

# MONTE CARLO SIMULATIONS OF THE 2D XY MODEL USING MARKOV-CHAINS

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## Introduction

The 2D XY model is a system that exhibits many interesting physical phenomena, for example phase transitions [2] which are studied in this task. These types of systems are studied with Monte Carlo simulations using the principles of the Metropolis algorithm, where Markov chains are used to generate spin configurations [1]. The focus was primarily on measuring the magnetization and energy as a function of temperature for the system and attempting to use the function called Binder cumulant to determine the critical temperature [1]. The 2D XY model is of special interest because of its topological properties resulting in the Berezinski - Kosterlitz - Thouless (BKT) transition which exhibits similar characteristics found in liquid films and 2D superconductors.

## Theory

The spins of the system is layered on a square lattice with nearest neighbour interactions and with periodic boundary conditions. The layout is illustrated in Figure (1) where each spin represented by a dot has two components  $S_x$  and  $S_y$  acting like a magnetic arrow.

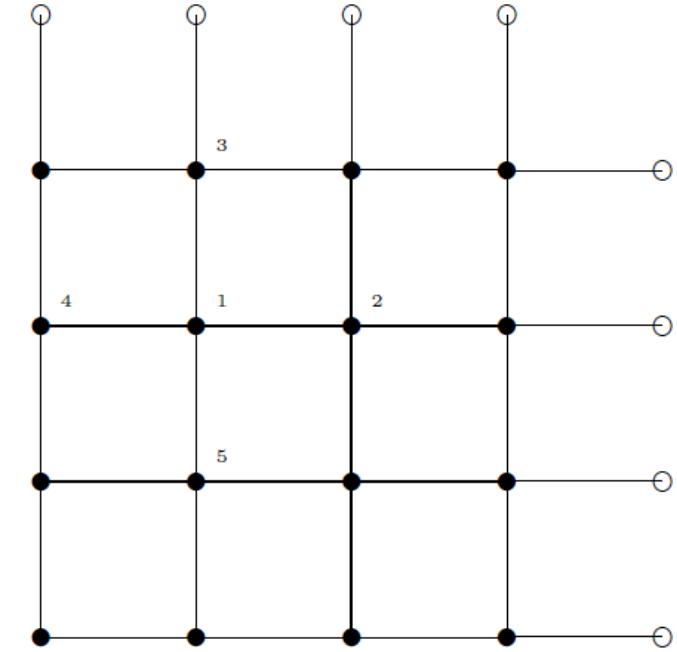


Fig. 1: Spins on a square lattice with nearest neighbour interaction.

This study uses an angular representation where each spin is assigned an angle  $\theta \in [0, 2\pi]$  in the XY-plane. The 2D XY model can be described by the Hamiltonian given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) + H \sum_i \cos(\theta_i), \quad (1)$$

where  $\langle i, j \rangle$  denotes summation over the nearest neighbours,  $H$  represents an external magnetic field ( $H = 0$  is used) and the variable  $J$  represents the coupling constant ( $J = 1$ ). Each new state is generated by changing the angle of one spin using a random walking algorithm to a new angle  $\theta \in [0, 2\pi]$  randomly and calculating the energy difference  $\Delta\mathcal{H} = \mathcal{H}_{\text{new}} - \mathcal{H}_{\text{old}}$ . The value for  $\Delta\mathcal{H}$  is then used with the Metropolis algorithm to determine if the new state is accepted or discarded. The system starts in a randomly generated state and is iterated on until the system is in equilibrium. Measurements of the magnetization and energy are done in the equilibrium state.

## Method

The critical temperature ( $\tau_c$ ) is determined by measuring the Magnetic susceptibility ( $\chi$ ), Specific heat capacity ( $C_v$ ) and the Binder cumulant ( $g$ ) [3, 1] defined as

$$\chi = \frac{L^2(\langle M^2 \rangle - \langle M \rangle^2)}{\tau} \quad (2), \quad C_v = \frac{\langle E^2 \rangle - \langle E \rangle^2}{L^2 \tau^2} \quad (3), \quad g = 1 - \frac{\langle M^4 \rangle}{3 \langle M^2 \rangle^2} \quad (4),$$

where  $L^2$  denotes lattice size,  $\tau$  is the fundamental temperature [2],  $\langle M \rangle$  and  $\langle E \rangle$  denotes the average magnetization and energy per spin.

The critical temperature is at the value  $\tau$  where the Binder cumulant for different lattices sizes cross, specifically for systems which undergoes a first or second order phase transition.

The critical exponents can be determined by using the finite size scaling relations according to [1]

$$g \sim \tilde{G} [L^{1/\nu} (\tau - \tau_c)] \quad (4), \quad \chi \sim L^{\gamma/\nu} \tilde{C} [L^{1/\nu} (\tau - \tau_c)] \quad (5)$$

where the values for the exponents  $\nu$  and  $\gamma$  are correct when the magnetic susceptibility and Binder cumulant form a universal curve for different lattice sizes. However, due to the nature of the phase transition in the XY model, it is not possible to extract the exponents with the usual power laws and is consequently excluded from the analysis.

## Results

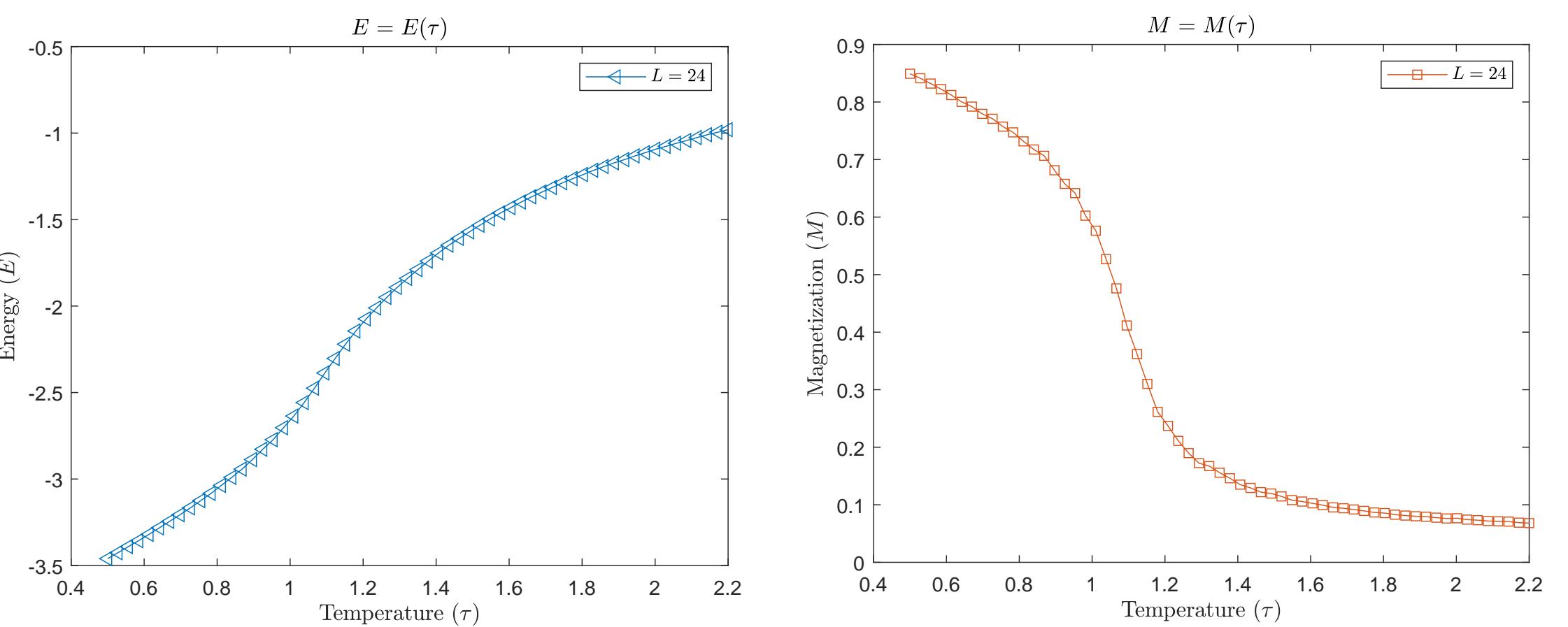


Fig. 2: Left: Average energy per spin as a function of temperature.  
Right: Average magnetization per spin as a function of temperature.

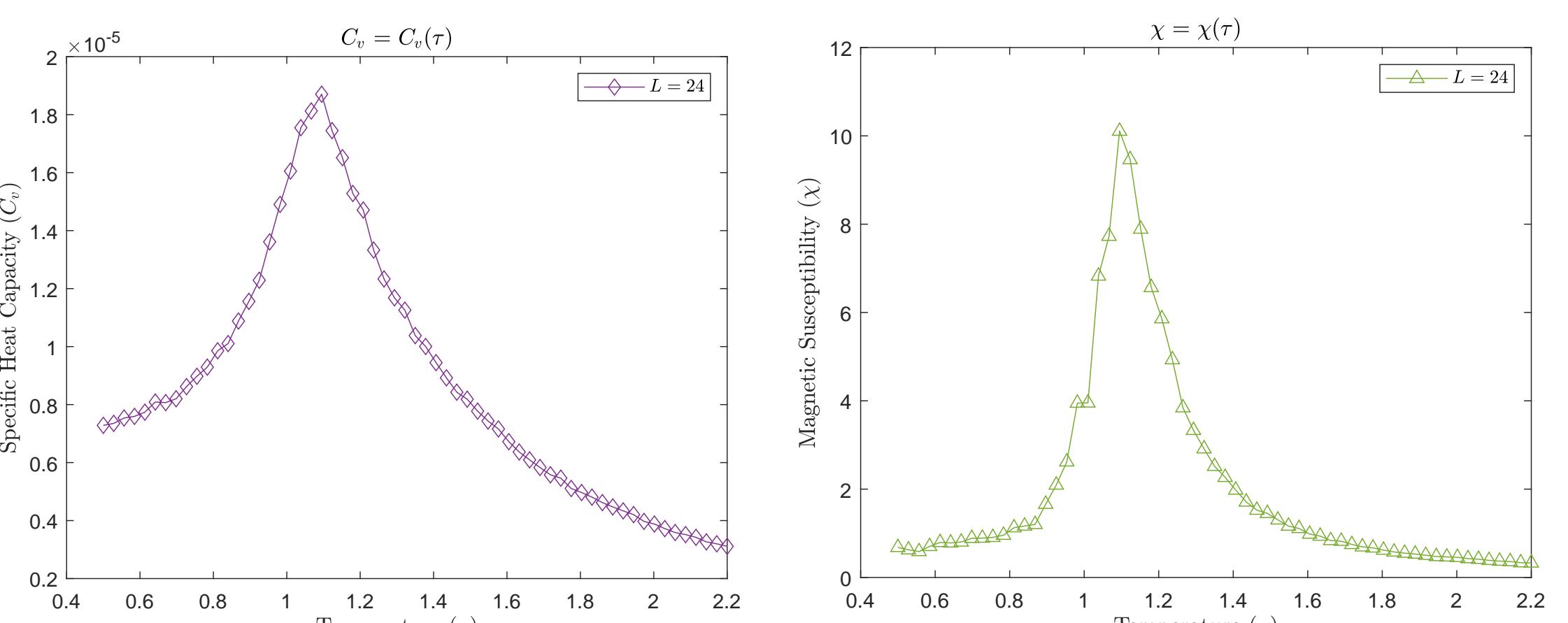


Fig. 3: Left: Average magnetic susceptibility per spin as a function of temperature.  
Right: Average specific heat capacity per spin as a function of temperature.

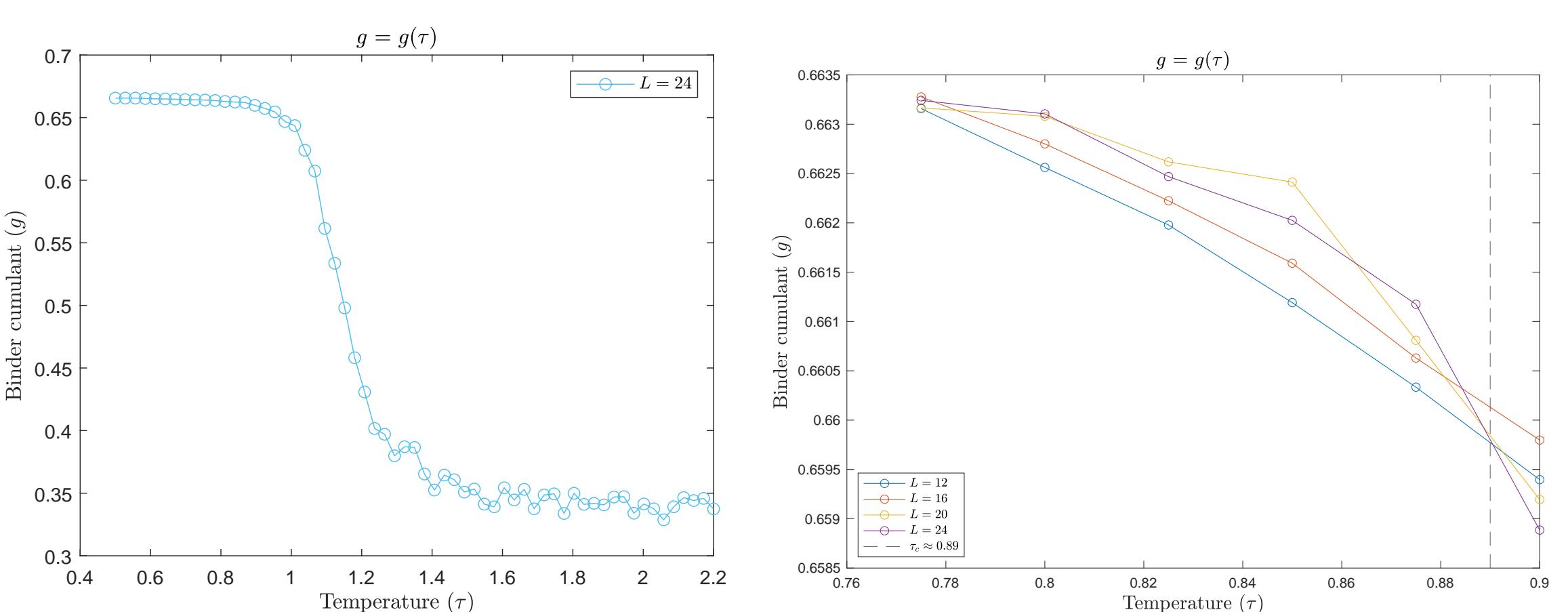


Fig. 4: Left: Binder cumulant as a function of temperature for lattice size  $L = 24$ .  
Right: Binder cumulant as a function of temperature for lattice sizes  $L = 12 - 24$ . The crossing value shows the critical temperature  $\tau_c \approx 0.89$ .

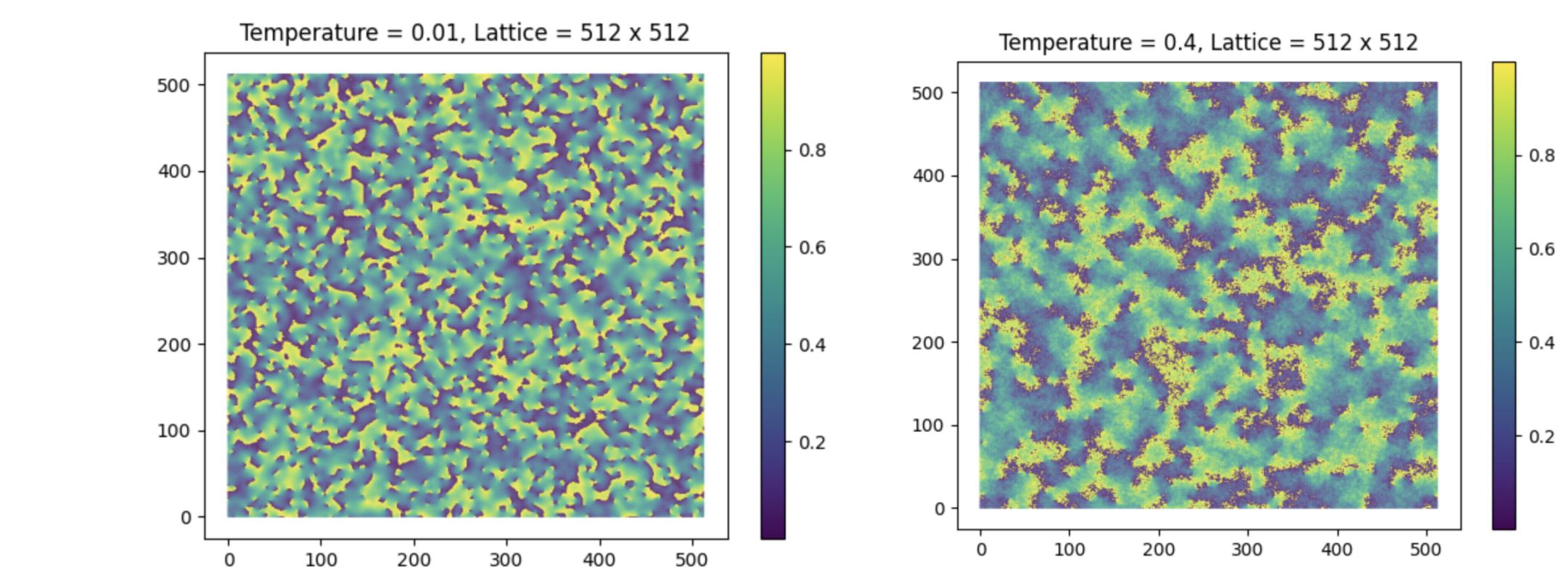


Fig. 5: The figures show the lattice with  $512 \times 512$  spins. The colorbar represents the scaled angle for each spin.  
The fundamental temperature for the left figure is  $\tau = 0.01$  and right figure  $\tau = 0.4$ .

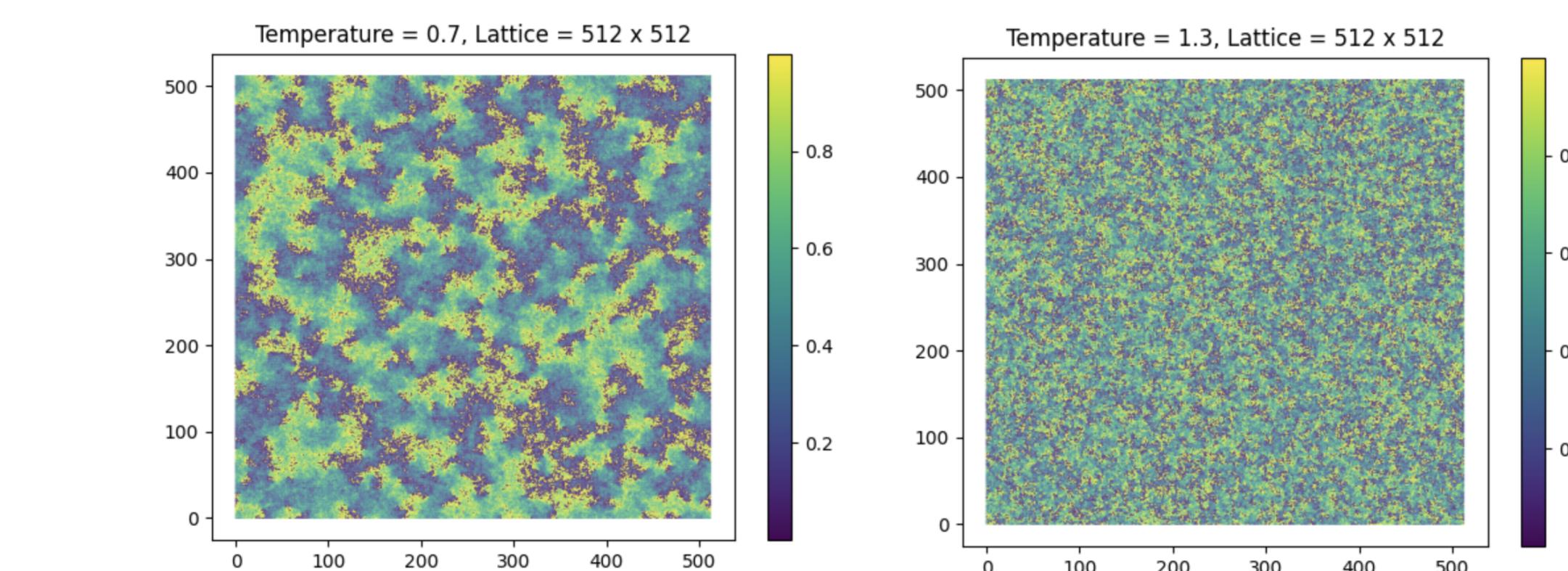


Fig. 6: The figures show the lattice with  $512 \times 512$  spins. The colorbar represents the scaled angle for each spin.  
The fundamental temperature for the left figure is  $\tau = 0.7$  and right figure  $\tau = 1.3$ .

## Conclusions

The graph in Figure (4) shows the Binder cumulant for different lattice sizes in the range  $L = 12 - 24$ . The critical temperature in the thermodynamic limit is at the value where the Binder cumulant crosses for different lattice sizes, which was determined to be  $\tau_c \approx 0.89$ . This value is in agreement with estimations of the critical temperature found in various sources, but the analysis of the critical temperature is not complete and should be complemented with an analysis of the helicity modulus  $\Upsilon$ .

The Figures (5) and (6) shows the spins for a system size of  $512 \times 512$  with colors representing the angle  $\theta \in [-\pi, \pi]$  for each spin. The BKT transition is characterised by the binding and unbinding of topological defects in the form of vortices and anti-vortices, with a vortex characterised by spins whirling around a point, and the antivortex being the same but having the spin orientation in the opposite direction. At low temperatures the vortices are bound in pairs leading to a "quasi-ordered" phase with long-range correlation of the spin orientations, seen in Figure (5) and (6). At the critical temperature  $\tau_c$ , a transition occurs where the vortex-antivortex pairs "unbind" leading to a disordered phase which destroys the long-range correlation.

The 2D XY model does not exhibit a normal second order transition leading to symmetry breaking, but instead undergoes a phase transition of infinite order from a quasi-ordered state characterized by topological defects.

The simulation was done in MATLAB and the source files can be found at <https://github.com/kimyliv>.

## References