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PROJECT REPORT

Lognormal Mixture Model for Option Pricing and Calibration to Market Volatility Surface

HE Junjie

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Abstract

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by HE Junjie

Risk Management Institute

The Black and Scholes formula has been the most important pricing formula since it was proposed in 1970s. The Black and Scholes model was derived under several bold assumptions, which were later proved not consistent with the observation from the market. One of the most significant inconsistency is the finding of "volatility smile" in the market, which deviates from the constant volatility assumption of the model. A large number of subsequent research has been trying to model the volatility on top of Black and Scholes framework. Nevertheless, some of the models have complex closed-form pricing formula, some are computationally expensive to calibrate to the market. In this project report, we introduce the Lognormal Mixture Model. One of the most appealing points of the model is that the pricing formula of the vanilla option under the Lognormal Mixture Model is straightforward, simply a linear combination of the prices of all its sub-processes, while being able to generate an endogenous volatility curve across moneyness dimension. In this project report, we first derive the Lognormal Mixture Model, explore its properties, then introduce two extensions from the model, then give a implied volatility solver and provide an end-to-end calibration process to the market data. As the main contribution of this report, we give an effective and stable end-to-end solution to generate a market implied volatility surface under the extended Lognormal Mixture framework, and evaluate the performance of the model to the real market. The above process provides a good reference for market volatility surface calibration.

Keywords: Lognormal Mixture Model, Lognormal Mixture Model with multiplicative means, implied volatility surface, calibration, option pricing

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Chapter 1

Introduction

In option pricing, the Black and Scholes Model (Black and Scholes, 1973; Merton, 1973) has been widely used since its publication and become one of the most fundamental pricing tools for option traders. In short, this model is built upon the no-arbitrage assumption and assumes that the stock price follows a Geometric Brownian Motion. But in practice, this framework has been exhibiting many aspects of inconsistencies with the observation from the market. The model assumes that the underlying price follows a Geometric Brownian Motion with constant drift and volatility, but pratical observation that is not covered by the Black and Scholes Model reveals the existence of the volatility smile, where the implied volatility of the market quotes is not constant, but bent upward at far moneyness. Moreover, the volatility curve is not even symmetric, but skewed to either side of the moneyness (which means the implied volatility at the at-the-money is not the lowest point).

To fit the model to real market observation, many different variants of Black and Scholes Model were proposed. Duan (Duan, 1995) proposed a GARCH Model to reflect the changes in the conditional volatility of the underlying asset in a parsimonious manner. Other ways to model the volatility smile/skew include stochastic models, local volatility model and Lognormal Mixture Model. Heston (Heston, 1993) proposed a model that set volatility as a mean-reverting stochastic process, where the stochastic part comes from the Brownian Motion that shows some level of correlation with the underlying. Local volatility model is developed by Emanuel Derman and Iraj Kani (Derman and Kani, 1994) on the basis of Dupire's PDE that links the contemporaneous prices of European call options of all strikes and maturities to the instantaneous volatility of the price process (Dupire, 1994). The local volatility model sees volatility as a function of both the current asset level and of time t. To get the local volatility, one needs to get the differenciation of market price with respect to time and strike(including first and second order) The model is increasingly widely used in practice recently. Brigo et al. (Brigo and Mercurio, 2002b) recently studies Lognormal Mixture Models and examined the analytical form of the mixture lognormal distribution to help generate desired volatility smiles. They also extended the model to fit into default cases and volatility skew, and gave some empirical examples to validate that the model effectively modelled the market quotes. On top of Brigo's work, Leisen (Leisen, 2004) utilized a sequence of Lognormal Mixture distributions to effectively approximate Black and Scholes model with jump-diffusion process and with stochastic volatility. Wilkens (Wilkens, 2005) compared the Lognormal Mixture Model with two and three components with Black and Scholes model and Gram-Charlier model in Deutsche Aktienindex (DAX) and Euro-Bond Future options and found that the Lognormal Mixture Model improved the fitting result in a quarter of the cases. Fang Mingyu (Fang, 2012) derived the closed form formulas of exotic options under the Lognormal Mixture Model. Wang et al (Peng Wang and Xinglin Yang, 2016) applied the 2-sub-process model to the China 50 ETF options and found that the model was effective for option pricing and calibration. Brigo et al. (Brigo, Mercurio, and Sartorelli, 2003) extended the Lognormal Mixture Model with different means and to hyperbolic-sine density-mixture dynamics to allow more complex properties on the model when fitting the model to the market.

In our project we examine the Lognormal Mixture Model and its extension. The following reasons drive us to this model: 1. the volatility smile and skew is endogenous and available to be in the model; 2. The closed form formula is simple enough that makes it feasible to solve volatility in satisfactory time; 3. The model is currently being researched and improved in the industry. This hints that the model is appealing enough in practice. As the main contribution of this report, we give an effective and stable end-to-end solution to generate a market implied volatility surface under the extended Lognormal Mixture framework, and evaluate the performance of the model to the real market. The above process provides a good reference for market volatility surface.

The rest of the project report is organized as follows: Chapter 2 provides a review of the Black and Scholes model, and then we specifies the Lognormal Mixture Model and its extensions, starting from the derivation of the mixture diffusion process, and finally we give a closed-form pricing formula of the model. Chapter 3 shows the detailed calibration process starting from the market price to implied volatility surface, using the closed-form Lognormal Mixture Model we get at Chapter 2. Chapter 4 exhibits the numerical result of our model and process and the comparison of our derived volatility surface with that provided by the market, and we give analysis according to the result. Chapter 5 wraps up our model and result and gives further potential improvement. To minimize confusion, we set the notation and parameters consistent throughout this report, and we number all of the equations and definitions shown in the report.

Chapter 2

Lognormal Mixture Model

The original Black and Scholes model shed light, for the first time, on pricing options effectively, though it is still far from perfect, it is our starting point to build our model, to a more complex world. In this Chapter we will start from the basic Black and Scholes model, and then introduce Mixed Lognormal probability function to model stock price. We will prove that there exist a unique bounded volatility, indicating the volatility in single process, that can be formulated by its sub-processes. Then we will derive that the option price of the alternative model is nothing but weighted average of the Black and Scholes price of its all sub- processes. Then we will extend the dynamics to include multiplicative means and default status.

2.1 Revisit Black and Scholes Model

There are several key assumptions of Black and Scholes framework, including:

- The rate of return on the risk-free asset is constant;
- The stock price follows a Geometric Brownian motion, with constant mean and volatility (To be exact, the instantaneous log return of the stock price is an infinitesimal Wiener process);
- No arbitrage in the market;
- The market is able to borrow and lend any amount of cash at the riskless rate, as well as any amount of stocks;
- There is no transaction fee.

Under the assumptions given above, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the stock price evolves following the process:

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) \tag{2.1}$$

where α the drift term, usually equal to risk-free rate minus dividend yield in risk neutral measurement for dividend-paying stocks, and σ is the volatility, and $dW(t)$ the Brownian Motion. Note that the σ here is a constant and does not change throughout the entire process of the stock price.

Let $V(t, S(t))$ be the value of an European vanilla call option at time t. Apparantly, the price of the option is a function of both time and underlying stock price, which is defined as a stochastic process at 2.1. By Ito's Lemma, the change of $V(t, S(t))$ from t to $t + dt$ is

$$dV(t, S(t)) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS(t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \tag{2.2}$$

If one simultaneously short $-\frac{\partial V}{\partial S}$ shares of the underlying stock at time t, the change of the portfolio $\Phi(t)$ from t to $t + dt$ is

$$d\Phi(t) = dV(t, S(t))dt - \frac{\partial V}{\partial S}dS(t) \tag{2.3}$$

Substitute the $dV(t, S(t))$ from equation 2.2 to 2.3, the dS term will be written out. Hence, this portfolio is not subject to any price risk at time t. Under no-arbitrage assumption mentioned above, the instantaneous change of the total portfolio at time t should equal to the riskless assets

$$d\Phi(t) = r\Phi(t)dt \tag{2.4}$$

hence one can get

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV(t, S(t)) = 0 \tag{2.5}$$

with terminal condition

$$V(T, S(T)) = (S(T) - K)^+ \quad \text{for all } S(T). \tag{2.6}$$

By Feynman-Kac theorem, the solution of 2.5,2.6 is equal to

$$V(t, S(t)) = \mathbb{E}[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}_t] \tag{2.7}$$

where K the strike price, $S(t)$ follows the process defined in 2.1, which can be re-written as a closed form lognormal stochastic process

$$S(t) \sim LN(\alpha - \frac{1}{2}\sigma^2(T - t), \sigma^2(T - t)). \tag{2.8}$$

Solving this equation, we arrive at the Black and Scholes formula

$$C_{BS} = e^{-rT}\left(FN(d_1) - KN(d_2)\right). \tag{2.9}$$

where $N(x)$ is the cumulative standard normal distribution function, S_0 the initial stock price,

$$d_1 = \frac{\ln\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and r the risk-free rate,

$$F = S_0 * e^{\alpha T}$$

the forward price of the stock.

In this formula, the parameters r, α, σ are all constant that is fixed for different strike level for a given process. These assumptions deviates from the market observation. One significant defect is its failure to model volatility smile, a phenomenon where the volatility fluctuates in different strike level and expiry dates. The Mixed Lognormal dynamics is a way to model the volatility variation that is not covered by original Black and Scholes model.

2.2 The Mixed Probability Density Function

On top of the single price process, Brigo et al. (Brigo and Mercurio, 2002b; Brigo and Mercurio, 2002a) introduced a mixture dynamics that served as improvement on the Black and Scholes model in its ability to generate implied volatility smiles that are able to simulate the empirical market data. Another advantage of this model is that there is a closed form Black and Scholes Style formula that is easy to implement pricing and calibration.

We still start from the one-dimensional stochastic process as in 2.1, but this time we let the volatility be a deterministic function of stock price and time, hence a "local volatility":

$$\frac{dS(t)}{S(t)} = \alpha dt + \nu(S(t), t)dW(t) \tag{2.10}$$

Note that we still set the drift term parameter be constant, while setting the volatility term be a deterministic function of time and the stochastic underlying price process. Next consider N independent one-dimensional time-homogeneous diffusion processes in the same probability space,

$$\frac{dS_i(t)}{S_i(t)} = \alpha dt + \nu_i dW(t) \quad \text{for all } i = 1, ..., N \tag{2.11}$$

where all $S_i(t)$ and $S(t)$ share a same mean α and same starting point

$$S(t) = S_i(t) = S(0) \quad \text{for all } i = 1, ..., N \tag{2.12}$$

Note that in the model, all sub-processes share a same starting point, a same drift term and diffusion motion, only to have different volatility terms. Hence, for each sub-process, we can still apply Black and Scholes formula respectively. No further variation for volatility is needed, because the mixture density function is enough to generate an implied volatility curve. We will derive this property in the following content.

Definition 2.1: *Let $p(y,t)$ the density function of $S(t)$, and $p_i(y,t)$ the density function of each $S_i(t)$ under the same probability space. The problem we want to address is the derivation of the single process volatility $\nu(t,S(t))$ such that the*

$$p(y,t) := \frac{d}{dy}P(S(t) \leq y) = \sum_{i=1}^{N} \lambda_i \frac{d}{dy} P(S_i(t) \leq y) = \sum_{i=1}^{N} \lambda_i p_i(y,t) \tag{2.13}$$

where each λ_i are strictly positive constants such that

$$\sum_{i=1}^{N} \lambda_i = 1. \tag{2.14}$$

Looking backward to the stochastic process, by applying Kolmogorov forward equation to equations 2.1 and 2.11, we have

$$\begin{cases} \frac{\partial}{\partial t} p(S,t) = -\alpha S \frac{\partial}{\partial S} p(S,t) + \frac{1}{2} \nu^2 (S,t) S^2 \frac{\partial^2}{\partial S^2} p(S,t) \\ \frac{\partial}{\partial t} p_i(S,t) = -\alpha S \frac{\partial}{\partial S} p_i(S,t) + \frac{1}{2} \nu_i^2 (S,t) S^2 \frac{\partial^2}{\partial S^2} p_i(S,t) \quad \text{for all } i = 1, ..., N \end{cases} \tag{2.15}$$

Multiplying each of the above second equations with λ_i respectively and take summation, the equation 2.15 is transformed to

$$\frac{\partial}{\partial t} \sum_{i=1}^{N} \lambda_i p_i(S,t) = -\alpha S \frac{\partial}{\partial S} \sum_{i=1}^{N} \lambda_i p_i(S,t) + \frac{1}{2} S^2 \frac{\partial^2}{\partial S^2} \sum_{i=1}^{N} \lambda_i \nu_i^2 (S,t) p_i(S,t) \tag{2.16}$$

Then substitute the left hand side and the first term of right hand side with equation 2.13, we have

$$\frac{\partial}{\partial t} p(S,t) = -\alpha S \frac{\partial}{\partial S} p(S,t) + \frac{1}{2} S^2 \frac{\partial^2}{\partial S^2} \sum_{i=1}^{N} \lambda_i \nu_i^2 (S,t) p_i(S,t) \tag{2.17}$$

Then Substitute the first equation of 2.15, the first term of right hand side can be written-out, the above equation is then transformed to

$$\nu^2 (S,t) S^2 \frac{\partial^2}{\partial S^2} p(S,t) = S^2 \frac{\partial^2}{\partial S^2} \sum_{i=1}^{N} \lambda_i \nu_i^2 (S,t) p_i(S,t) \tag{2.18}$$

By integrating both sides w.r.t S twice, finally we obtain that the expression for $\nu(S,t)$ under whose condition the objective equation of marginal density 2.13 can be achieved.

Proposition 2.1: *Assume that each ν_i is continuous and bounded from below by a positive constant, and that there exists an $\epsilon > 0$ such that $\nu_i(t) = \nu_0 > 0$, for each* *t in* $[0, \epsilon]$ *and i = 1, . . . , N. Then, if* $\nu(S,, t)$ *satisfies*

$$\nu(S,t) = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \nu_i^2 p_i(S,t)}{p(S,t)}} = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \nu_i^2 p_i(S,t)}{\sum_{i=1}^{N} \lambda_i p_i(S,t)}} \tag{2.19}$$

Hence, the stock price SDE as in 2.10 can be expressed as

$$\frac{dS(t)}{S(t)} = \alpha dt + \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \nu_i^2 p_i(S,t)}{\sum_{i=1}^{N} \lambda_i p_i(S,t)}} dW(t) \tag{2.20}$$

which means a mixture diffusion process can be precisely determined with a set of equations as in 2.11 and has a strong solution.

Brigo et al. (Brigo and Mercurio, 2000) also proved that the local volatility of the model is bounded,

$$\nu^2(S,t)S^2 \leq \mathcal{L}(1 + S^2) \tag{2.21}$$

where $\mathcal{L}$ is a finite positive constant, such that the strong solution exists. And they also proved that the representation is unique. These are necessary conditions to derive the valuation of the option with a mixture diffusion process. But as this condition is almost always satisfied in calibration process, we do not further discuss the detail of this condition.

The Figure 2.1 illustrates a stock price process, denoted with a line, with 4 sub-processes, denoted as dash lines, with parameters $r = 10\%, S_0 = 1, \lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.3, \lambda_4 = 0.4$ and $\sigma_1 = 0.12, \sigma_2 = 0.2, \sigma_3 = 0.1, \sigma_4 = 0.25$, and how they are correlated. This gives a good heuristic example of how the Lognormal Mixture Model works.

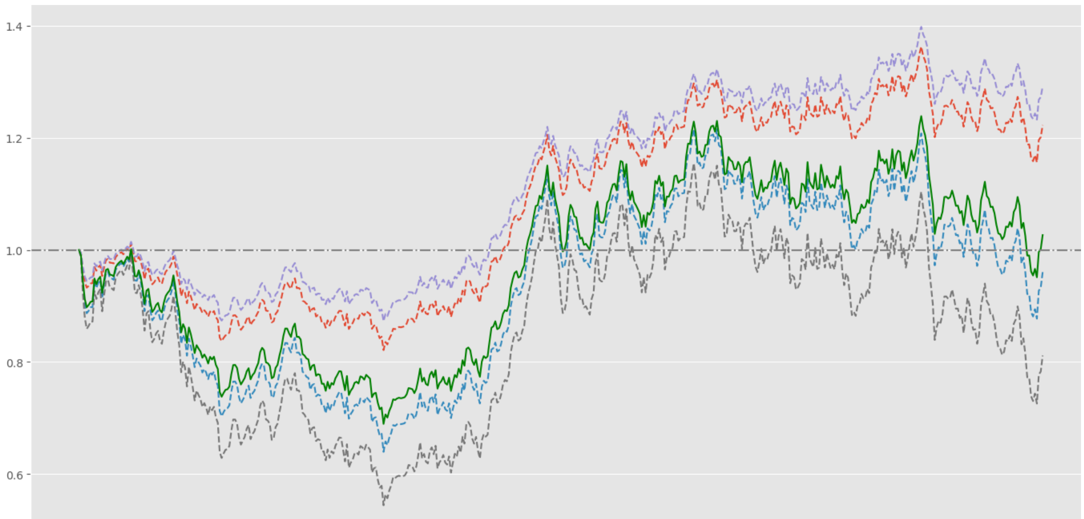

FIGURE 2.1: An illustration of the stock price process and all its sub-processes.

2.3 The Mixture of Lognormal Model

In this section we consider the lognormal density for the density function, as it is simple and consistent with most established option pricing frameworks. Assume the density function of the stock price process is the density function based on the lognormal distribution conditioning on the latest information. From equations 2.10, 2.11, after some basic algebra, their probability density function is given by

$$p(S,t) = LN(S,t,\nu(S,t),\alpha) \tag{2.22}$$

and

$$p_i(S,t) = LN(S,t,\nu_i,\alpha) \quad \text{for all } i = 1,...,N \tag{2.23}$$

where

$$LN(y,t,x,\alpha) = \frac{1}{xy\sqrt{2\pi t}} \exp\left(-\frac{1}{2x^2 t}\left(\ln\frac{y}{S_0} - \alpha t + \frac{1}{2}x^2 t\right)^2\right) \tag{2.24}$$

Given the above Mixture probability function in lognormal density, we incorporate the definition as in 2.11, the probability density function of the process

$$f_X(y) = \frac{d}{dy}F_X(y) = \frac{d}{dy}\sum_{i=1}^{N}\lambda_i F_{X_i}(y) = \sum_{i=1}^{N}\lambda_i f_{X_i}(y) \tag{2.25}$$

can be represented as weighted sum of its marginal probability function.

Next we will derive the pricing formula with mixture lognormals. To simplify calculation, we adopt the risk-neutral pricing technique, which is a pricing method to first transform the probability space to a risk-neutral world and then conduct calculation, at last transform the result back to the real-world prcing. Under the risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, the drift terms of the process and each sub-processes, α, are all replaced by the risk free rate r, to calculate the discount factor the payoff at expiry.

Let us consider a call option with maturity T, strike K, the expected value of the option under risk-neutral measure C_{LM} is

$$\begin{aligned}
C_{LM} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\left(S(T)-K\right)^+\right] \\
&= e^{-rT}\int_0^\infty (y-K)^+\, p(y,T)dy \\
&= e^{-rT}\int_0^\infty (y-K)^+ \sum_{i=0}^{N}\lambda_i p_i(y,T)dy \\
&= \sum_{i=0}^{N}\left(\lambda_i e^{-rT}\int_0^\infty (y-K)^+\, p_i(y,T)dy\right) \\
&= \sum_{i=0}^{N}\left(\lambda_i C_{BS}^i\right)
\end{aligned} \tag{2.26}$$

The equation 2.26 indicates that the pricing of the option under lognormal mixture dynamics is nothing but the linear combination of the Black and Scholes price shown as 2.9 of its all sub-processes. This equation is also valid for vanilla put option. This equation makes the pricing for options under lognormal dynamics easy to implement, because each sub-process still applies to the naïve Black and Scholes pricing framework. This attribute also makes the derivative calculation easy to implement, which is essential for our model solver in next chapter.

Proposition 2.2: *For a European option with maturity T, strike K with N independent one-dimensional time-homogeneous diffusion processes, each follows lognormals probability distribution, with risk free rate r, the option value at initial time $t = 0$ is given by*

$$V_{LM} = \varphi e^{-rT} \sum_{i=1}^{N} \lambda_i \left[FN\left(\varphi \frac{\ln \frac{F}{K} + \frac{1}{2}\nu_i^2 T}{\nu_i \sqrt{T}}\right) - KN\left(\varphi \frac{\ln \frac{F}{K} - \frac{1}{2}\nu_i^2 T}{\nu_i \sqrt{T}}\right)\right] \tag{2.27}$$

where $\varphi = 1$ for call option and -1 for put option.

To wrap up, in this chapter we proved that the local volatility of a diffusion process can be uniquely determined by its mixture probability density functions if the density functions follow the definition 2.1 and all sub-processes are time-homogeneous at the same probability space. Then we derived that the option pricing is simply a linear combination of the Black and Scholes prices of all its sub-processes. In next section we will take a step furture, by loosening some assumptions, to reflect more characteristics of the volatility surface in the market.

2.4 Extension of the Lognormal Mixture Model

The endogenous volatility smile looks satisfactory, but some extension can further improve the performance of the model. The first additional condition is to take the default status of the stock as consideration; The second is to imply the multiplicative means to enable a larger volatility skew, which is already an observation in most of the options at the market.

2.4.1 The Default Probability

As all stocks are probable to default, where the stock prices drop to zero and thus the option prices. Another reason to introduce this amendment is to facilitate calibration to steep, short-term equity put skews far out of the money by loading a point mass at zero with finite probability.

Let us consider the hazard rate (also called default density) $\gamma(t)$ at time t, which is defined so that $\gamma(t)\Delta t$ is the conditional default probability for a short period between t and $t + \Delta t$. The default probability $Q(t)$ then is given by

$$Q(t) = 1 - \sum_{i=0}^{N} p_i(t) = 1 - e^{-\bar{\gamma}(t)t} \tag{2.28}$$

where

$$\bar{\gamma}(t) = \frac{1}{t}\int_0^t \gamma(\tau)d\tau \tag{2.29}$$

defined as the average hazard rate. As the cumulative hazard rate $\bar{\gamma}(t)t$ needs to be monotonically increasing, for the time $t_1 < ... < t_M$, an additional constraint to the original surface must apply:

$$1 > \sum_{i=0}^{N} p_i(t_1) \geq ... \geq \sum_{i=0}^{N} p_i(t_M) > 0 \quad \text{for each } t = t_1, ..., t_M \tag{2.30}$$

where ν_i are different constants satisfying the conditions of proposition 2.1, and α_i are multiplicative means specified within each sub-process.

2.4.2 The Multiplicative Means

As is shown at 2.11, the mean of each diffusion process is the same. In this section we will discuss the an alternative of the equations to enable a multiplicative means, where the densities $p_i(t)$ are still lognormal, for each sub-process. We implement this change to allow steeper and more skewed local volatility curve with minima that can be shifted far away, usually to a side with larger strike, from the at-the-money level. This phenomenon is constantly observed at the stock derivative markets. Hence, we include different means in each sub-process.

Consider N independent one-dimensional time-homogeneous diffusion processes in the same probability space

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i dt + \nu_i dW(t) \quad \text{for all } i = 1, ..., N \tag{2.31}$$

Note that in this case, each sub-process has its own drift term, apart from its independent constant volatility parameter. Nevertheless, they still share the same starting point and Brownian Motion. Mercurio et al.(Brigo, Mercurio, and Sartorelli, 2002) proved that the volatility in the case of multiplicative means has a unique strong solution whose marginal density is given by the mixture of lognormals at Definition 2.1. The unconditional density of S_i at time t is thus given by

$$p_i(S,t) = LN(S, t, \nu_i, \alpha_i) \quad \text{for all } i = 1, ..., N \tag{2.32}$$

where the right hand side function is defined at 2.24. Apparently, one cannot arbitrary set the α_i, some of the mean should be larger, others need to be lower. α_i is subject to certain constraint. If we take the expectation of both sides in Definition 2.1, then

$$\begin{aligned}\mathbb{E}[p(y,t)] &= \sum_{i=1}^{N} \mathbb{E}[\lambda_i p_i(y,t)] \\ e^{\alpha t} &= \sum_{i=1}^{N} \lambda_i e^{\alpha_i t} \\ \sum_{i=1}^{N} \lambda_i \xi_i &= 1 \quad \text{for all } i = 1, ..., N\end{aligned} \tag{2.33}$$

where

$$\xi_i = e^{(\alpha_i - \alpha)t}$$

in other words, the linear combination of forward price of each sub-process must equal to the forward of the single process. Moreover, if we differentiate both sides with respect to t, we get

$$\sum_{i=1}^{N} \lambda_i (\alpha_i - \alpha) e^{\alpha_i t} = 0 \tag{2.34}$$

which implies that some α_i must be larger and some smaller than (or equal to) α.

Still, the local volatility $\nu(S,t)$ can be uniquely characterized by the new parameters in the stochastic equations 2.31. Applying Kolmogorov forward equation to 2.31 and 2.1, and making similar arithmetic simplification as in last section, we get

$$\nu(S,t)^2 = \frac{\sum_{i=1}^{N} \lambda_i \nu_i^2 p_i(S,t)}{\sum_{i=1}^{N} \lambda_i p_i(S,t)} + \frac{2e^{\alpha t} \sum_{i=1}^{N} \lambda_i \xi_i S_0 \ln \xi_i N\left(\frac{\ln \frac{\xi_i S_0}{S} + (\alpha + \frac{1}{2}\nu_i^2)t}{\nu_i \sqrt{t}}\right)}{tS^2 \sum_{i=1}^{N} \lambda_i p_i(S,t)} \tag{2.35}$$

However, the result of the right-hand side equation is not necessarily always positive, as some of the α_i needs to be smaller than α, some of the ξ_i needs to be smaller than 1. Brigo et al. (Brigo and Mercurio, 2000) proved the conditions under which strict positivity of $\nu(S,t)^2$ is guaranteed. We do not delve into the detail derivation at this section.

The derivation of a call option price at 2.26 still holds in the case of multiplicative means, meaning the price of the option is also the linear combination of the Black and Scholes price of all its sub-processes.

Proposition 2.3: *For a European option with maturity T, strike K with N independent one-dimensional time-homogeneous diffusion processes, each follows lognormals probability distribution with different means $(\ln \xi_i)/t + \alpha$, with risk free* *rate r, the option value at initial time $t = 0$ is given by*

$$V_{LM} = \varphi e^{-rT} \sum_{i=1}^{N} \lambda_i \left[F_i N \left(\varphi \frac{\ln \frac{F_i}{K} + \frac{1}{2}\nu_i^2 T}{\nu_i \sqrt{T}} \right) - KN \left(\varphi \frac{\ln \frac{F_i}{K} - \frac{1}{2}\nu_i^2 T}{\nu_i \sqrt{T}} \right) \right] \tag{2.36}$$

where

$$F_i = \xi_i S_0 * e^{\alpha T} \quad \text{for all } i = 1, ..., N$$

The Figure 2.2 illustrates the comparison of a stock price processes with 4 sub-processes with constant mean and with multiplicative means. The parameters are set to $r = 10\%, \lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.3, \lambda_4 = 0.4$ and $\sigma_1 = 0.12, \sigma_2 = 0.2, \sigma_3 = 0.1, \sigma_4 = 0.25$, and how they are correlated. The right hand side represents the multiplicative extension of the model with, $\xi_1 = 1.2, \xi_2 = 1.5, \xi_3 = 1.0, \xi_4 = 0.7$, and the left hand side represents the equivalent model with globally constant mean. This gives a good heuristic example of how the Lognormal Mixture Model with multiplicative means works differently to the base model.

The Figure 2.3 shows the probability functions of the Lognormal Mixture Model, the multiplicative means extension of the model, and the equivalent lognormal density with the same mean and variance as the Lognormal Mixture Model. The x axis is the moneyness level, and the y axis is the probability density. The parameters are set to $r = 0, F = 1, \lambda_1 = 0.25, \lambda_2 = 0.25, \lambda_3 = 0.25, \lambda_4 = 0.25$ and$\xi_1 = 0.9, \xi_2 = 0.9, \xi_3 = 1.1, \xi_4 = 1.1$ and $\sigma_1 = 0.4, \sigma_2 = 0.3, \sigma_3 = 0.2, \sigma_4 = 0.3$. In this specification, the multiplicative means model has a lowest mode, but a heaviest tail, followed by the Lognormal Mixture Model, and then the Black and Scholes equivalent density function. This property is able to reflect the flat tail effect, which was observed in the stock market from time to time and was measured as nearly impossible to happen.

The Figure 2.4 illustrates the volatility curves generated by the Lognormal Mixture Model at maturity of 1 year and the multiplicative means extension of the model with specification same as in 2.3. The x axis is the moneyness level, and the y axis is the Black Scholes equivalent implied volatility. The horizontal dash line is the implied volatility at at-the-money strike. First, the Lognormal Mixture Model is able to generate an endogenous volatility smile on top of the Black and Scholes model, and a dip lower than the ATM implied volatility. Second, the multiplicative means model exhibits the smile rightward. This reflects the observation on the market that when stock price drops, the market tends to price the option at a higher price, hence a higher implied volatility level. The Figure 2.5 illustrates the curves when the risk free rate is set to 10%. When the risk rate increases, the implied volatility curve will shifts rightward and downward, becomes flatter. THis change is in line with the observation that when interest rate increases or when the maturity date is set at later date, the implied volatility curves become flatter.

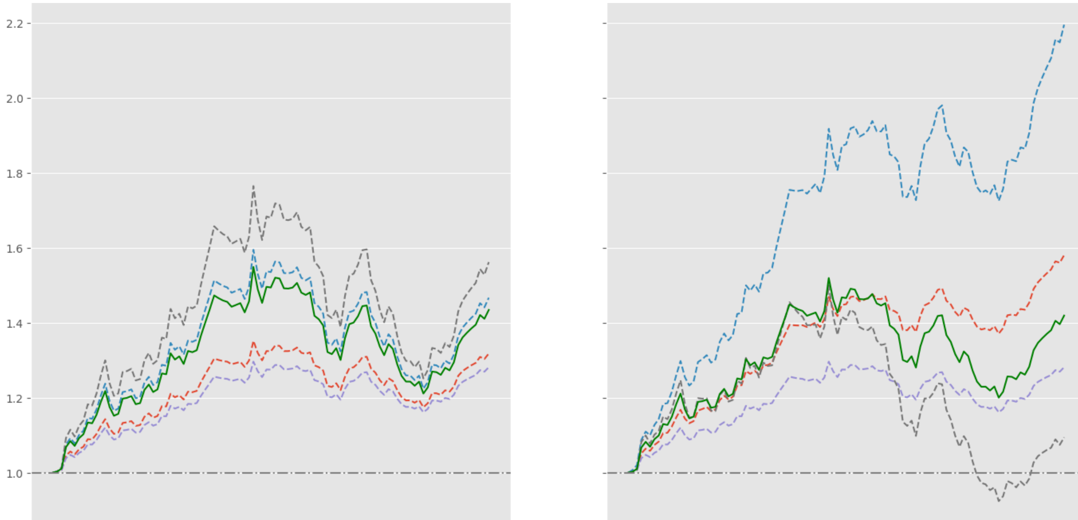

FIGURE 2.2: An illustration of the stock price process and all its sub-processes with multiplicative means.

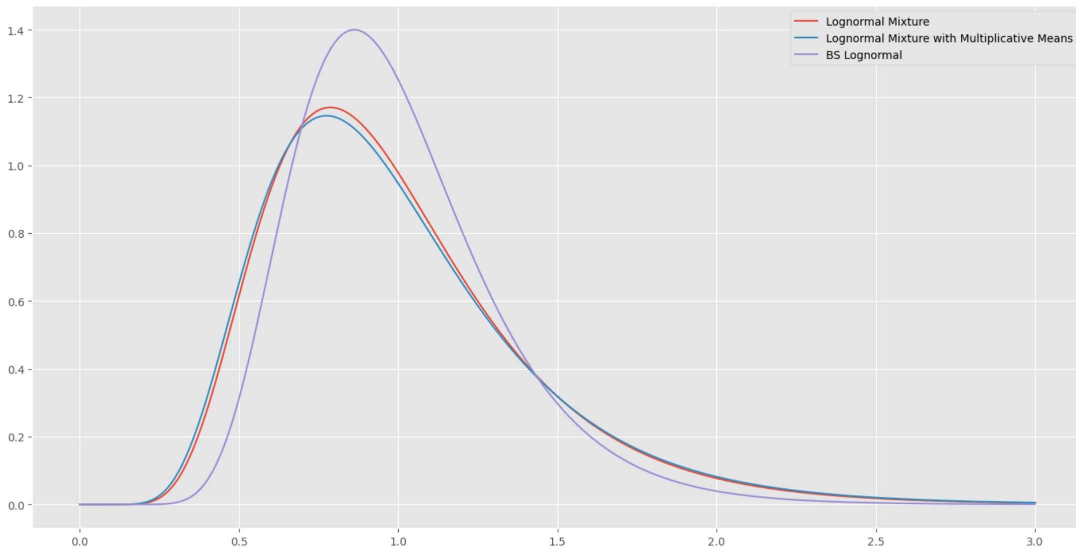

FIGURE 2.3: The probability functions of the Black and Scholes model, the Lognormal Mixture Model, and the multiplicative means extension of the model.

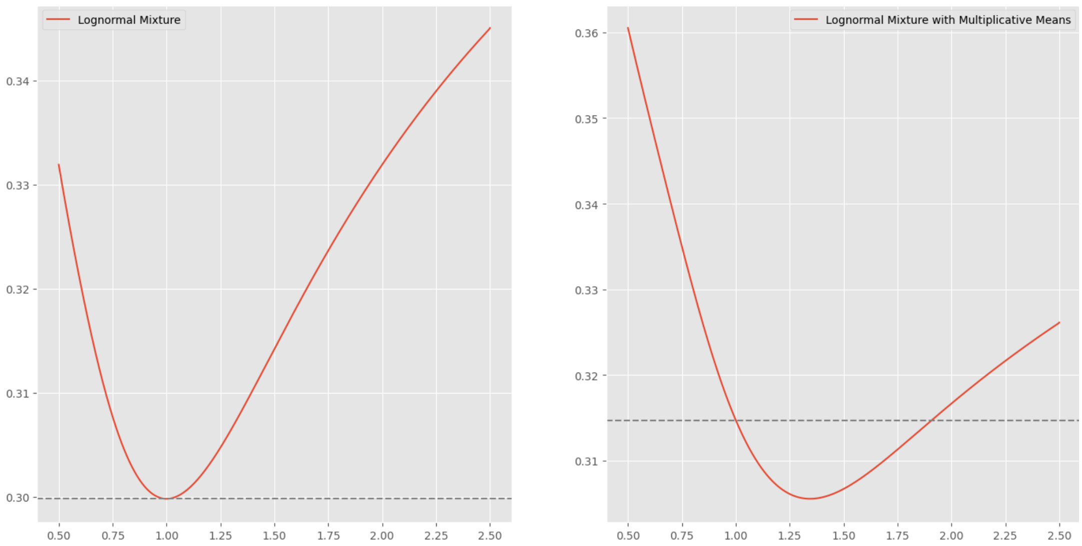

FIGURE 2.4: The volatility curve generated by the Lognormal Mixture Model at r=0, and the multiplicative means extension of the model, and their equivalent implied volatility at at-the-money strike.

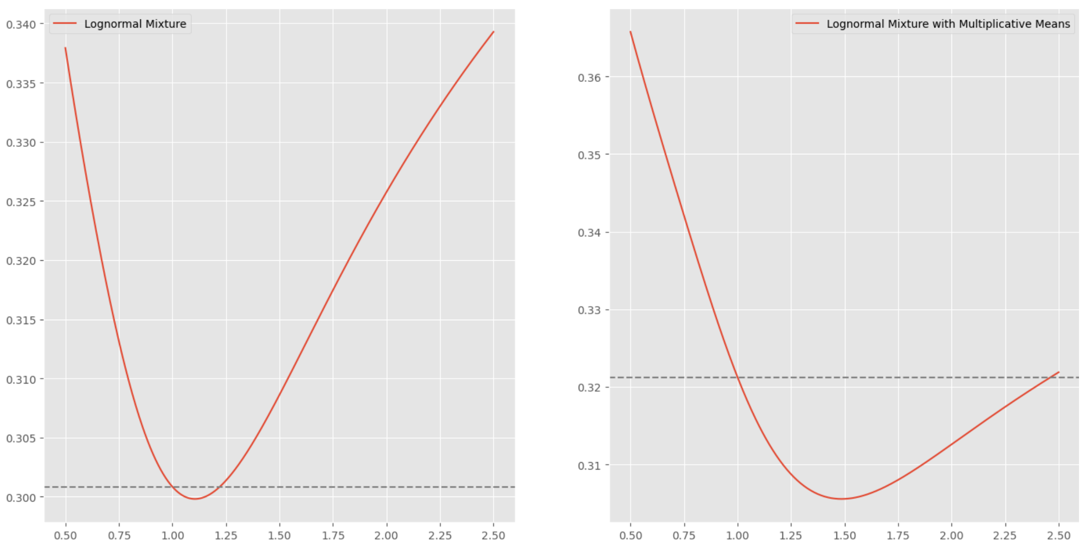

FIGURE 2.5: The volatility curve generated by the Lognormal Mixture Model at r=10%, and the multiplicative means extension of the model, and their equivalent implied volatility at at-the-money strike.

Chapter 3

Volatility Surface Calibration

Before we start our work to calibrate the model to the market, there is still a procedure. Now the most heavily traded and fluid equity product is American style option. What we have derived in the last section is the closed-form pricing formula of an European-style option under lognormal mixture assumption. Hence we need to "transform" the American-style quote price to European-style at first, which will be discussed in detail in the first section. The second input to consider is the interpolation and extrapolation techniques across different expiry date other than the market expiry date. The third input is the calculation of implied forward, the forward price implied by the market quotes, and back out the implied dividend yield, which is an input to the lognormal model.

As for the calculation of implied dividend, we apply the approach introduced by Koster et al.(Koster, Menn, and Oeltz, 2017), by iterately calculating the forward price to fit into the put-call-parity equation until the difference between two consecutive iteration shrinks to an acceptable level (0.1% in our assessment). After we get the implied forward for each maturity, an cumulative implied dividend as of that maturity is backed out. Then we determine the dividend payout month according to the dividend forecast result, and allocate the additional dividend at this maturity to the predicted payout month. For example, suppose the implied dividend of an option expired at June 10, 2020 is 3, and December 10, 2020 is 6. And suppose the predicted dividend payout is 2.2 at July 15 and 1.1 at October 15, then the allocated dividend is 2 at July 15 and 1 at October 15. After that, the continuous implied dividend yield curve available to the last expiry can be derived.

The overall framework to implement the Lognormal Mixture Model is summarized as follows:

1. Fetch market quote data and pre-process the data.
2. Calculate the implied forward price of each maturity and back out implied dividend. Then calculate the continuous the implied yield curve.
3. "Transform" the American option price to the equivalent European price that shares the same Black implied volatility (Skip this step for European options) (we perform this step for bid and ask price separately).

4. Calculate the mid equivalent European prices of each of the options.
5. Feed the data into the model and solve all the parameters, subject to certain constraints and bounds.
6. Solve the Black and Scholes implied volatility that has the same price as calculated with Lognormal Mixture Model for each strike at the same maturity.
7. Apply interpolation and extrapolation to the variance across different time.
8. Generate the result to a table or surface plotting.

3.1 De-Americanization

There are two main ways to derive the volatility surface from market quotes of American options. The first is an analytical way, by directly deriving the pricing formula of American option under lognormal dynamics. Barone Adesi (Barone-Adesi, 2005) summarized the fundamental problem regarding the American option and reviewed some numerical method to solve American option. The downside for this method is that the closed form formula or even the numerical way to solve the related PDEs is remarkably complex, and hence is even harder to process to the implied volatility solver with the formulas. The second way is more "tricky", by transforming the American-style option to European-style one that has the same binomial tree, then calculating the expected value of the European tree under all same parameters. What's more appealing is that the calibration process will be far less computational expensive by transforming the calculation of an American option to an volatility-equivalent European option.

We apply the binomial tree method the carry out the de-Americanization process as is introduced by Mahlstedt (Mahlstedt, 2017). The core idea of de–Americanization method is to convert the market American option quotes to pseudo-European option prices before we process it to calibration. The pricing of American options is simple and straightforward. Consider S_0 the starting point of an underlying stock price, the underlying can either go up by a factor of u or down by a factor of $\frac{1}{u}$ at the next stage. As such, the risk-neutral probability for the underlying to move up is given by

$$q = \frac{e^{r\Delta t} - \frac{1}{u}}{u - \frac{1}{u}}. \tag{3.1}$$

Notice that the upward movement factor is uniquely corresponding to a volatility level and the relation is given by

$$u = e^{\nu\sqrt{\Delta t}} \tag{3.2}$$

and the difference converges when t reaches zero.

For each of the node, the tree splinters in the same logic. Once the end of the step is reached, the whole tree is set up. Then we calculate backward from the end of each final nodes and apply risk-neutral average of upward and downward nodes, and take the higher value of the average price and the value if it is immediately exercised as the expected option value at this node. At the end we reach the root of the tree, which is the expected option value at starting point. In next step, the moving-up factor u^* is determined such that the expected price of the American option calculated by the binomial tree method is consistent with the actual market data. Hence, the expected payoff of the option at time 0 is given by

$$P^A = \sup_{\tau \in \Delta t_i} \mathbb{E}^Q[e^{r\tau} \left(\varphi \left(S_\tau^{u^*} - K \right) \right)^+ |\mathcal{F}_0] \tag{3.3}$$

where Δt_i denotes each discrete time where the tree notes is located, and τ a stopping time.

The early exercise characteristics of American options are reflected in the fact that the the supremum is used in all discrete time steps. Once $S_\tau^{u^*}$ is determined, the corresponding Black and Scholes implied volatility is uniquely determined, and European option with the same strike and maturity P^E as the American option is uniquely corresponded. We calculate one option at a time, by founding a corresponding European option P^E for each of the American options P^A. These two schemes share the same Black and Scholes implied volatility. And the European option P^E is the price to be processed to the next calibration step.

3.2 The Lognormal Mixture Model Solver

In this section we discuss the way to solve the Black and Scholes implied volatility under the assumption of the model that is derived in Chapter 2, given the pseudo-European option price we got in last section. We denote

$$\begin{aligned}
\Lambda(t) &= \{\lambda_1(t), ..., \lambda_N(t)\}, \\
\Xi(t) &= \{\xi_1(t), ..., \xi_N(t)\}, \\
\Sigma(t) &= \{\nu_1(t), ..., \nu_N(t)\}
\end{aligned}$$

as the set of all the parameters we used at the lognormal mixture model. And we set

$$\begin{aligned}
V_{LM}(t, K; \Lambda(t), \Xi(t), \Sigma(t), r) &= LM(\Lambda(t), \Xi(t), \Sigma(t), K, r, t) \\
&= \varphi e^{-rT} \sum_{i=1}^{N} \lambda_i(t) \left[F_i N \left(\varphi \frac{\ln \frac{F_i(t)}{K} + \frac{1}{2}\nu_i(t)^2 t}{\nu_i(t)\sqrt{t}} \right) - KN \left(\varphi \frac{\ln \frac{F_i(t)}{K} - \frac{1}{2}\nu_i(t)^2 t}{\nu_i \sqrt{t}} \right) \right]
\end{aligned} \tag{3.4}$$

as we have derived in Proposition 2.3. We have the function to optimize and the corresponding bounds and constraints, hence the next step for calibration is then formulated as a constrained, nonlinear least squares problem, where the optimal values of the model parameters are determined to match, in the least squares sense, the chosen pseudo-European option prices at a given maturity:

$$
\begin{aligned}
&\min_{\Lambda(t),\Xi(t),\Sigma(t)} \sum_{t_j}\sum_{K_m} \left(P(t_j,K_m) - V_{LM}(t_j,K_m;\Lambda(t_j),\Xi(t_j),\Sigma(t_j),r)\right)^2 \\
&\qquad s.t. \quad 0 < \sum_{i=1}^{N} \lambda_i(t_j) < 1 \\
&\qquad\qquad \sum_{i=1}^{N} \lambda_i(t_j)\xi_i(t_j) = 1 \\
&\qquad\qquad 0 < \nu_i(t_1)\sqrt{t_1} < ... < \nu_i(t_n)\sqrt{t_n} \quad \text{for } i = 1,...,N \\
&\qquad\qquad \sum_{i=1}^{N} \lambda_i(t_1) > ... > \sum_{i=1}^{N} \lambda_i(t_n) \\
&\qquad\qquad 0 \le \lambda_i(t_j) < 1 \quad \text{for } i = 1,...,N \\
&\qquad\qquad \xi_i(t_j) > 0, \quad \nu_i(t_j) > 0 \quad \text{for } i = 1,...,N
\end{aligned}
\tag{3.5}
$$

for all $t_j \in \{t_1,...t_n\}$ the maturities of the options that are actively quoted in the market, and $P(t_j,K_m)$ the pseudo-European option prices derived from the market quotes with strike K_m and maturity t_j. We set the third constraint to guarantee monotonically increasing of variance implied by the market quotes to prevent calendar arbitrage. [1]

We use Sequential Least Squares Programming (SLSQP) method to perform the nonlinear optimization with bounds and constraints. The SLSQP method was developed and refined by Kraft (Kraft, 1988) and Boggs et al. (Boggs and Tolle, 1995) proposed some effective algorithms to implement this method. SLSQP combines two fundamental algorithms for solving non-linear optimization problems: an active set method and Newton's method. It is an iterative procedure, for a given iterate x_k, to first solve the nonlinear subproblem, which contains the calculation of the derivatives of both the objective functions and each of the constraints (to construct Jacobian matrices), and then uses the solution to construct a new iterate x_{k+1}. This construction is done in such a way that the sequence $\{x_k\}$ converges to a local optima x^* as $k \to \infty$. We do not go deep to the algorithm in this report, but briefly an analytical form of the first derivative formulas of both the objective function and each of the constraint equations. Here we give the derivatives of the objective function, denoted as F_t, with respect to the model parameters $\{\Lambda(t),\Xi(t),\Sigma(t)\}$,

$$
\frac{\partial}{\partial \lambda_i(t)} F_t = g_t \varphi e^{-rt} \left[F_i N\left(\varphi \frac{\ln \frac{F_i(t)}{K} + \frac{1}{2}\nu_i(t)^2 t}{\nu_i(t)\sqrt{t}}\right) - KN\left(\varphi \frac{\ln \frac{F_i(t)}{K} - \frac{1}{2}\nu_i(t)^2 t}{\nu_i\sqrt{t}}\right)\right]
\tag{3.6}
$$

[1] The first and the forth constraint is simply from equation 2.30, and the second constraint is from equation 2.33.

and

$$\frac{\partial}{\partial \xi_i(t)} F_t = g_t \varphi e^{-rt} \lambda_i(t) F_i N \left(\varphi \frac{\ln \frac{F_i(t)}{K} + \frac{1}{2}\nu_i(t)^2 t}{\nu_i(t)\sqrt{t}} \right) \tag{3.7}$$

and

$$\frac{\partial}{\partial \nu_i(t)} F_t = g_t K e^{-rt} N' \left(\frac{\ln \frac{F_i(t)}{K} - \frac{1}{2}\nu_i(t)^2 t}{\nu_i \sqrt{t}} \right) \sqrt{t} \tag{3.8}$$

for each $i \in \{1, 2...N\}$ sub-processes, where

$$g_t = 2\left(P(t_j, K) - F_t\right).$$

It is obvious that the derivative functions of the Lognormal Mixture pricing formula are also simple variations of Black and Scholes formula derivatives (*the Greeks*). Hence it makes the SLSQP algorithm feasible for this nonlinear system. The derivatives of the constraints are straight forward, so we don't give the specific formulas for this section. In practice, the nonlinear system would be too large and computationally expensive if we solve the entire volatility surface at a time. Hence we solve the model one maturity at a time, from short term to long term, and set the bounds of each maturity according to the value carried out in last maturity. For example, if we get the $\lambda_1 = 0.01$ at t_1, then the lower bound of the λ_1 at t_2 is set to 0.01. The same iteration logic applies to the parameters of volatilities. For the most fluent options, there are usually a large number of strikes available to trade at a single maturity, hence the Jacobian is too large for an iteration step. In implementation, we divide the objective function by M where M is the number of data points at a maturity.

3.3 Solving Implied Volatility

As is shown in Chapter 2, the Black and Scholes implied volatility exists and is uniquely determined by the dynamics in lognormal mixture assumption. Once we solve all the parameters of the model, the next step is to solve the Black and Scholes implied volatility given the full model formula. The direct analytical formula is complex, one possible way of approximation is to use the Taylor expansion. We tried this approximation, and found that if we only used a polynomial of degree two to approximate the result, the implied volatility curve at deep out-of-money deviates significantly from the accurate result; but if we add on more terms to the approximation, the overall formula would be clumsy and has no advantage to the iteration method. At last we chose the iteration method. To illustrate, the method equals the fitted price carried out by the model with the Black and Scholes price calculated by the selected implied volatility. Hence the process for the next step is to solve the following equation

$$LM(\Lambda(t), \Xi(t), \Sigma(t), K, r, t) = e^{-rT} \left(FN(\frac{\ln \frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) - KN(\frac{\ln \frac{F}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}) \right) \tag{3.9}$$

where the left hand side is the option price we get with the Lognormal Mixture Model, and the right hand side is simply the Black and Scholes price. We get the implied volatility σ by solving the Black and Scholes formula. There are many engines and algorithms available to solve the equation (e.g. Newton's method, Brent's method, bisection method, etc.), and the performance is satisfactory. In practice we found the different implied volatility solvers make no significant difference on the performance of the program.

3.4 Interpolation and Extrapolation Across Time

After the calibration is done and implied volatility is solved for each maturity, we now consider the interpolation and extrapolation across each listed maturity. For tenors at the range of $T_1 < t < T_n$ where T_1 is the shortest tenor that is listed, and T_n is the longest tenor that is listed. For a given strike K_l and maturity t_l, we first calculate the implied volatility at the same forward moneyness

$$\frac{F_1}{K_1} = ... = \frac{F_l}{K_l} = ... = \frac{F_n}{K_n} = m \tag{3.10}$$

where m is the forward moneyness derived by F_l and K_l, and F_l is the interpolated forward price. Then we can get the strikes for each maturity if we have the same forward moneyness m and implied forward price for each maturity. Then for each maturity we can get the implied volatility by again solving the equation 3.9, and then apply a Hermite interpolation that preserves the monotonicity on the total variances $\sigma_t^2 t$ to get σ_t. We apply Hermite interpolation here to prevent a calendar arbitrage. For extrapolation, we apply a flat volatility level for all sub-process. In other words, we make the $\Sigma_t = \Sigma_{T_n}$ for $t > T_n$, and $\Sigma_t = \Sigma_{T_1}$ for $0 < t < T_1$, then the implied volatility can be solved if we applied the the equation 3.9, the short-term side of extrapolation will be done if no valid short-term listed option prices are available. When we finish the interpolation and extrapolation, The whole surface is carried out.

Chapter 4

Numerical Result

In this chapter we will apply the Lognormal Mixture Model to the live option market, and analyze the performance of the model and the solver. We choose US market to test, because it is the most liquid stock option market. And to evaluate the performance of the model more objectively, we devided our sample to 2 subset, one is the most fluid and actively quoted options (H set), and other is the least fluid options (L set). Finally, we chose 20 stocks, with 10 in the first set and 10 in the second set. We fetch all the listed options of each of the stock underlyings, and use the total dataset to fit the model. We set the testing date to 20 October, 2020. The statistics of the stock options grouped by 2 sets are stated as Table 4.1.[1] From the table we can see that the stocks in H set have around twice the quoted option number of the L set stocks. The average number of the quoted maturities and the strikes in each maturity of the H set are higher than that of the L set. Hence, it is reasonable that the average standard deviation of the option prices at each maturity is higher for H set than for L set. In contrast, the average spread level of H set is lower than that of L set, because the options in H set are more actively quoted.

The raw market quotes are usually noisy and need to be pre-processed before they are fed to the model. We filter the data according to the following criteria:

1. Drop the options that are mature at shorter than 1 day.

[1] the N stands for the total data points of each set, the Std stands for the average standard deviation of option prices for each stock, the Spread stands for the average spread level in percentage. The Ks stands for the average number of strikes, and Ts stands for the average number of maturities for each stock.

Sets	N	Std	Spread(%)	Ks	Ts
H set	29680	160.41	10.58	95.08	17.33
L set	14672	28.19	37.23	62.19	13.00

TABLE 4.1: The statistics of the options selected, as at the closing price at 2 Oct. 2020.

Sets	Min	Max	Mean	Median	Std	Skew	Kurt
H set call	0.0149	713.6013	64.9303	15.8863	110.6288	2.7059	8.5189
H set put	0.0150	707.6519	57.2425	14.1834	97.0900	2.8386	9.5884
L set call	0.0149	68.5337	4.2080	1.2198	7.8418	3.7622	18.0724
L set put	0.0150	67.3950	4.2026	1.2549	7.4281	3.4907	15.9890

TABLE 4.2: The statistics of the option prices after pre-processing at 2 Oct. 2020. Result is presented in average for each set.

2. To guarantee a feasible result at the option solver and 1 unique point for each strike, we filter out all in-the-money options and use only the out-of-money call and put options to fit the model.
3. Eliminate abnormal spread of the prices that are significantly too wide or too low. In particular, if the ask price is greater than 3 times of bid price, or if the spread is over 3 standard deviation away the average spread level of that maturity, or if any of the bid/ask prices are less or equal to 0.01.
4. Filter out prices that are too high or too low compared to the other quotes according to their implied volatilities. Implied volatility that is over 15 times or smaller than 1/15 of the median of the global implied volatility are eliminated.

Table 4.2 illustrates the statistics of option prices, grouped by H&L set and option type (call or put), after the above 4 steps of pre-processing. The table shows that the L set has higher skewness and higher kurtosis, which indicates that the L set has fewer far out-of-money options being quoted than the H set options. And the higher standard deviation of call options, both in H set and L set, indicates that the quotes of out-of-money call options are wider than the out-of-money put options.

Then we need to determine the number N of lognormal densities in the mixture.It is a delicate choice since if N is too small, the volatility smile would not have enough flexibility and does not fit well to the relevant market data as many as possible. If N is too large, the model will overfit the data. Sometimes the number of parameters to be fit nearly equal the data points, hence the model will be significantly influced by single abnormal data point, and may largely fluctuate in interpolated volatilities. At last, we choose $N = 4$ to obtain enough flexibility to calibrate tightly to market data, while prevent a large number of parameters in the model.

Next, for the range of the surface, we consider the moneyness range and set in $[80\%, 120\%]$ and then compute the corresponding strikes for each moneyness, considering that the strikes outside this range are rarely quoted. In comparison to market's implied volatility surface, we set the points at

	MSD at each tenor (10^{-3})								
Sets	**1M**	**2M**	**3M**	**6M**	**9M**	**1Y**	**18M**	**2Y**	**Mean**
H set	0.44	0.11	0.12	0.11	0.04	0.05	0.12	0.12	0.14
L set	1.50	0.52	0.37	0.09	0.08	0.11	0.14	0.28	0.39

TABLE 4.3: Mean Square Difference (MSD) of each tenor between the model output and the market volatility surface.

$\{80\%, 90\%, 95\%, 97.5\%, 100\%, 102.5\%, 105\%, 110\%, 120\%\}$. At tenor's dimension, we set the fixed tenor at $\{1, 2, 3, 6, 9, 12, 18, 24\}$ months. Note that we set these grid points only for comparison with market's quotes. The model is able to generate a continuous full volatility surface.

The last parameter for the calibration is the risk-free rate. To get the result as close as possible, we use market yield curve in stead of constant rate to feed in the model. For maturity shorter than 1 year, we use USD Libor rate; for maturity equal or longer than 1 year, we use the par swap rate, and then we bootstrap the zero rate for each time point, and apply linear interpolation in between the zero rates. Then the discount factor at each specific point within the volatility surface can be derived from the zero rates. Hence we have a continuous and smooth curve for discount factor for interest rate. The risk-free rates in the model in last section can then be transferred to discount factor.

The Table 4.3 summarizes the average differences between the implied volatility generated by the Lognormal Mixture Model and that from the market. We show the result grouping by 2 different sets.[2] The result shows, in general, the model gives a satisfactory approximation to market volatility surface, and the calibration is successful in all of the option chains, even when the option chain is sparse at the L set sample. In detail, the model works better in more fluid option chains than in less fluid option chain on average. And the performance of the model is better when the tenor is longer than 3 months. This is because there are multiple listed maturities within 3 months' tenor, hence the difference is affected largely by the method of interpolation scheme across time.

The Table 4.4 shows in detail the average calibrated parameters to the curve of a selected underlying and analyze the steadiness of the model. We select Apple Inc. in H set and Coca-Cola Co. in L set to perform the test, and we calibrate the target curve at the close market snapshots at 2 October 2020 to 7 October 2020. We compare each parameter respectively accross time. The table shows that the difference of the parameters hold in relatively similar interval, though some smaller λs fluctuate in a larger scale. But considering they having little impact on the final result, we regard this situation as a

[2]The *Mean* stands for global square difference for the set.

	Apple Inc.				Coca-Cola Co.			
	2 Oct.	**5 Oct.**	**6 Oct.**	**7 Oct.**	**2 Oct.**	**5 Oct.**	**6 Oct.**	**7 Oct.**
S	113.02	116.05	113.16	115.08	49.36	49.38	48.94	49.56
λ_1	0.3198	0.3913	0.2968	0.3890	0.4757	0.5992	0.8501	0.5093
λ_2	0.4936	0.3654	0.5791	0.5019	0.4542	0.1688	0.1206	0.3827
λ_3	0.1805	0.2214	0.1119	0.0997	0.0618	0.2204	0.0253	0.1002
λ_4	0.0042	0.0181	0.0103	0.0076	0.0043	0.0112	0.0008	0.0057
ξ_1	0.9385	0.9504	1.0128	1.1717	0.9763	1.0144	1.0118	1.0286
ξ_2	0.9921	1.0937	0.9913	0.9975	0.9879	0.7488	0.7643	0.9631
ξ_3	0.8523	1.4258	1.0599	1.4127	1.1605	1.2991	1.0347	1.0708
ξ_4	4.4260	1.1122	2.4712	7.8314	1.3302	0.6625	0.9911	0.4779
ν_1	0.2609	0.2207	0.2799	0.2798	0.1781	0.1504	0.2059	0.1576
ν_2	0.3153	0.3342	0.3466	0.4001	0.2594	2.9855	0.7474	0.2508
ν_3	0.7542	0.9773	0.8815	4.9741	1.0652	4.2141	1.6020	0.8137
ν_4	6.6849	2.9406	10.3980	17.6738	8.1076	/	3.6334	/

TABLE 4.4: The stock prices and parameters from calibration from 2 Oct. 2020 to 7 Oct. 2020, each parameter is shown as the average of all listed maturities.

degration to a lower number of sub-processes (4 to 3). In the degraded sub-process, which has a low λ (usually lower than 0.05), the other corresponding parameters (ξ and σ) fluctuate in a large range. The fluctuation is due to the high sensitivity of the sub-process to the original process. But this situation does not necessarily happen in other stocks. In total, we can conclude from the table that the model is able to maintain a stable set of parameters, and when it degrades the number of sub-processes, it will maintain the number of sub-processes across different time.

The surface plotting graphs of Apple Inc. and Coca-Cola Co. at 2 October 2020 are displayed as in Figure 4.1, Figure 4.2. Notice that the tenors of the surface can be extended to shorter than 1M, and also larger than 2Y. In practice, the MSD is smaller when extends to shorter maturity than to larger maturity, because there are more quoted options in these maturities, hence more reference points for the model to calibrate. Hence it needs more work to adjust the model for long maturity in the future.

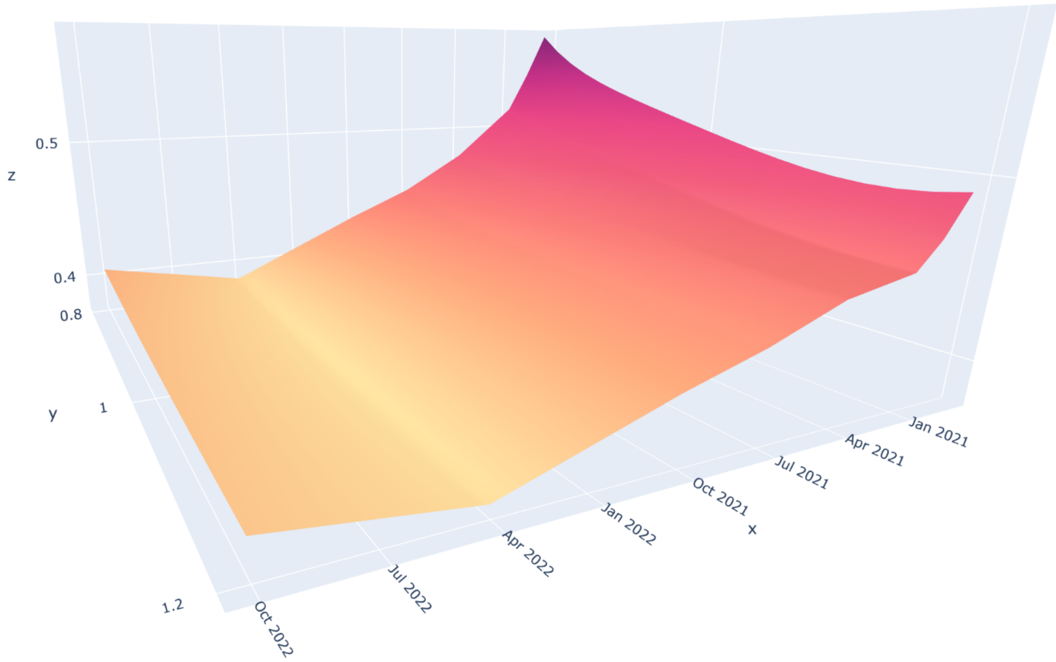

FIGURE 4.1: The implied volatility surface of Apple Inc. as at 2 Oct. 2020

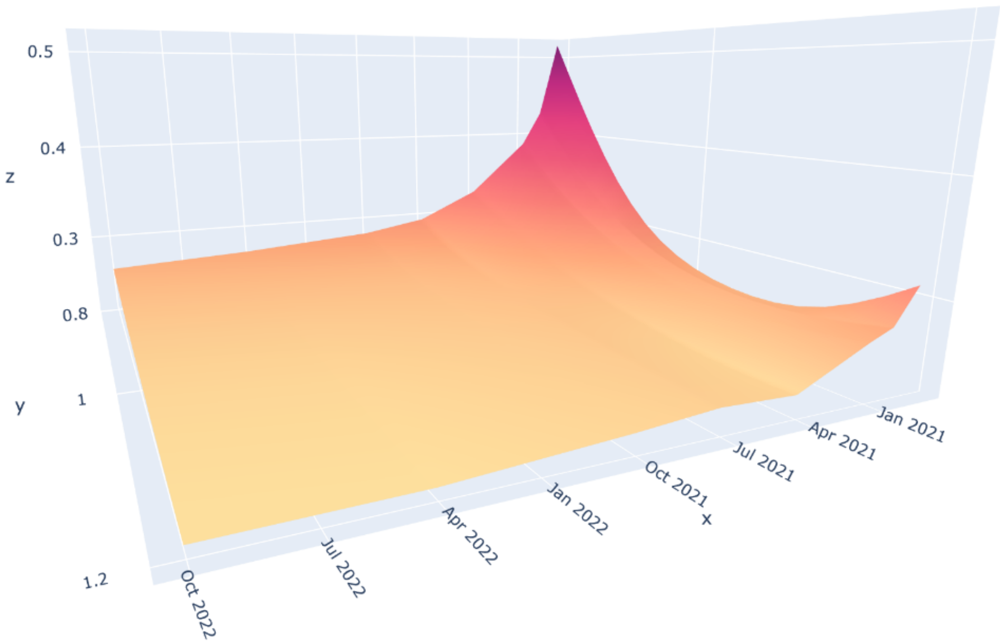

FIGURE 4.2: The implied volatility surface of Coca-Cola Co. as at 2 Oct. 2020

Chapter 5

Conclusion

In conclusion, we have built the Lognormal Mixture Model starting from Black and Scholes model, and examine the properties of the model, and provided 2 extensions of the model. After reviewing the Black and Scholes model, we started our analytics by defining the mixture diffusion process. Then with the mixture diffusion process, we built the stock price Lognormal Mixture model with time-invariant volatility in each sub-process. Then we gave the closed form European option pricing formula under the Lognormal Mixture dynamics, and proved the important property that the pricing formula of the model is simply the linear combination of each of its sub-processes pricing models. Then we extended the model by introducing a default status, and multiplicative means. We gave the extended stock price Lognormal Mixture model, and gave a new European option pricing formula under the Lognormal Mixture dynamics with multiplicative means.

In calibration part, we first introduced a way to *de-Americanize*, a practical way that is used by many traders, the American style option prices to European style prices. Then we integrated what we had got so far, the pricing formulas of Lognormal Mixture Model with 2 extensions, and formulated an optimization function with respect to all the parameters and subject to all constraints. We leveraged the SLSQP algorithm to carry out the solution of all the parameters. This solver is followed by another implied volatility solver that carries out the Black and Scholes implied volatility. We performed this step for each of the maturities respectively, and set the bound of the next maturity according to the last maturity found by the model. At last, for different tenors in between the listed maturities, we used Hermite interpolation and flat variance extrapolation to fill the tenors in between. This guarantees the monotonic increasing of variance with respect to time. Hence, we have the complete workflow for calibrating the volatility surface under Lognormal Mixture framework.

To evaluate the model in more liquid and in less liquid situation, we selected 10 stocks in H set and in L set respectively and calibrated the model to the sample. We analyzed the average difference between the market and the model, and found that all calibration succeeded, and the difference to the market surface was satisfactory. And we analyzed the parameters of a stock from H set and from L set, and evaluated the fluctuation in consecutive 4 days. This evaluation shows the steadiness of the result of the model.

As a major contribution of this report, we proposed an end-to-end feasible and effective solver to leverage the Lognormal Mixture Model with extensions, and to calibrate the model to the market. We also evaluate the result of the model and proved its steadiness. Nevertheless, further improvements for the model are still possible. First, we found that the result of the model in far out-of-money surface at L set deviated relatively large from the market, because the far end of calibrated volatility curve of the narrow-quoted option chain at a maturity is usually too steep, hence it is necessary to apply some adjustment at these "extrapolation" area across moneyness dimension. Second, we apply the extrapolation across time with flat parameters in all sub-processes, which is a naive assumption. Possible improvement might take the default probability function to consideration. Third, the interpolation across time at short maturities (shorter than 3 months) is also possible for improvement, in order to further lower the MSD of the implied volatility surface generated by the model in short tenors.

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