Ch 34. NP-completeness
Ch 35. Approximation Algorithms

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Grigori Perelman

From Wikipedia, the free encyclopedia

In this [Eastern Slavic name], the [patronymic] is Yakovlevich *and the [family name] is* Perelman.

Grigori Yakovlevich Perelman (Russian: Григорий Яковлевич Перельман, IPA: [ɡrʲɪˈɡorʲɪj ˈjakəvlʲɪvʲɪtɕ pʲɪrʲɪlʲˈman] (listen); born 13 June 1966) is a Russian mathematician who is known for his contributions to the fields of geometric analysis, Riemannian geometry, and geometric topology.

In the 1990s, partly in collaboration with Yuri Burago, Mikhael Gromov, and Anton Petrunin, he made influential contributions to the study of Alexandrov spaces. In 1994, he proved the soul conjecture in Riemannian geometry, which had been an open problem for the previous 20 years. In 2002 and 2003, he developed new techniques in the analysis of Ricci flow, thereby providing a detailed sketch of a proof of the Poincaré conjecture and Thurston's geometrization conjecture, the former of which had been a famous open problem in mathematics for the past century. The full

Grigori Perelman

Grigori Perelman in 1993

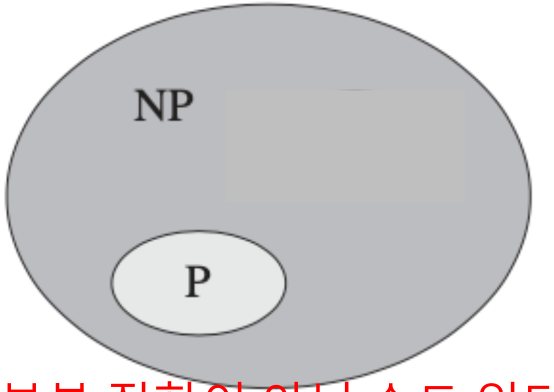
Born 13 June 1966 (age 54)
Leningrad, Soviet Union

class P problems

- class P (Polynomial time class)

  input size n 에 대해서 $O(n^k)$ 에 해를 찾을 수 있는 문제들

class NP problems



진부분 집합이 아닐 수도 있다.

- class P

  input size n 에 대해서 $O(n^k)$ 에 해를 찾을 수 있는 문제들
- class NP : Nondeterministic Polynomial time class

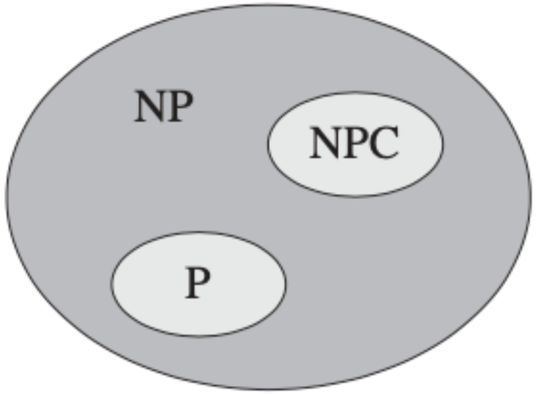
  $O(n^k)$ 에 verifiable 한 문제들

  “verifiable” : 해의 후보가 주어지면 그것이 해인지를 확인할 수 있음

  = nondeterministic Turing machine 으로 polynomial time 에 풀 수 있는 decision problems

$$P \subseteq NP$$

NP-completeness(class NPC)



하나의 가능성

- class P (Polynomial time class)

  input size n 에 대해서 $O(n^k)$ 에 풀 수 있는 문제들

- class NP : Nondeterministic Polynomial time class

  $O(n^k)$ 에 verifiable 한 문제들

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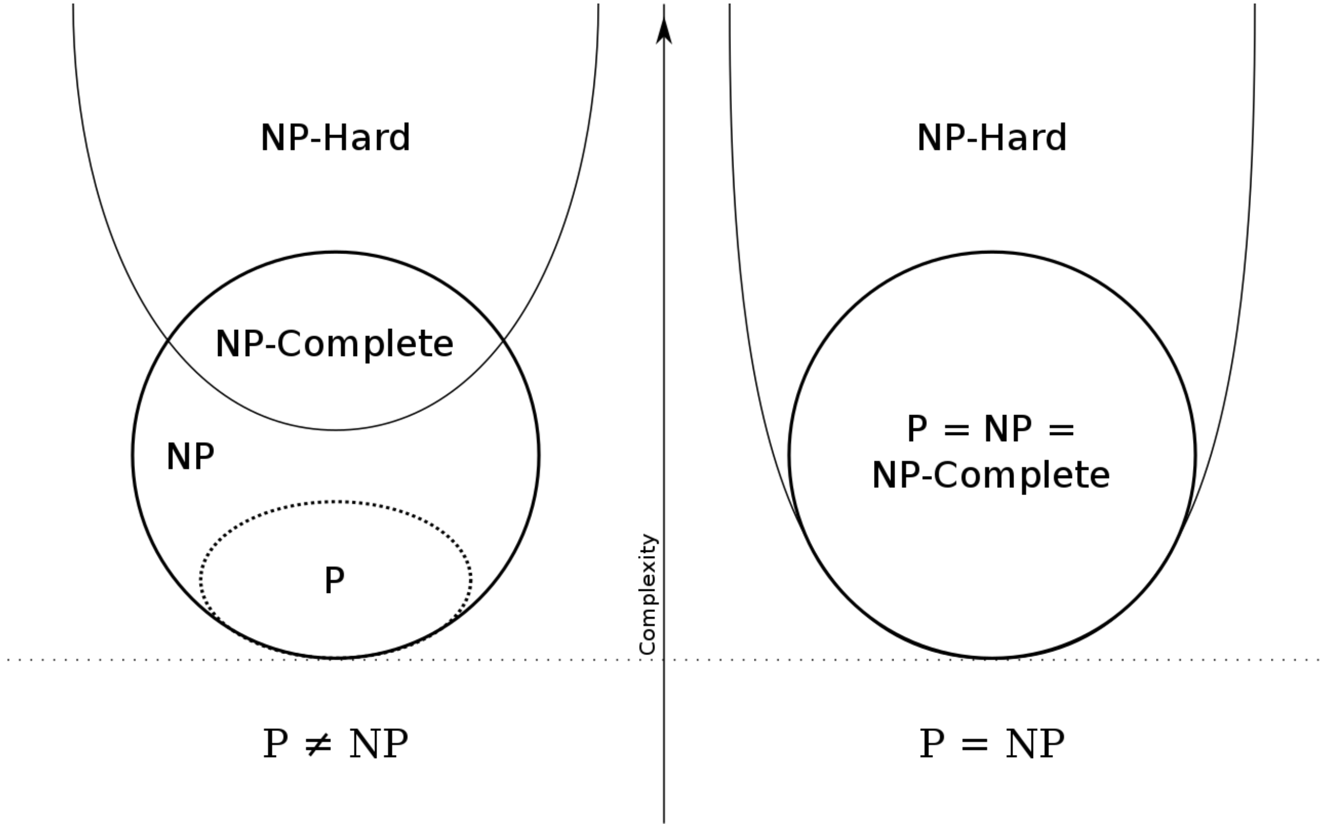
$$P \subseteq NP$$

- class NPC : NP 중에서 P 인지 P가 아닌지 현재까지 알 수 없는 문제들

P = NP? we don’t know yet.
(one of millennium prize problems)

- 하나의 NP-complete 문제가 polynomial time 에 풀린다면 모든 NP-complete 문제가 polynomial time 에 풀린다.



tractability

- polynomial algorithm 으로 풀 수 있는 문제 : tractable
- polynomial algorithm 으로 풀 수 없는 문제 : intractable
- NPC 의 문제들은 NP 중에서 가장 어려운 문제들 (intractable 할 것으로 추정되나 실제로 intractable 한지 증명되지 않았음)

shortest vs. longest simple paths in a graph with negative edge weights

- shortest path from a single source = $O(VE)$
- 주어진 수 이상의 edge 를 갖는 단순 경로가 그래프에 존재하는지를 판별하는 문제 = NP-complete

Euler tour vs. Hamiltonian cycle

- Euler tour : a connected, directed graph 에서 vertex를 한 번 이상 방문해도 되지만 모든 edge를 정확히 한 번씩 방문하는 cycle 을 찾기 = $O(E)$
- hamiltonian cycle : 모든 vertex 를 포함하는 simple cycle 이 존재하는지를 판별하는 문제 = NP-complete

2-CNF satisfiability vs. 3-CNF satisfiability

- CNF(Conjunctive Normal Form) : ANDs of ORs

i.e. $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)$

- k-CNF(Conjunctive Normal Form) : CNF 의 각 clause 들이 정확히 k 개의 변수 혹은 그 부정을 갖는 형태

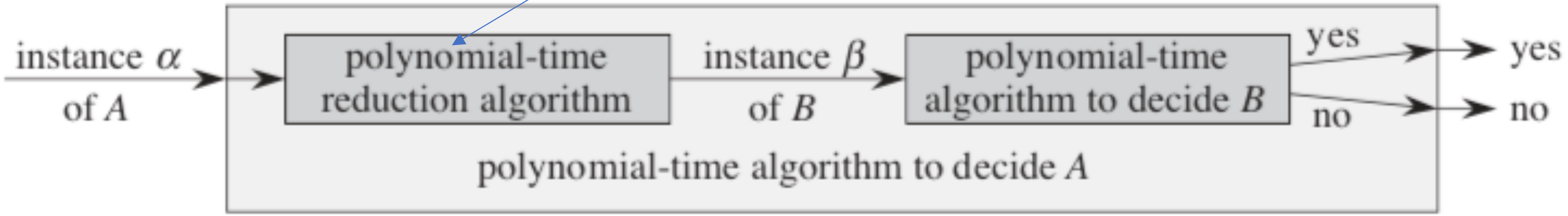
i.e. $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge \boxed{(\neg x_2 \vee \neg x_3)}$ 는 2-CNF

- satisfiability : 논리식의 값이 1이 되게 하는 변수들의 진리값 조합의 존재 여부 i.e. $x_1 = 1, x_2 = 0, x_3 = 1.$
- 2-CNF satisfiability = P class
- 3-CNF satisfiability = NP-complete

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

How to show a problem is NP-complete

Using a polynomial time reduction algorithm



problem B 가 P면 problem A 도 P 다.

→problem A 가 NP-complete 이면 problem B 도 NP-complete 이다.

(proof by contradiction)

problem A = circuit satisfiability problem (proven to be NP-complete)

에 대하여 B로의 polynomial reduction algorithm 이 있음을 보이면 B 가 NPC 에 속함을 증명한 것이다.

The problem A : the first NP-complete problem

- circuit satisfiability problem : "Given a boolean combinational circuit composed of AND, OR, and NOT gates, is it satisfiable?"

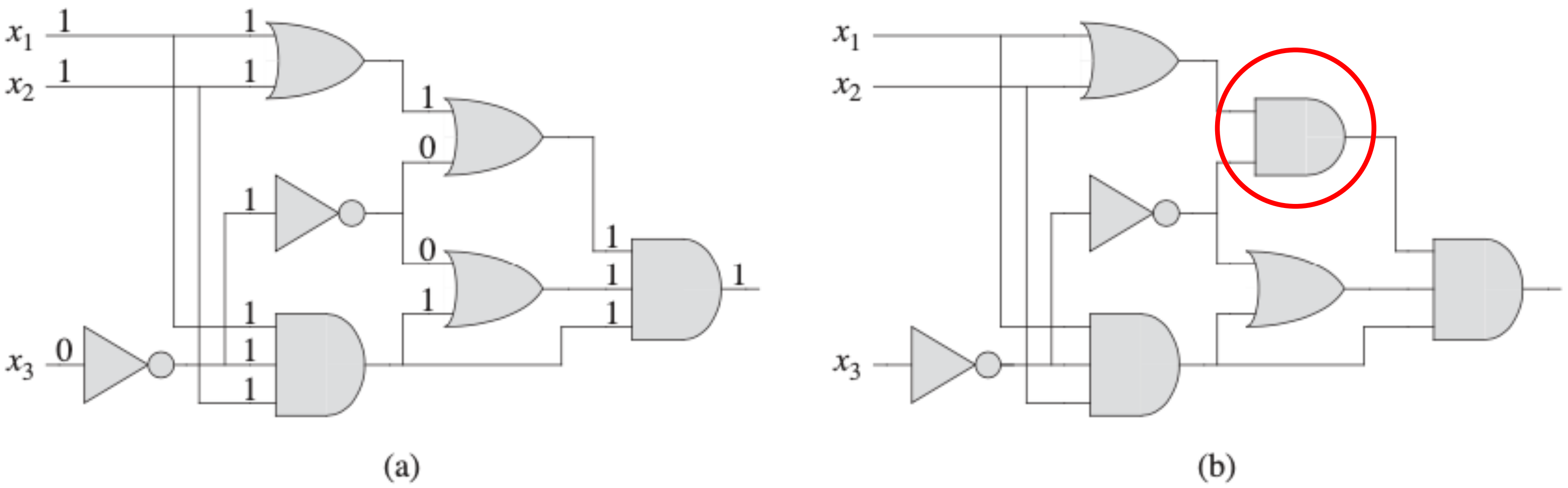
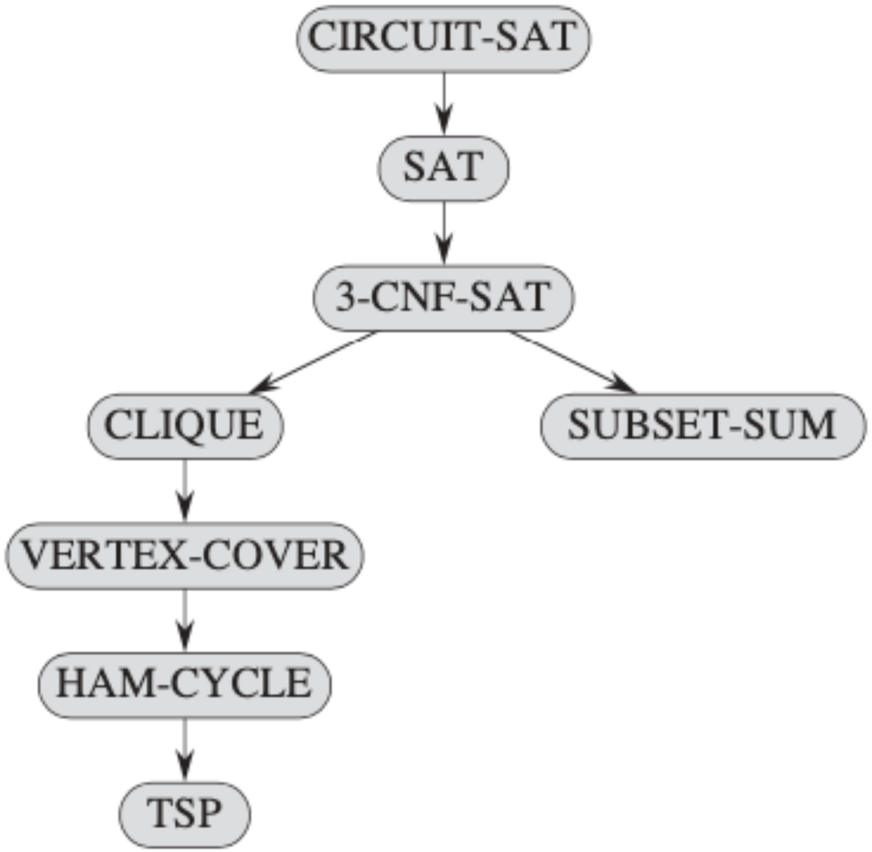


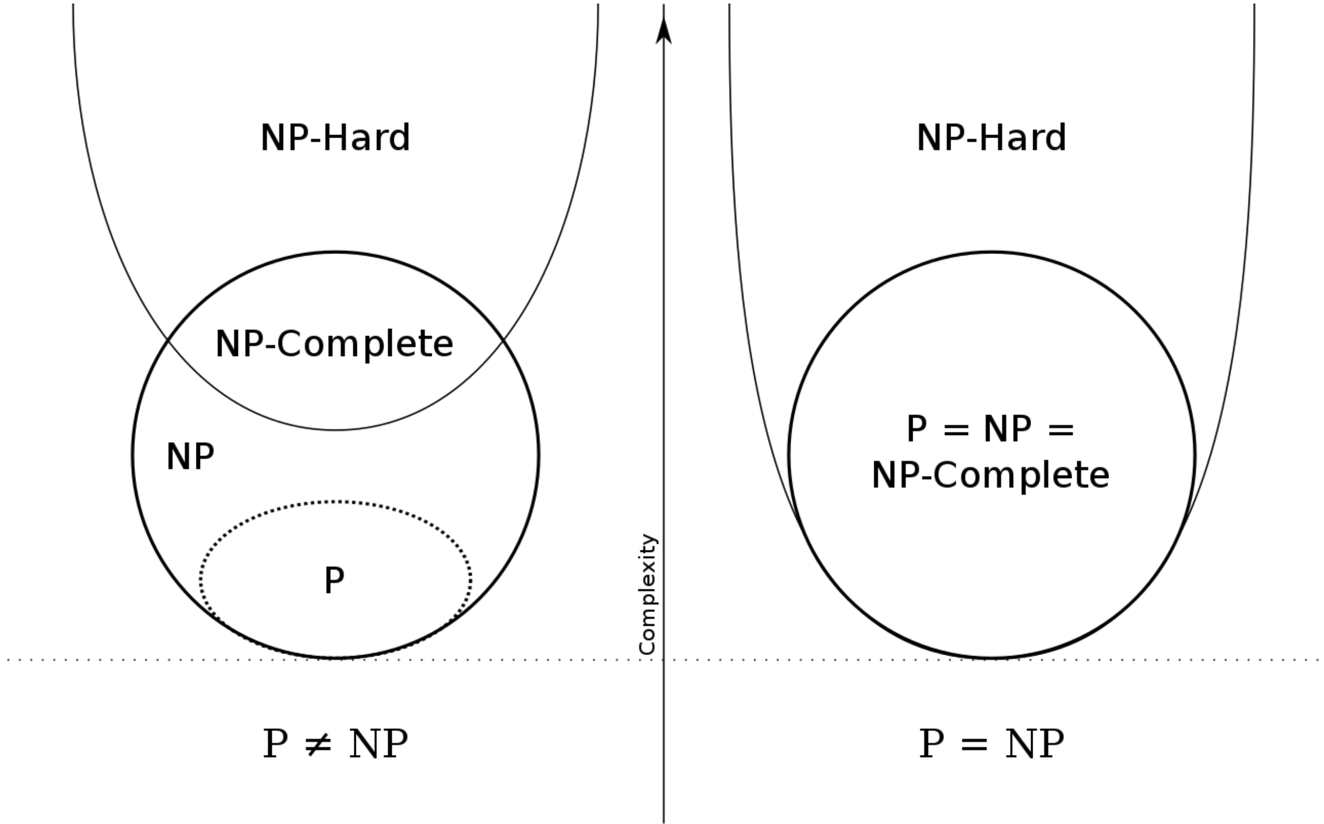
Figure 34.8 Two instances of the circuit-satisfiability problem. **(a)** The assignment $\langle x_1 = 1, x_2 = 1, x_3 = 0 \rangle$ to the inputs of this circuit causes the output of the circuit to be 1. The circuit is therefore satisfiable. **(b)** No assignment to the inputs of this circuit can cause the output of the circuit to be 1. The circuit is therefore unsatisfiable.

Some NPC problems



P = NP? we don't know yet.

- 하나의 NP-complete 문제가 polynomial time 에 풀린다면 모든 NP-complete 문제가 polynomial time 에 풀린다.



NP-hard

- a problem *H* is NP-hard when every problem *L* in NP can be reduced in polynomial time to *H.*
- *H* is "at least as hard as the hardest problems in NP".

- example :

single source shortest path problem → all-pairs shortest path problem

- 주어진 문제가 NPC class 의 문제로 reduce 된다면 exact solution 을 찾는 알고리즘을 포기하고 approximation algorithm 을 구현한다.

Approximation Algorithm : TSP

- TSP (Traveling Salesman Problem) : Finding a Hamiltonian cycle of minimum weight $c(A) = \sum_{(u,v)\in A} c(u,v)$ in a complete undirected graph G

- 삼각 부등식 (triangle inequality) 가 성립하면 approximation algorithm

$$c(u,w) \le c(u,v) + c(v,w)$$

- 삼각 부등식 (triangle inequality) 가 성립하지 않으면 P=NP 이어야만 좋은 approximation algorithm 이 있다.

APPROX-TSP-TOUR(G, c)

```
select a vertex r ∈ G.V to be a "root" vertex
compute a minimum spanning tree T for G from root r
    using MST-PRIM(G, c, r)
let H be a list of vertices, ordered according to when they are first visited
    in a preorder tree walk of T
return the hamiltonian cycle H
```

MST-PRIM(G, w, r)

```
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

preorder tree walk prints the root before the values in either subtree.

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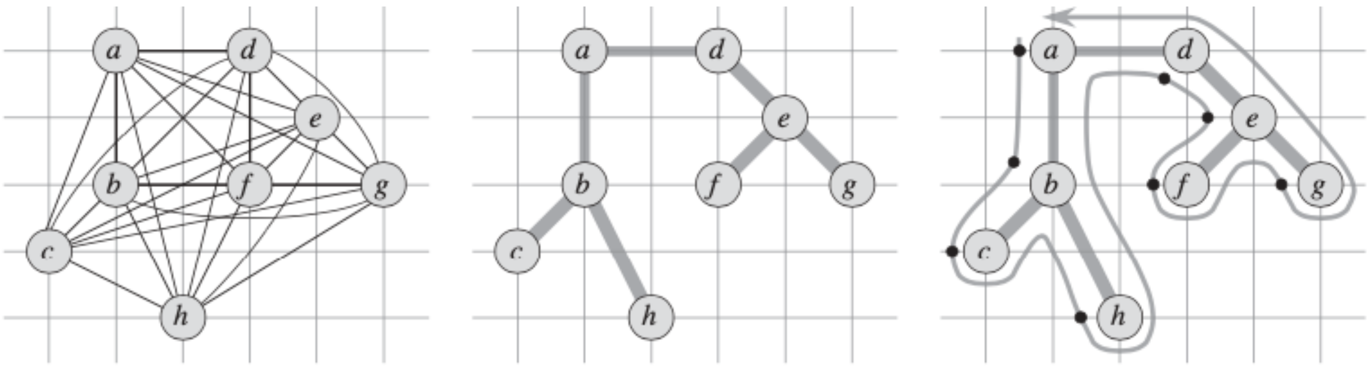
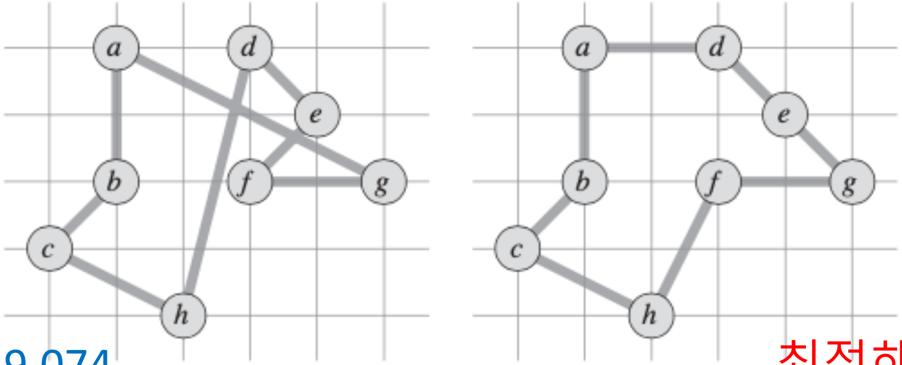


(a) (b) (c)

G is a complete undirected graph.



(d) (e)

- vertex 간 edge weight 는 직선거리이므로 삼각 부등식이 성립
- vertex 들이 MST 에 추가되는 순으로 알파벳을 썼음

c(A) = 19.074

최적해 c(A) = 14.715

기말 고사

- 12/8 화 저녁 7:30 시에 각자 school programmers 에서 기말고사 응시, 9시 종료
- 절대평가
- offline 자료 open everything
- computer 로는 school programmers 만 사용
- 다른 website 방문, 프로그램 실행, 통신 행위 : 부정 행위로 F 학점
- 사용하는 monitor 전체 화면 공유
- 휴대폰으로 모니터 화면이 보여야 함

한 학기 동안 즐거웠습니다.
즐거운 겨울 방학 보내세요!