FE 5222 Group Project

Spread Option

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1 Introduction to Spread Option

A spread call option with expiry T and strike K has the payoff

$$\max\{S_1(T) - S_2(T) - K, 0\}$$

where $S_1(t)\, S_2(t)$ are the price of two stocks and follow

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i dW_i(t) \quad i = 1, 2$$

and

$$dW_1 dW_2 = \rho dt$$

2 Pricing Formula of Spread Option

Let

$$Y(t) = S_2(t) + Ke^{-r(T-t)}$$

then for $S_2 >> K$

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= \frac{dS_2(t)}{Y(t)} + \frac{dKe^{-r(T-t)}}{Y(t)} \\ &= \left(\frac{Ke^{-r(T-t)} + S_2(t)}{Y(t)}\right) rdt + \frac{\sigma_2 S_2(t)}{Y(t)} dW_2(t) \\ &= rdt + \hat{\sigma} dW_2(t) \end{aligned} \tag{1}$$

where

$$\hat{\sigma} = \frac{S_2(t)}{Y(t)}\sigma_2.$$

Let

$$dW_2(t) = \rho dW_1(t) + \sqrt{1-\rho^2}\, dW_3(t)$$

where $W_3(t)$ is an independent and uncorrelated Brownian motion with $W_1(t)$. The spread option can be converted into

$$V_s(t) = e^{-rT}\mathbb{E}[(S_1(T) - Y(T))^+]$$

where

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma dW(t), \quad \frac{dY(t)}{Y(t)} = rdt + \hat{\sigma} dW(t)$$

$$\sigma = (\sigma_1, 0), \quad \hat{\sigma} = (\frac{S_2(t)}{S_2(t) + Ke^{-r(T-t)}}\sigma_2\rho, \frac{S_2(t)}{S_2(t) + Ke^{-r(T-t)}}\sigma_2\sqrt{1-\rho^2}),$$

$$dW(t) = (dW_1(t), dW_3(t))$$

in vector form, with $W(t)$ a two dimentional Geometric Bronian motion.
Then by rewriting the price formula,

$$\begin{aligned} V_s(t) &= e^{-rT}\mathbb{E}[(S_1(T) - Y(T))^+] \\ &= e^{-rT}\mathbb{E}[Y(T)(\frac{S_1(T)}{Y(T)} - 1)^+] \end{aligned}$$

by Change of Numeraire formula

$$V_s(t) = Y(0)\mathbb{E}^{(Y)}[Y(T)(S_1^{(Y)}(T) - 1)^+] \tag{2}$$

and by Change of Numeraire Thereom

$$\frac{dS_1^{(Y)}(t)}{S_1^{(Y)}(t)} = (\sigma - \hat{\sigma})\dot{d}W^{(Y)}.$$

Let

$$dB(t) = \frac{1}{||\sigma - \hat{\sigma}||_2}(\sigma - \hat{\sigma})\dot{d}W^{(Y)}(t)$$

witch is a Brownian Motion. Let

$$\nu = ||\sigma - \hat{\sigma}||_2 = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2\left(\frac{Ke^{-r(T-t)} + S_2(t)}{Y(t)}\right) + \sigma_2^2\left(\frac{Ke^{-r(T-t)} + S_2(t)}{Y(t)}\right)^2}$$

Then

$$\frac{dS_1^{(Y)}(t)}{S_1^{(Y)}(t)} = \nu dB(t)$$

$\frac{dS_1^{(Y)}(t)}{S_1^{(Y)}(t)}$ is a Geometric Brownian motion under $\mathbb{P}^{(Y)}$.
Hence,

$$V_s(t) = S_1(0)\Phi(d_1) - Y(0)\Phi(d_2) \tag{3}$$

where

$$d_{1,2} = \frac{ln(\frac{S_1(0)}{Y(0)}) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2\left(\frac{Ke^{-r(T-t)} + S_2(t)}{Y(t)}\right) + \sigma_2^2\left(\frac{Ke^{-r(T-t)} + S_2(t)}{Y(t)}\right)^2}$$

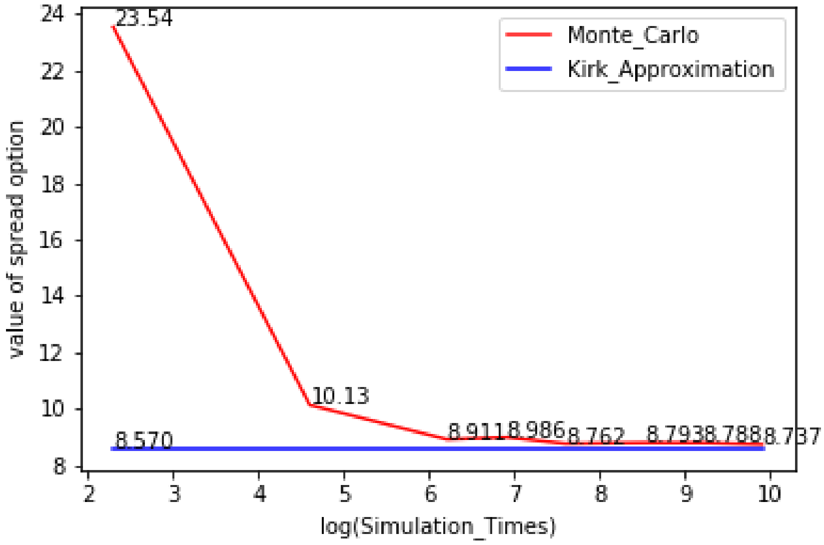
which is the same result as Kirk's approximation formula (1995).

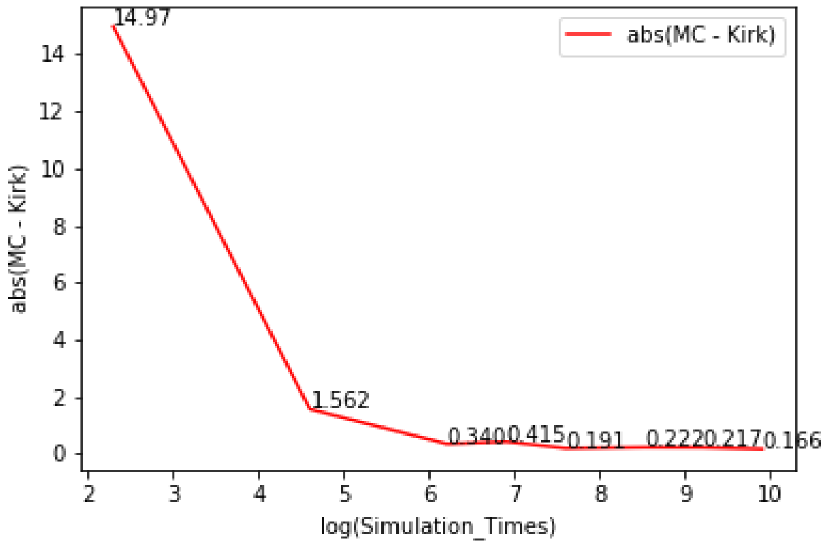
3 Verifying the pricing formula with Monte Carlo simulation

The summary of the Monte Carlo Simulation is shown as below:

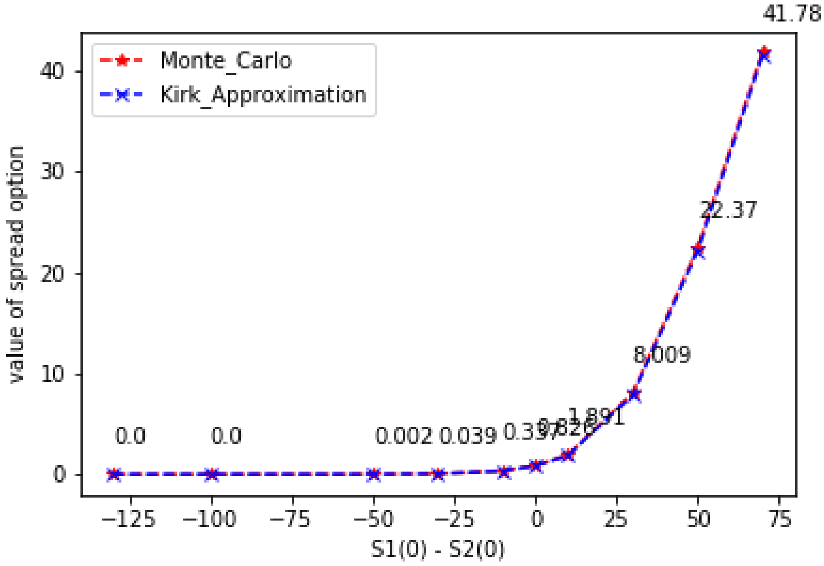
#	Experiment	Type	Value
1	Runtimes	Constant	$S_1(0) = 110$; $S_2(0) = 80$; $K = 30$; $\sigma_1 = 0.2$; $\sigma_2 = 0.2$; $\rho = 0.5$
		Variable	Runtimes=[10, 100, 500, 1000, 2000, 5000, 10000, 20000]
2	Distance Between $S_1(0)$ and $S_2(0)$	Constant	Runtimes=10,000; $S_1(0) = 110$; $K = 30; \sigma_1 = 0.2$; $\sigma_2 = 0.2$; $\rho = 0.5$
		Variable	$S_2(0)$=[30, 50, 70, 90, 100, 110, 130, 150, 200, 230]
3	σ_i	Constant	Runtimes=10,000; $S_1(0) = 110; S_2(0) = 80; K = 30; \rho = 0.5$
		Variable	σ_i=[0.1, 0.2, 0.5, 0.7, 0.9], i=1,2
4	Distance Between σ_1 and σ_2	Constant	Runtimes=10,000; $S_1(0) = 110$; $S_2(0) = 80$; $K = 30$; $\sigma_1 = 0.5$; $\rho = 0.5$
		Variable	σ_2=[0.1, 0.2, 0.5, 0.7, 0.9]
5	Correlation of 2 Stocks	Constant	Runtimes=10,000; $S_1(0) = 110$; $S_2(0) = 80$; $K = 30; \sigma_1 = 0.2$; $\sigma_2 = 0.2$
		Variable	ρ=[-0.9, -0.5, -0.2, 0, 0.2, 0.5, 0.9, 0.95, 0.98, 0.99, 0.999]
6	Strike Price	Constant	Runtimes=10,000; $S_1(0) = 110$; $S_2(0) = 80$; $\sigma_1 = 0.2$; $\sigma_2 = 0.2$; $\rho = 0.5$
		Variable	K=$S_2(0) * \%K$, %K=[0.1, 0.3, 0.5, 0.7, 0.9, 1.1]

3.1 Effect of simulation steps

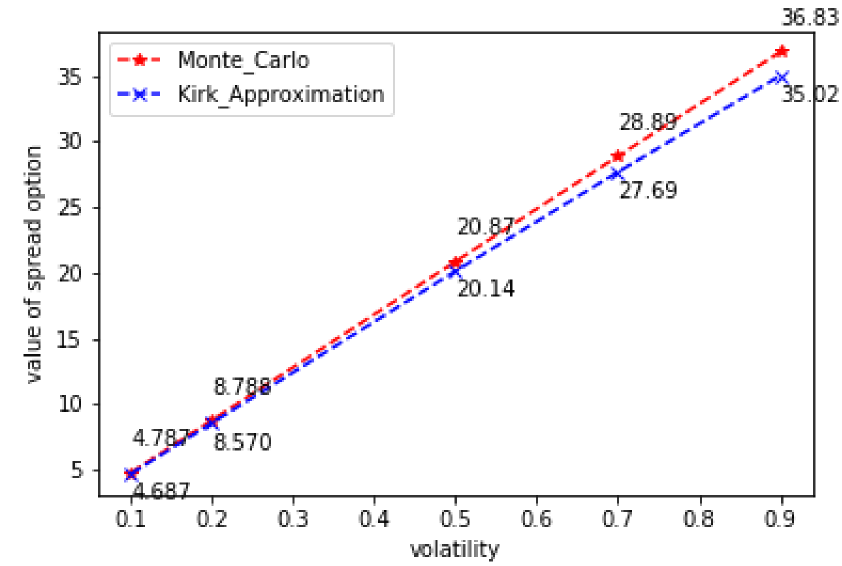


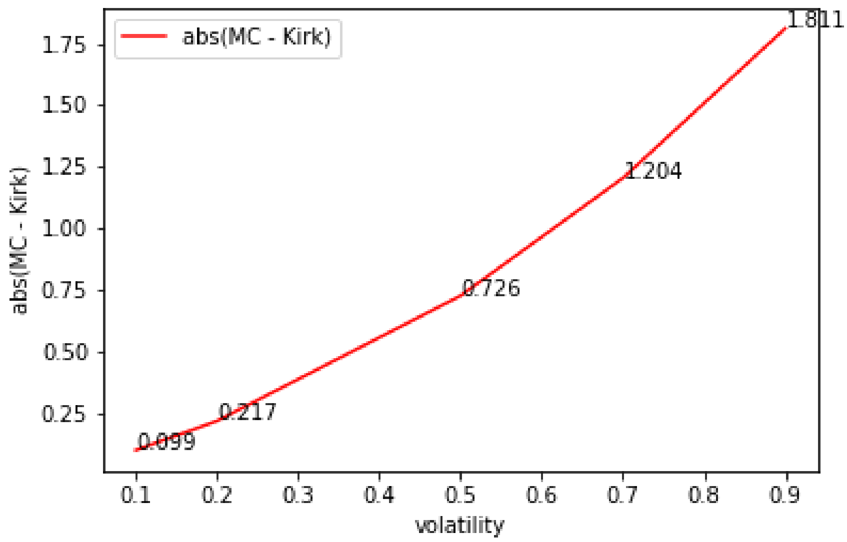


3.2 Effect of different prices $(S_1(0) - S_2(0))$

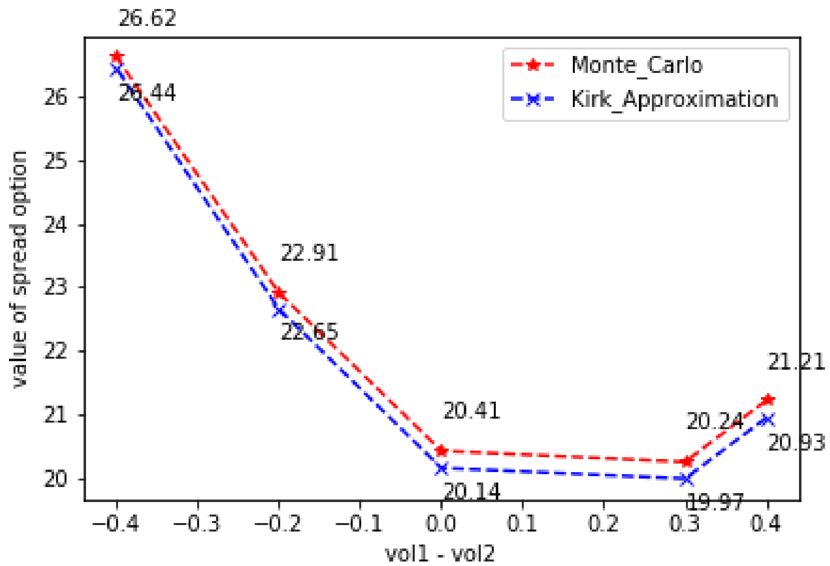


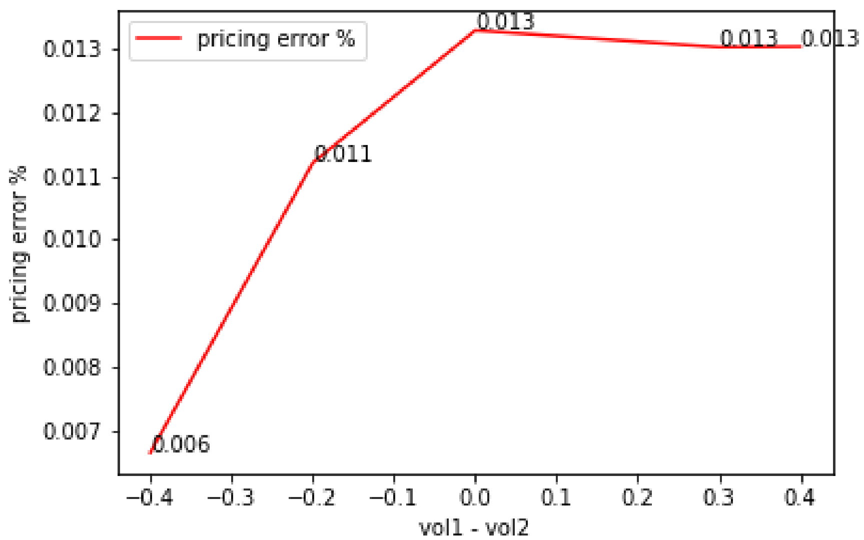
3.3 Effect of stocks volatility (σ)



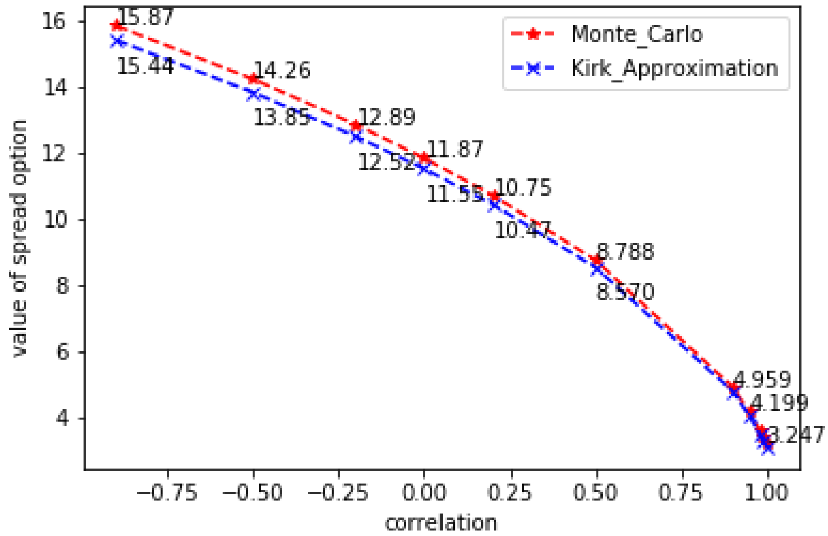


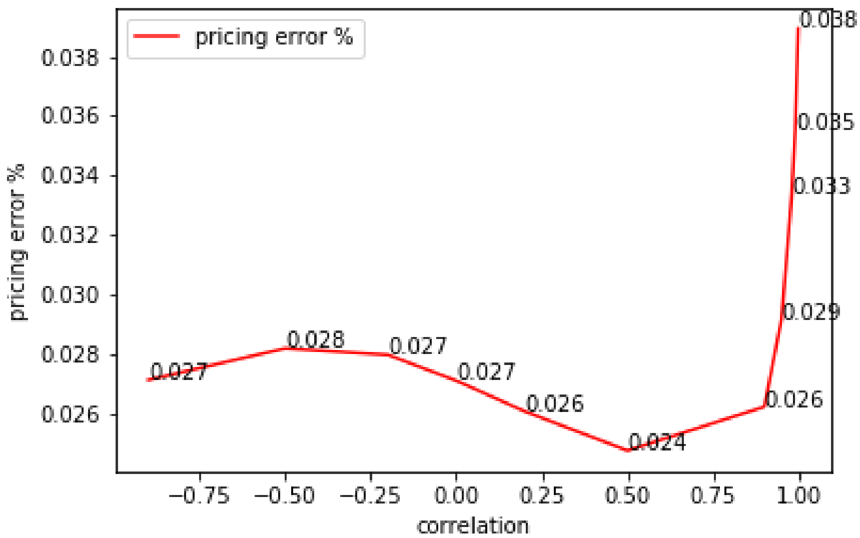
3.4 Effect of stock volatility difference $(\sigma_1 - \sigma_2)$



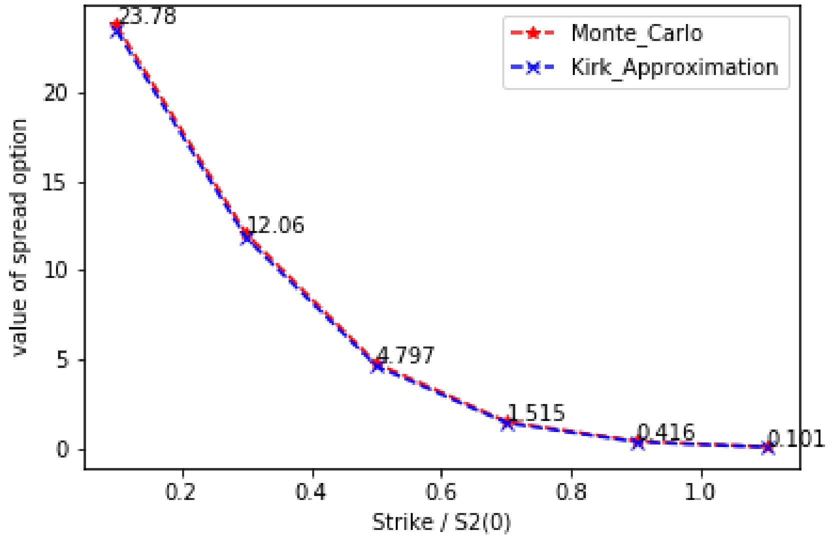


3.5 Effect of Correlation (ρ)





3.6 Effect of strike ratio ($K/S_2(0)$)



4 Conclusion

From the graphs above, we come to the following conclusion:

- When the steps of simulation increase, the Monte Carlo simulation result will be close to the Kirk Approximation;
- When $S_1(0)$ increases, the probability of Spread Option expiring in the money becomes larger and thus the option is more valuable;
- When the number of simulation times is constant, the larger sigma will contribute larger simulation error and thus the $|MC - Kirk|$ becomes larger. Besides, the Monte Carlo simulated values are always larger than the Kirk's ones, which may due to the computer random number generator error; [1]
- The same conclusion as (iii), the larger σ_2 will contribute larger simulation error and thus the pricing error becomes larger;
- When ρ becomes smaller, sigma used in the Kirk's price formula is larger and thus the option is more valuable. In addition, when ρ is large and close to 1, the simulation error increases considerably. The main reason is that when ρ is close to 1, the implied volatility of the option is highly skewed, hence the volatility apporximated by Kirk's formula is highly biased; [2]
- When K becomes larger, obviously the option is cheaper. However, the pricing error ($\frac{|MC-Kirk|}{Kirk}$) becomes larger too, which means that Kirk's approximation formula is valid under the condition $K << S_2$.

[1] Bingqian Lu, Monte Carlo simulations and option pricing,
http://www.personal.psu.edu/alm24/undergrad/bingqianMonteCarlo.pdf

[2] Elisa Alos et. al. On the goodness of fit of Kirk's formula for spread option prices.
https://pdfs.semanticscholar.org/a939/e75f987784bafdbf4c85eec3a631ab15e73a.pdf