# Universal Coefficient Theorem for Cohomology

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## 1 Free Modules are Projective

Let $R$ be a commutative ring, and let $M, N$ be two-sided $R-$modules. Let $F$ be a free $R-$module such that

$$\mathbf{0} \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow \mathbf{0}$$

is a short exact sequence. Then for any $R-$module $L$,

$$\mathbf{0} \longrightarrow \mathrm{Hom}(F, L) \longrightarrow \mathrm{Hom}(N, L) \longrightarrow \mathrm{Hom}(M, L) \longrightarrow \mathbf{0}$$

is also a short exact sequence. Therefore, all free modules are projective. It follows that

$$\mathrm{Ext}^i_R(F, \_\_) = 0$$

## 2 Ext Measures the Non-exactness of Hom

We take the short exact sequence

$$\mathbf{0} \longrightarrow \mathrm{im}\,\partial_n \xrightarrow{\iota_n} \ker\partial_{n-1} \longrightarrow H_n(C) \longrightarrow \mathbf{0}$$

Now, we can make a long exact sequence by the derived functor Ext. However, as $\ker\partial_{n-1}$ is a subgroup of the free group $C_n$, it is also free. Therefore,

$$\mathrm{Ext}^1_R(\ker\partial_{n-1}, G) = 0$$

Then, the derived functor *Ext* conjures the following sequence

$$\mathbf{0} \longrightarrow \mathrm{Hom}(H_n(C), G) \longrightarrow \mathrm{Hom}(\ker\partial_{n-1}, G) \xrightarrow{\iota_n^*} \mathrm{Hom}(\mathrm{im}\,\partial_n, G) \longrightarrow \mathrm{Ext}^1_R(H_n(C), G) \longrightarrow \mathbf{0}$$

. The short exact sequence gives the relations

$$\ker\iota_n^* \cong \mathrm{Hom}(H_n(C), G)$$
$$\mathrm{coker}\,\iota_n^* \cong \mathrm{Ext}^1_R(H_n(C), G)$$

## 3 The Universal Coefficient Theorem for Cohomology

**Proposition.** *(Universal Coefficient Theorem)*

$$\mathbf{0} \longrightarrow \operatorname{Ext}^1_R(H_{n-1}(C), G) \longrightarrow H^n(C, G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \longrightarrow \mathbf{0}$$

*is a split exact sequence.*

*Proof.* We consider the long exact sequence in cohomology

$$\xleftarrow{\iota_n^*} \operatorname{Hom}(Z_n, G) \longleftarrow H^n(C, G) \longleftarrow \operatorname{Hom}(B_{n-1}, G) \xleftarrow{\iota_{n-1}^*} \operatorname{Hom}(Z_{n-1}, G)$$

. This leads to a sequence of short exact sequences

$$\mathbf{0} \longrightarrow \operatorname{coker} \iota_{n-1}^* \longrightarrow H^n(C, G) \longrightarrow \ker \iota_n^* \longrightarrow \mathbf{0}$$

We have shown above that

$$\ker \iota_n^* \cong \operatorname{Hom}(H_n(C), G)$$
$$\operatorname{coker} \iota_n^* \cong \operatorname{Ext}(H_n(C), G)$$

. Therefore we obtain the desired short exact sequence

$$\mathbf{0} \longrightarrow \operatorname{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C, G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \longrightarrow \mathbf{0}$$

.

Now we show that the short exact sequence above is split. We define $Z_n = \ker \partial_{n-1}$, $B_n = \operatorname{im} \partial_n$. Then, the short exact sequence

$$\mathbf{0} \longrightarrow Z_n \xrightarrow{\iota_n} C_n \longrightarrow B_n \longrightarrow \mathbf{0}$$

is split, because $B_n$ is a subgroup of the free group $C_n$. Therefore, we can find a left-inverse $\pi$ for $\iota_n$ such that $\pi \circ \iota_n = id_{Z_n}$. Now

$$\operatorname{Hom}(H_n(C, G)) \longrightarrow \ker \partial_n^* \xrightarrow{\pi} H^n(C, G)$$

is a right-inverse for $h$. Therefore, we can see that the exact sequence is split. □

## References


[1] Paolo Aluffi. *Algebra: Chapter 0.* Graduate Studies in Mathematics, 104, AMS, 2009.

[2] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[3] Charles Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, 1995.