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall f \in N_0(\epsilon)$ on h(K), $f \circ h \in N_0(\delta)$ on K.



Proof. Take coordinate charts $(\phi, (x^1, .., x^n))$,$(\psi, (y^1, .., y^n))$, such that $h = \psi \circ \phi^{-1}$. Now, we calculate the partial derivatives of $f \circ h$.

$$\frac{\partial}{\partial x^i}(f \circ h) = \Sigma \frac{\partial f}{\partial y^j} \frac{\partial h^j}{\partial x^i}$$

$$\begin{aligned}
\frac{\partial^2}{\partial x^i \partial x^j} |_p (f \circ h) &= \frac{\partial}{\partial x^j} |_p (\Sigma \frac{\partial f}{\partial y^j} |_{h(p)} \frac{\partial h^j}{\partial x^i} |_p) \\
&= \Sigma \frac{\partial f}{\partial y^k} |_{h(p)} \frac{\partial^2 h^k}{\partial x^i \partial x^j} |_p + \Sigma \frac{\partial}{\partial x^j} (\frac{\partial f}{\partial y^k}) \mid h(p) \frac{\partial h^k}{\partial x^i} |_p \\
&= \Sigma \frac{\partial f}{\partial y^k} |_{h(p)} \frac{\partial^2 h^k}{\partial x^i \partial x^j} |_p + \Sigma\Sigma \frac{\partial^2 f}{\partial y^k \partial y^l} \frac{\partial h^l}{\partial x^j} \frac{\partial h^k}{\partial x^i}
\end{aligned}$$

Now, for $f \in N_0(\epsilon)$,

$$|\frac{\partial f}{\partial y^j}|, |\frac{\partial^2 f}{\partial y^k \partial y^l}| < \epsilon$$

And K is a compact region, so any continuous function on K is bounded. Therefore, we can find suitable $\delta > 0$ such that $f \circ h \in N_0(\delta)$ on K. $\square$

Now, we show that on a compact manifold M with the C^2 topology on C^∞(M), the Morse functions occupy an open dense set.

Proposition. *The set of Morse functions is open in C^∞(M) with the C^2 topology.*

Proof. We first show that for a compact coordinate neighborhood (K, ϕ) and a Morse function f on K, $\exists\ \delta > 0$ such that all of $N_f(\delta)$ are Morse functions on K.

Since K is compact, we can find $\exists\ \epsilon > 0$ s.t.

$$\Sigma\ |\frac{\partial f}{\partial x^i}| + |det(\frac{\partial^2 f}{\partial x^i \partial x^j})| > 2\epsilon$$

Now, we can take δ small enough so that $\forall g \in N_f(\delta)$,

$$\Sigma\ |\frac{\partial g}{\partial x^i}| + |det(\frac{\partial^2 g}{\partial x^i \partial x^j})| > \epsilon$$

on K. At any critical point of g in K, $\frac{\partial g}{\partial x^i} = 0$, so

$$|det(\frac{\partial^2 g}{\partial x^i \partial x^j})_p| > 0$$

Therefore, $\forall g \in N_f(\delta)$ is a Morse function on K.

Now, for any Morse function f, we can find an open nbd $N_1 \underset{open}{\subset} C^\infty$(M) such that $\forall g \in N_1$ is Morse on K_1. Repeat for $K_2, ..., K_n$, and we have an open neighborhood of $f \in C^\infty$(M) such that all elements are Morse functions. $\square$

Proposition. *Let $f : U \underset{open}{\subset} \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Then, for almost every $L \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $f + L$ is a Morse function.*

Proof. Define $\Omega : \{(p, L) | d(f + L)|_p = 0\} \subset U \times Hom(\mathbb{R}^n, \mathbb{R})$, and $\pi : \Omega \to Hom(R^n, R) : (p, L) \mapsto L$. Now, assume $f + L$ has a degenerate critical point p. Then,

$$(\frac{\partial^2 (f + L)}{\partial x^i \partial x^j})\ |_p = (\frac{\partial^2 f}{\partial x^i \partial x^j})\ |_p$$

is a singular matrix. Meanwhile on Ω, $L = dL = -df$, so $\pi : (p, L) \mapsto L = -df\ |_p$, and

$$d\pi\ |_{(p,L)} = (-\frac{\partial^2 f}{\partial x^i \partial x^j})\ |_p$$

Therefore, $f + L$ having a degenerate critical point p is equivalent to L being a critical value for π. By Sard's theorem, the set of L that are critical values of π have measure zero.

We conclude that almost every $f + L$ is a Morse function. $\square$

Proposition. *$\forall f \in C^\infty(M)$, open nbd $N_f \subset C^\infty(M)$, there is a Morse function $g \in N_f$.*

Proof. Let (U, ϕ) be an open coordinate nbd in M, and let $K \subset U$ be a compact subset of M. Let ρ be a bump function such that $\rho(K) = 1$, and $\rho(M \setminus U) = 0$. Now, for small $L : \mathbb{R}^n \to \mathbb{R}$, we show that

$$f_1(p) := f(p) + \rho(p)L(\phi(p))$$

is Morse on K. Assume $L(x_1, ..., x_n) = \Sigma l_i x_i$. Then,

$$L(\phi(p)) = \Sigma l_i x^i(p)$$

Now, we show that by taking l_i to be small enough, we can put f_1 in any open nbd of $f \in C^\infty$(M). We first calculate the partial derivatives of $f_1 - f$.

$$\frac{\partial}{\partial x^i} |_p (f_1 - f) = \frac{\partial \rho}{\partial x^i} |_p L(\phi(p)) + \rho(p) l_i$$

$$\frac{\partial^2}{\partial x^i \partial x^j} |_p (f_1 - f) = \frac{\partial^2 \rho}{\partial x^i \partial x^j} |_p L(\phi(p)) + l_i \frac{\partial \rho}{\partial x^j} |_p + l_j \frac{\partial \rho}{\partial x^i} |_p$$

On a compact region, all terms but l_i, l_j are continuous, therefore bounded. Therefore by taking l_i small, we can put f_1 in any open nbd $N_f \subset C^\infty$(M). Now, openness of Morse functions guarantee

$$N_{f_1} \underset{open}{\subset} C^\infty(M), f \in N_{f_1}$$

that is Morse on K.

To summarize, for any compact coordinate nbd (K, ϕ), we can find an open set $\exists N_1 \ni f$ of Morse functions on K.

Repeat this for compact coordinate nbd $K_1, ..., K_n$ to find a Morse function on $K_1 \cup ... \cup K_n = M$. □

3 Homotopy Type of Level Sets

Now we can finally check how the shape of the manifold changes along the level sets of the Morse function. However, we first have to alter the Morse function slightly so that the critical points occur at different critical values.

Proposition
Let M be a compact manifold. Let f be a Morse function on M. Then for any $\epsilon > 0$, we can find a Morse function g ϵ-close to f such that

(i). g *and* f *have the same critical points* $p_1, ..., p_n$

(ii). $g(p_i) \neq g(p_j)$ $(\forall i, j)$

Proof. We know that there are only finitely many critical points. So we modify the function locally at each critical point to obtain different critical values. Now assume $f(p_1) = f(p_2)$. Take open $U_1 \subset V_1$ such that $\bar{V_1}$ contains critical point only p_1, and a bump function ρ having constant value 1 on U_1, and support in $\bar{V_1}$. Define

$$f_1 = f + \epsilon\rho$$

Since critical values are finite, we can take ϵ so small that $f_1(p_1)$ has different value from all the other critical values. Also, since the Morse functions are open dense in the C^2 topology on C^∞(M), we can take ϵ small enough that f_1 is a Morse function. So we need to check only if f_1 and f has the same critical points. Now,

$$\frac{\partial f_1}{\partial x^i} = \frac{\partial f}{\partial x^i} + \epsilon \frac{\partial \rho}{\partial x^i}$$

On ${\bar{V_1}}^c$ and U_1, ρ is constant, and f_1 and f have the same critical points.

Now, $\bar{V_1} \underset{closed}{\subset} M$, so $\bar{V_1} \setminus U_1$ is compact. Therefore, we can find $\epsilon > 0$ s.t.

$$min(\Sigma|\frac{\partial f}{\partial x^i}|) > \epsilon \; max(\Sigma|\frac{\partial \rho}{\partial x^i}|)$$

on $\bar{V_1} \setminus U_1$. For that ϵ,

$$\frac{\partial f_1}{\partial x^i} = \frac{\partial f}{\partial x^i} + \epsilon \frac{\partial \rho}{\partial x^i} > 0$$

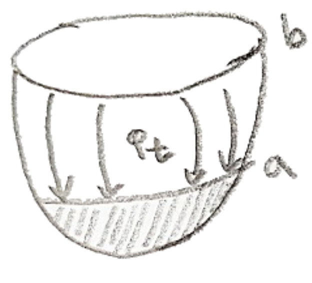
so f_1 has no critical points in $\bar{V_1} \setminus U_1$.

Therefore we have constructed f_1 that has same critical points with f, and $f_1(p_1) \neq f_2(p_2)$. We can repeat this for all the critical points to get our desired function. □

The next step to take now is to use this Morse function with different critical values to analyze the homology of the level sets of the Morse function. We first study the case where the change in the level set does not pass through a critical point.

Proposition

For a Morse function f on a manifold M, if $f^{-1}[a,b]$ is compact, then the inclusion map $\iota : M^a \to M^b$ is a deformation retract.



Proof. Take a Riemmanian metric on M. Then, the Riemmanian metric induces a canonical pairing between TM and TM^*. Now define $\nabla f := (df)^*$. And take compact K that surrounds $f^{-1}[a,b]$. Then we can find a function ρ that has support in K, and takes value $1/\| \nabla f \|^2$ in $f^{-1}[a,b]$. Now define a vector field $X_q := \rho(q)\nabla f(q)$. Then this vector field has support in K, therefore has compact support. Therefore, this vector field induces a global flow ϕ_t. Now,

$$df(\phi_t(q))/dt = < d\phi_t(q)/dt, \nabla f > = < X_q, \nabla f > = 1$$

Therefore, $\phi_{b-a}(M^b)$=M^a, and $\iota : M^a \to M^b$ is a deformation retract. □

Proposition

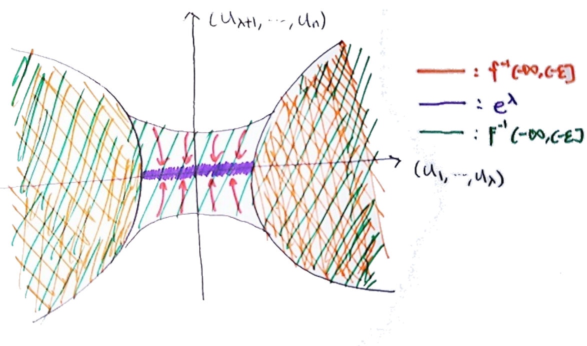
Let f be a Morse function on M, and let p be a critical point of index λ. Let $f(p) = c$. Assume $\epsilon > 0$ such that $f^{-1}[c - \epsilon, c + \epsilon]$ is compact and contains no other critical point than p. Then,

$$M^{c+\epsilon} \cong M^{c-\epsilon} \sqcup e^{\lambda}$$

Proof. Construct $\mu : \mathbb{R} \to \mathbb{R}$ so that

(i). $\mu(0) > 0$, $\mu(r) = 0$ ($\forall r > 2\epsilon$)

(ii). $-1 < \mu^{'} \leq 0$



Now let $F = f - \mu(u_1^2 + ... + 2(u_{\lambda+1}^2 + ... + u_n^2))$, where $u^1, .., u^n$ satisfy $f = -u_1^2 - ... - u_\lambda^2 + u_{\lambda+1}^2 + ... + u_n^2 + f(p)$ locally near p. We show that F has same critical points as f, and $F^{-1}(-\infty, c + \epsilon] = f^{-1}(-\infty, c + \epsilon]$.

(1) F and f have the same critical points.
We calculate the partial derivatives of F.

$$\frac{\partial F}{\partial u^i} |_q = \begin{cases} -2u^i - 2\mu^{'}(q)u^i & \text{if } 1 \leq i \leq \lambda \\ 2u^j - 4\mu^{'}(q)u^j & \text{if } \lambda + 1 \leq j \leq n \end{cases}$$

Because $-1 < \mu^{'} \leq 0$, we can see that

$$\frac{\partial F}{\partial u^i} |_q = 0 \iff u^i = 0$$

(2) $F^{-1}(-\infty, c + \epsilon] = f^{-1}(-\infty, c + \epsilon]$
Since $f \geq F$, it is enough to show $f > c + \epsilon \implies F = f$. But,

$$\begin{aligned} f > c + \epsilon \iff & \sum_{\lambda+1}^{n} u_j^2 - \sum_{1}^{\lambda} u_i^2 > \epsilon \\ & \implies u_1^2 + ... + 2(u_{\lambda+1}^2 + ... + u_n^2) > 2\epsilon \\ & \implies F = f - \mu(u_1^2 + ... + 2(u_{\lambda+1}^2 + ... + u_n^2)) = f \end{aligned}$$

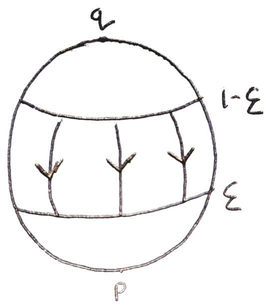
Therefore, we have shown that $F^{-1}(-\infty, c + \epsilon] = f^{-1}(-\infty, c + \epsilon]$.

Now, we see in the picture that

$$\begin{aligned} f^{-1}(-\infty, c+\epsilon] &= F^{-1}(-\infty, c+\epsilon] \\ &\cong F^{-1}(-\infty, c-\epsilon] \\ &\cong f^{-1}(-\infty, c-\epsilon] \sqcup e^{\lambda} \end{aligned}$$

□

Theorem (The Sphere Lemma). *Let M be a compact manifold of dimension n. Then, if a Morse function with only 2 critical points exists, $M \cong S^n$.*



Proof. We can assume that the 2 critical points p, q are where the function f takes its min, max values. Let $f(p) = 0$, $f(q) = 1$. Now, the Morse Lemma shows that $M^{\epsilon} \cong D^n, f^{-1}[1-\epsilon, 1] \cong D^n$.

Now since there are no critical values in $[\epsilon, 1-\epsilon]$,

$$M^{1-\epsilon} \cong M^{\epsilon} \cong D^n$$

Therefore, M is homeomorphic to 2 n-discs attached around their boundary, which is homeomorphic to the n-sphere. □

4 Examples

We now use the machinery that we developed to determine the CW structure of several important manifolds.

4.1 $\mathbb{RP}^n$

We will briefly review the manifold structure of $\mathbb{RP}^n$. Define a coordinate chart on $\mathbb{RP}^n$ by

$$\begin{aligned} &U_i := \{[x_1, ..., x_{n+1}] \in \mathbb{RP}^n \mid x_i \neq 0\} \\ &\phi_i : U_i \to \mathbb{R}^n : [x_1, ..., x_{n+1}] \mapsto (\frac{x_1}{x_i}, ..\frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, .., \frac{x_{n+1}}{x_i}) \end{aligned}$$

We can also treat $\mathbb{RP}^n$ as the continuous image of

$$\pi : S^n \to \mathbb{RP}^n$$

so $\mathbb{RP}^n$ is compact, as the continuous image of a compact set. Now define $f : \mathbb{RP}^n \to \mathbb{R}$ by

$$f([x_1, ..., x_{n+1}]) = \frac{\Sigma a_i x_i^2}{\Sigma x_i^2}$$

for $a_1 < ... < a_{n+1}$. We claim that f is a Morse function.

For (U_i, ϕ_i), we have

$$f \circ \phi_i^{-1} : (x_1, ..., x_n) \mapsto \frac{a_1 x_1^2 + ... + a_i + a_{i+1} x_i^2 + ... + a_{n+1} x_n^2}{\sum x_k^2 + 1}$$

Therefore, we can calculate the partial derivatives for $\alpha < i$ as

$$\frac{\partial f}{\partial x_\alpha} = \frac{2x_\alpha((a_\alpha - a_1)x_1^2 + ... + a_\alpha - a_i + ... + (a_\alpha - a_{n+1})x_n^2)}{(\sum x_k^2 + 1)^2} \tag{1}$$

The subscripts differ slightly for $\alpha \geq i$. Now, when $\alpha = n+1$, $a_1 < ... < a_{n+1}$ means that

$$\frac{\partial f}{\partial x_n} = 0 \iff x_n = 0$$

Repeating this for $\alpha = n-1, ..., 1$, we find that

$$\frac{\partial f}{\partial x_n} = \frac{\partial f}{\partial x_{n-1}} = ... \frac{\partial f}{\partial x_1} = 0 \iff x_1 = ... = x_n = 0$$

Therefore, the critical points of f are [0, 0, ..., 1, 0, ..., 0]. Now calculate the Hessian at each critical point.

$$\frac{\partial^2 f}{\partial {x_\alpha}^2} = \frac{(2(a_\alpha - a_1)x_1^2 + .. + a_\alpha - a_i + .. + (a_\alpha - a_{n+1})x_n^2)(\sum x_k^2 + 1) - 4x_\alpha^2 \triangle}{(\sum x_k^2 + 1)^3} \tag{2}$$

where $\triangle$ is the numerator in (1). Therefore, we evaluate (2) at [0, 0, ..., 1, 0, ..., 0] to get

$$\frac{\partial^2 f}{\partial {x_\alpha}^2} \mid_{(0,...,0)} = 2(a_\alpha - a_i)$$

We omit the other calculations. So, the Hessian is a $n \times n$ diagonal matrix

$$\begin{pmatrix} 2(a_0 - a_i) & & & \\ & 2(a_1 - a_i) & & \\ & & \ddots & \\ & & & 2(a_{n+1} - a_i) \end{pmatrix}$$

Since $a_q < ... < a_{n+1}$, we can see that the diagonal entries are all nonzero. Therefore, all the critical points are nondegenerate, and f is a Morse function. For this f, $\mathbb{RP}^n$ has $n+1$ critical points with index $0, 1, ..., n$. We conclude that $\mathbb{RP}(n)$ has a CW structure with a cell in each dimension.

4.2 SO(n)

Since SO(n) is a closed, bounded subset of $\mathbb{R}^{n^2}$, it is compact. Now we review the manifold structure of SO(n) . Define

$$B_\theta^{(ik)} = \begin{pmatrix} \cos\theta & & -\sin\theta \\ & \ddots & \\ \sin\theta & & \cos\theta \end{pmatrix}$$

where the nonzero entries are where the i,k-th columns meet the i,k-th rows. Now we define for $\frac{n(n-1)}{2}$ variables $\theta, \varphi, \psi, ...$

$$\Phi_A : \mathbb{R}^{n(n-1)/2} \to \mathrm{SO}(n) : (\theta, \varphi, \psi, ...) \mapsto A \cdot B_\theta^{(ik)} \cdot B_\varphi^{(jl)} ...$$

We can check that Φ_A indeed form an atlas for $SO(n)$. Now we define f : $\mathrm{SO}(n) \to \mathbb{R}$ by

$$f(A) = \sum c_i x_{ii}$$

for fixed $1 < c_1 < c_2 < ... < c_n$. We claim that f is a Morse function on SO(n). First, calculate the partial derivatives relative to Φ_A.

$$\begin{aligned} \frac{\partial f \circ \Phi^{-1}}{\partial \theta} &= \frac{d}{d\theta} f(A \cdot B_\theta^{(ik)}) \mid_{\theta=0} \\ &= f(A \frac{d}{d\theta} B_\theta^{(ik)} \mid_{\theta=0}) \\ &= f(A \begin{pmatrix} -\sin\theta & & -\cos\theta \\ & \ddots & \\ \cos\theta & & -\sin\theta \end{pmatrix}) \\ &= c_i x_{ik} - c_k x_{ki} \end{aligned}$$

Therefore, we conclude that the critical points of f satisfy

$$c_i x_{ik} = c_k x_{ki}$$

We likewise obtain that at a critical point,

$$\frac{d}{d\theta} f(B_\theta^{(ik)} \cdot A) \mid_{\theta=0} = -c_i x_{ki} + c_k x_{ik} = 0$$

Since $c_i < c_k$, we conclude that the critical points of f satisfy $x_{ik} = x_{ki}$ for $\forall i < k$. Therefore, the critical points of f in SO(n) are the diagonal matrices. Now, we calculate the Hessian.

Take a critical point given as a diagonal matrix with diagonal entries $\epsilon_1, ..., \epsilon_n$. Now

$$\begin{aligned} &\frac{\partial^2 f}{\partial\theta\partial\varphi}(A \cdot B_\theta^{(ik)} \cdot B_\varphi^{(jl)}) \mid_{(\theta,\varphi)=(0,0)} \\ &= f(A \cdot \frac{d}{d\theta} B_\theta^{(ik)} \mid_{\theta=0} \cdot \frac{d}{d\varphi} B_\varphi^{(jl)} \mid_{\varphi=0}) \\ &= \begin{cases} 0 & \text{if } 1(i,k) \neq (j,l) \\ -c_i\epsilon_i - c_i\epsilon_k & \text{if } (i, k = (j,l)) \end{cases} \end{aligned}$$

Therefore, the Hessian at a critical point

$$\begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_n \end{pmatrix}$$

is the $n(n-1)/2 \times n(n-1)/2$ matrix

$$\begin{pmatrix} -c_1\epsilon_1 - c_2\epsilon_2 & & & \\ & -c_1\epsilon_1 - c_3\epsilon_3 & & \\ & & \ddots & \\ & & & -c_{n-1}\epsilon_{n-1} - c_n\epsilon_n \end{pmatrix}$$

and its diagonal entries are nonzero because $1 < c_1 < c_2 < ... < c_n$. Therefore, all critical points are nondegenerate, and f is a Morse function on SO(n).

Now we calculate the index of each critical point. Assume that $\epsilon_{i_1} = .. = \epsilon_{i_k} = 1$, and the other entries have value -1. Then there are $(i_1-1)+...+(i_k-1)$ negative entries in the Hessian. Therefore, we can determine the number of m-cells in the CW structure of SO(n) by a combinatorical argument. For example, SO(3) has a CW structure with one cell in each dimension $0, 1, 2, 3$. This matches the CW structure of $\mathbb{RP}(3)$ we calculated in the previous section. In fact, $\mathbb{RP}(3)$ and SO(3) are diffeomorphic.

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