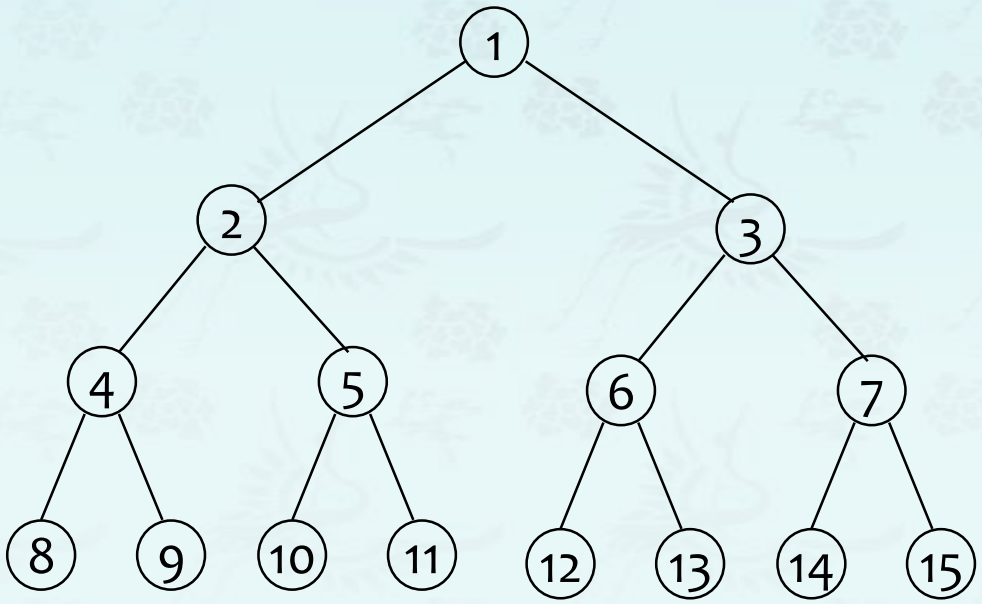
heap

- ***complete binary tree (review)***
- *heap and priority queues (Chapter 9)*
- *binary heap and minheap*
- *maxheap demo*
- *maxheap coding*
- *heap sort (Chapter 7)*

Binary Trees – Properties

Definition: *A **full** binary tree of **level k*** is a binary tree having **$2^k - 1$ nodes**, $k \geq 0$.



*A **full** binary tree*

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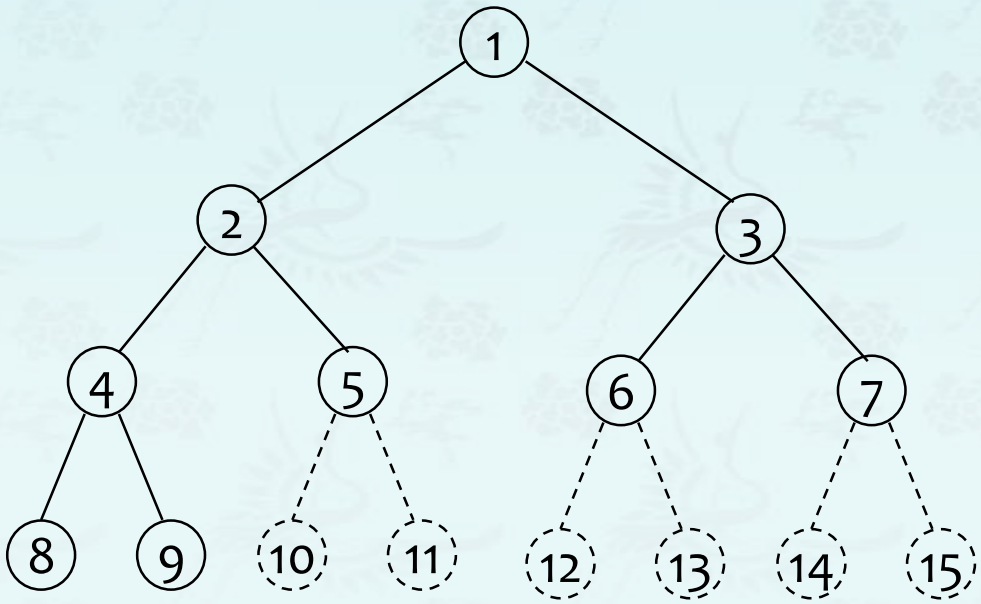
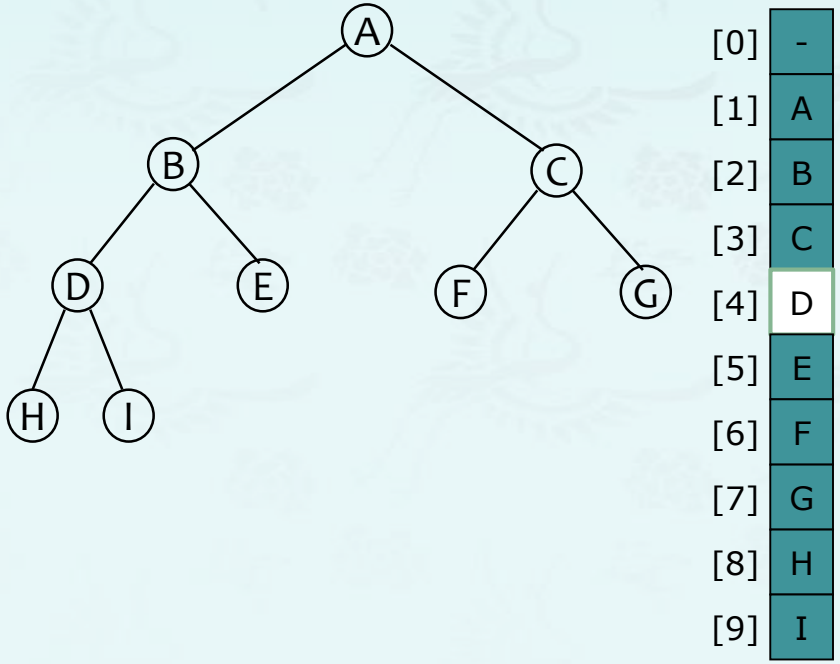


*A **complete** binary tree*

Binary Trees – Array representation

Property: a **complete** binary tree with n nodes, any node index i, $1 \leq i \leq n$, we have

(1) $parent(i)$ is at $\boxed{\lfloor i/2 \rfloor}$ $if\ i = 1$. $If\ i = 1$, i is at the root and has no parent.
(2) $leftChild(i)$ is at $\mathbf{2i}$ $if\ 2i <= n$. $If\ 2i > n$, then i has no left child.
(3) $rightChild(i)$ is at $\mathbf{2i + 1}$ $if\ 2i + 1 <= n$. $If\ 2i + 1 > n$, then i has no right child.



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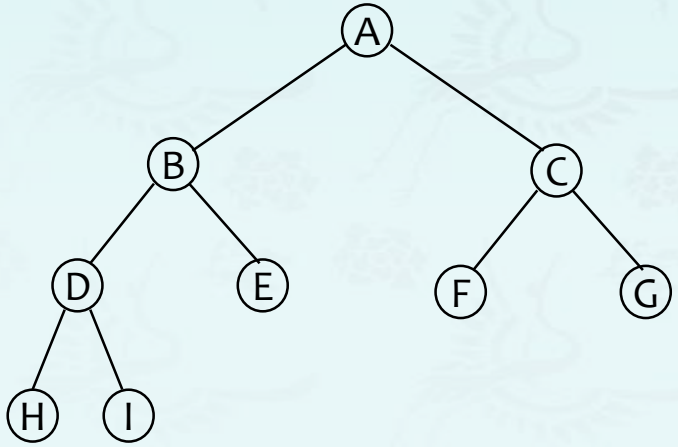

[0]	-
[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
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Example:
Find its parent, left child and right child at node D

Solution:

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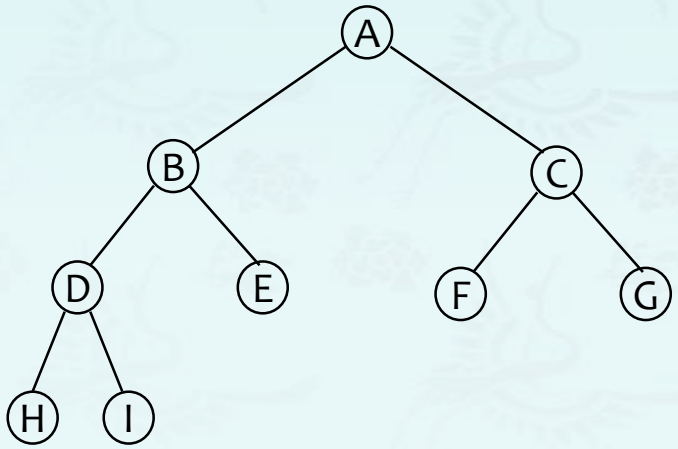

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Example:
Find its parent, left child and right child at node D

Solution:
$parent(i = 4)$ is at 4/2 = 2
$leftChild(4)$ is at 2x4 = 8
$rightChild(4)$ is at 2x4 + 1 = 9

How do you like this property of the tree?

heap

- ***complete binary tree (review)***
- ***heap and priority queues (Chapter 9)***
- *binary heap and minheap*
- *heap coding*
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Heaps & Priority Queues

Heaps are frequently used to implement **priority queues.**

- Because it provides an efficient implementation for **priority queues.**

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Priority queues.

- Queues with priorities associated to.
- **Example:** A line waiting to be served at a bank and served FIFO except if a senior or a disabled person arrives in the line. They are served first. Seniors and disabled persons have higher priority than others.

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- Get the top priority element (min or max)
- Insert an element
- Delete the top priority element
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A typical ADT for Priority Queue

• Get the top priority element (min or max)	• O(1)
• Insert an element	• O(log n)
• Delete the top priority element	• O(log n)
• Decrease the priority of an element	• O(log n)

Priority queue applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Number theory. [sum of powers]
- Artificial intelligence. [A* search]
- Statistics. [maintain largest M values in a sequence]
- Operating systems. [load balancing, interrupt handling]
- Discrete optimization. [bin packing, scheduling]
- Spam filtering. [Bayesian spam filter]

Heaps & Priority Queues

Challenge: Find the largest **M** items in a stream of **N** items.

- Fraud detection: isolate $$ transactions.
- Hacking: KT's customer DB access by their sales agents
- File maintenance: find biggest files, directories, or emails.

N huge,
M large

Constraints: Not enough memory to store N items.

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%more trans.txt

```
Turing       6/17/1990      644.08
vonNeumann   3/26/2002      4121.85
Dijkstra     8/22/2007      2678.40
vonNeumann   1/11/1999      4409.74
Dijkstra     11/18/1995     837.42
Hoare        5/10/1993      3229.27
vonNeumann   2/12/1994      4732.35
Hoare        8/18/1992      4381.21
Turing       1/11/2002      66.10
Thompson     2/27/2000      4747.08
Turing       2/11/1991      2156.86
Hoare        8/12/2003      1025.70
vonNeumann   10/13/1993     2520.97
Dijkstra     9/10/2000      708.95
Turing       10/12/1993     3532.36
Hoare        2/10/2005      4050.20
```

%java TopM 5 < trans.txt

```
Thompson     2/27/2000      4747.08
vonNeumann   2/12/1994      4732.35
vonNeumann   1/11/1999      4409.74
Hoare        8/18/1992      4381.21
vonNeumann   3/26/2002      4121.85
```

Sort key

Challenge: Find the largest **M** items in a stream of **N** items.
Constraints: Not enough memory to store N items.

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Order of growth of finding the largest M **in a stream of N items**

implementation	time	space
sort	N log N	N
binary heap	N log M	M
best in theory	N	M

Challenge: Find the largest **M** items in a stream of **N** items.
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N huge,
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Order of growth of finding the largest M **in a stream of N items**

implementation	insert	delete	min/max
unordered array	1	N	N
ordered array	N	1	1
goal	log N	log N	log N

Mission Impossible?

heap

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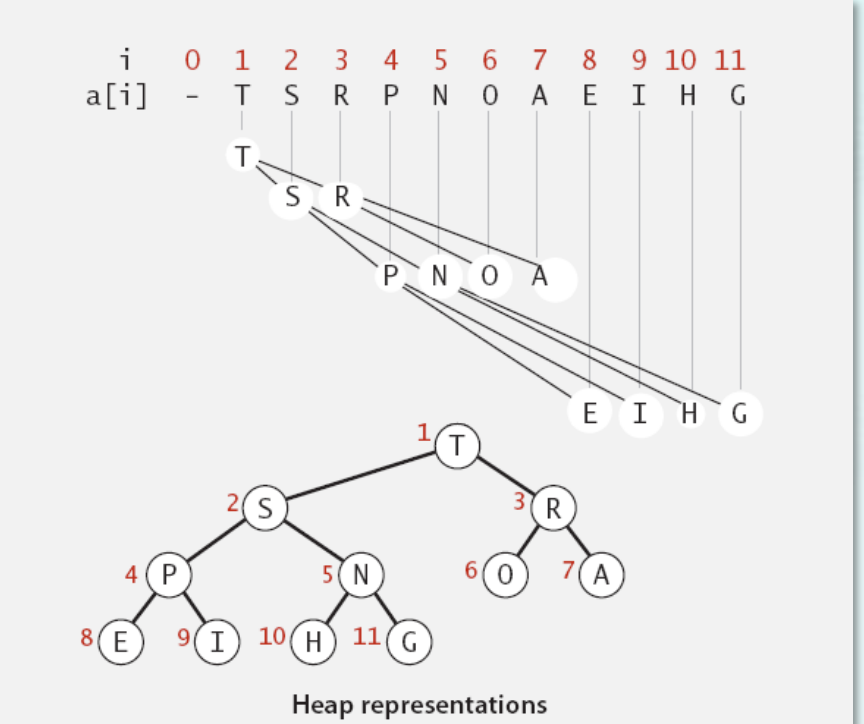
Heaps & Priority Queues

Binary heap: array representation of a **heap-ordered** complete binary tree

- **Properties:**

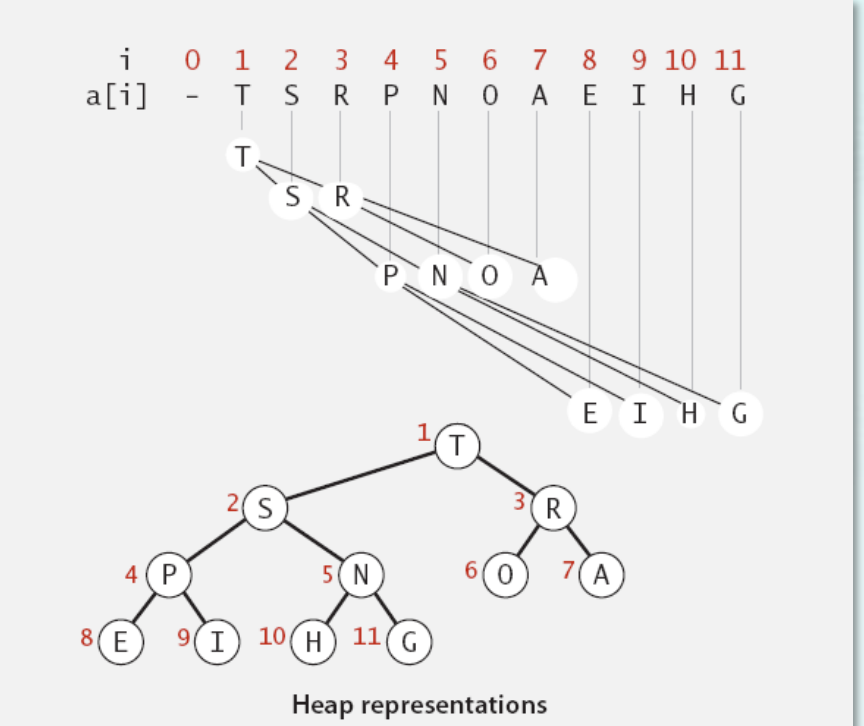
- Array representation



Heap representations

Binary heap: array representation of a **heap-ordered** complete binary tree

- **Properties:**
  - **Heap-ordered:**
    Parent's key no smaller than children's keys. [maxheap]
  - **Heap-structure**:
    A complete binary tree
- Array representation



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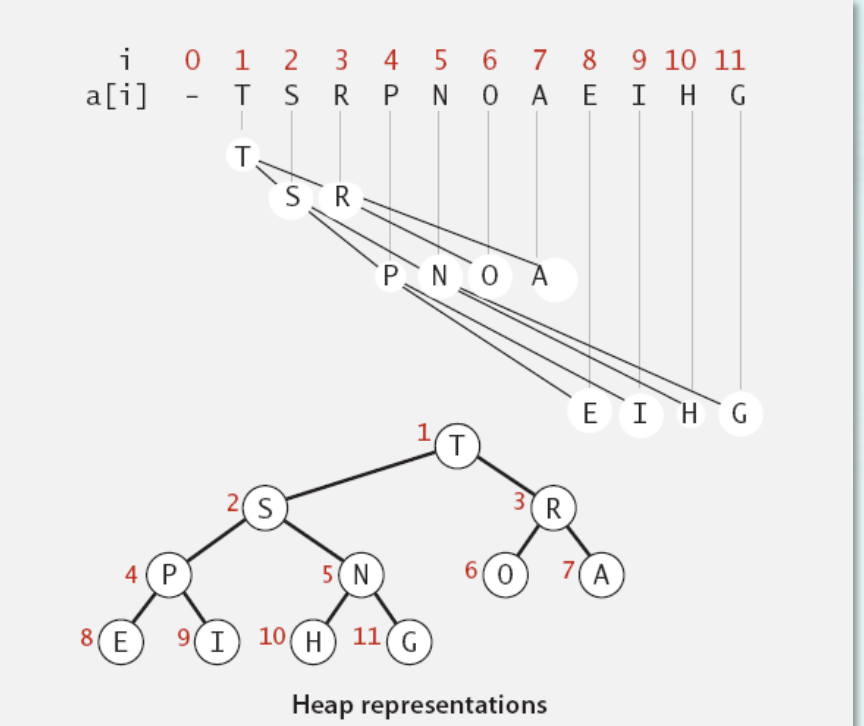
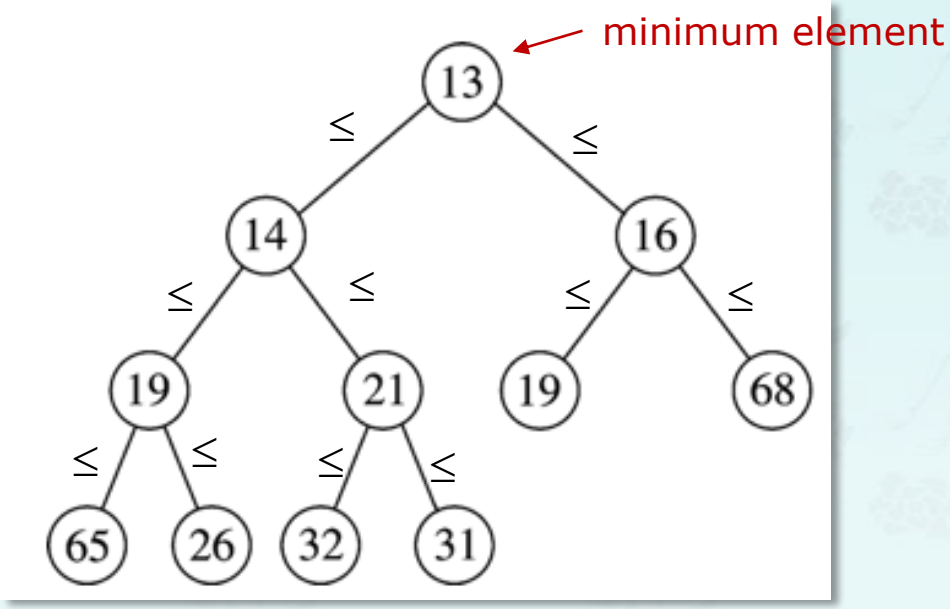
- **Properties:**
  - **Heap-ordered:**
    Parent's key no smaller than children's keys. [maxheap]
  - **Heap-structure**:
    A complete binary tree

- Array representation
  - Indices start at 1.
  - Take nodes in level order.
    - Parent at k is at k/2.
    - Children at k are at 2k and 2k+1.
  - No explicit links needed!



Heap representations

minheap example



- Duplicates are allowed
- No order implied for elements which do not share ancestor-descendant relationship

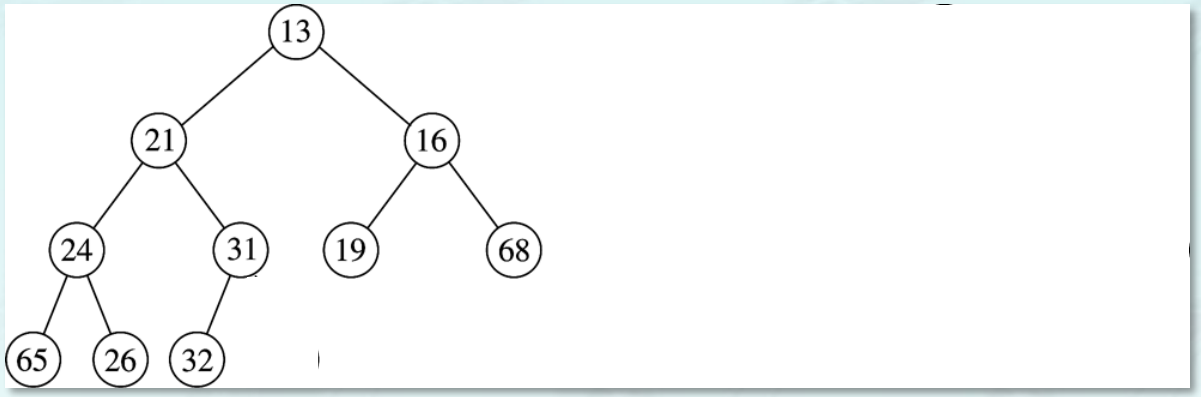
minheap example

insertion:

- Insert a new element **while maintaining a heap-structure**
- Move the element up the heap **while not satisfying heap-ordered**

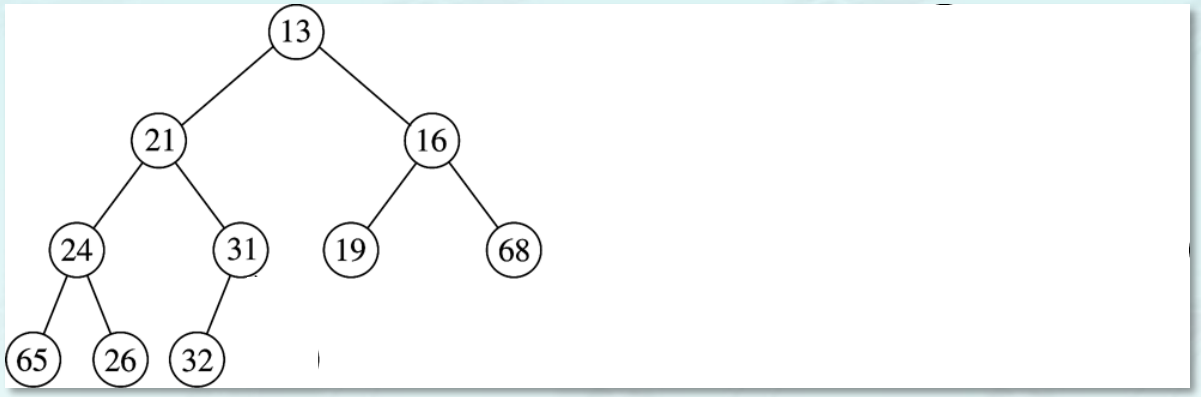
minheap example

insertion: Insert a node 14



minheap example

insertion: Insert a node 14 Where is an empty node to start?



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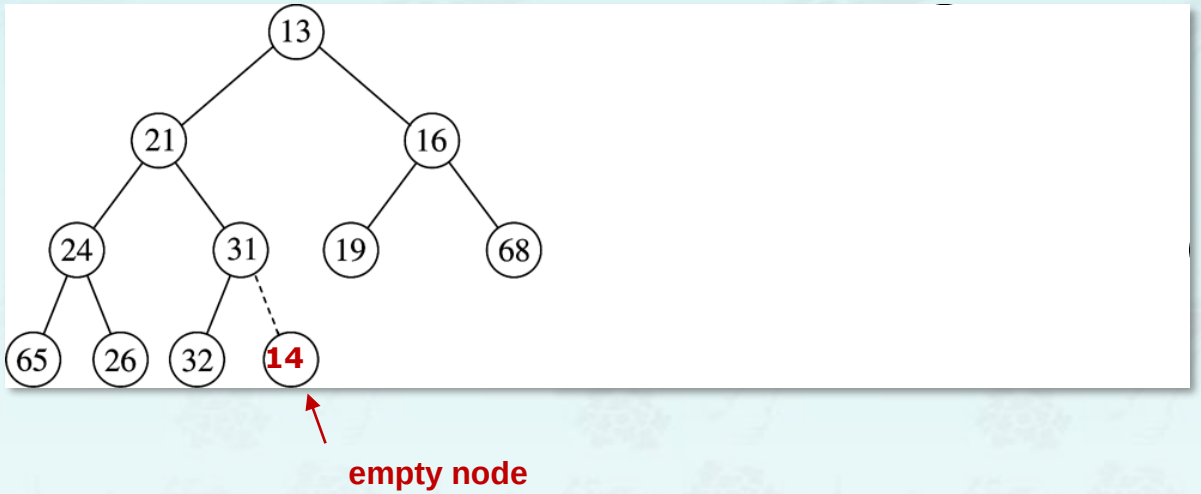

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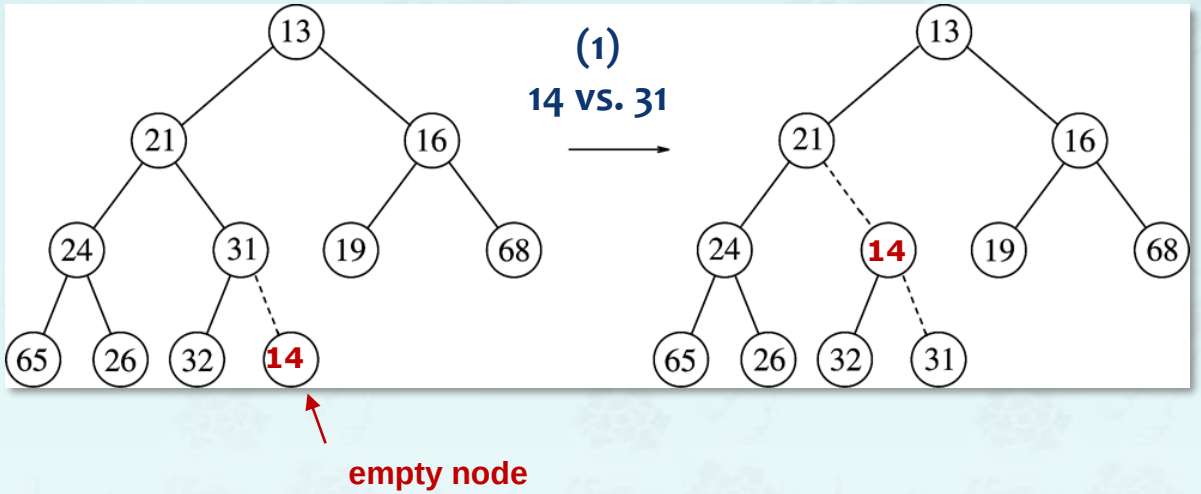

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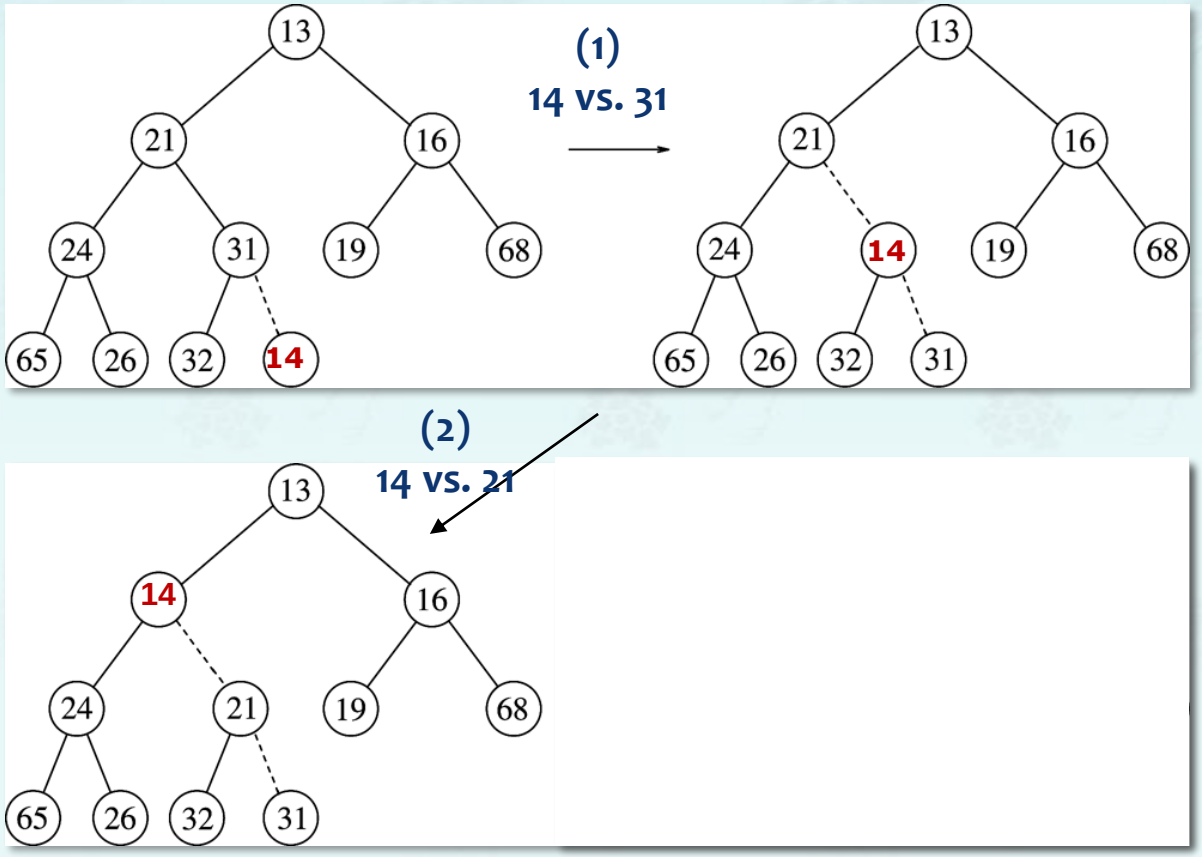

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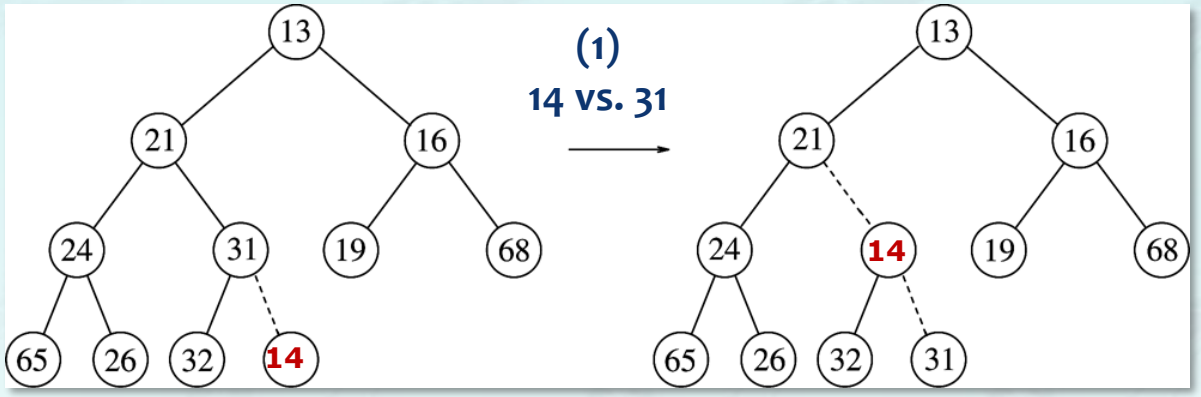
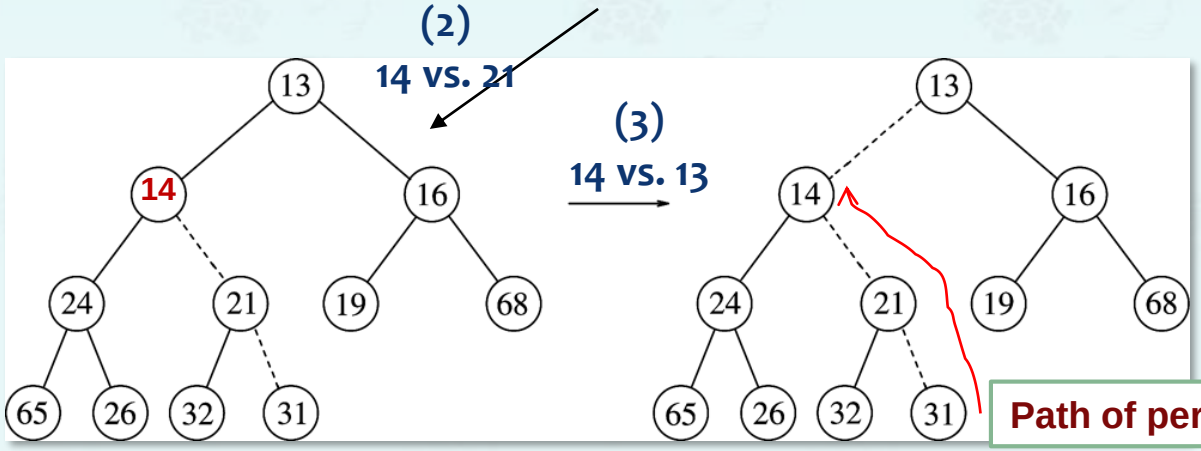




✓ Heap-ordered

✓ Heap-Structure

Path of percolation (swim) up

minheap example

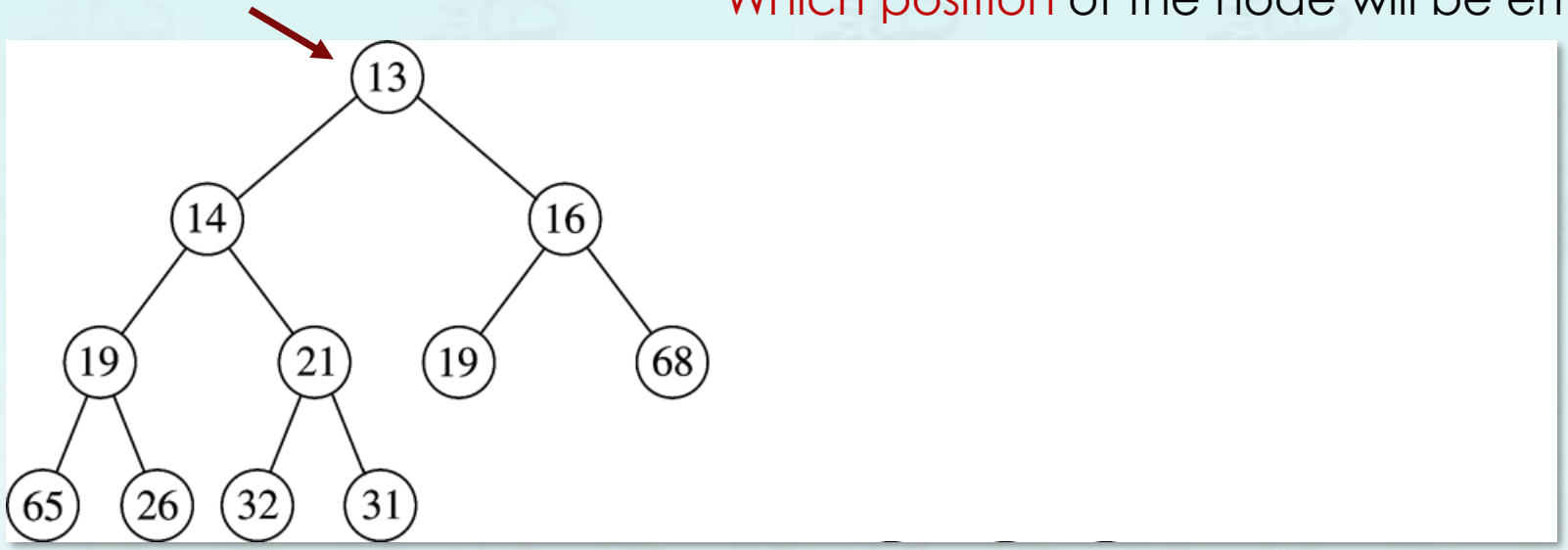
deletion: dequeue – delete the root

- Swap the root and the the last element.
- Heap decreases by one in size.
- ***Move down (sink) the root*** while not satisfying heap-ordered.
  - Minimum element is always at the root (by minheap definition).

minheap example

deletion: dequeue – delete the root

Which position of the node will be empty?



minheap example

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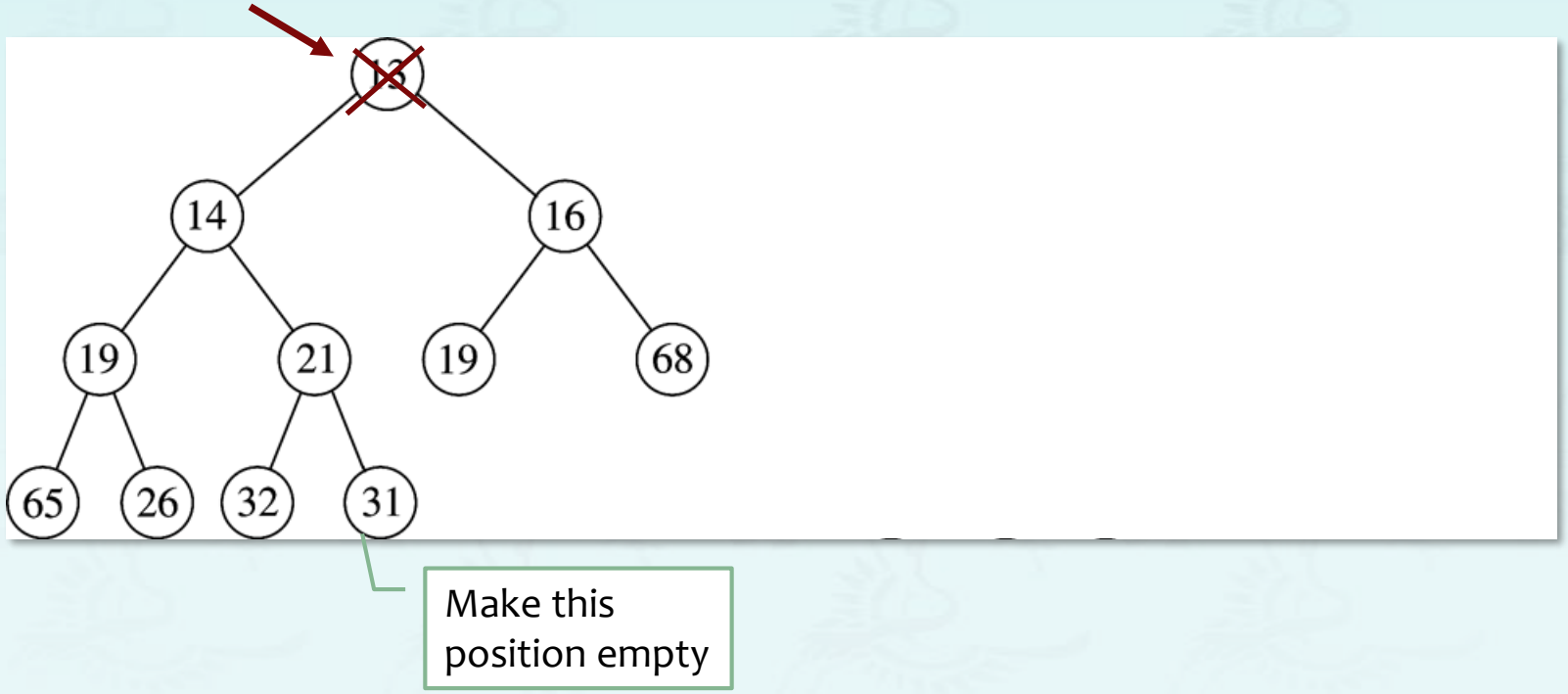
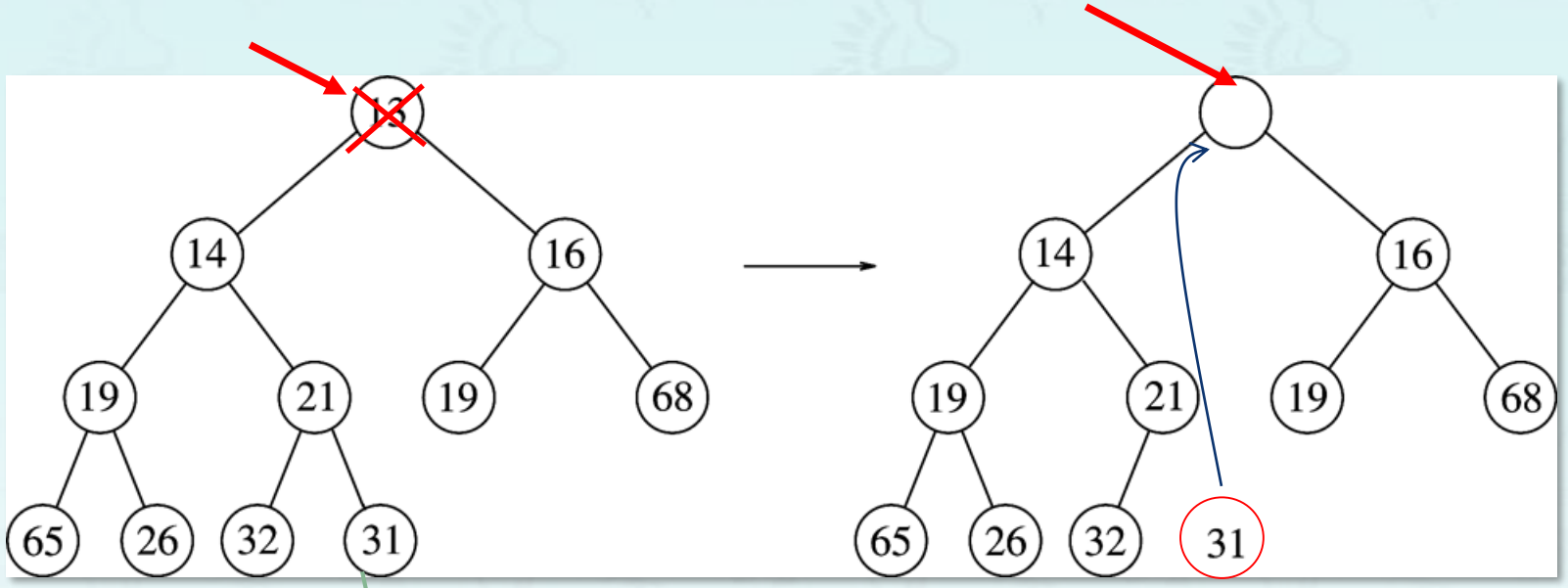


minheap example

deletion: dequeue – delete the root

Copy 31 temporarily here.

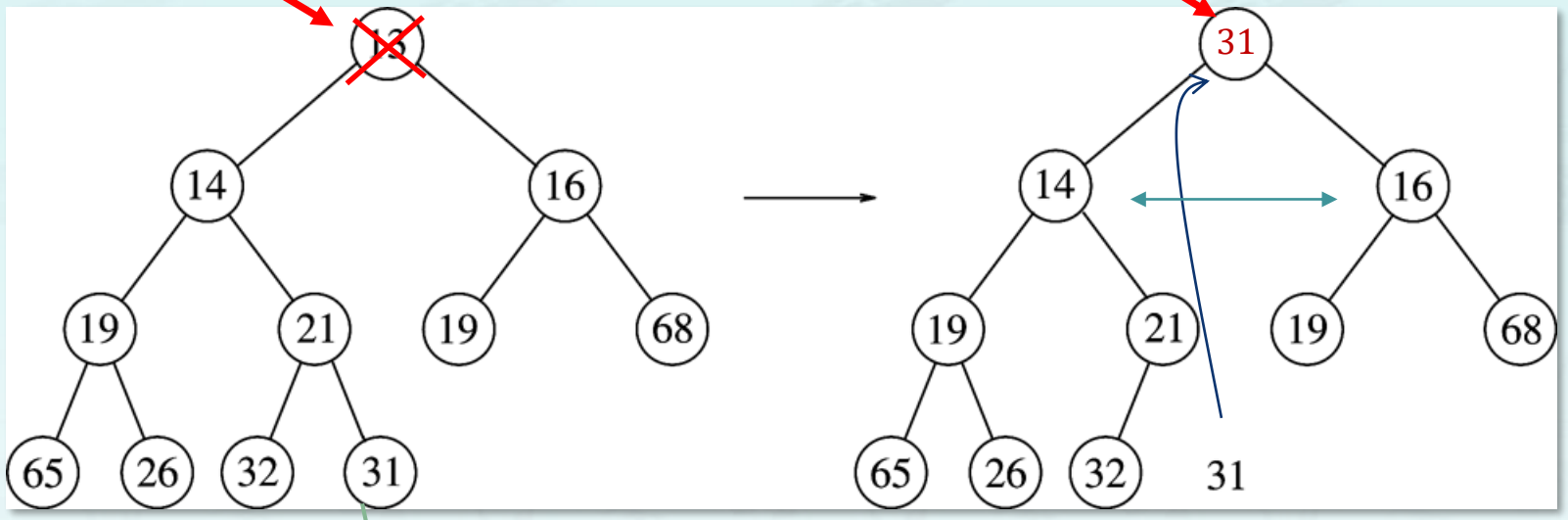


Make this position empty

minheap example

deletion: dequeue – delete the root

Copy 31 temporarily here and ask **heap-ordered?**



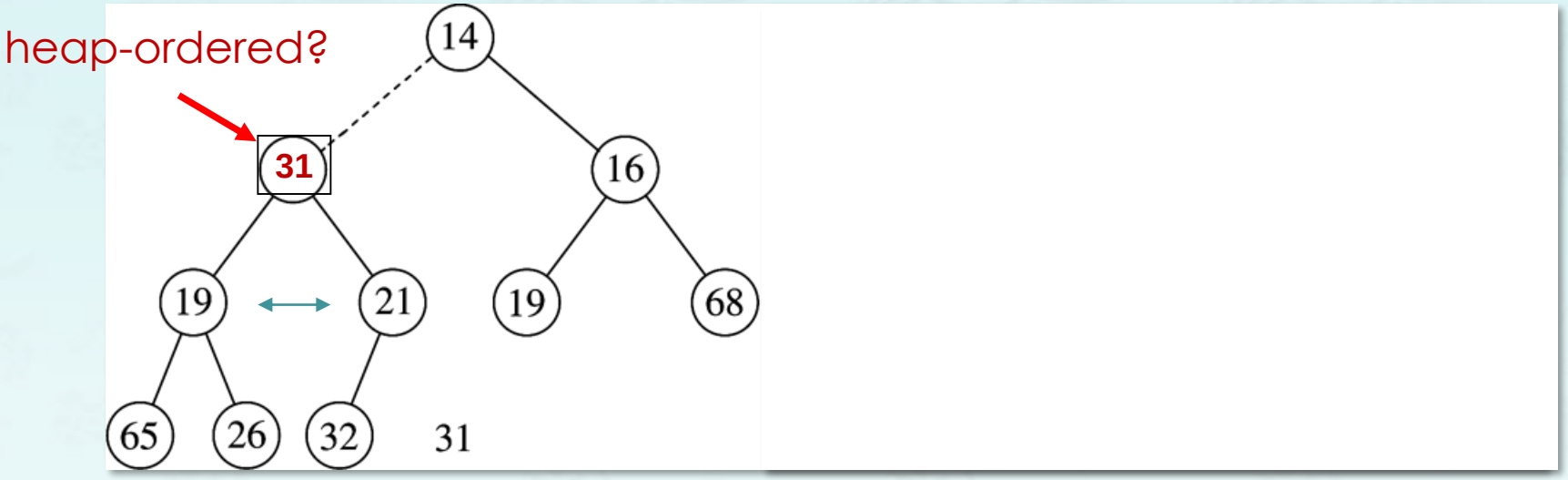
Make this position empty

Is 31 > min(14,16)?
- Yes - swap 31 with min(14,16)

sink…

minheap example

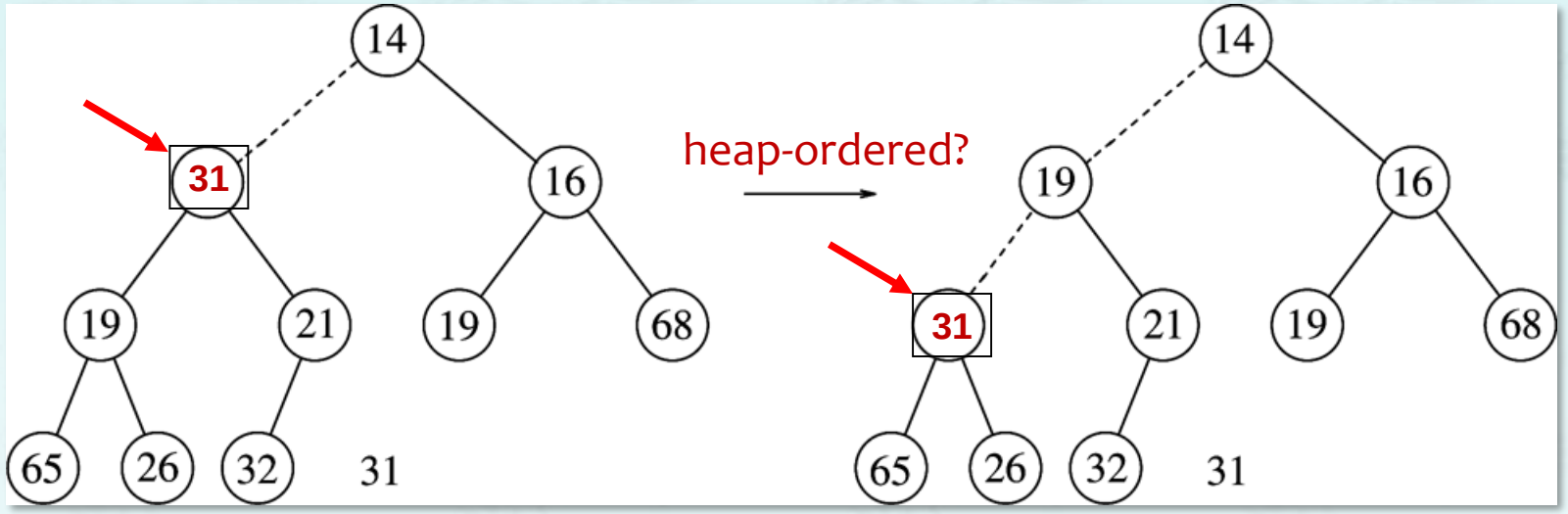
deletion: dequeue – delete the root



Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

minheap example

deletion: dequeue – delete the root

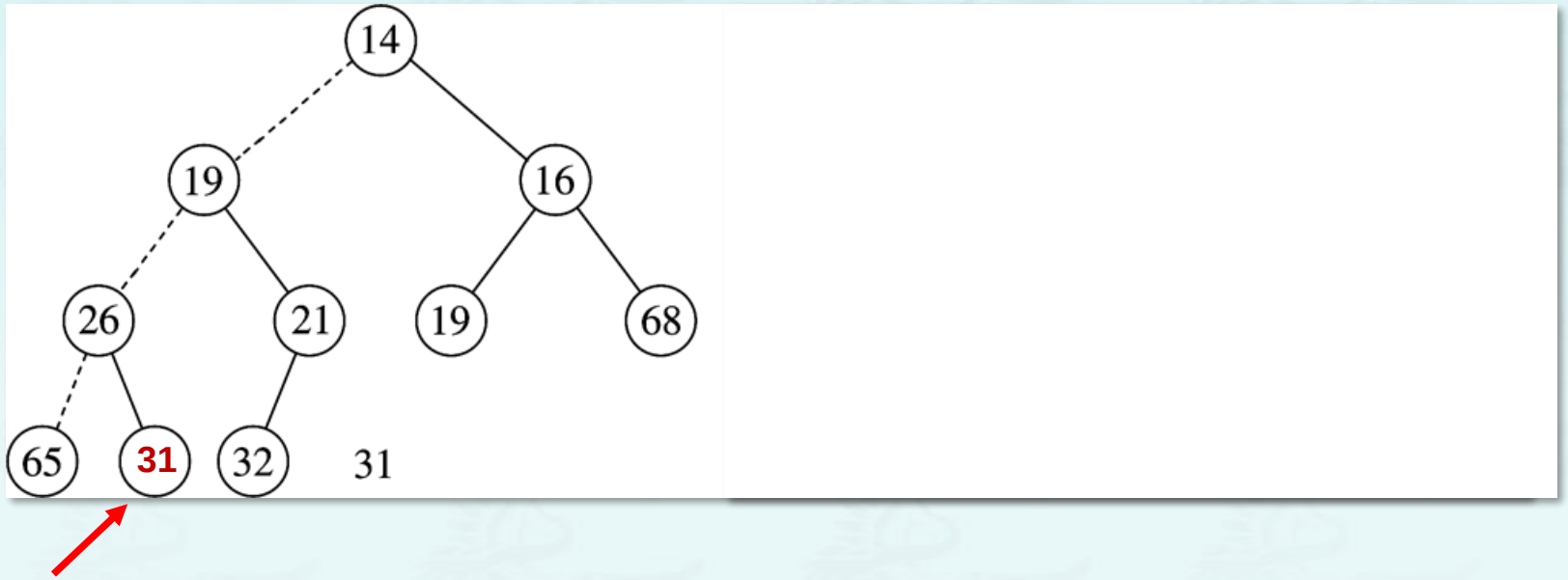


Is 31 > min(19,21)?
- Yes - swap 31 with min(19,21)

Is 31 > min(65,26)?
- Yes - swap 31 with min(65,26)

minheap example

deletion: dequeue – delete the root



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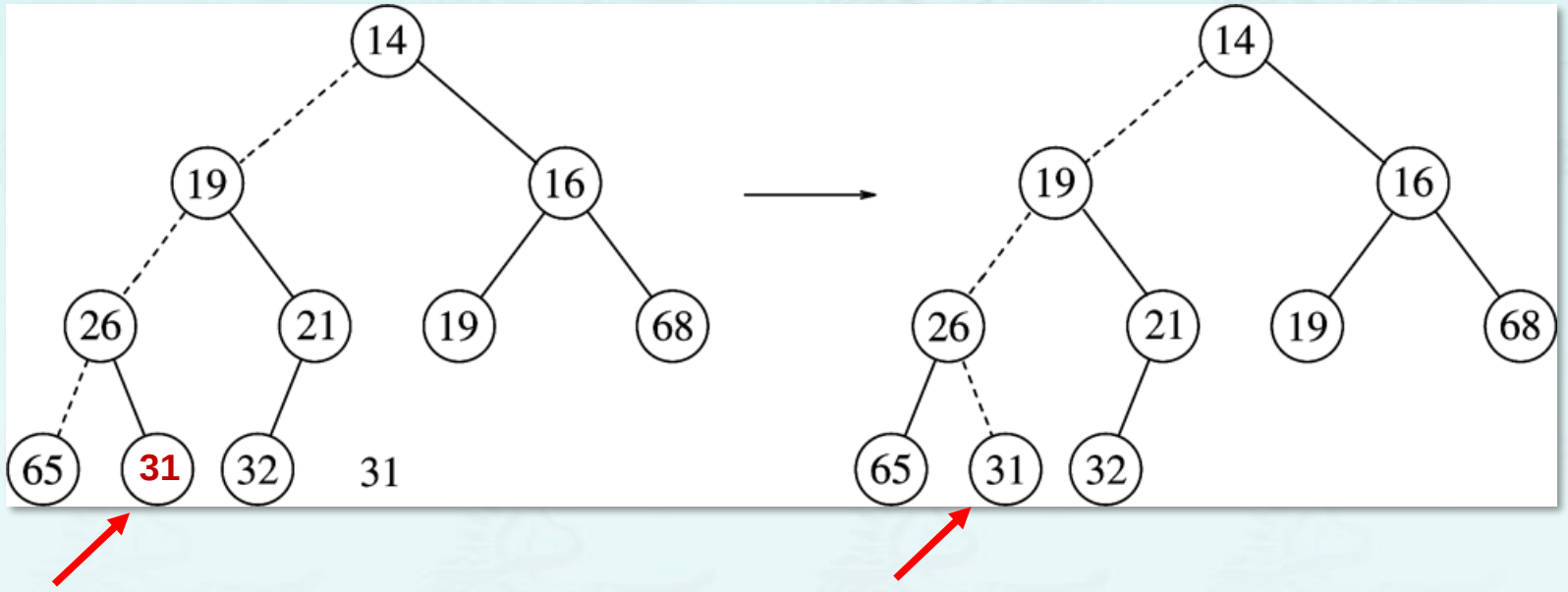

- ✓ Heap-ordered
- ✓ Heap-structure

Binary heap operations time complexity:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node in min/max heap
- decreaseKey: O(log N)
- increaseKey: O(log N)
- remove: O(log N) // removing a node in any location

Binary heap operations time complexity with N items:

Implementation	Insert	Delete	max
Unordered array	1	N	N
Ordered array	N	1	1
Binary heap	**log N**	**log N**	1

Mission Completed

heap

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maxheap example

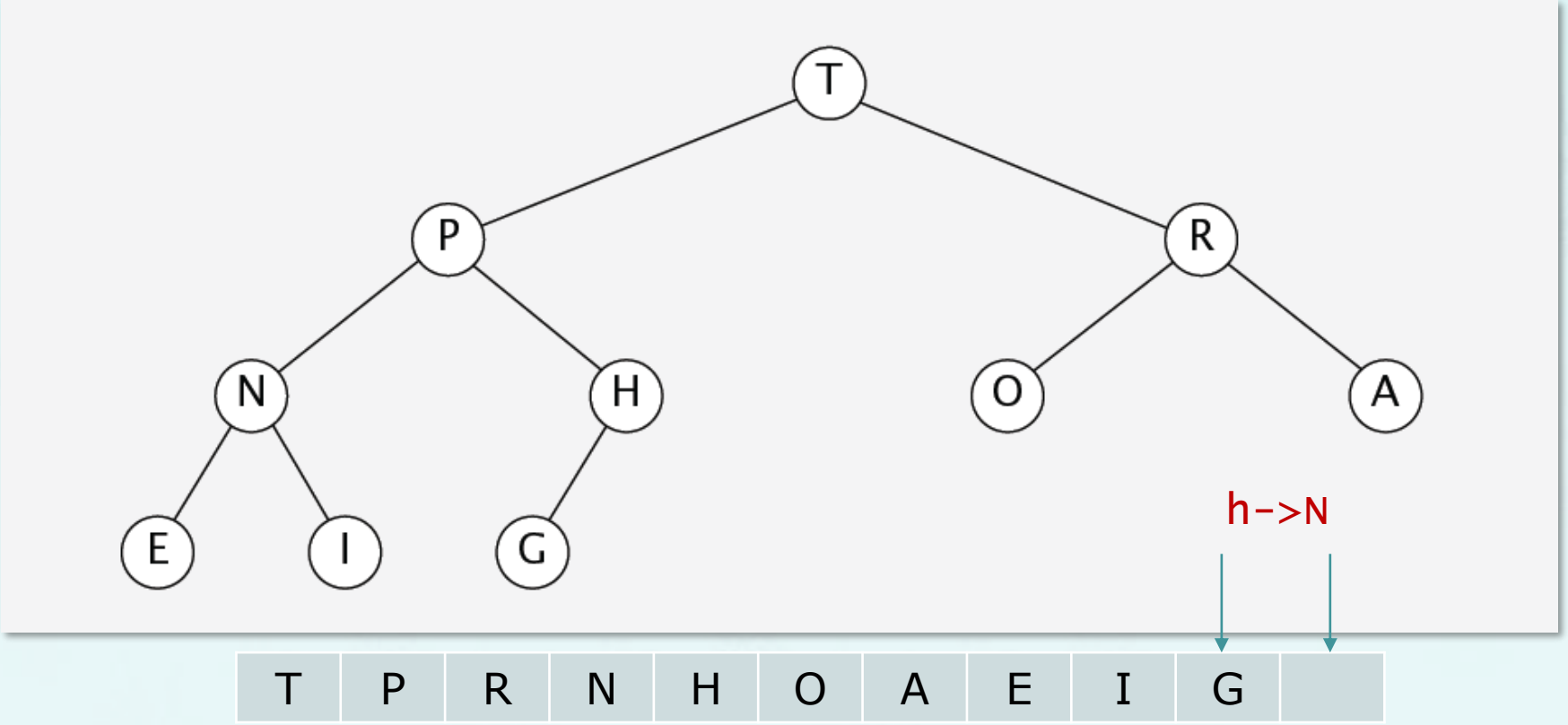
- **Insert:** Add node at end, then swim it up.

T	P	R	N	H	O	A	E	I	G	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

Heap ordered



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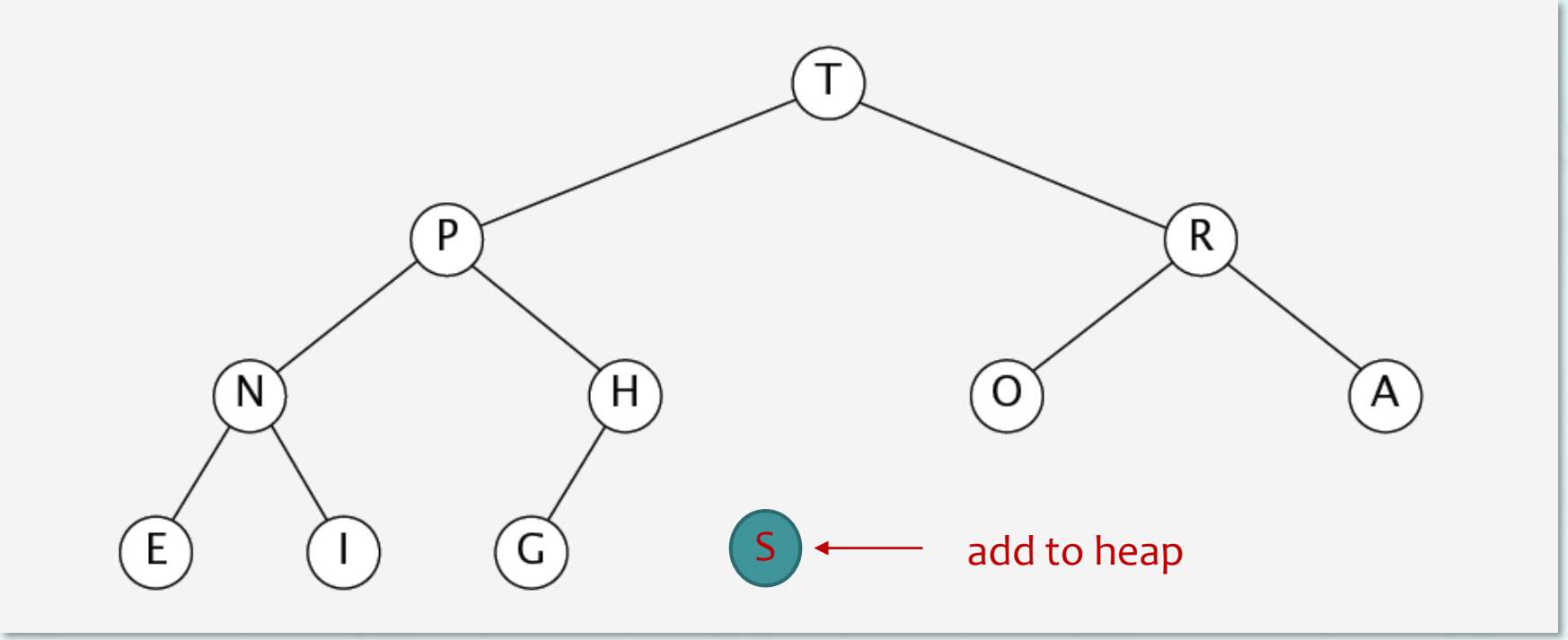
insert S



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h->N

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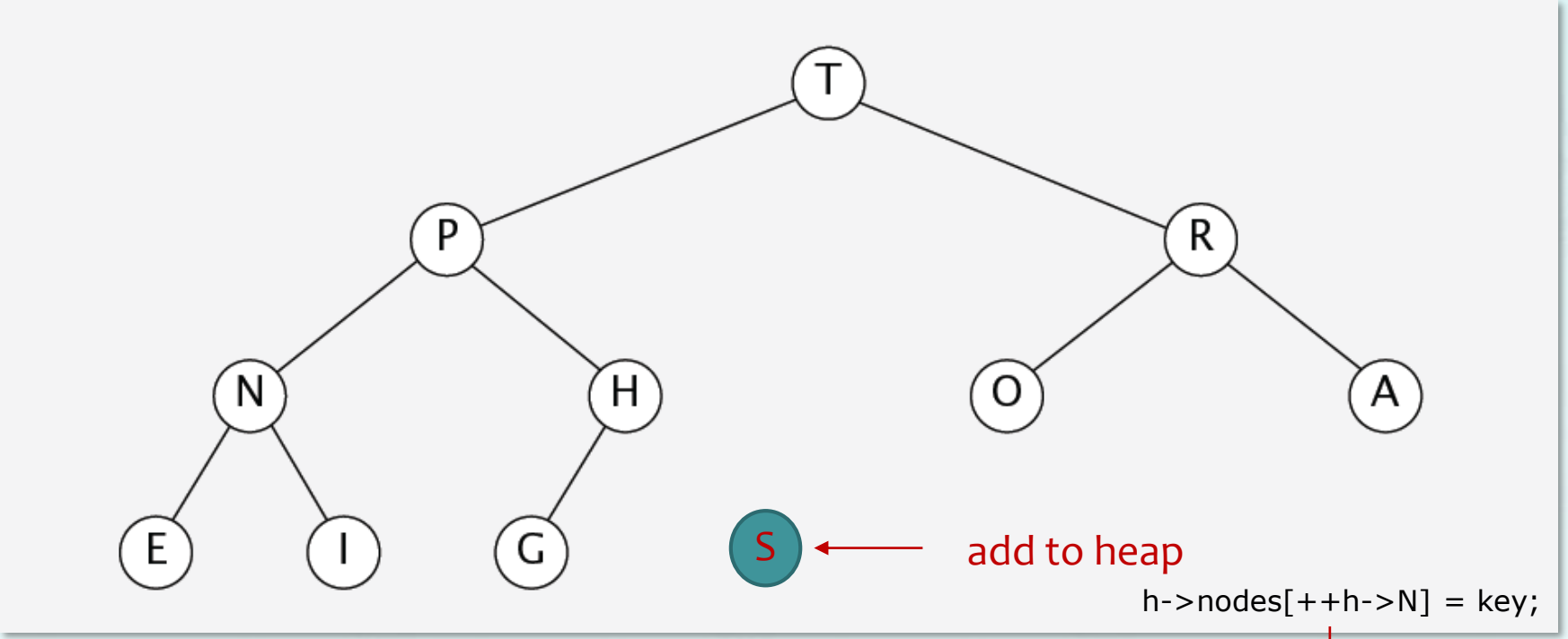

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h->N

maxheap example

- **Insert:** Add node at end, then swim it up.
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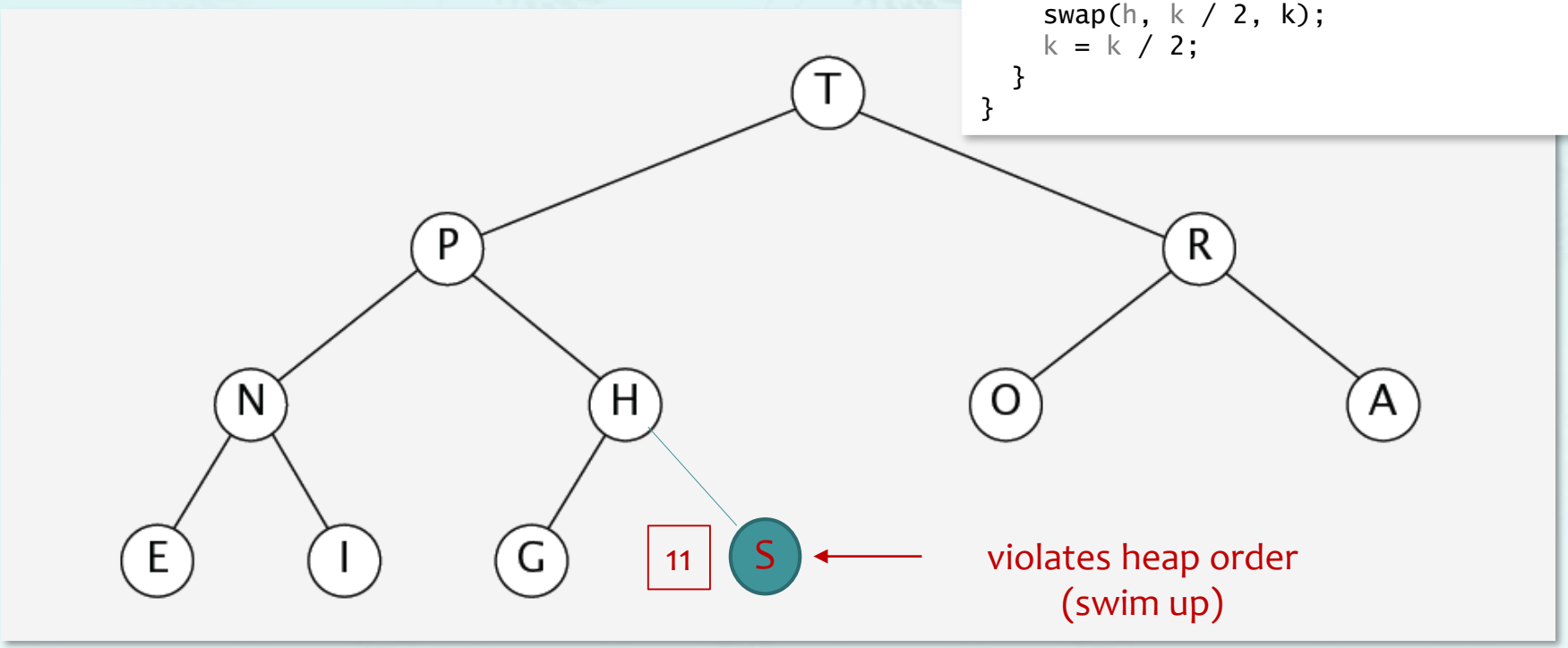
```
void swim(heap h, int k) {
  while (k > 1 && less(h, k / 2, k)) {
    swap(h, k / 2, k);
    k = k / 2;
  }
}
```

insert S



T	P	R	N	H	O	A	E	I	G	S
										11

maxheap example

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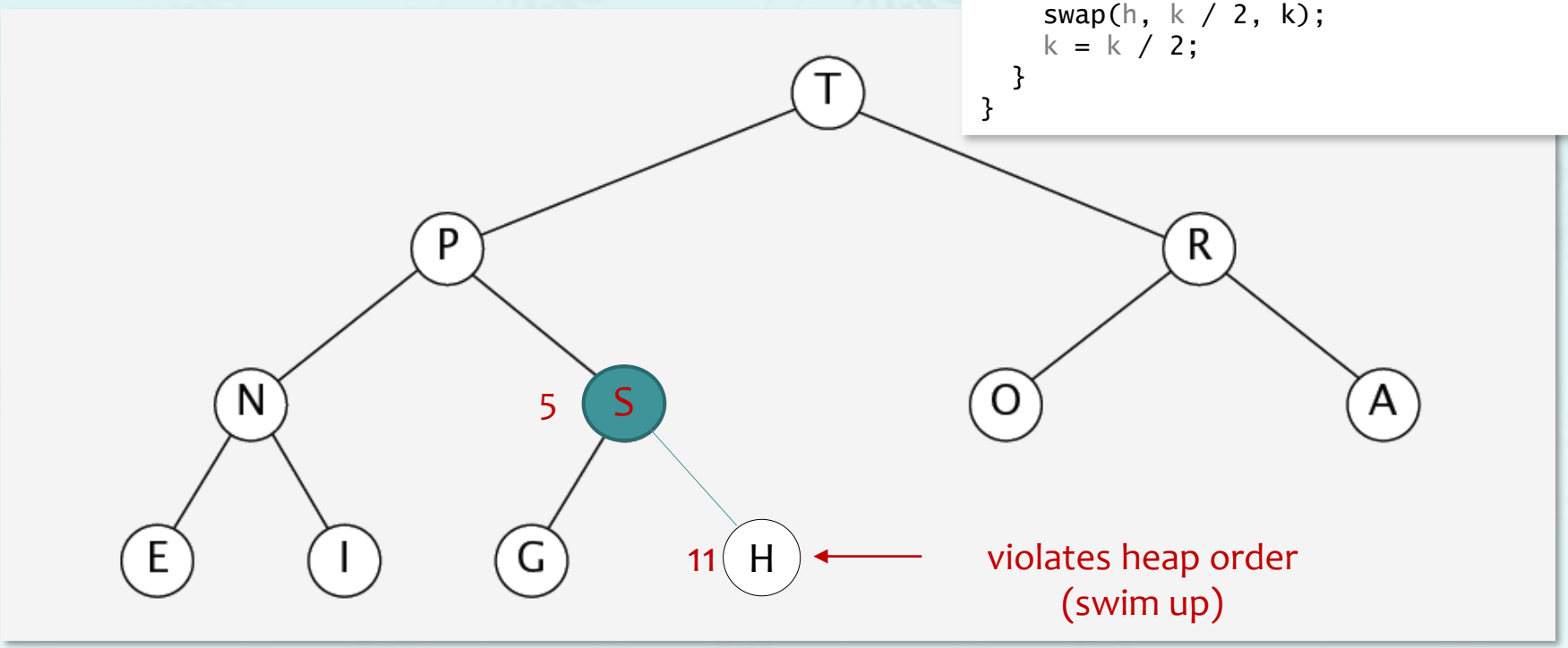
```
void swim(heap h, int k) {
  while (k > 1 && less(h, k / 2, k)) {
    swap(h, k / 2, k);
    k = k / 2;
  }
}
```

insert S



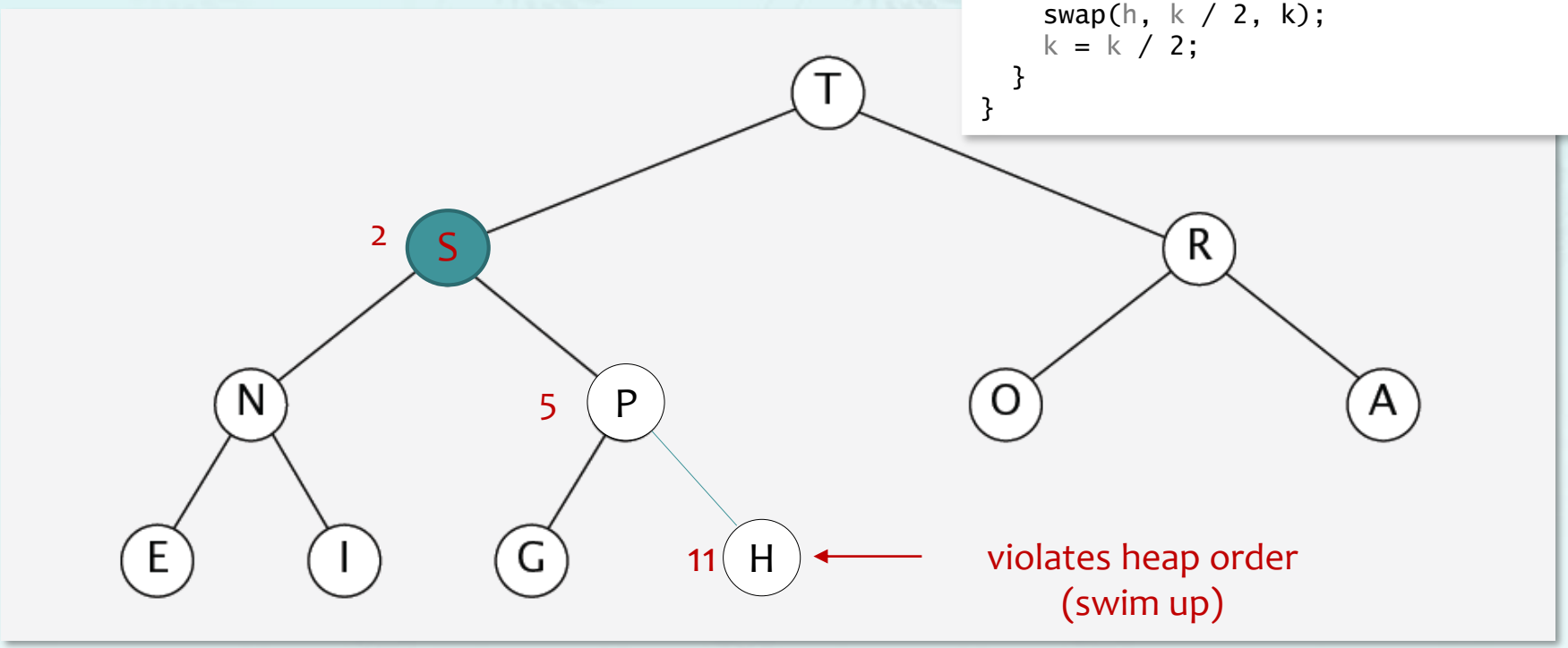
T	P	R	N	S	O	A	E	I	G	H
				5						11

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

```
void swim(heap h, int k) {
  while (k > 1 && less(h, k / 2, k)) {
    swap(h, k / 2, k);
    k = k / 2;
  }
}
```

insert S

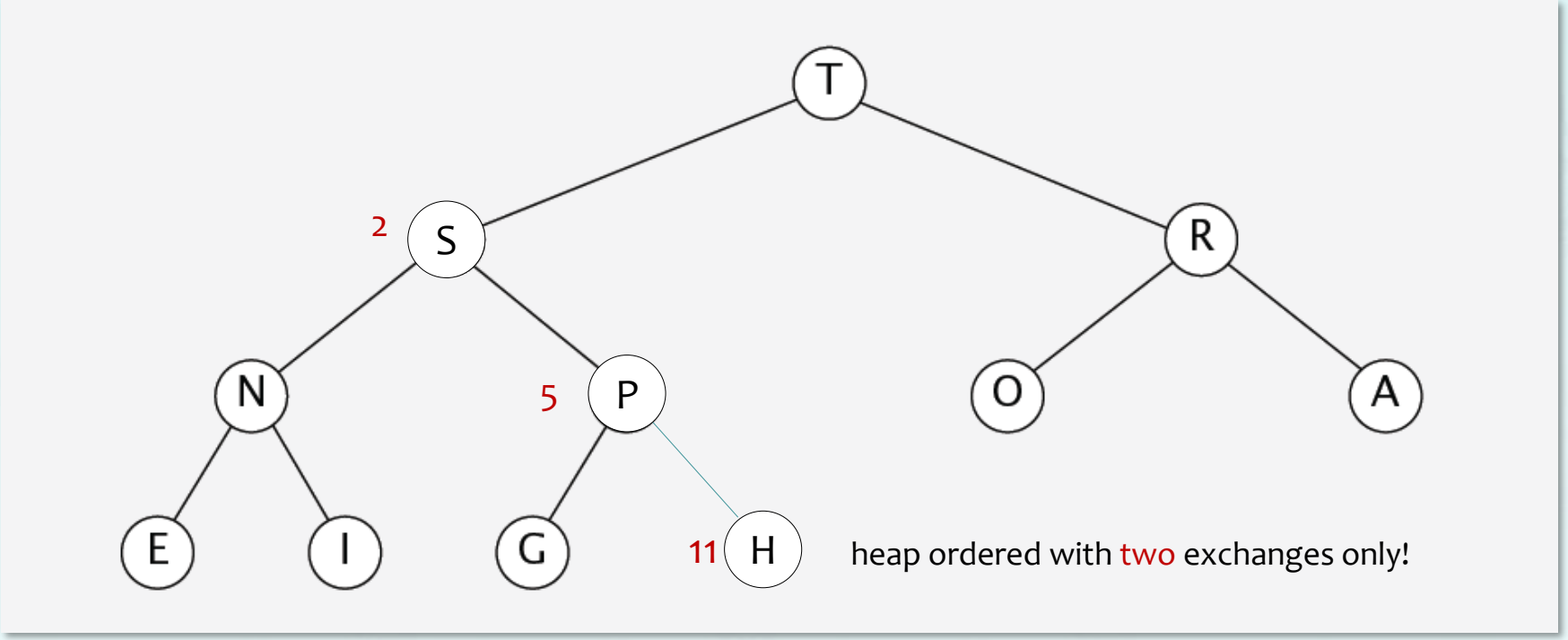


T	S	R	N	P	O	A	E	I	G	H
	2			5						11

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

heap ordered



T	S	R	N	P	O	A	E	I	G	H
	2			5						11

maxheap example

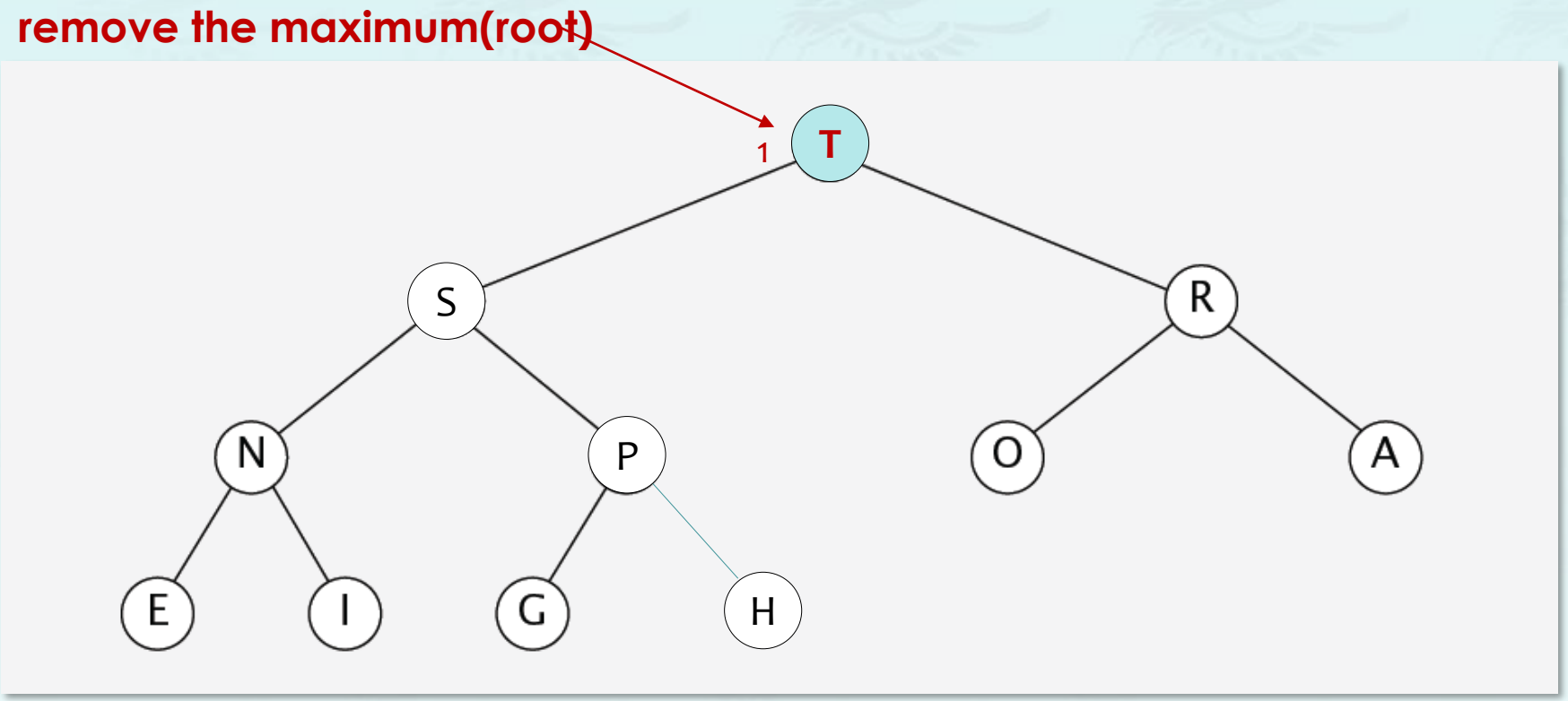
- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

T	S	R	N	P	O	A	E	I	G	H
1										

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

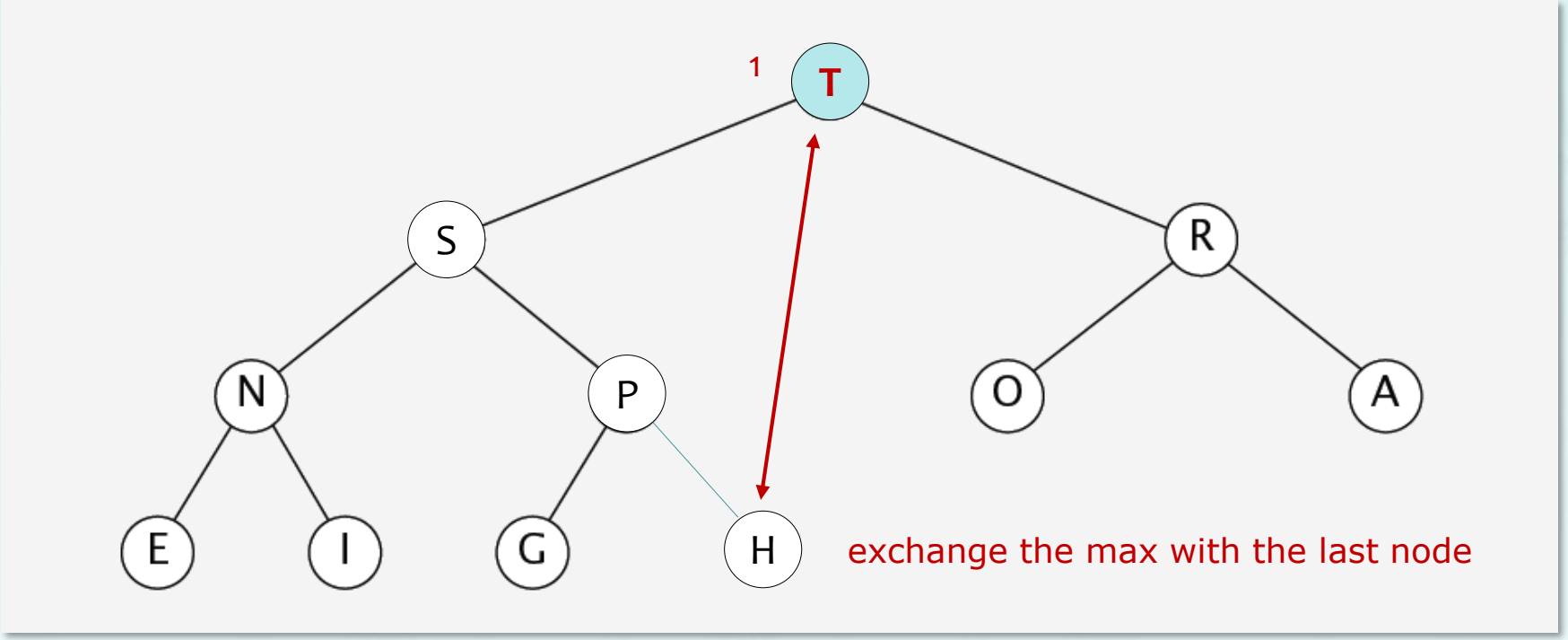


T	S	R	N	P	O	A	E	I	G	H
1										

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



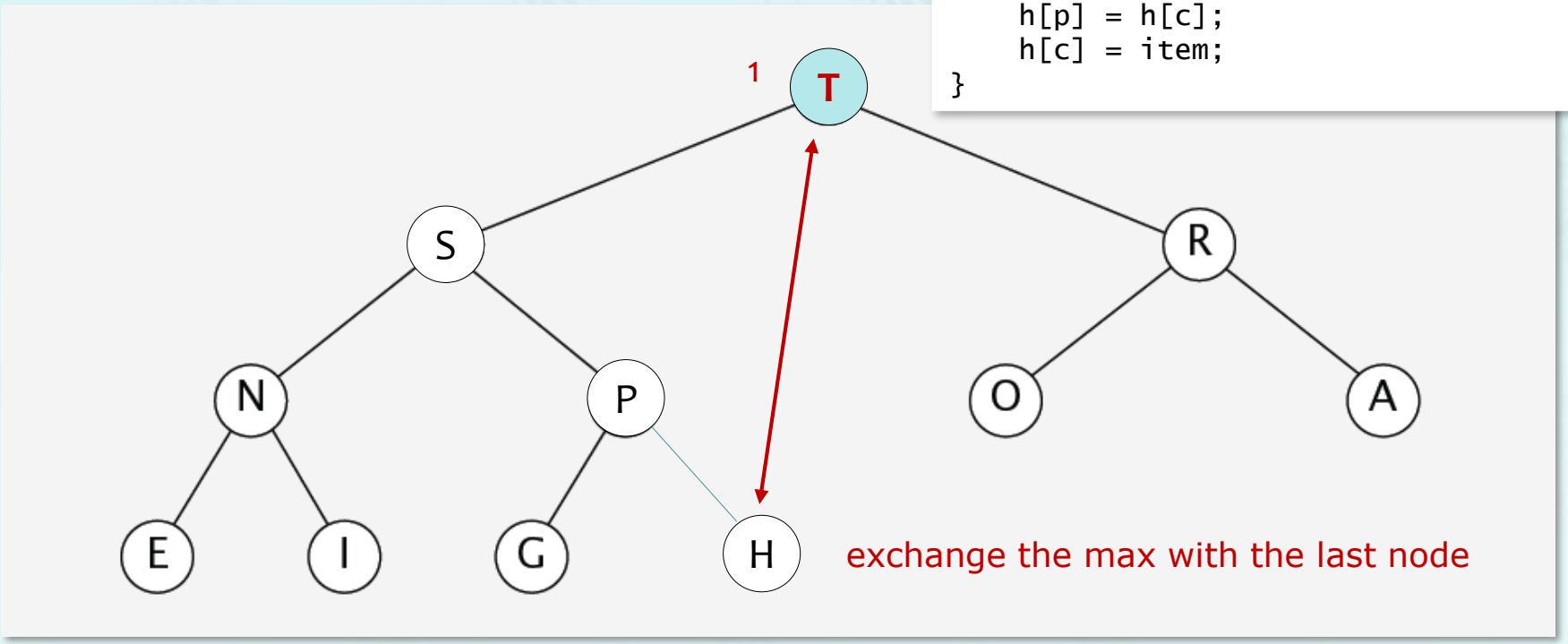
T	S	R	N	P	O	A	E	I	G	H
1										h->N

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

```
void swap(heap h, int p, int c) {
    Key item = h[p];
    h[p] = h[c];
    h[c] = item;
}
```

remove the maximum(root)



T	S	R	N	P	O	A	E	I	G	H

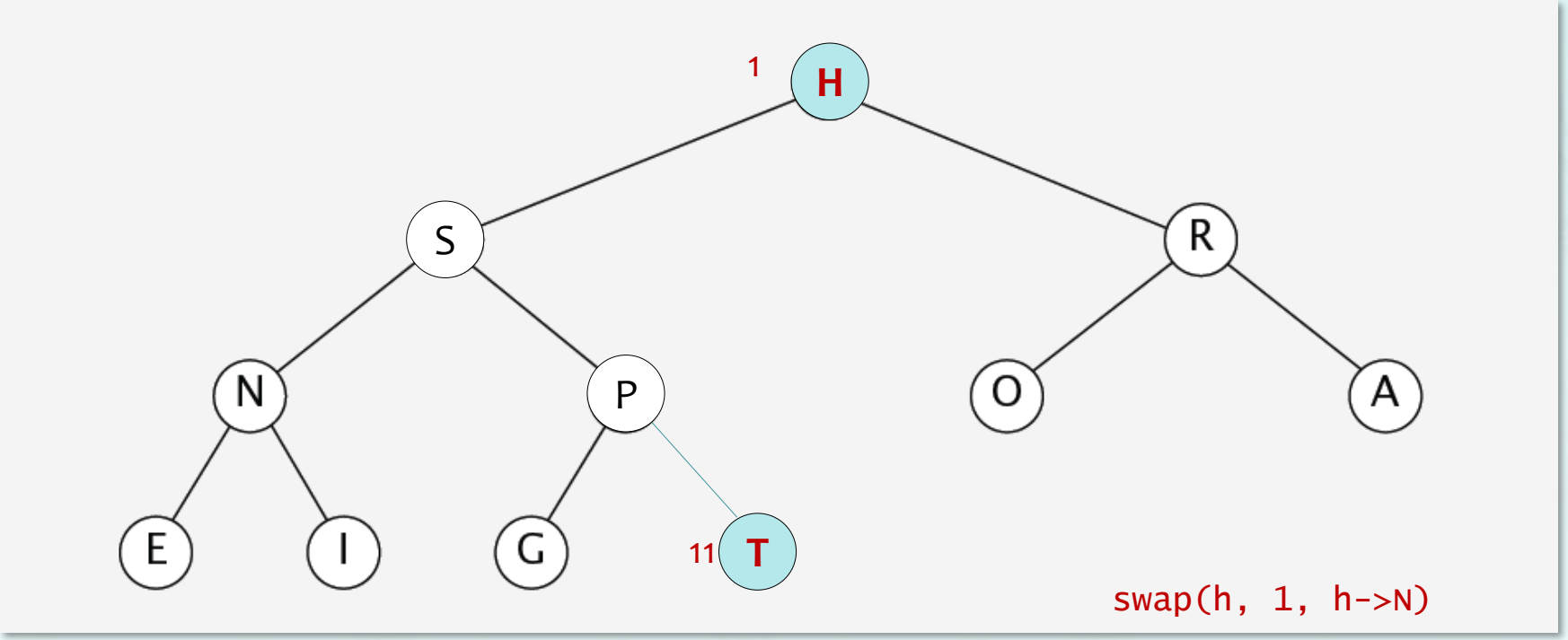
1

h->N

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

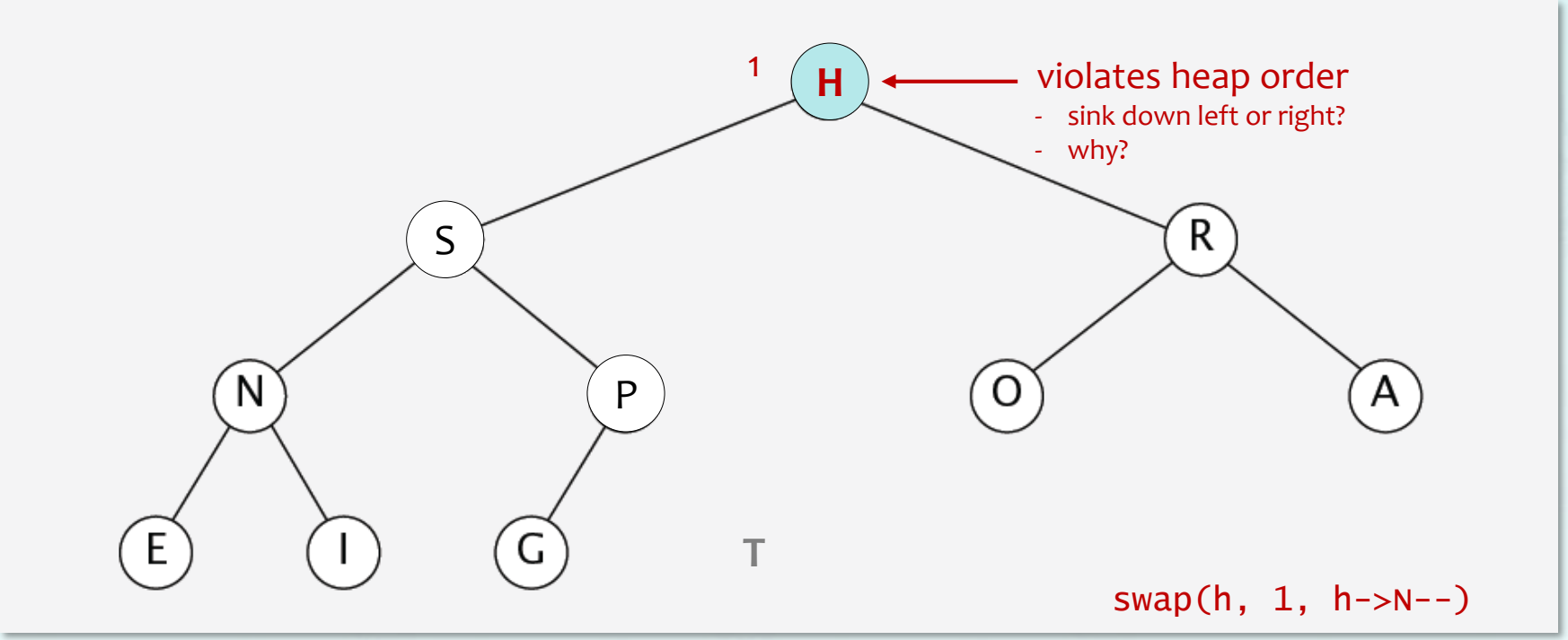


H	S	R	N	P	O	A	E	I	G	T
1										11

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



H	S	R	N	P	O	A	E	I	G	T

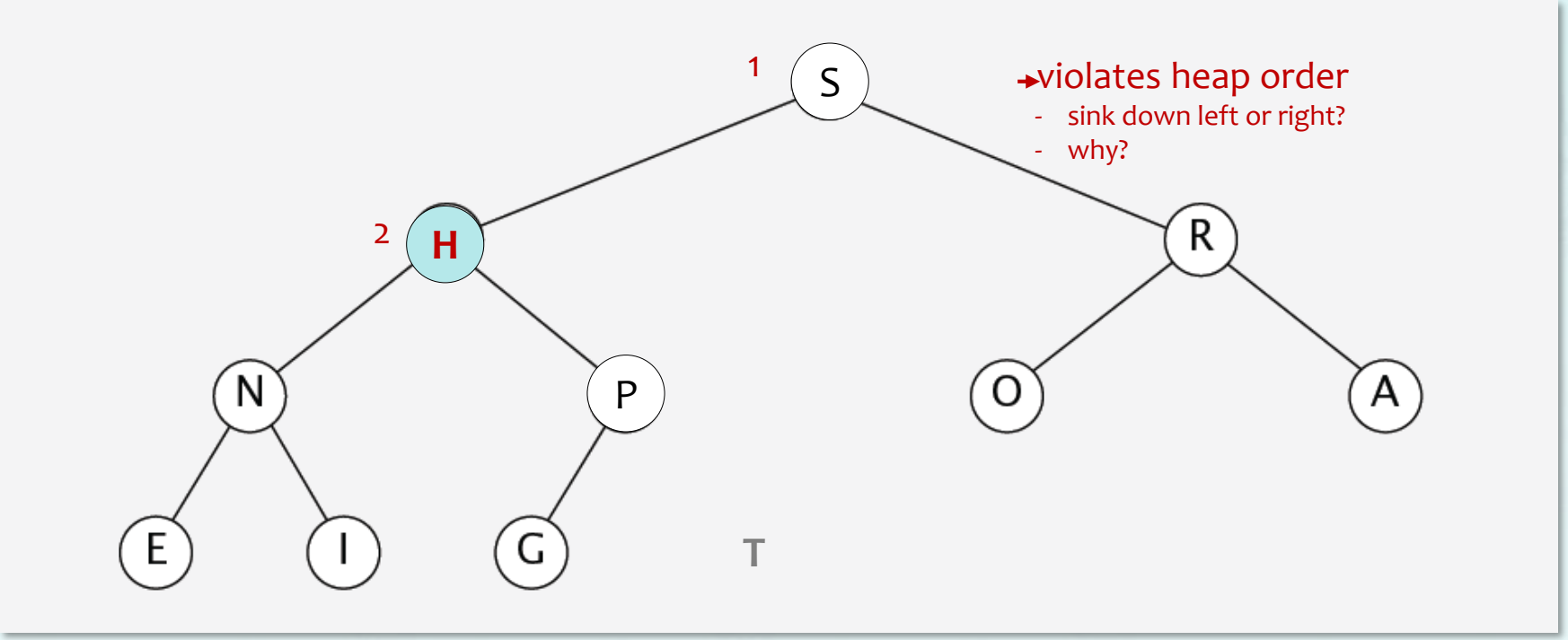
1

heap size decreased by one

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

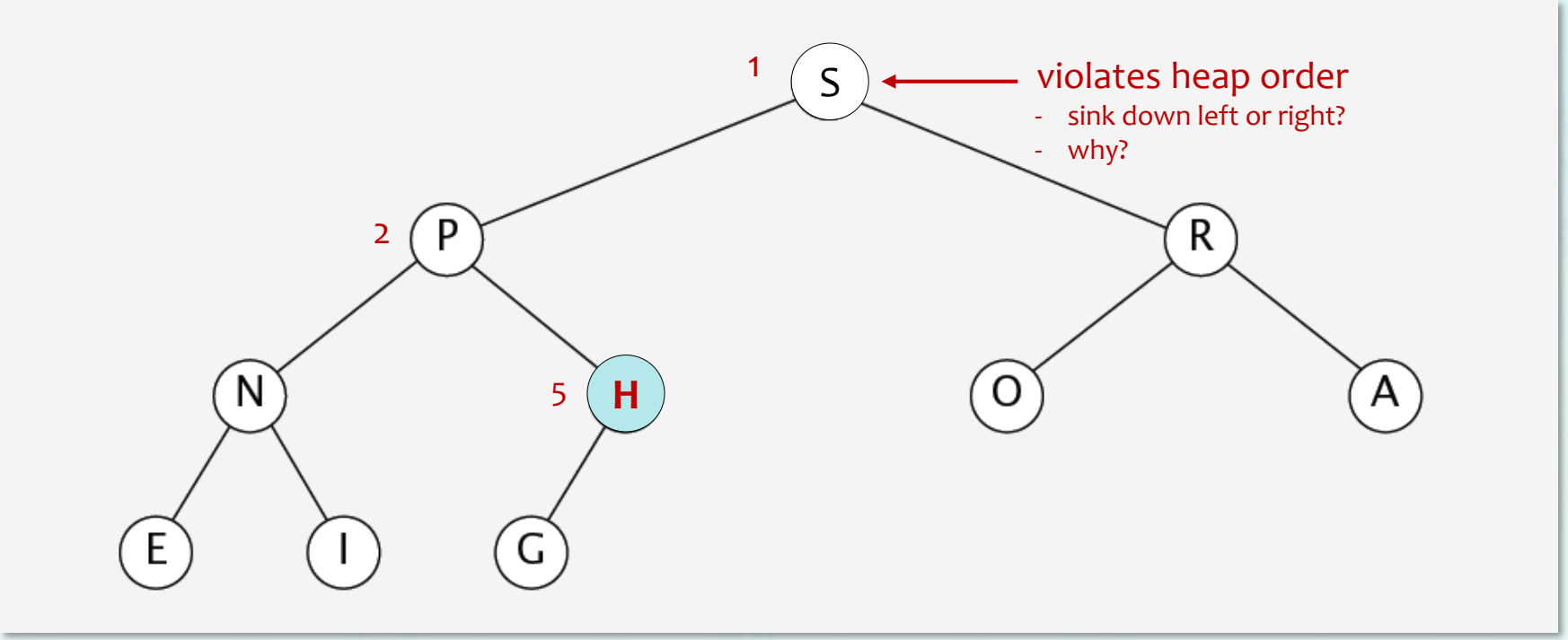


S	H	R	N	P	O	A	E	I	G	T
1	2									

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

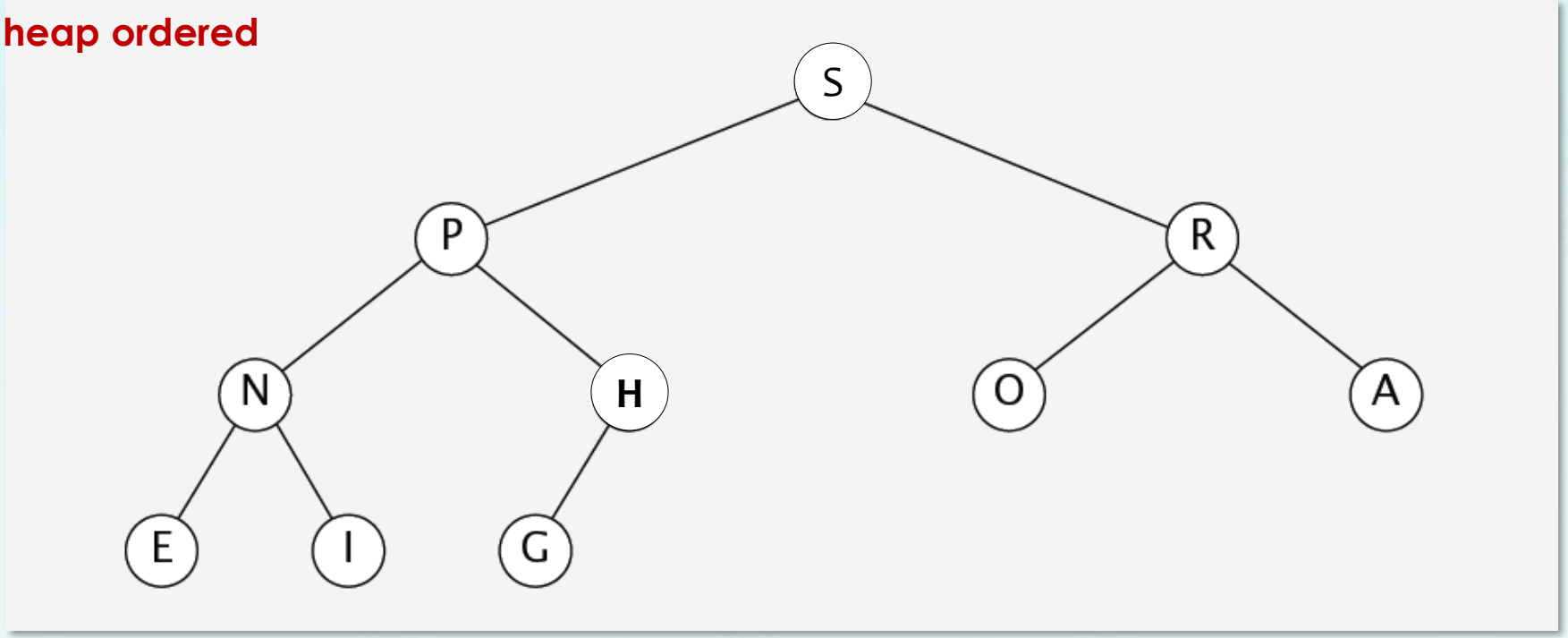
remove the maximum(root)



S	P	R	N	H	O	A	E	I	G	
1	2			5						

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

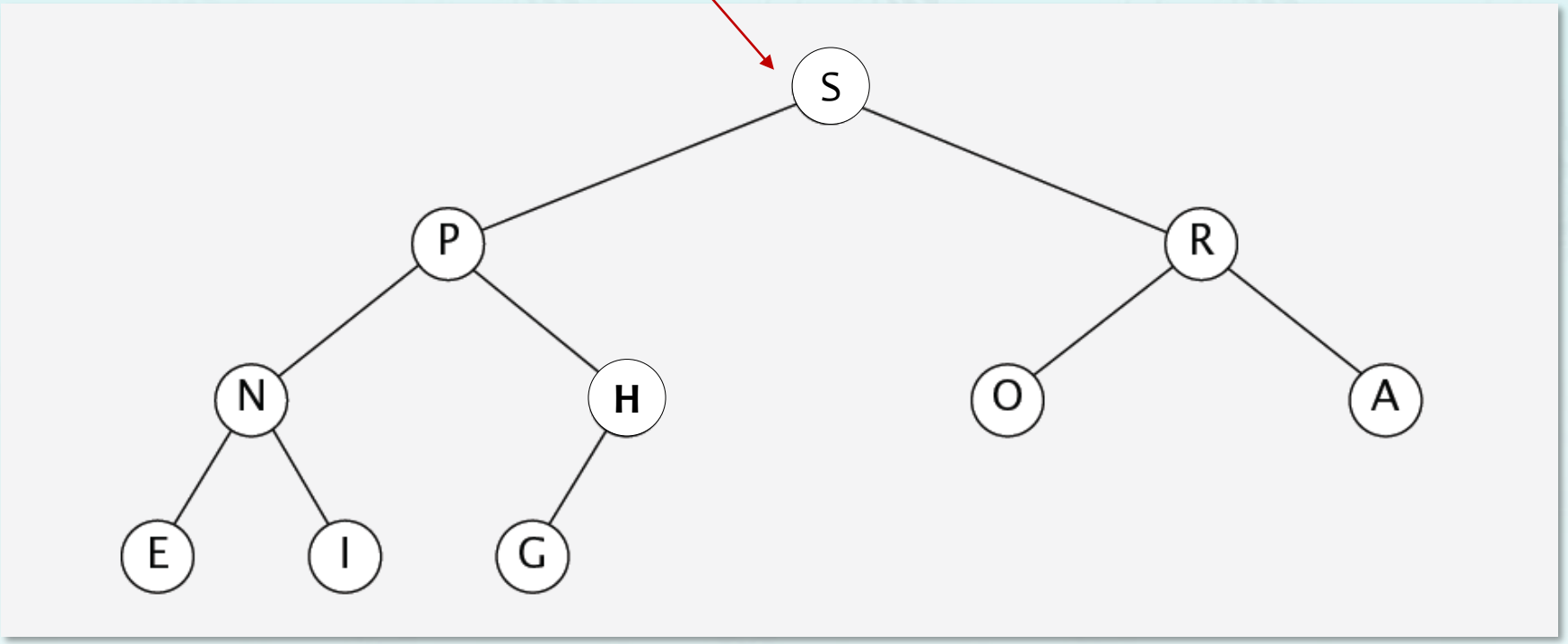


S	P	R	N	H	O	A	E	I	G	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



S	P	R	N	H	O	A	E	I	G	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

```
void swap(heap h, int p, int c) {
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}
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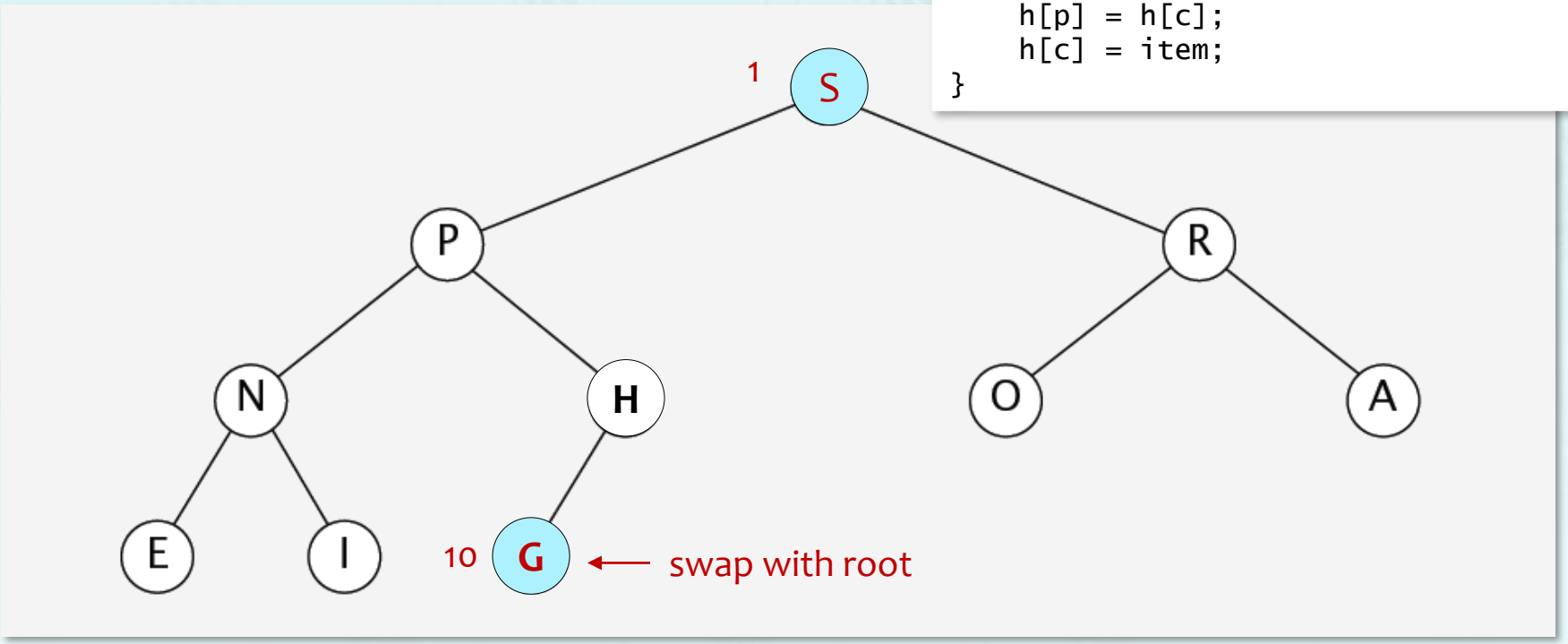

S	P	R	N	H	O	A	E	I	G	
1									10 h->N	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

```
void swap(heap h, int p, int c) {
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    h[p] = h[c];
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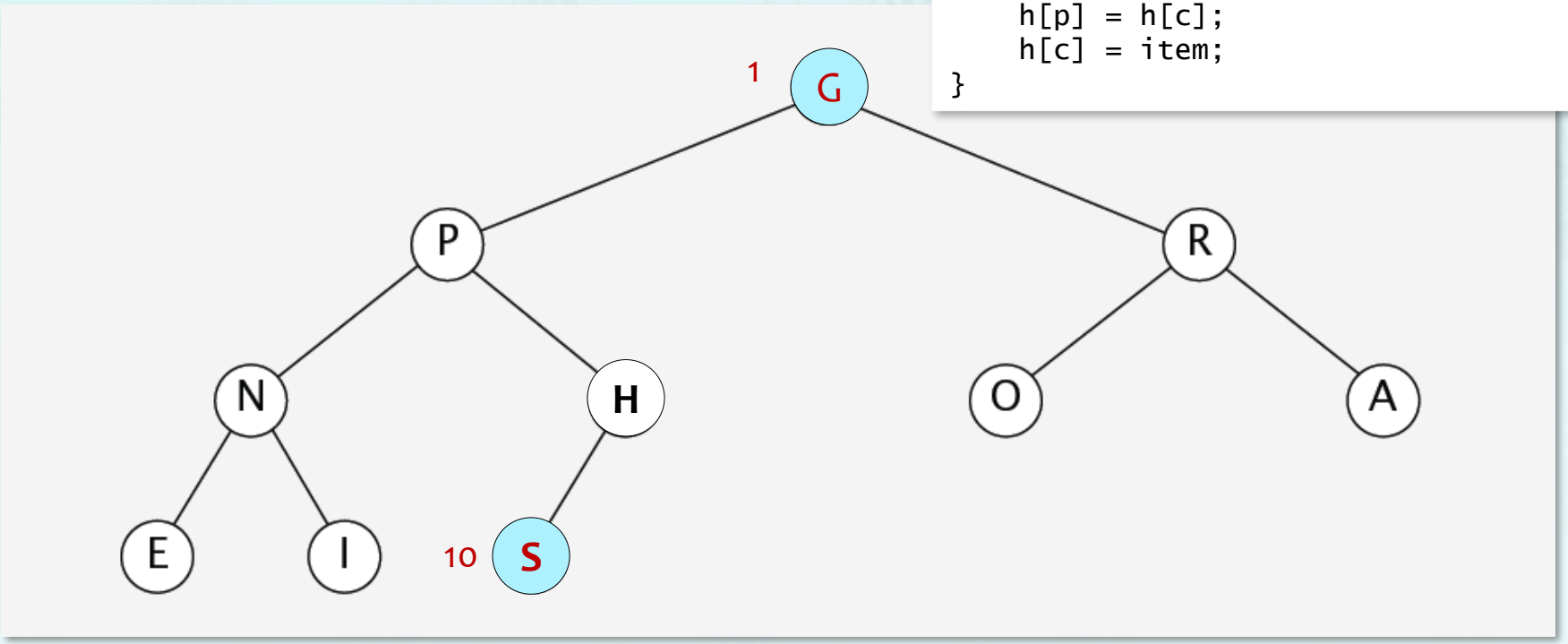
}
```

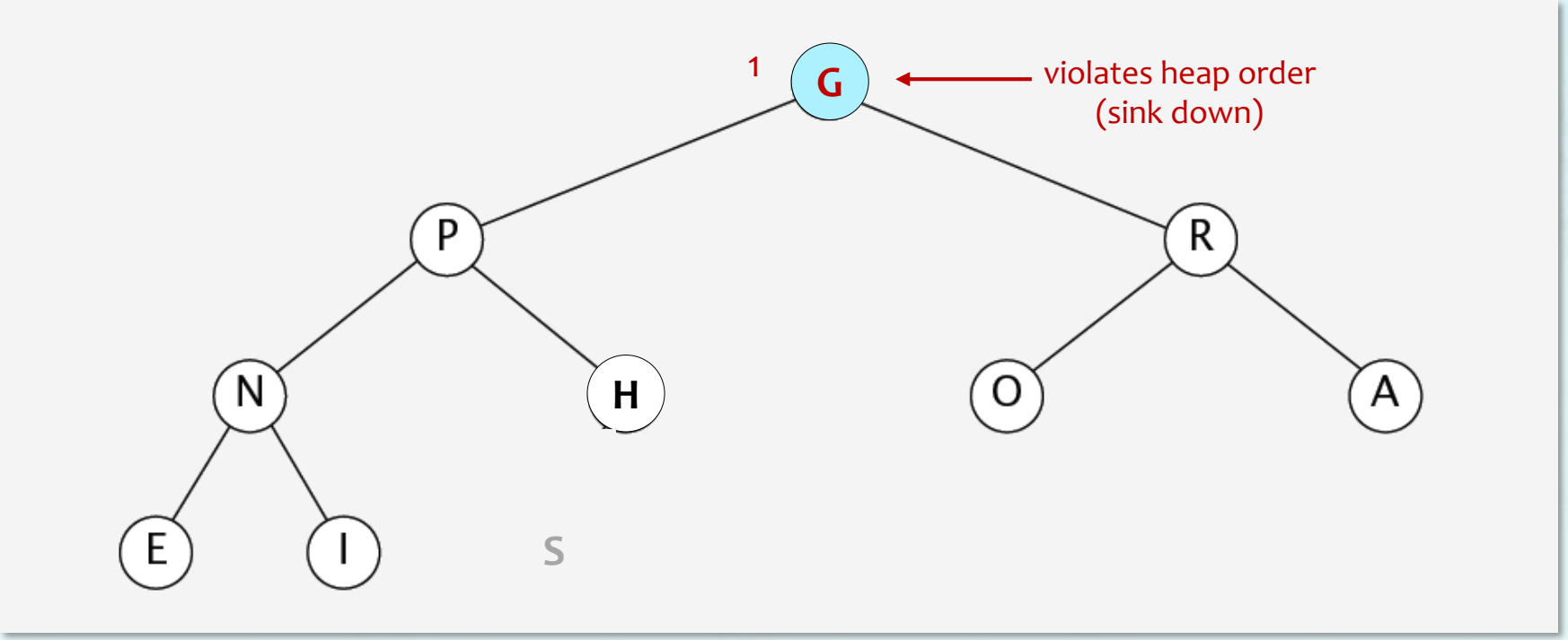


G	P	R	N	H	O	A	E	I	S	
1									10	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)



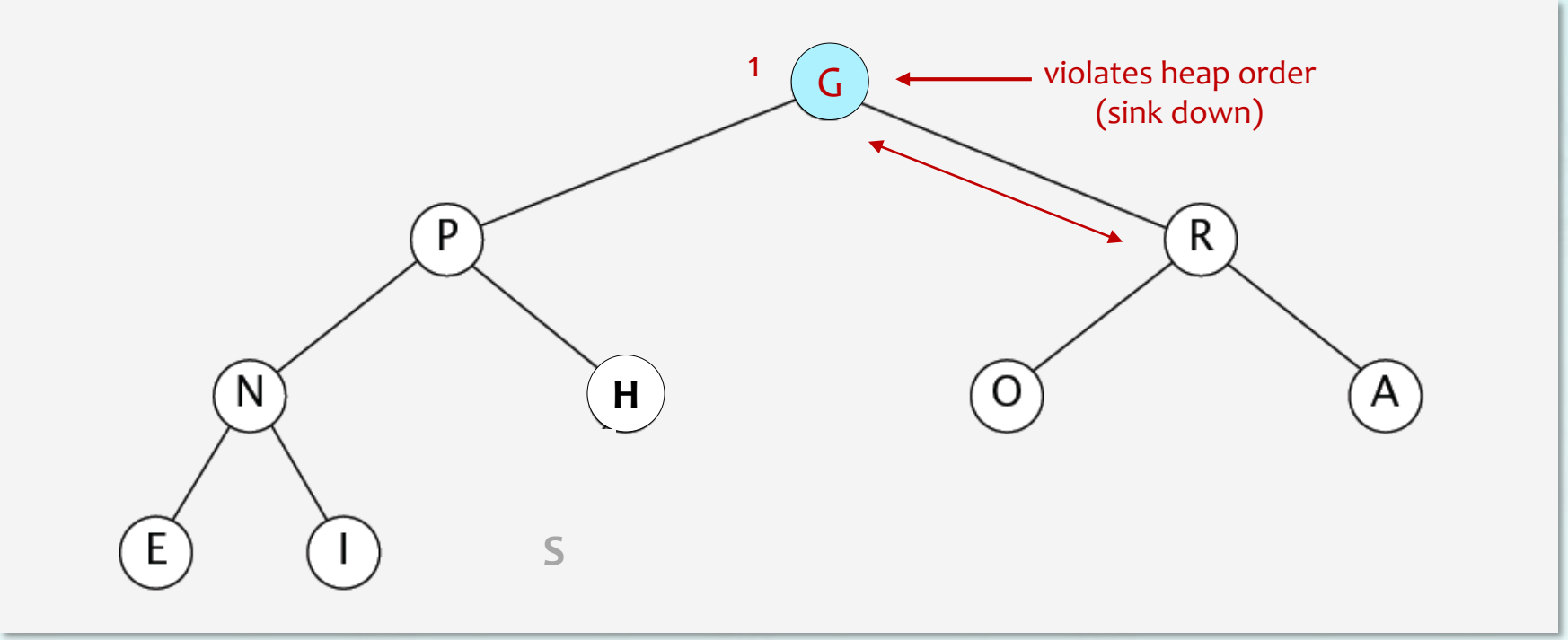
G	P	R	N	H	O	A	E	I	S	
1										

heap size decreased by one

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

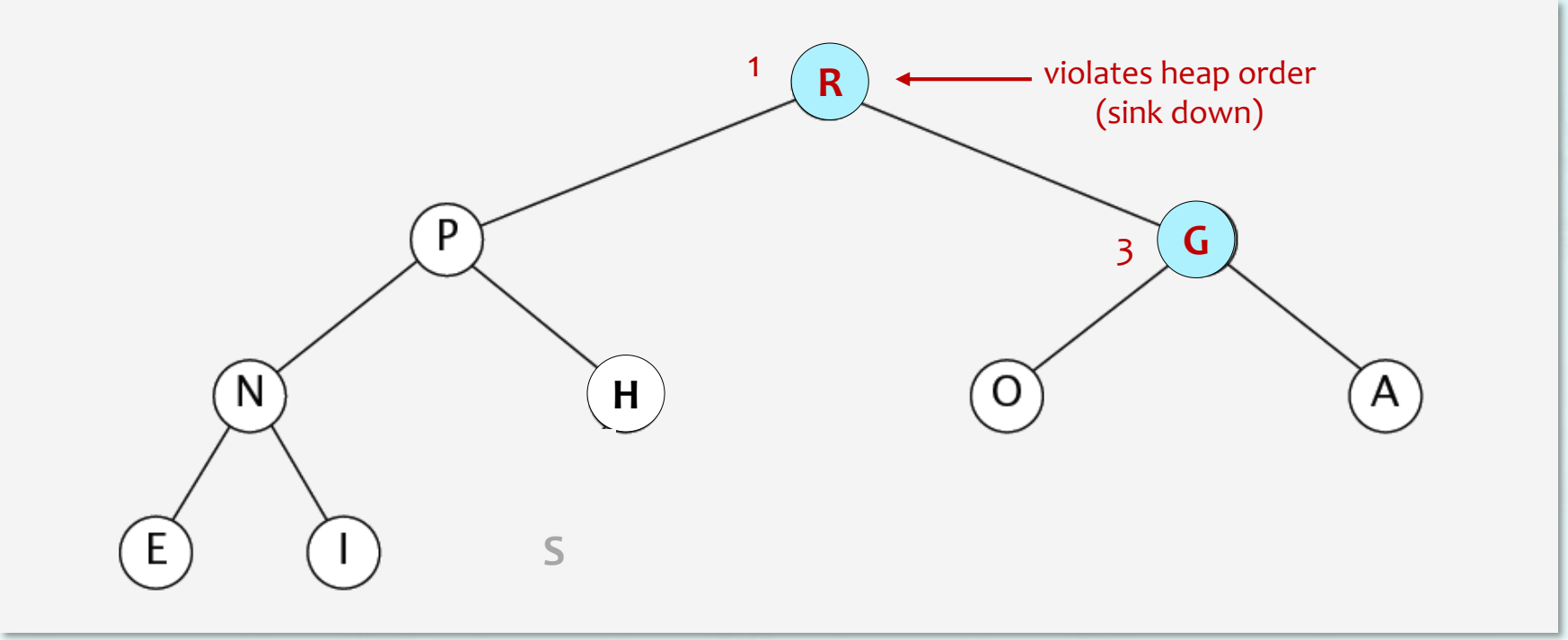


G	P	R	N	H	O	A	E	I	S	
1								N=9		

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

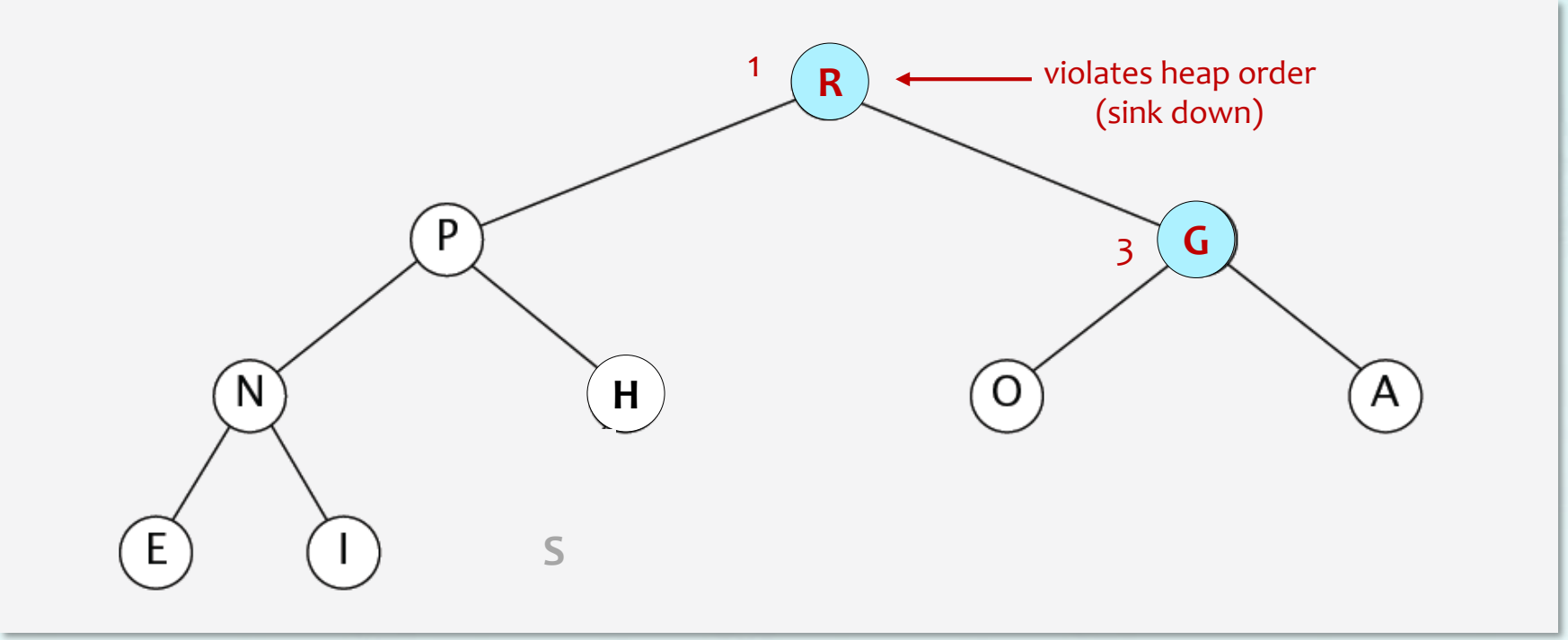


R	P	G	N	H	O	A	E	I	S	
1		3						N=9		

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

remove the maximum(root)

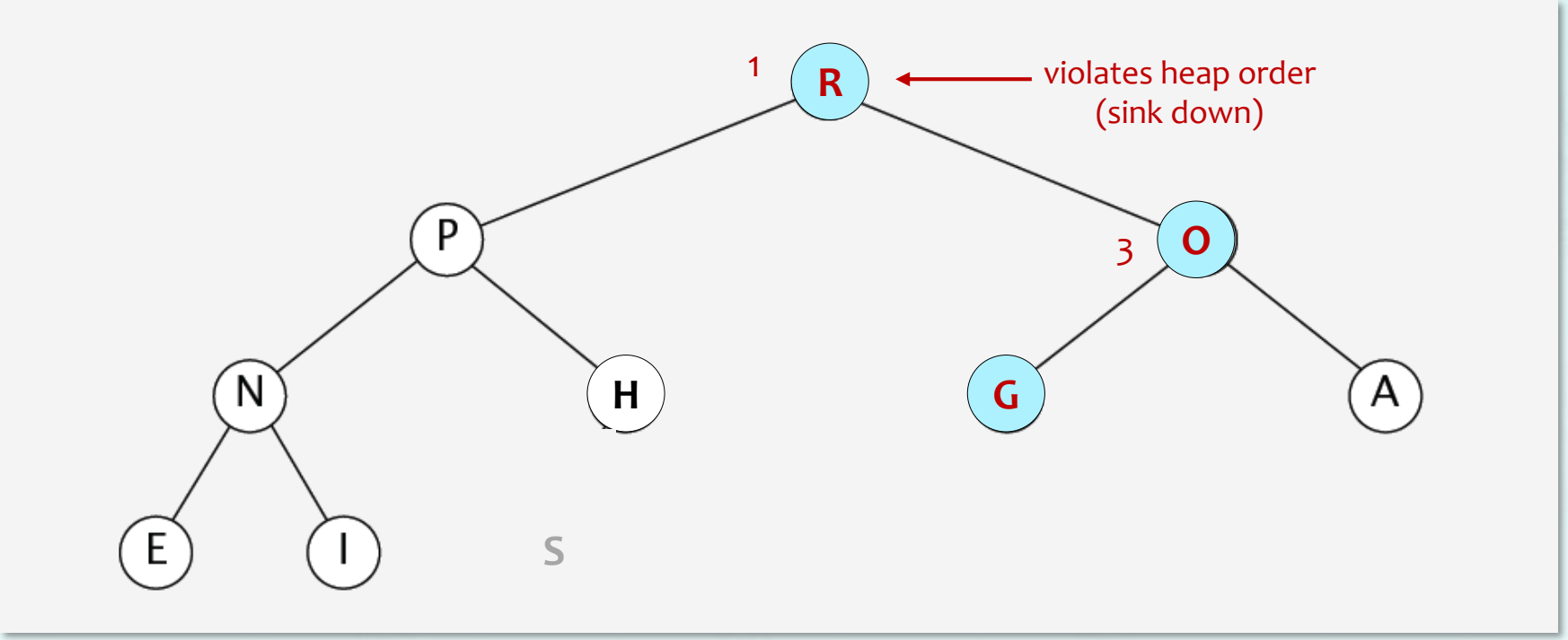


R	P	G	N	H	O	A	E	I	S	
1		3						N=9		

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

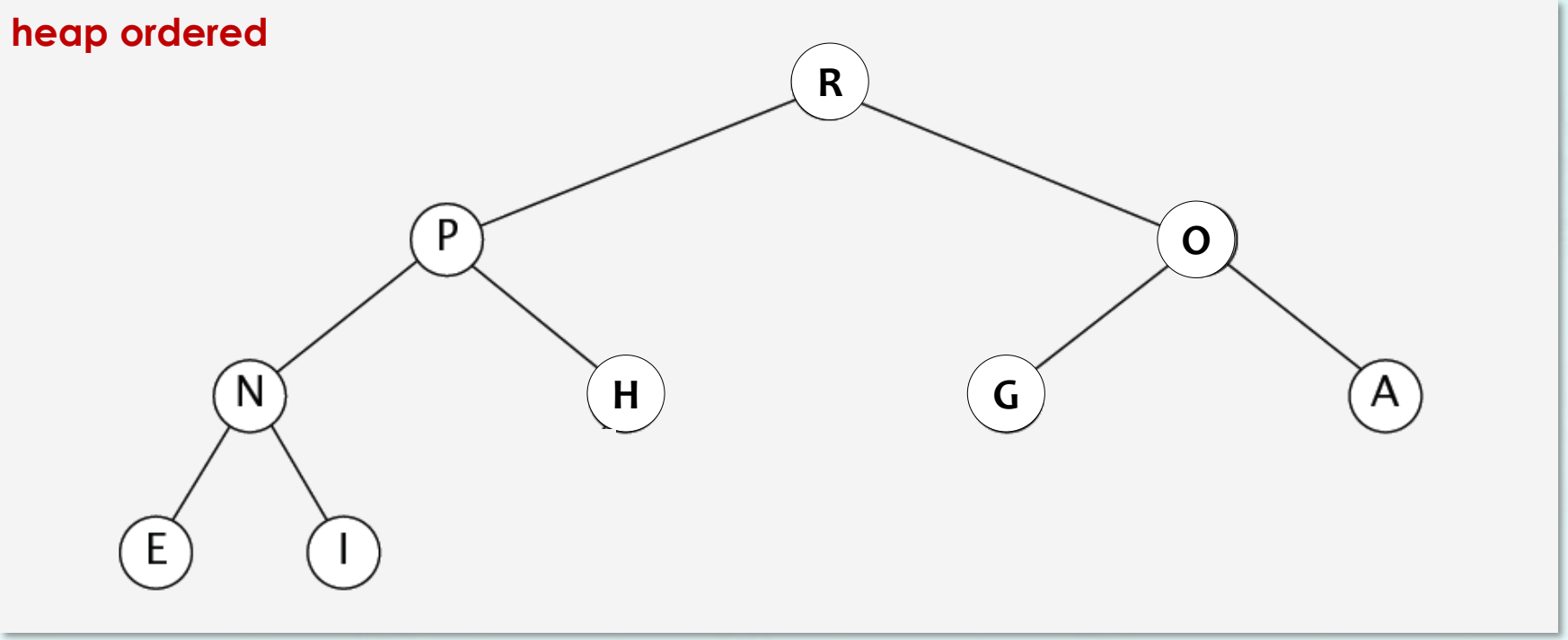
remove the maximum(root)



R	P	O	N	H	G	A	E	I	S	
1		3			6			N=9		

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.



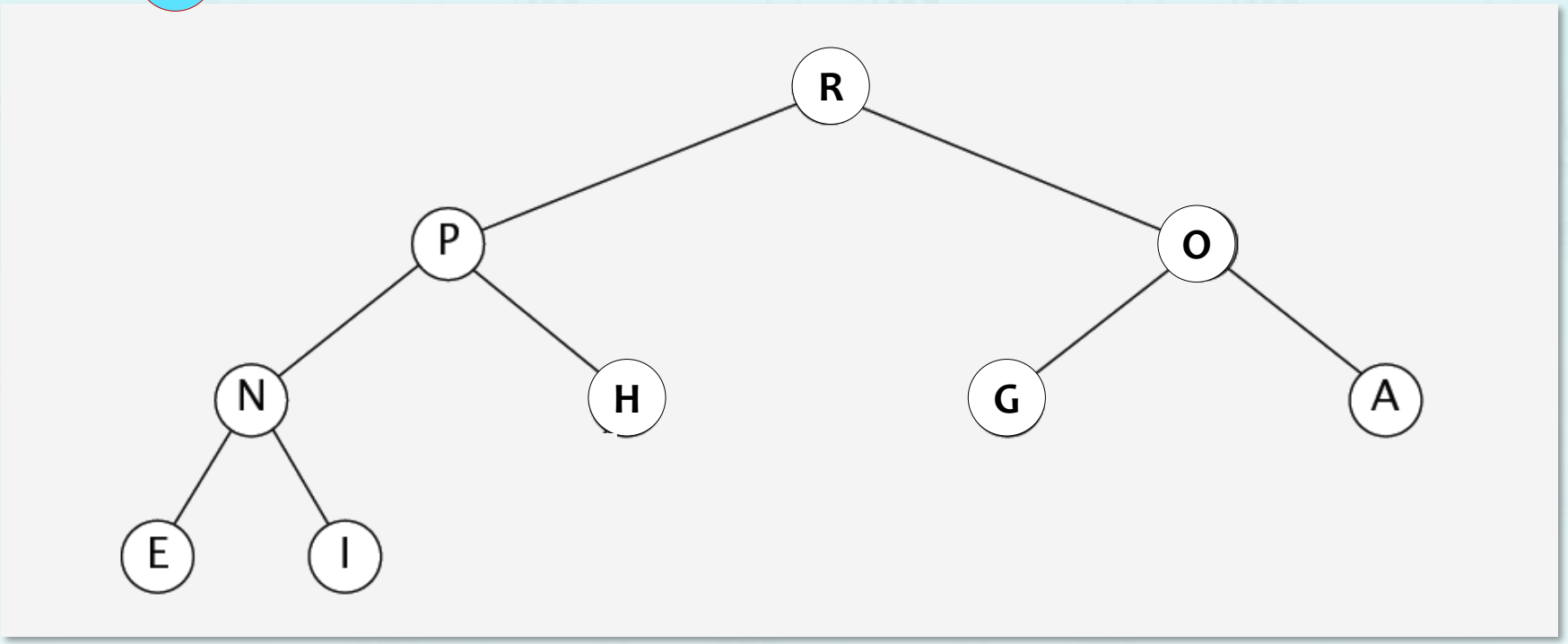
R	P	O	N	H	G	A	E	I		

N=9

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S

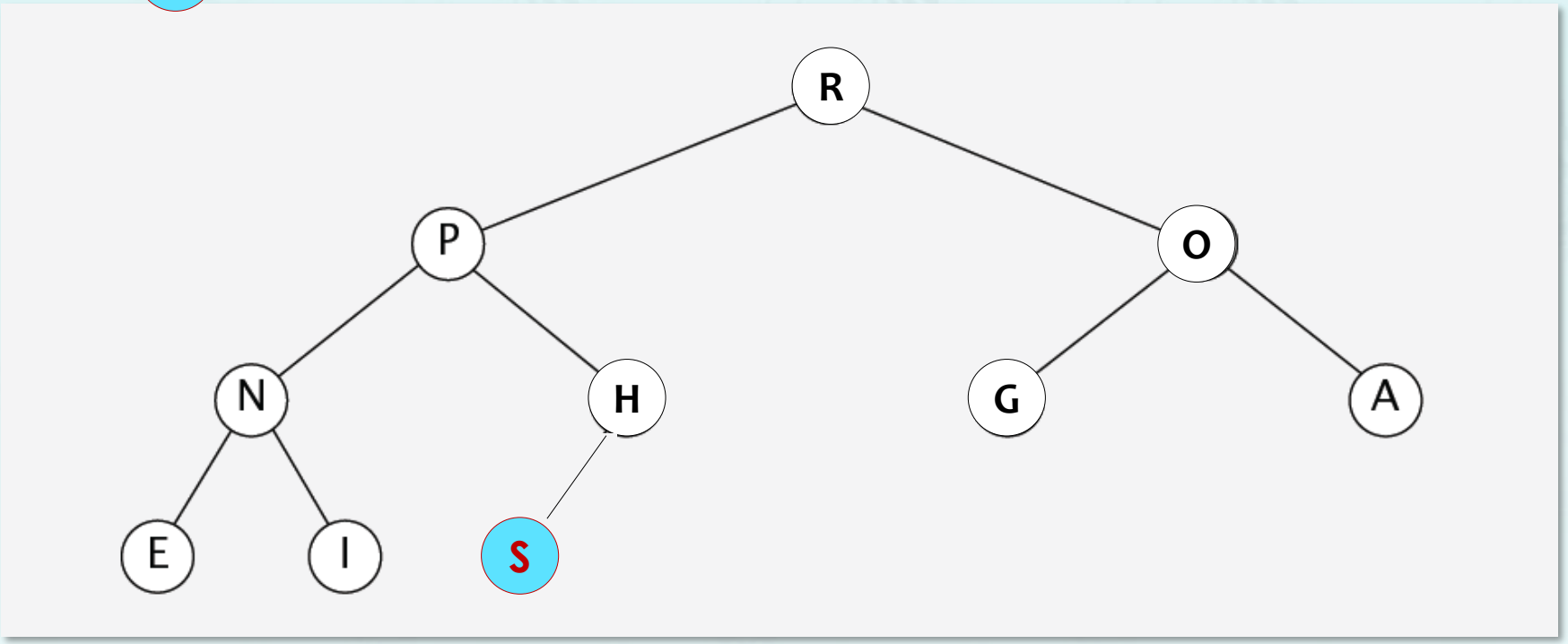


R	P	O	N	H	G	A	E	I		

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.

insert S

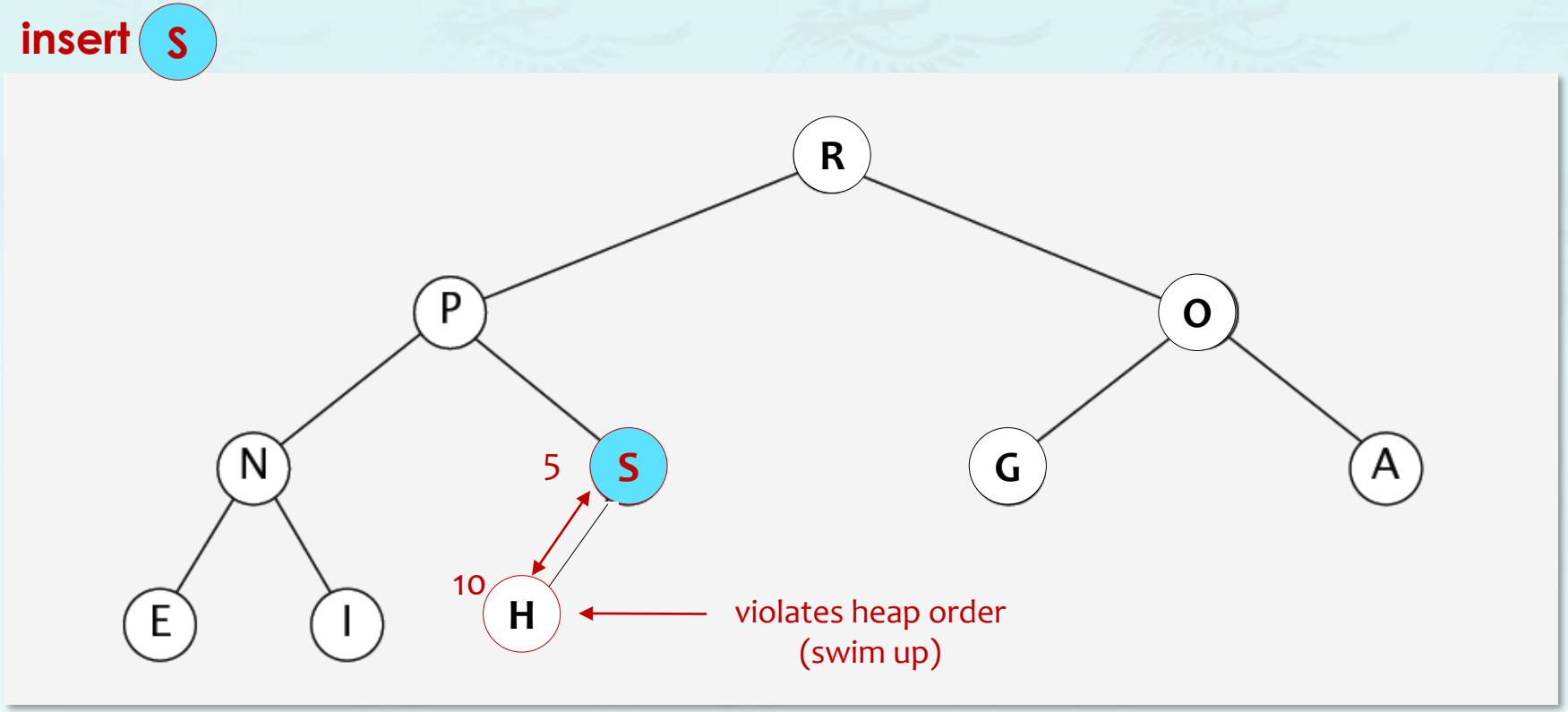


R	P	O	N	H	G	A	E	I	S	

N=10

maxheap example

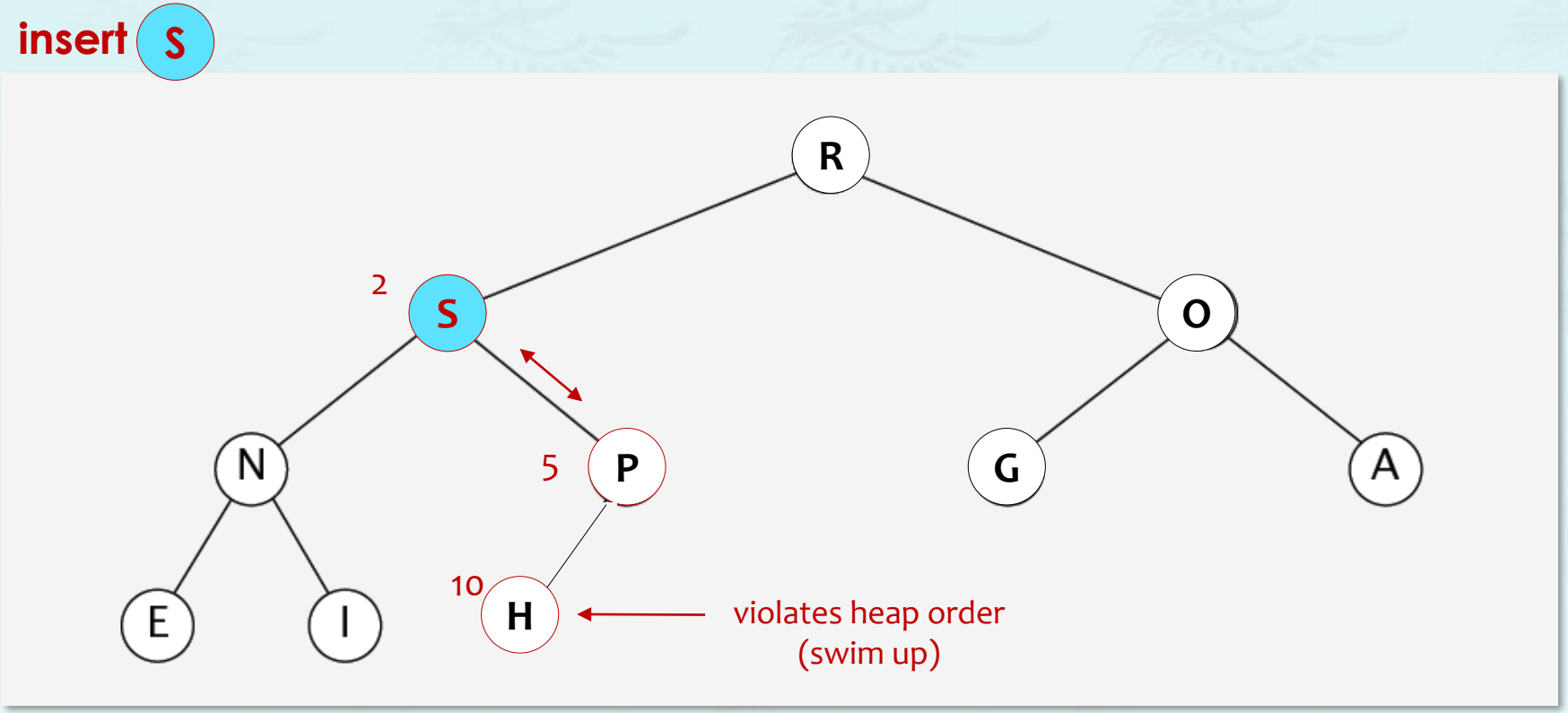
- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.



R	P	O	N	S	G	A	E	I	H	
				5					N=10	

maxheap example

- **Insert:** Add node at end, then swim it up.
- **Remove the root/max:** Swap root with node at end, then sink it down.



R	S	O	N	P	G	A	E	I	H	
	2			5					N=10	

maxheap example

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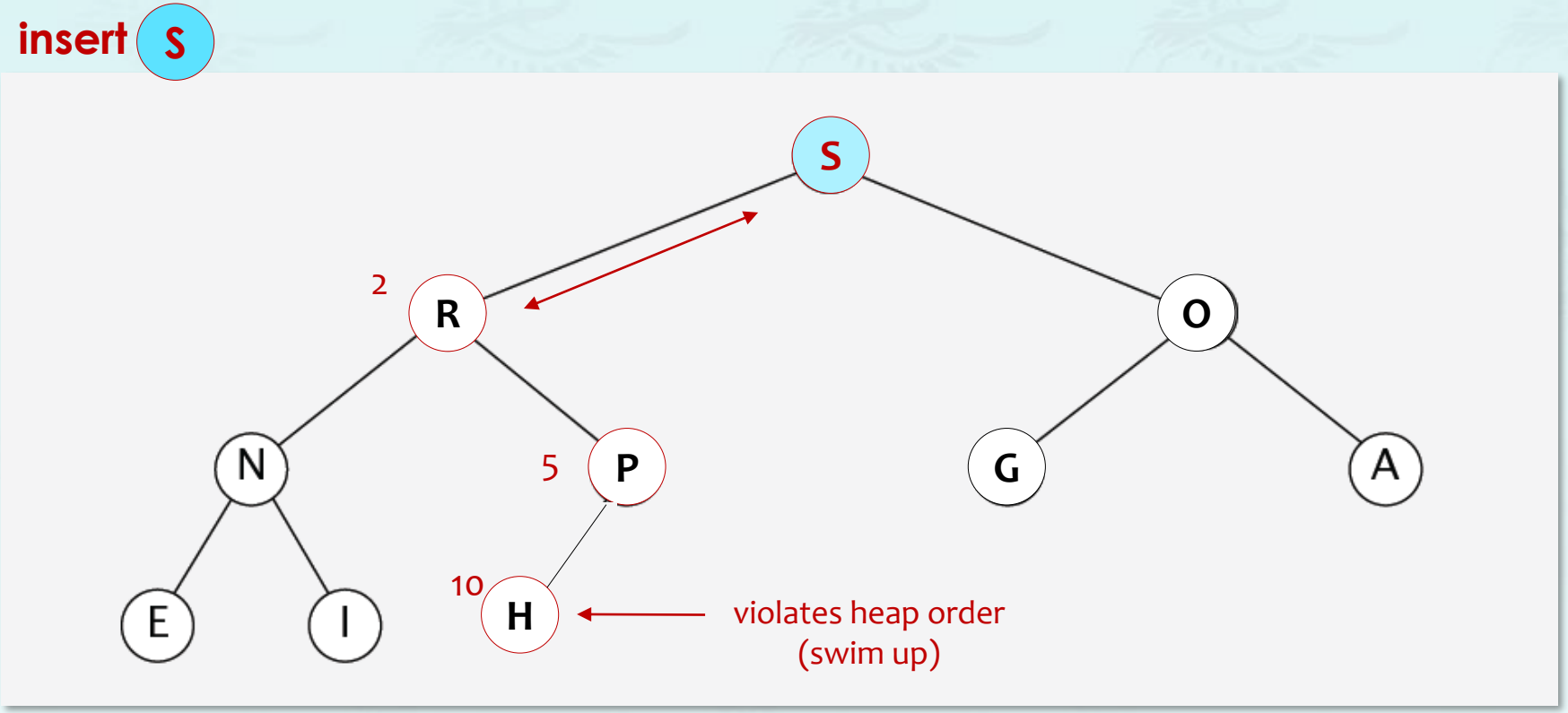
- **Remove the root/max:** Swap root with node at end, then sink it down.



S	R	O	N	P	G	A	E	I	H	
1	2			5					N=10	

maxheap example

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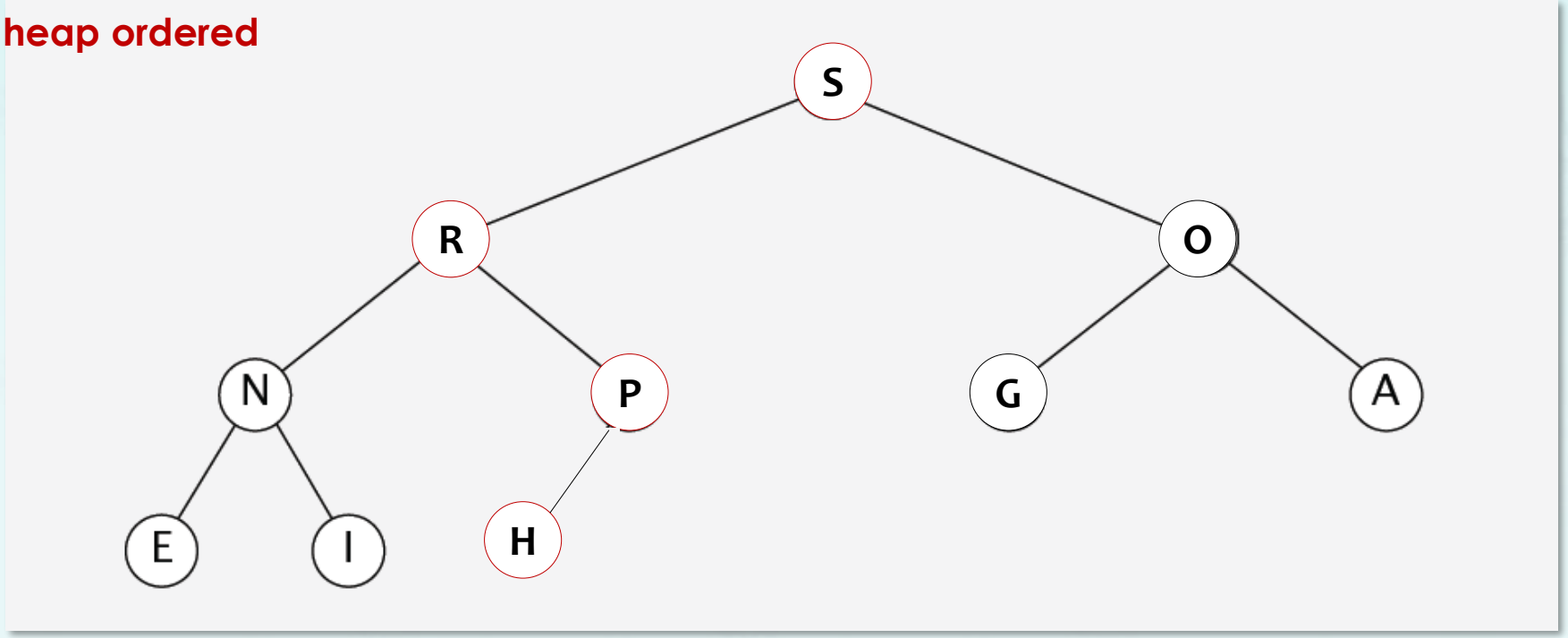


S	R	O	N	P	G	A	E	I	H	

Binary heap operations time complexity with N items:

- Level of heap is $\lfloor \log_2 N \rfloor$
- insert: O(log N) for each insert
  - In practice, expect less
- delete: O(log N) // deleting root node in min/max heap
- decreaseKey: O(log N)
- increaseKey: O(log N)
- remove: O(log N) // removing a node in any location

Heapify(): O(N)

Heapsort(): O(n log n)

Because O(N) heapify + O(n log n) remove nodes = O(n log n)

https://stackoverflow.com/questions/9755721/how-can-building-a-heap-be-on-time-complexity

Binary heap operations time complexity with N items:

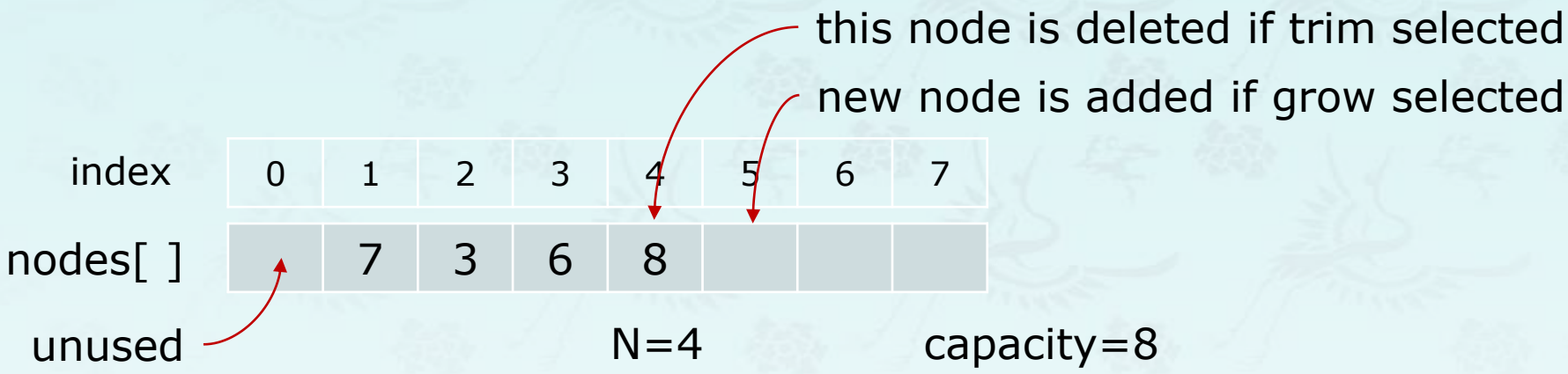
Implementation	Insert	Delete	max
Unordered array	1	N	N
Ordered array	N	1	1
Binary heap	**log N**	**log N**	1

Mission Completed

heap

- *complete binary tree (review)*
- *heap and priority queues (Chapter 9)*
- *binary heap and minheap*
- *maxheap demo*
- *maxheap coding*
- *heap sort (Chapter 7)*

Chapter 7


this node is deleted if trim selected
new node is added if grow selected
index
0
1
2
3
4
5
6
7
nodes[ ]
7
3
6
8
unused
N=4
capacity=8