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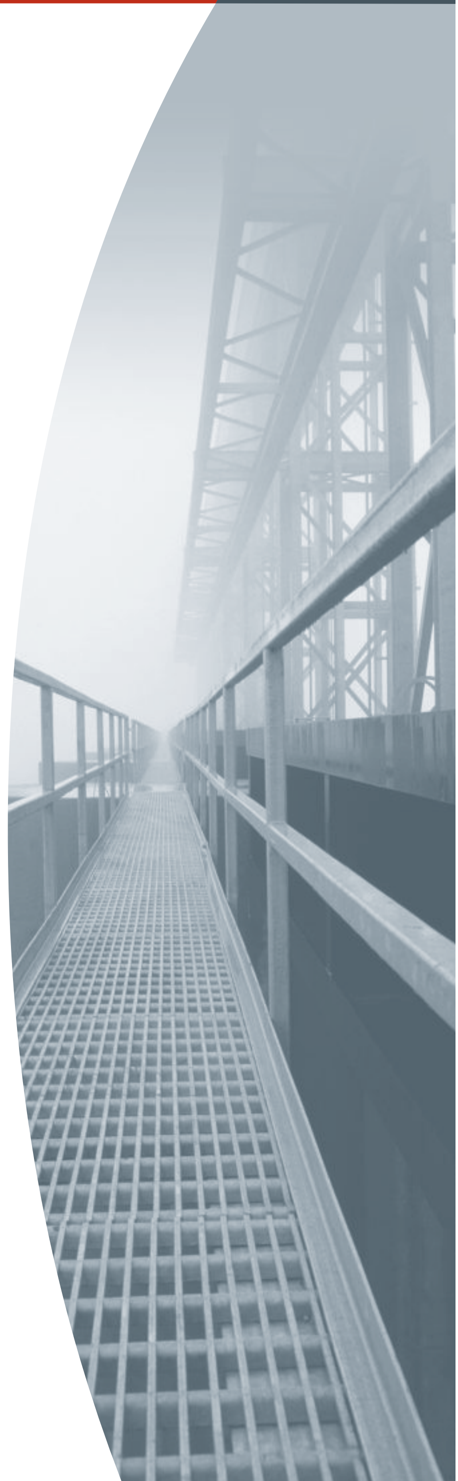
Advanced Calculus

Second Edition

Patrick M. Fitzpatrick



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Advanced Calculus

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Patrick M. Fitzpatrick

American Mathematical Society
Providence, Rhode Island

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This book is dedicated to Benjamin Patrick Evans

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In order to put his system into mathematical form at all, Newton had to devise the concept of differential quotients and propound the laws of motion in the form of differential equations—perhaps the greatest advance in thought that a single individual was ever privileged to make.

Albert Einstein
from an essay
On the one hundredth anniversary of Maxwell's birth
James Clerk Maxwell: A Commemorative Volume

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A

B

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PREFACE

The goal of this book is to rigorously present the fundamental concepts of mathematical analysis in the clearest, simplest way, within the context of illuminating examples and stimulating exercises. I hope that the student will assimilate a precise understanding of the subject together with an appreciation of its coherence and significance. The full book is suitable for a year-long course; the first nine chapters are suitable for a one-semester course on functions of a single variable.

I cannot overemphasize the importance of the exercises. To achieve a genuine understanding of the material, it is necessary that the student do many exercises. The exercises are designed to be challenging and to stimulate the student to carefully reread the relevant sections in order to properly assimilate the material. Many of the problems foreshadow future developments. The student should read the book with pencil and paper in hand and actively engage the material. A good way to do this is to try to prove results before reading the proofs.

Mathematical analysis has been seminal in the development of many branches of science. Indeed, the importance of the applications of the computational algorithms that are a part of the subject often leads to courses in which familiarity with implementing these algorithms is emphasized at the expense of the ideas that underlie the subject. While these techniques are very important, without a genuine understanding of the concepts that are at the heart of these algorithms, it is possible to make only limited use of these computational possibilities. I have tried to emphasize the unity of the subject. Mathematical analysis is not a collection of isolated fact and techniques, but is, instead, a coherent body of knowledge. Beyond the intrinsic importance of the actual subject, the study of mathematical analysis instills habits of thought that are essential for a proper understanding of many areas of pure and applied mathematics.

In addition to the absolutely essential topics, other important topics have been arranged in such a way that selections can be made without disturbing the coherence of the course. Chapters and sections containing material that is not subsequently referred to are labeled by asterisks.

At the beginning of this course it is necessary to establish the properties of real numbers on which the subsequent proofs will be built. It has been my experience that in order to cover, within the allotted time, a substantial amount of analysis, it is not possible to provide a detailed construction of the real numbers starting with a serious treatment of set theory. I have chosen to codify the properties of the real numbers as three groups of axioms. In the Preliminaries, the arithmetic and order properties are codified in the Field and Positivity Axioms: a detailed discussion of the consequences of these axioms, which certainly are familiar to the student, is provided in Appendix A. The least familiar of these axioms, the Completeness Axiom, is presented in the first section of the first chapter, Section 1.1.

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The first four chapters contain material that is essential. In Chapter 2 the properties of convergent sequences are established. Monotonicity, linearity, sum, and product properties of convergent sequences are proved. Three important consequences of the Completeness Axiom are proved: The Monotone Convergence Theorem, the Nested Interval Theorem, and the Sequential Compactness Theorem. Chapter 2 lays the foundation for the later study of continuity, limits and integration which are approached through the concept of convergent sequences. In Chapter 3 continuous functions and limits are studied. Chapter 4 is devoted to the study of differentiation.

Chapter 5 is optional. The student will be familiar with the properties of the logarithmic and trigonometric functions and their inverses, although, most probably, they will not have seen a rigorous analysis of these functions. In Chapter 5, the natural logarithm, the sine, and the cosine functions are introduced as the (unique) solutions of particular differential equations; on the provisional assumption that these equations have solutions; an analytic derivation of the properties of these functions and their inverses is provided. Later, after the differentiability properties of functions defined by integrals and by power series have been established, it is proven that these differential equations do indeed have solutions, and so the provisional assumptions of Chapter 5 are removed. I consider Chapter 5 to be an opportunity to develop an appreciation of the manner in which the basic theory of the first four chapters can be used in the study of properties of solutions of differential equations. Not all of the chapter need be covered and certainly the viewpoint that the basic properties of the elementary functions are already familiar to the student and therefore the chapter can be skipped is defensible.

Chapter 6 is devoted to essential material on integration. The fundamental properties of the Riemann integral are developed exploiting the properties of convergent real sequences through an integrability criterion called the Archimedes–Riemann Theorem. Chapter 7 contains further topics in integration that are optional: later developments are independent of the material in Chapter 7.

The study of the approximation of functions by Taylor polynomials is the subject of Chapter 8. In Chapter 9, we consider a sequence of functions that converges to a limit function and study the way in which the limit function inherits properties possessed by the functions that are the terms of the sequence; the distinction between pointwise and uniform convergence is emphasized. Depending on the time available and the focus of a course, selections can be made in Chapters 8 and 9; the only topic in these chapters that is needed later is the several variable version of the second-order Taylor Approximation Theorem. I always cover the first three sections of Chapter 8 and one or two of the particular jewels of analysis such as the Weierstrass Approximation Theorem, the example on an infinitely differentiable function that is not analytic, or the example of nowhere differentiable continuous functions.

The study of functions of several variables begins in Chapter 10 with the study of Euclidean space $\mathbb{R}^n$. The scalar product and the norm are introduced. There is no class of subsets of $\mathbb{R}^n$ that plays the same distinguished role with regard to functions of several variables as do intervals with regard to functions of a single variable. For this reason, the general concepts of open and closed subsets of $\mathbb{R}^n$ are introduced and their elementary properties examined. In Chapter 11, we study the manner in which the results about sequences of numbers and functions of a single variable extend to sequences of points in $\mathbb{R}^n$, to functions defined on subsets of Euclidean space, and to mappings between such

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spaces. The concepts of sequential compactness, compactness, pathwise connectedness and connectedness are examined for sets in $\mathbb{R}^n$ in the context of the special properties possessed by functions that have as their domains such sets. Chapters 10 and 11 are extensions to functions of several variables of the material covered in Chapters 1, 2, and 3 for functions of one variable.

Chapter 12, on metric spaces, is optional. The student will have already seen important specific realizations of the general theory, namely the concept of uniform convergence for sequences of functions and the study of subsets of Euclidean space, and with these examples in mind can better appreciate the general theory. The Contraction Mapping Principle is proved and used to establish the fundamental existence result on the solvability of nonlinear scalar differential equations for a function of one variable. This serves as a powerful example of the use of brief, general theory to furnish concrete information about specific problems. None of the subsequent material depends on Chapter 12.

The material related to differentiation of functions of several variables is covered in Chapters 13 and 14. The central point of these chapters is that a function of several variables that has continuous partial derivatives has directional derivatives in all directions, the Mean Value Theorem holds, and therefore the function has good local approximation properties.

The study of mappings between Euclidean spaces that have continuously differentiable component functions is studied in Chapter 15. At each point in the domain of a continuously differentiable mapping there is defined the derivative matrix, together with the corresponding linear mapping called the differential. Approximation by linear mapping is studied and the chapter concludes with the Chain Rule for mappings. Here, and at other points in the book, it is necessary to understand some linear algebra. As one solution of the problem of establishing what a student can be expected to know, the entire Section 15.1 is devoted to the correspondence between linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices. As for the other topics that involve linear algebra, in Appendix B basic topics in linear algebra are described, and using the cross product of two vectors full proofs are provided for the case of vectors and linear mappings in $\mathbb{R}^3$: in particular, the relation between the determinant and volume is established.

The Inverse Function Theorem and the Implicit Function Theorem are the focus of Chapters 16 and 17, respectively. I have made special effort to clearly present these theorems and related materials, such as the minimization principle for studying nonlinear systems of equations, not as isolated technical results but as part of the theme of understanding what properties a mapping can be expected to inherit from its linearization. These two theorems are surely the clearest expression of the way that a nonlinear object (a mapping or a system of equations) inherits properties from a linear approximation. In a course in which there is very limited time and it is decided that a significant part of integration for functions of several variables must be covered, the material in Chapters 16 and 17, except for the Inverse Function Theorem in the plane, can be deferred and the course can proceed directly from Chapter 15 to Chapter 18.

The theory of integration of functions of several variables occupies the last three chapters of the book. In Chapter 18, the integral is first defined for bounded functions defined on generalized rectangles. Most of the results for functions of a single variable carry over without change of proof. The Archimedes–Riemann Theorem is proved as

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the principal criterion for integrability. We prove that a bounded function defined on a generalized rectangle is integrable if its set of discontinuities has Jordan content 0. Then integration for bounded functions defined on bounded subsets of $\mathbb{R}^n$ is considered, in terms of extensions of such functions to generalized rectangles containing the original domain. Familiar properties of the integral of a function of a single variable (linearity, monotonicity, additivity over domains, and so forth) are established for the integral of functions of several variables. In Chapter 19, Fubini's Theorem on iterated integration is proved and the Change of Variables Theorem for the integral of functions of several variables is proved. In Chapter 20, the book concludes with the study of line and surface integrals. Our goal is to clearly present a description and proof of the way in which the First Fundamental Theorem of Calculus (Integrating Derivatives) for functions of a single variable can be lifted from the line to the plane (Green's Formula) and then how Green's Formula can be lifted from the plane to three-space (Stokes's Formula). I have resisted the temptation to present the general theory of integration of manifolds. In order to make the analytical ideas transparent, rather than present the most general results, emphasis has been placed on a careful treatment of parameterized paths and parameterized surfaces, so that the essentially technical issues associated with patching of surfaces are not present.

Comments on the New Edition

The first edition was thoroughly scrutinized in the light of almost ten years of experience and much comment from users. More than two hundred new exercises were added, of varying levels of difficulty. A multitude of small changes have been made in the exposition to make the material more accessible to the student. Moreover, quite substantial changes were made that need to be taken into consideration in developing a syllabus for a course. Some comment on these changes is in order.

Chapter 1: Sections 1.1 and 1.2 have been rewritten. Auxiliary material has been pruned or placed in the exercises. The Dedekind Gap Theorem is no longer present since it is no longer needed in the development of the integral. The theorem that any interval of the form $[c, c+1)$ contains exactly one integer has become the basis of the proof of the density of the rationals.

Chapter 2: Lemma 2.9, which we call the Comparison Lemma, has replaced the Squeezing Principal of the first edition as a frequently used tool to establish convergence of a sequence. Proofs of the product and quotient properties of convergent sequences have been simplified. The material from the preceding Chapter 2 has been regathered differently among the Sections 2.1, 2.2, 2.3, and 2.4 and some additional topics and examples have been included. A new optional section, Section 2.5, on compactness has been added with a novel proof of the Heine-Borel Theorem. The theorem formerly called the Bolzano–Weierstrass Theorem is now consistently called the Sequential Compactness Theorem.

Chapter 3: Section 3.4 is now an independent brief section on uniform continuity in which a novel sequential definition of uniform continuity is used. Section 3.5 is now a brief independent section in which the sequential definitions of continuity at a point and uniform continuity are reconciled with the corresponding ϵ-δ criteria.

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Section 3.6 is a significantly altered version of Section 3.4 of the first edition in which the main results regarding continuity of inverse functions are derived from the fact that a monotone function whose image is an interval must be continuous. A more careful treatment of rational power functions is provided.

Chapter 4: Section 4.2 on the differentiation of inverse functions and compositions has been amplified and clarified. A better motivated proof of the Mean Value Theorem is provided that is an easily recognized model for the later Cauchy Mean Value Theorem. The Darboux Theorem regarding the intermediate value property possessed by the derivative of a differentiable function and what was called the Fundamental Differential Equation are not present in this edition. Material has been inserted in the section on the Fundamental Theorem of Calculus (Differentiating Integrals) which emphasizes the points previously made in now absent Section 4.5 of the first edition.

Chapter 5: The basic material remains the same as in the first edition but many more details have been added and the material has been divided into more easily digestible subsections.

Chapters 6 and 7: The material in these two chapters on integration is very different from the corresponding material in the first edition. First, the essential material, including both Fundamental Theorems of Calculus, has been gathered together in the single Chapter 6 while the auxiliary material is now in Chapter 7. More importantly, the basis of the development of the integral is different. In the first edition, a function was defined to be integrable provided that there was exactly one number that lay between each lower and upper Darboux sum and then the Dedekind Gap Theorem was used to establish an integrability criterion. We now immediately introduce the concept of lower and upper integrals and define a function to be integrable provided that the upper integral equals the lower integral. We define the concept of an Archimedean sequence of partitions for a bounded function on a closed bounded interval and prove a basic integrability criterion we call the Archimedes–Riemann Theorem. This accessible sequential convergence criterion together with the results we have established for sequences provides a well motivated method to establish the basic properties of the integral. Finally, the gap of a partition P is now denoted by gap P rather that $\|P\|$.

Chapter 8: A number of smaller changes have been made: for instance, a crude initial estimate of e is now obtained by a transparent comparison with Darboux sums rather than the previous subtle change of variables computation and the treatment of Euler's constant is clarified. The proofs of Newton's Binomial Theorem and the Weierstrass Approximation Theorem are simplified.

Chapter 9: Section 9.3 on the manner in which continuity, differentiability and integrability is inherited by the limit of a sequence of functions has been sharpened. The discussion in Section 9.6 of the example of a continuous, nowhere differentiable function has been greatly simplified by the introduction of the geometric concept of a tent function of base length 2ℓ.

Chapter 10: In this and succeeding chapters the distance between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is now denoted by $\text{dist}(\mathbf{u}, \mathbf{v})$ rather than $d(\mathbf{u}, \mathbf{v})$, what was called a

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"symmetric neighborhood" of a point is now called "an open ball about" a point and the notation changes from $\mathcal{N}_r(\mathbf{u})$ to $\mathcal{B}_r(\mathbf{u})$. The material in Chapter 10 is essentially unchanged.

Chapter 11: Uniform continuity is now defined in terms of differences of sequences as it was for functions of a single variable. What was Section 11.3 in the first edition is now split into two sections. Section 11.3 is devoted to pathwise connectedness and the intermediate value property while Section 11.4 is devoted to connectedness and the intermediate value property. Both sections are labeled as optional since the only use made later regarding connectedness is that the image of a generalized interval under a continuous function is an interval. A very short independent proof of this fact can be provided when it is needed.

Chapter 12: This chapter remains essentially unchanged and is still optional.

Chapters 13 and 14: In these and succeeding chapters for a function of several variables f at the point $\mathbf{u}$ in its domain the classic notation $\nabla f(\mathbf{u})$ is now consistently used for the derivative vector and $\nabla^2 f(\mathbf{u})$ is used for the Hessian. The exposition in Section 14.1 has been abbreviated and the succeeding two sections labeled as optional.

Chapters 15, 16, and 17: The exposition has been tightened and clarified in a number of places but the material remains essentially the same as in the first edition.

Chapter 18: The development of the integral has been substantially changed in order to parallel the new treatment of integration of functions of a single variable in Chapter 6. This has led to considerable simplification and clarification.

Chapter 19 and Chapter 20: These contain material that was in Chapter 19 of the first edition. The material has not been substantially changed.

Acknowledgments for the First Edition

Preliminary versions of this book, in note form, have been used in classes by a number of my colleagues. The book has been improved by their comments about the notes and also by suggestions from other colleagues. Accepting sole responsibility for the final manuscript, I warmly thank Professors James Alexander, Stuart Antman, John Benedetto, Ken Berg, Michael Boyle, Joel Cohen, Jeffrey Cooper, Craig Evans, Seymour Goldberg, Paul Green, Denny Gulick, David Hamilton, Chris Jones, Adam Kleppner, John Millson, Umberto Nero, Jacobo Pejsachowicz, Dan Rudolph, Jerome Sather, James Schafer, and Daniel Sweet. I would like to thank the following reviewers for their comments and criticisms: Bruce Barnes, University of Oregon; John Van Eps, California Polytechnic State University-San Luis Obispo; Christopher E. Hee, Eastern Michigan University; Gordon Melrose, Old Dominion University; Claudio Morales, University of Alabama; Harold R. Parks, Oregon State University; Steven Michael Seubert, Bowling Green State University; William Yslas Velez, University of Arizona; Clifford E. Weil, Michigan State University; and W. Thurmon Whitney, University of New Haven. It is a pleasure to thank Ms. Jaya Nagendra for her excellent typing of various versions of the manuscript, and also to thank the editorial and production personnel at PWS Publishing Company for their considerate and expert assistance in making the manuscript into a book. I am especially grateful to a teacher of mine and to a student of mine. As an undergraduate at

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Rutgers University, I was very fortunate to have Professor John Bender as my teacher. He introduced me to mathematical analysis. Moreover, his personal encouragement was what led me to pursue mathematics as a lifetime study. It is not possible to adequately express my debt to him. I also wish to single out for special thanks one of the many students who have contributed to this book. Alan Preis was a great help to me in the final preparation of the manuscript. His assistance and our stimulating discussions in this final phase made what could have been a very tiresome task into a pleasant one.

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However, for the publication of the second edition a quite special acknowledgment is due my friend and colleague Professor James A. Yorke. During the academic year 2003–4, Jim taught a year-long course from my book. It was the first time he had taught such a course. He dove into the task with great vigor and he critically thought about all aspects of the development of the material. We talked for many hours each week. He had numerous suggestions for improving the exposition ranging from small, important additions that made the material more understandable for the student to more global suggestions that significantly altered parts of the main exposition. He worked extensively with students to find the points of the text they found difficult to read or lacking in motivation. Together we worked at making more natural the proofs that seemed to the students like pulling rabbits from a hat. We sought proofs they could imagine creating themselves. Jim's creative enthusiasm and energy encouraged me to actually complete the second edition despite the persistent pressure of other work.

I warmly thank my colleagues in publishing, Bob Pirtle, Katherine Cook, Cheryll Linthicum, and Merrill Peterson, for their highly professional, accommodating, and friendly partnership in preparing this second edition.

Patrick M. Fitzpatrick

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Patrick M. Fitzpatrick (Ph.D., Rutgers University) held post-doctoral positions as an instructor at the Courant Institute of New York University and as an L. E. Dickson Instructor at the University of Chicago. Since 1975 he has been a member of the Mathematics Department at the University of Maryland at College Park, where he is now Professor of Mathematics and Chair of the Department. He has also held Visiting Professorships at the University of Paris and the University of Florence. Professor Fitzpatrick's principal research interest, on which he has written more than fifty research articles, is nonlinear functional analysis.

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PRELIMINARIES

SETS AND FUNCTIONS

For a set A, the membership of the element x in A is denoted by $x \in A$ or x in A, and the nonmembership of x in A is denoted by $x \notin A$. A member of A is often called a *point* in A. Two sets are the same if and only if they have the same members. Frequently sets are denoted by braces, so that $\{x \mid \text{statement about } x\}$ is the set of all elements x such that the statement about x is true.

If A and B are sets, then A is called a *subset* of B if and only if each member of A is a member of B, and we denote this by $A \subseteq B$ or by $B \supseteq A$. The *union* of two sets A and B, written $A \cup B$, is the set of all elements that belong either to A or to B; that is, $A \cup B = \{x \mid x \text{ is in } A \text{ or } x \text{ is in } B\}$. The word *or* is used here in the nonexclusive sense, so that points that belong to both A and B belong to $A \cup B$. The *intersection* of A and B, denoted by $A \cap B$, is the set of all points that belong to both A and B; that is, $A \cap B = \{x \mid x \text{ is in } A \text{ and } x \text{ is in } B\}$. Given sets A and B, the *complement* of A in B, denoted by $B \backslash A$, is the set of all points in B that are not in A. In particular, for a set B and a point x_0, $B \backslash \{x_0\}$ denotes the set of points in B that are not equal to x_0. The set that has no members is called the *empty set* and is denoted by $\emptyset$.

Given two sets A and B, by a *function* from A to B we mean a correspondence that associates with each point in A a point in B. Frequently we denote such a function by $f: A \to B$, and for each point x in A, we denote by $f(x)$ the point in B that is associated with x. We call the set A the *domain* of the function $f: A \to B$, and we define the *image* of $f: A \to B$, denoted by $f(A)$, to be $\{y \mid y = f(x) \text{ for some point } x \text{ in } A\}$. If $f(A) = B$, the function $f: A \to B$ is said to be *onto*. If for each point y in $f(A)$ there is exactly one point x in A such that $y = f(x)$, the function $f: A \to B$ is said to be *one-to-one*. A function $f: A \to B$ that is both one-to-one and onto is said to be *invertible*. For an invertible function $f: A \to B$, for each point y in B there is exactly one point x in A such that $f(x) = y$, and this point is denoted by $f^{-1}(y)$; this correspondence defines the function $f^{-1}: B \to A$, which is called the *inverse function* of the function $f: A \to B$.

THE FIELD AXIOMS FOR THE REAL NUMBERS

In order to rigorously develop analysis, it is necessary to understand the foundation on which it is constructed; this foundation is the set of real numbers, which we will denote by $\mathbb{R}$. Of course, the reader is quite familiar with many properties of the real numbers. However, in order to clarify the basis of our development, it is very useful to codify the properties of $\mathbb{R}$. We will assume that the set of real numbers $\mathbb{R}$ satisfies three groups

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of axioms: the Field Axioms, the Positivity Axioms, and the Completeness Axiom. A discussion of the Completeness Axiom, which is perhaps the least familiar to the reader, will be deferred until Chapter 1. We will now describe the Field Axioms and the Positivity Axioms and some of their consequences.

For each pair of real numbers a and b, a real number is defined that is called the *sum* of a and b, written $a + b$, and a real number is defined that is called the *product* of a and b, denoted by ab. These operations satisfy the following collection of axioms.

The Field Axioms

Commutativity of Addition: For all real numbers a and b,

$$a + b = b + a.$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

The Additive Identity: There is a real number, denoted by 0, such that

$$0 + a = a + 0 = a \qquad \text{for all real numbers } a.$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0.$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba.$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc).$$

The Multiplicative Identity: There is a real number, denoted by 1, such that

$$1a = a1 = a \qquad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

The Nontriviality Assumption:

$$1 \neq 0.$$

The Field Axioms are simply a record of the properties that one has always assumed about the addition and multiplication of real numbers.

From the Field Axioms it follows[1] that there is only one number that has the property attributed to 0 in the Additive Identity Axiom. Moreover, it also follows that for each

[1] Verification of these and of subsequent assertions in these Preliminaries is provided in Appendix A.

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real number a,

$$a0 = 0a = 0,$$

and that for any real numbers a and b,

$$\text{if } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

The Additive Inverse Axiom asserts that for each real number a, there is a solution of the equation

$$a + x = 0.$$

One can show that this equation has only one solution; we denote it by $-a$ and call it the *additive inverse* of a. For each pair of numbers a and b, we define their *difference*, denoted by $a - b$, by

$$a - b \equiv a + (-b).$$

The Field Axioms also imply that there is only one number having the property attributed to 1 in the Multiplicative Identity Axiom. For a real number $a \neq 0$, the Multiplicative Inverse Axiom asserts that the equation

$$ax = 1$$

has a solution. One can show there is only one solution; we denote it by a^{-1} and call it the *multiplicative inverse* of a. We then define for each pair of numbers a and $b \neq 0$ their *quotient*, denoted by a/b, as

$$\frac{a}{b} \equiv ab^{-1}.$$

It is a somewhat tedious algebraic exercise to verify the implications of the Field Axioms that we have just mentioned and also to verify the following familiar consequences of these same axioms: For any real numbers a and $b \neq 0$,

$$-(-a) = a, \qquad (b^{-1})^{-1} = b, \qquad \text{and} \qquad (-b)^{-1} = -b^{-1}.$$

THE POSITIVITY AXIOMS FOR THE REAL NUMBERS

In the real numbers there is a natural notion of order: greater than, less than, and so on. A convenient way to codify these properties is by specifying axioms satisfied by the set of positive numbers.

The Positivity Axioms

There is a set of real numbers, denoted by $\mathcal{P}$, called the set of *positive numbers*. It has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are also positive.

P2 For a real number a, exactly one of the following three alternatives is true:

$$a \text{ is positive}, \qquad -a \text{ is positive}, \qquad a = 0.$$

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The Positivity Axioms lead in a natural way to an ordering of the real numbers: For real numbers a and b, we define $a > b$ to mean that $a - b$ is positive, and $a \geq b$ to mean that $a > b$ or $a = b$. We then define $a < b$ to mean that $b > a$, and $a \leq b$ to mean that $b \geq a$.

Using the Field Axioms and the Positivity Axioms, it is possible to establish the following familiar properties of inequalities (Appendix A):

i. For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
ii. For each positive number a, its multiplicative inverse a^{-1} is also positive.
iii. If $a > b$, then

$$ac > bc \qquad \text{if } c > 0,$$

and

$$ac < bc \qquad \text{if } c < 0.$$

Interval Notation

For a pair of real numbers a and b such that $a < b$, we define

$$(a, b) \equiv \{x \text{ in } \mathbb{R} \mid a < x < b\},$$
$$[a, b] \equiv \{x \text{ in } \mathbb{R} \mid a \leq x \leq b\},$$
$$(a, b] \equiv \{x \text{ in } \mathbb{R} \mid a < x \leq b\},$$

and

$$[a, b) \equiv \{x \text{ in } \mathbb{R} \mid a \leq x < b\}.$$

Moreover, it is convenient to use the symbols ∞ and $-\infty$ in the following manner. We define

$$[a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a \leq x\},$$
$$(-\infty, b] \equiv \{x \text{ in } \mathbb{R} \mid x \leq b\},$$
$$(a, \infty) \equiv \{x \text{ in } \mathbb{R} \mid a < x\},$$
$$(-\infty, b) \equiv \{x \text{ in } \mathbb{R} \mid x < b\},$$

and

$$(-\infty, \infty) \equiv \mathbb{R}.$$

The reader should be very careful to observe that although we have defined, say, $[a, \infty)$, we have *not* defined the symbols ∞ and $-\infty$. In particular, we have *not* adjoined additional numbers to $\mathbb{R}$.

It is also convenient to set $[a, a] \equiv \{a\}$. In general, when we write $[a, b]$ or (a, b), unless another meaning is explicitly mentioned, it is assumed that a and b are real numbers such that $a < b$.

Each of the sets listed above is called an *interval*. In the analysis of functions $f : A \to \mathbb{R}$, where A is a set of real numbers, a special role is played by those functions that have an interval as their domain A. In particular, intervals of the form (a, b), which we call *open intervals*, or of the form $[a, b]$, which we call *closed intervals*, will frequently be the domains of the functions that we will study.

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CHAPTER 1

TOOLS FOR ANALYSIS

1.1 THE COMPLETENESS AXIOM AND SOME OF ITS CONSEQUENCES

A rigorous understanding of mathematical analysis must be based on a proper understanding of the set of real numbers. The purpose of this first chapter is to establish the fundamental properties of the set $\mathbb{R}$ of real numbers, to describe the properties possessed by the special subsets of the real numbers consisting of the natural numbers, the integers, and the rational and irrational numbers, and to establish some basic inequalities and algebraic identities.

The properties of addition and multiplication of real numbers have been codified in the Preliminaries as the Field Axioms. The set of real numbers is also equipped with the concept of order, and the properties of order and inequality have been codified in the Preliminaries as the Positivity Axioms. Many interesting properties of the real numbers are consequences of the Field and Positivity Axioms. However, an additional axiom is necessary. To explain why this is so, we now introduce some special subsets of $\mathbb{R}$.

We begin by defining the natural numbers. Of course, these are long familiar to the reader. The natural numbers are the numbers 1, 2, 3, and so on. However, it is necessary to make this statement more precise, and a convenient way of doing so is to first introduce the concept of an *inductive set.*

Definition A set S of real numbers is said to be *inductive* provided that

i. the number 1 is in S
ii. if the number x is in S, the number $x + 1$ is also in S.

The whole set of real numbers $\mathbb{R}$ is inductive. Also, using just the fact that the number 1 is greater than the number 0, it follows that the set $\{x \text{ in } \mathbb{R} \mid x \geq 0\}$ is inductive, as is the set $\{x \text{ in } \mathbb{R} \mid x \geq 1\}$. The set of *natural numbers*, denoted by $\mathbb{N}$, is defined to be *the intersection of all inductive subsets* of $\mathbb{R}$. The set $\mathbb{N}$ itself is inductive. To see this, observe that the number 1 belongs to $\mathbb{N}$ since 1 belongs to every inductive set. Furthermore, if the number k belongs to $\mathbb{N}$, then k belongs to every inductive set; thus,

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$k+1$ belongs to every inductive set, and therefore $k+1$ belongs to $\mathbb{N}$. Therefore, $\mathbb{N}$ is inductive and, by definition, it is contained in every other inductive set. Thus,

$$\text{if } A \text{ is a set of natural numbers that is inductive, then } A = \mathbb{N}. \tag{1.1}$$

Arguments based on this property occur so frequently that it is useful to formalize them as follows.

Principle of Mathematical Induction

For each natural number n, let $S(n)$ be some mathematical assertion. Suppose that $S(1)$ is true. Also suppose that whenever k is a natural number such that $S(k)$ is true, then $S(k+1)$ is also true. Then $S(n)$ is true for every natural number n.

Proof

Define $A \equiv \{k \text{ in } \mathbb{N} \mid S(k) \text{ is true}\}$. The assumptions mean precisely that A is an inductive subset of $\mathbb{N}$. According to (1.1), $A = \mathbb{N}$. Thus, $S(n)$ is true for every natural number n. ■

Example 1.1 For each natural number n,

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

We use the Principle of Mathematical Induction to prove this. For a natural number n, let $S(n)$ be the assertion that the above formula holds. Clearly, $S(1)$ is true. Suppose that k is a natural number such that $S(k)$ is true; that is,

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}.$$

Then

$$\sum_{j=1}^{k+1} j = \left[\sum_{j=1}^{k} j\right] + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2};$$

that is, $S(k+1)$ is true. By the Principle of Mathematical Induction, the above summation formula holds for all natural numbers n. ■

As we expect, for natural numbers m and n,

- The sum, $m+n$, is a natural number, and
- The product, mn, is a natural number.

We leave it as an exercise for the reader to use the Principle of Mathematical Induction to prove the sum and product properties of the natural numbers (Exercise 6).

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We define the set of *integers*, denoted by $\mathbb{Z}$, to be the set of numbers consisting of the natural numbers, their negatives, and the number 0. Again, as we expect, for integers m and n,

- The sum, $m + n$, is an integer,
- The difference, $m - n$, is an integer, and
- The product, mn, is an integer.

We again leave it as an exercise for the reader to use the sum and product properties of the natural numbers to establish the above three properties of the integers (Exercise 9).

The set of *rational numbers*, denoted by $\mathbb{Q}$, is defined to be the set of quotients of integers, that is, numbers x of the form $x = m/n$, where m and n are integers and $n \neq 0$. A real number is called *irrational* if it is not rational. At present, we have no evidence that there are any irrational numbers.

Now it is a little tedious, but not really difficult, to show that since sums, differences, and products of integers are again integers, then the set $\mathbb{Q}$ of rational numbers satisfies the Field Axioms (Exercise 10). However, despite possessing this coherent algebraic structure, it is not possible to develop calculus using only rational numbers. For instance, it is necessary to conclude that a polynomial that attains both positive and negative values must also attain the value 0. This is not true if one considers only rational numbers. For instance, consider the polynomial defined by $p(x) = x^2 - 2$ for all real numbers x. Then $p(0) < 0$ and $p(2) > 0$. However, as has been known since antiquity, there is no rational number x having the property that $x^2 = 2$; that is, there is no rational number x such that $p(x) = 0$. Before giving the classical proof of this assertion, we first note the following two properties of the integers:

i. Each rational number x can be expressed as $x = m/n$, where m and n are integers and m or n is odd.
ii. An integer n is even if its square n^2 is even.

The proofs of the basic properties of the integers, including the two above, are outlined in Exercise 26 of Section 1.3.

Proposition 1.2 There is no rational number whose square equals 2.

Proof

The proof relies on the two properties of the integers noted above. We will suppose that the proposition is false and derive a contradiction. Suppose there is a rational number x such that $x^2 = 2$. The above property (i) asserts that we can express x as $x = m/n$, where m and n are integers and either m or n is odd. Since $m^2/n^2 = 2$, we have $m^2 = 2n^2$. Thus, m^2 is even, so by the above property (ii), m is also even. We now express m as $m = 2k$, where k is an integer. Since $m^2 = 2n^2$, we have $4k^2 = 2n^2$. Thus, n^2 is even, so once more using the above property (ii), we conclude that n is also even. Hence both m and n are even. But we chose these integers so that at least one of them was odd.

The assumption that the proposition is false has led to a contradiction, so the proposition must be true. ∎

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Thus, there is no rational number x such that $x^2 = 2$, and hence it is not possible to prove even the simplest geometric result concerning the intersection of the graph of a polynomial and the x-axis (that is, points where $x^2 - 2 = 0$) if we restrict ourselves to rational numbers. Worse yet, even the Pythagorean Theorem fails if we restrict ourselves to rational numbers: If r is the length of the hypotenuse of a right-angled triangle whose other two sides have length 1, then $r^2 = 2$, and so the length of the hypotenuse is not a rational number.

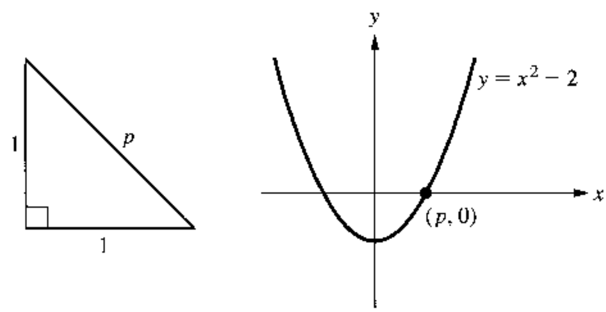

FIGURE 1.1 $p^2 = 2$ and p is not a rational number.

We need an additional axiom for the real numbers that, at the very least, assures us that there is a real number whose square equals 2. The final axiom will be the Completeness Axiom. To state this axiom, we need to introduce the concept of boundedness.

Definition A nonempty set S of real numbers is said to be *bounded above* provided that there is a number c possessing the property that

$$x \leq c \qquad \text{for all } x \text{ in } S.$$

Such a number c is called an *upper bound* for S.

It is clear that if a number c is an upper bound for a set S, then every number greater than c is also an upper bound for this set. For a nonempty set S of numbers that is bounded above, among all the upper bounds for S it is not at all obvious why there should be a smallest, or least, upper bound. In fact, the assertion that there is such a least upper bound will be the final axiom for the real numbers.

The Completeness Axiom

Suppose that S is a nonempty set of real numbers that is bounded above. Then, among the set of upper bounds for S there is a smallest, or least, upper bound.

For a nonempty set S of real numbers that is bounded above, the *least upper bound* of S, the existence of which is asserted by the Completeness Axiom, will be denoted by l.u.b. S. The least upper bound of S is also called the *supremum* of S and is denoted by sup S. Thus,

$$\sup S \text{ or l.u.b. } S \text{ denotes the least upper bound of the set } S.$$

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It is worthwhile to note explicitly that if the number b is an upper bound for the set S, then in order to verify that $b = \sup S$, it is necessary to show that b is less than any other upper bound for S. This task, however, is equivalent to showing that each number smaller than b is not an upper bound for S.

At first glance, it is not at all apparent that the Completeness Axiom will help our development of mathematical analysis. In fact, the Completeness Axiom is indispensable to the development of mathematical analysis. As one instance of its importance, while Proposition 1.2 states that there is no rational number whose square equals 2, the Completeness Axiom guarantees that there is a number, necessarily irrational, whose square equals 2. Indeed, the set

$$S \equiv \{x \text{ in } \mathbb{R} \mid x \geq 0, x^2 < 2\}$$

is nonempty. Moreover, S also is bounded above since if $x \geq 0$ and $x^2 < 2$, then $x \geq 0$ and $x^2 < 2^2$, so $x < 2$. Thus, 2 is an upper bound for S. The Completeness Axiom assures us that the set S has a least upper bound b. It is not obvious, but, in fact, $b^2 = 2$. In Exercise 17 we outline a proof of this last assertion. Moreover, we have the following general proposition regarding the existence of square roots.

Proposition 1.3 Let c be a positive number. Then there is a (unique) positive number whose square is c; that is, the equation

$$x^2 = c, \qquad x > 0$$

has a unique solution.

We outline a proof of the existence part of this proposition in Exercise 17. We will see in Chapter 3 that the existence part is a corollary of a much more general result called the Intermediate Value Theorem. The proof of the uniqueness part of the above proposition is as follows. Observe that if a and b are positive numbers each of whose square is c, then $0 = a^2 - b^2 = (a - b)(a + b)$. Since $a + b > 0$, it follows that $a = b$. As usual, we denote the positive number whose square is c by $\sqrt{c}$. We define $\sqrt{0} \equiv 0$.[1]

Definition A nonempty set S of real numbers is said to be *bounded below* provided that there is a number b with the property that

$$b \leq x \qquad \text{for all } x \text{ in } S.$$

Such a number b is called a *lower bound* for S. The set S is said to be *bounded* if it is both bounded below and bounded above.

[1] We proved that $\sqrt{2}$ is irrational. In fact, a much more general result holds: For any natural numbers n and m, if the mth root of n, $\sqrt[m]{n}$, is not a natural number, then it must be irrational. The proof of this depends on the Prime Factorization Theorem: Any natural number can be uniquely expressed as the product of powers of primes. For a proof of this theorem see the excellent book, *Topics in Algebra*, 3rd. ed., by I. N. Herstein (New York: John Wiley and Sons, 1996).

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It is clear that if a number b is a lower bound for a set S, then every number less than b is also a lower bound for S. We will now use the Completeness Axiom to show that for a nonempty set of numbers S that is bounded below, among the lower bounds for the set there is a *greatest lower bound,* denoted by g.l.b. S. Sometimes the greatest lower bound of S is called the *infimum* of S and is denoted by $\inf S$.

Theorem 1.4 Suppose that S is a nonempty set of real numbers that is bounded below. Then among the set of lower bounds for S there is a largest, or greatest, lower bound.

Proof
We will consider the set obtained by "reflecting" the set S about the number 0; that is, we will consider the set $T \equiv \{x \text{ in } \mathbb{R} \mid -x \text{ is in } S\}$.

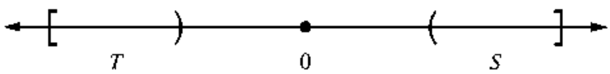

FIGURE 1.2 The reflection of a set S about 0.

For any number x, $b \leq x$ if and only if $-x \leq -b$. Thus, a number b is a lower bound for S if and only if the number $-b$ is an upper bound for T. Since the set S has been assumed to be bounded below, it follows that the set T is bounded above. The Completeness Axiom asserts that there is a least upper bound for T, which we denote by c. Since lower bounds of S occur as negatives of upper bounds for T, the number $-c$ is the greatest lower bound for S. ∎

EXERCISES FOR SECTION 1.1

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. The set of irrational numbers is inductive.
   b. The set of squares of rational numbers is inductive.
   c. The sum of irrational numbers is irrational.
   d. The product of irrational numbers is irrational.
   e. If n is a natural number and n^2 is odd, then n is odd.
2. For each of the following statements, determine whether it is true or false and justify your answer.
   a. Every nonempty set of real numbers that is bounded above has a largest member.
   b. If S is a nonempty set of positive real numbers, then $0 \leq \inf S$.
   c. If S is a set of real numbers that is bounded above and B is a nonempty subset of S, then $\sup B \leq \sup S$.
3. Use the Principle of Mathematical Induction to prove that for a natural number n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

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4. Let n be a natural number. Find a formula for $\sum_{j=1}^{n} j(j+1)$.
5. Let n be a natural number. Prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

6. Let m and n be natural numbers.
   a. Prove that the sum, $m+n$, also is a natural number. (*Hint:* Fix m and define $S(n)$ to be the statement that $m+n$ is a natural number.)
   b. Prove that the product, mn, also is a natural number. (*Hint:* Fix m and define $S(n)$ to be the statement that mn is a natural number.)
7. Prove that if n is a natural number greater than 1, then $n-1$ is also a natural number. (*Hint:* Prove that the set $\{n \mid n = 1 \text{ or } n \text{ in } \mathbb{N} \text{ and } n-1 \text{ in } \mathbb{N}\}$ is inductive.)
8. Prove that if n and m are natural numbers such that $n > m$, then $n-m$ is also a natural number. (*Hint:* Prove this by induction on m, making use of Exercise 7.)
9. Use Exercise 8 to prove that the sum, difference, and product of integers also are integers.
10. Use Exercise 9 to prove that the rational numbers satisfy the Field Axioms.
11. a. Prove that the sum of a rational number and an irrational number must be irrational.
    b. Prove that the product of two nonzero numbers, one rational and one irrational, is irrational.
12. Use Proposition 1.2 to show that there is no rational number whose square equals $2/9$.
13. Suppose that S is a nonempty set of real numbers that is bounded. Prove that $\inf S \le \sup S$.
14. Suppose that S is a nonempty set of real numbers that is bounded and that $\inf S = \sup S$. Prove that the set S consists of exactly one number.
15. For a set S of numbers, a member c of S is called the *maximum* of S provided that it is an upper bound for S. Prove that a set S of numbers has a maximum if and only if it is bounded above and $\sup S$ belongs to S. Give an example of a set S of numbers that is nonempty and bounded above but has no maximum.
16. Prove that $\sqrt{3}$ is not a rational number. (*Hint:* Follow the idea of the proof of Proposition 1.2.)
17. (Outline of the proof of Proposition 1.3) Define

$$S \equiv \{x \mid x \text{ in } \mathbb{R}, x \ge 0, x^2 < c\}$$

   a. Show that $c+1$ is an upper bound for S and therefore, by the Completeness Axiom, S has a least upper bound that we denote by b.
   b. Show that if $b^2 > c$, then we can choose a suitably small positive number r such that $b-r$ is also an upper bound for S, thus contradicting the choice of b as the *least* upper bound of S.
   c. Show that if $b^2 < r$, then we can choose a suitably small positive number r such that $b+r$ belongs to S, thus contradicting the choice of b as *an* upper bound of S.
   d. Use parts (b) and (c) and the Positivity Axioms for $\mathbb{R}$ to conclude that $b^2 = c$.

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18. Prove that there is a positive number x such that $x^3 = 5$. (*Hint:* Follow the proof outlined in Exercise 17.)

19. Define $S \equiv \{x \text{ in } \mathbb{R} \mid x^2 < x\}$. Prove that $\sup S = 1$.

20. **a.** For real numbers a and b, suppose that the number x is a solution to the equation

$$(x-a)(x-b) = 0.$$

Prove that either $x = a$ or $x = b$.

b. For a positive number c, show that if x is any number such that $x^2 = c$, then either $x = \sqrt{c}$ or $x = -\sqrt{c}$.

c. Let a, b, and c be real numbers such that $a \neq 0$, and consider the quadratic equation

$$ax^2 + bx + c = 0.$$

Prove that a number x is a solution of this equation if and only if

$$(2ax + b)^2 = b^2 - 4ac.$$

Suppose that $b^2 - 4ac > 0$. Prove that the quadratic equation has exactly two solutions, given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

d. In part (c) now suppose that $b^2 - 4ac < 0$. Prove that there is no real number that is a solution of the quadratic equation.

1.2 THE DISTRIBUTION OF THE INTEGERS AND THE RATIONAL NUMBERS

Our principal goal in this section is to establish the following two theorems regarding the distribution among the real numbers of the integers and the rational numbers.

Distribution of the Integers

For any number c,

$$\text{there is a unique integer in the interval } [c, c+1). \tag{1.2}$$

Distribution of the Rational Numbers

For any numbers a and b, with $a < b$,

$$\text{there is a rational number in the interval } (a, b). \tag{1.3}$$

There is a property of the real numbers called the Archimedean Property that underlies both (1.2) and (1.3). Our approach is to first prove the Archimedean Property and then use this property as a cornerstone in construction of the proofs of (1.2) and (1.3).

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Theorem 1.5 The Archimedean Property[2] The following two equivalent properties hold:

i. For any positive number c, there is a natural number n such that $n > c$.
ii. For any positive number ϵ, there is a natural number n such that $1/n < \epsilon$.

Proof
First, let us observe that the above two properties are equivalent. Indeed, for two positive numbers c and ϵ related by

$$\epsilon = 1/c,$$

for a natural number n,

$$n > c \qquad \text{if and only if } 1/n < \epsilon.$$

Thus, property (i) holds if and only if property (ii) holds.

We will establish property (i) by assuming that this property does not hold and deriving a contradiction. So suppose there is a positive number c for which there is no natural number greater than c. Then, using the Positivity Axioms, we conclude that

$$n \leq c \qquad \text{for every natural number } n.$$

Thus, the set $\mathbb{N}$ of natural numbers is bounded above. The Completeness Axiom asserts that $\mathbb{N}$ has a least upper bound. Denote the least upper bound of $\mathbb{N}$ by b.

Since b is the smallest upper bound for $\mathbb{N}$, the number $b - 1/2$ is not an upper bound for $\mathbb{N}$. Thus, we can choose a natural number n such that $n > b - 1/2$, and therefore

$$n + 1 > (b - 1/2) + 1 > b.$$

So $n + 1$ is a natural number that is larger than b. This contradicts the choice of b as an upper bound of $\mathbb{N}$. This contradiction proves the result. ■

Proposition 1.6 For any integer n,

there is no integer k in the open interval $(n, n + 1)$.

Proof
First, we consider the case $n = 0$. The set $\{k \mid k \in \mathbb{N}, k \geq 1\}$ is an inductive set of natural numbers. Thus, it is equal to $\mathbb{N}$, and hence, for any natural number k, $k \geq 1$. Since every positive integer is a natural number, we conclude that the open interval $(0, 1)$ does not contain an integer. Now we consider a general integer n. We

[2] This property is traditionally called the Archimedean Property, and we will continue to follow tradition. However, in his writings, Archimedes explicitly stated that this property was due to Eudoxos, another Greek mathematician who lived 100 years earlier than Archimedes. Moreover, the property is explicitly listed as a proposition in Euclid's fundamental books on geometry in which it is stated as follows: For any two positive number a and b, there is a natural number n such that $na > b$.

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will argue by contradiction. Suppose there is an integer k in the interval $(n, n+1)$. Then

$$n < k < n+1, \text{ so that } 0 < k-n < 1.$$

As we have already observed, the difference of two integers is again an integer. Thus, $k - n$ is an integer in the interval $(0, 1)$. But we just showed that this is not possible. The assumption that the interval $(n, n+1)$ contains an integer has led to a contradiction. Thus, the interval $(n, n+1)$ does not contain any integers. ∎

Proposition 1.7 Suppose that S is a nonempty set of integers that is bounded above. Then S has a maximum.

Proof
According to the Completeness Axiom, the set S has a least upper bound. Define

$$a \equiv \sup S.$$

Since the number a is the smallest upper bound for the set S, $a - 1$ is not an upper bound for S and so there is a member m of S such that $a - 1 < m$. Hence $a < m + 1$ and since a is an upper bound for S we have the set inclusion

$$S \subset (-\infty, m+1).$$

Since S is a set of integers, m is an integer and moreover, by Proposition 1.6, the interval $(m, m+1)$ contains no members of S. Therefore, using the above set inclusion we have the improved set inclusion

$$S \subset (-\infty, m].$$

Thus, m is the maximum of the set S. ∎

It should be explicitely noted that the above theorem is an assertion about a set of integers. In general, the Completeness Axiom states that any nonempty set of real numbers that is bounded above has a supremum: Such a set need not have a maximum since the supremum of a set need not be a member of the set.

Theorem 1.8 For any number c, there is exactly one integer k in the interval $[c, c+1)$.

Proof
Define

$$S \equiv \{n \mid n \text{ in } \mathbb{Z}, n < c+1\}.$$

We first show that the set S is nonempty. If $c + 1 \geq 0$, then 0 belongs to S. If $c + 1 < 0$, then the Archimedean Property asserts that there is a natural number n such that $n > -(c+1)$, so that $-n < c + 1$, and $-n$ is an integer. Hence $-n$ belongs to S. Thus, the set S is nonempty. By its very definition, $c + 1$ is an upper

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bound for S, so S is bounded above. The preceding proposition asserts that there is a largest member of S. Let k be this maximum member of S. Then $k \geq c$, for otherwise $k < c$, so that $k + 1 < c + 1$, contradicting the choice of k as being the largest integer less than $c + 1$. Thus, k belongs to the interval $[c, c + 1)$.

There is only one integer in the interval $[c, c + 1)$. Indeed, otherwise, there would be integers k and k' in the interval $[c, c + 1)$, with $k < k'$. Then

$$0 < k' - k \text{ since } k < k',$$

while

$$k' - k < [c + 1] - c = 1 \text{ since } k' < c + 1 \text{ and } k \geq c.$$

It follows that $k' - k$ is an integer in the interval $(0, 1)$, which contradicts Proposition 1.6. Thus, there is exactly one integer in the interval $[c, c + 1)$. ■

Definition A set S of real numbers is said to be *dense in* $\mathbb{R}$ provided that every interval $I = (a, b)$, where $a < b$, contains a member of S.

Theorem 1.9 The set of rational numbers is dense in $\mathbb{R}$.

Proof
Let a and b be real numbers such that $a < b$. We need to show that the interval (a, b) contains a rational number. By the Archimedean Property, we can choose a natural number n such that

$$1/n < b - a,$$

so $1/n$ is less than the length of the interval (a, b). By Theorem 1.8, applied to $c \equiv nb - 1$, there is an integer m in the interval $[nb - 1, nb)$. Thus,

$$nb - 1 \leq m < nb,$$

which, after dividing by n, gives

$$b - 1/n \leq m/n < b. \tag{1.4}$$

But $1/n < b - a$, so

$$a = b - (b - a) < b - 1/n \tag{1.5}$$

From the inequalities (1.4) and (1.5) we conclude that the rational number m/n belongs to the interval (a, b). ■

Corollary 1.10 The set of irrational numbers is dense in $\mathbb{R}$.

Proof
The density of the irrationals follows from the density of the rationals and the existence of positive irrational numbers. Indeed, given an interval (a, b), choose

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any positive irrational number z; for instance, choose $z = \sqrt{2}$. By the density of the rationals there is a rational number x in the interval $(a/z, b/z)$ so that zx lies in the interval (a, b) and zx is irrational since it is the product of an irrational number and a rational number. ■

EXERCISES FOR SECTION 1.2

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. The set $\mathbb{Z}$ of integers is dense in $\mathbb{R}$.
   b. The set of positive real numbers is dense in $\mathbb{R}$.
   c. The set $\mathbb{Q} \setminus \mathbb{N}$ of rational numbers that are not integers is dense in $\mathbb{R}$.
2. Suppose that S is a nonempty set of integers that is bounded below. Show that S has a minimum. In particular, conclude that every nonempty set of natural numbers has a minimum.
3. Let S be a nonempty set of real numbers that is bounded below. Prove that the set S has a minimum if and only if the number $\inf S$ belongs to S.
4. For each of the following two sets, find the maximum, minimum, infimum, and supremum if they are defined. Justify your conclusions.
   a. $\{1/n \mid n \text{ in } \mathbb{N}\}$
   b. $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$
5. Suppose that the number a has the property that for every natural number $n, a \leq 1/n$. Prove that $a \leq 0$.
6. Given a real number a, define $S \equiv \{x \mid x \text{ in } \mathbb{Q}, x < a\}$. Prove that $a = \sup S$.
7. Show that for any real number c, there is exactly one integer in the interval $(c, c+1]$.
8. Show that the Archimedean Property is a consequence of the assertion that for any real number c, there is an integer in the interval $[c, c+1)$.
9. Show that the Archimedean Property is a consequence of the assertion that every interval (a, b) contains a rational number.

1.3 INEQUALITIES AND IDENTITIES

Recall that for a real number x, its *absolute value*, denoted by $|x|$, is defined by

$$|x| \equiv \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Directly from this definition and from the Positivity Axioms for $\mathbb{R}$, it follows that if c and d are any numbers such that $d \geq 0$, then

$$|c| \leq d \quad \text{if and only if } -d \leq c \leq d. \tag{1.6}$$

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Moreover, we also have, for any number x,

$$-|x| \le x \le |x|. \tag{1.7}$$

Given a pair of real numbers a and b, we often need to estimate the size of $|a+b|$. The following inequality is a basic tool.

Theorem 1.11 The Triangle Inequality For any pair of numbers a and b,

$$|a+b| \le |a| + |b|.$$

Proof
Using (1.6), we see that the Triangle Inequality is equivalent to the assertion that

$$-|a| - |b| \le a + b \le |a| + |b|. \tag{1.8}$$

However, setting $x = a$ and then $x = b$ in (1.7), we have

$$-|a| \le a \le |a| \qquad \text{and} \qquad -|b| \le b \le |b|$$

from which, by addition, we obtain (1.8) and hence the Triangle Inequality. ∎

It is useful to explicitly record the following proposition.

Proposition 1.12 For a number a and a positive number r, the following three assertions about a number x are equivalent:

i. $|x - a| < r$.
ii. $a - r < x < a + r$.
iii. x belongs to the open interval $(a - r, a + r)$.

Proof
The equivalence of (i) and (ii) follows from (1.6), while the equivalence of (ii) and (iii) is simply the very definition of the interval $(a - r, a + r)$. ∎

At the heart of many arguments in analysis lies the problem of estimating the sizes of various quotients, differences, and sums and of simplifying various algebraic expressions. As a companion tool to the Triangle Inequality we now establish three useful algebraic identities.

For a natural number n and any number a, as usual, we write a^n to denote the product of a multiplied by itself n times.

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Observe that we have the following formulas for the difference of squares and the difference of cubes:

$$a^2 - b^2 = (a - b)(a + b) \qquad \text{and} \qquad a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

These are special cases of the following formula.

The Difference of Powers Formula

For any natural number n and any numbers a and b,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

It is easy to verify this formula just by expanding the right-hand side. Indeed,

$$\begin{aligned}
&(a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\
&= a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} \\
&\qquad - a^{n-1}b - a^{n-2}b^2 - \cdots - a^2b^{n-2} - ab^{n-1} - b^n \\
&\qquad\qquad = a^n - b^n.
\end{aligned}$$

In the Difference of Powers Formula, if we take $a \equiv 1$, set $b \equiv r \neq 1$, and replace n by $n + 1$, then after division by $1 - T$ we obtain the following important identity.

The Geometric Sum Formula

For any natural number n and any number $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n-1}}{1 - r}.$$

This formula is the essential tool underlying the frequent possibility of expressing functions as power series that we will consider in Chapters 8 and 9. It also plays an essential role in verifying many computational algorithms.

It will be useful to have a formula that expresses powers of the sum of the numbers a and b in terms of the powers of a and of b. In order to state this formula, we need to introduce factorial notation. For each natural number n, we define the symbol $n!$, which is called *n factorial*, as follows: We define $1! \equiv 1$, and if k is any natural number for which $k!$ has been defined, we then define $(k+1)! \equiv (k+1)k!$. By the Principle of Mathematical Induction, the symbol $n!$ is defined for all natural numbers n. It is convenient to define $0! \equiv 1$. We also need to introduce, for each pair of nonnegative integers n and k such that $n \geq k$, the *binomial coefficient* $\binom{n}{k}$, which is defined by the formula

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}.$$

We have the following formula for $(a + b)^n$, a proof of which is outlined in Exercises 21 and 22.

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The Binomial Formula

For each natural number n and each pair of numbers a and b,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

We close this discussion on algebraic identities by recalling the summation notation. For a natural number n and numbers $a_0, a_1, \ldots, a_n$, we define

$$\sum_{k=0}^{n} a_k \equiv a_0 + a_1 + \cdots + a_n.$$

This notation shortens many formulas. For instance, using this summation notation, the three algebraic formulas we have described become the following.

The Difference of Powers Formula

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k.$$

The Geometric Sum Formula

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n-1}}{1-r} \qquad \text{if } r \neq 1.$$

The Binomial Formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

EXERCISES FOR SECTION 1.3

1. Write out the Difference of Powers Formula explicitly for $n = 4$ and 5.
2. Write out the Binomial Formula explicitly for $n = 2, 3$, and 4.
3. Show that the Triangle Inequality becomes an equality if a and b are of the same sign.
4. Let $a > 0$. Prove that if x is a number such that $|x - a| < a/2$, then $x > a/2$.
5. Let $b < 0$. Prove that if x is a number such that $|x - b| < |b|/2$, then $x < b/2$.
6. Which of the following inequalities hold for all numbers a and b? Justify your conclusions.
   a. $|a + b| \geq |a| + |b|$.
   b. $|a + b| \leq |a| - |b|$.
7. By writing $a = (a + b) + (-b)$ use the Triangle Inequality to obtain $|a| - |b| \leq |a + b|$. Then interchange a and b to show that

$$||a| - |b|| \leq |a + b|.$$

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Then replace b by $-b$ to obtain

$$||a| - |b|| \le |a - b|.$$

8. Let a and b be numbers such that $|a - b| \le 1$. Prove that $|a| \le |b| + 1$.

9. For a natural number n and any two nonnegative numbers a and b, use the Difference of Powers Formula to prove that

$$a \le b \qquad \text{if and only if } a^n \le b^n.$$

10. For a natural number n and numbers a and b such that $a \ge b \ge 0$, prove that

$$a^n - b^n \ge nb^{n-1}(a - b).$$

11. (Bernoulli's Inequality) Show that for a natural number n and a nonnegative number b,

$$(1 + b)^n \ge 1 + nb.$$

(*Hint:* In the Binomial Formula, set $a = 1$.)

12. Use the Principle of Mathematical Induction to provide a direct proof of Bernoulli's Inequality for all $b > -1$, not just for the case where $b \ge 0$ which, as outlined in Exercise 11 follows from the Binomial Formula.

13. For a natural number n and a nonnegative number b show that

$$(1 + b)^n \ge 1 + nb + \frac{n(n-1)}{2}b^2.$$

14. (Cauchy's Inequality) Using the fact that the square of a real number is nonnegative, prove that for any numbers a and b,

$$ab \le \frac{1}{2}(a^2 + b^2).$$

15. Use Cauchy's Inequality to prove that if $a \ge 0$ and $b \ge 0$, then

$$\sqrt{ab} \le \frac{1}{2}(a + b).$$

16. Use Cauchy's Inequality to show that for any numbers a and b and a natural number n,

$$ab \le \frac{1}{2}\left(na^2 + \frac{1}{n}b^2\right).$$

(*Hint:* Replace a by $\sqrt{n}a$ and b by $b/\sqrt{n}$ in Cauchy's Inequality.)

17. Let a, b, and c be nonnegative numbers. Prove the following inequalities:

a. $ab + bc + ca \le a^2 + b^2 + c^2$.

b. $8abc \le (a + b)(b + c)(c + a)$.

c. $abc(a + b + c) \le a^2b^2 + b^2c^2 + c^2a^2$.

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18. A function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly increasing* provided that $f(u) > f(v)$ for all numbers u and v such that $u > v$.

a. Define $p(x) = x^3$ for all x. Prove that the polynomial $p : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

b. Fix a number c and define $q(x) = x^3 + cx$ for all x. Prove that the polynomial $q : \mathbb{R} \to \mathbb{R}$ is strictly increasing if and only if $c \geq 0$. (*Hint:* For $c < 0$, consider the graph to understand why it is not strictly increasing and then prove it is not increasing.)

19. Let n be a natural number and $a_1, a_2, \ldots, a_n$ be positive numbers. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n$$

and that

$$(a_1 + a_2 + \cdots + a_n)\left(a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1}\right) \geq n^2.$$

20. Use the Geometric Sum Formula to find a formula for

a. $\dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots + \dfrac{1}{(1 + x^2)^n}$.

b. Also, show that if $a \neq 0$, then

$$\frac{1}{a} = 1 + (1 - a) + (1 - a)^2 + \frac{(1 - a)^3}{a}.$$

21. Prove that if n and k are natural numbers such that $k \leq n$, then

$$\binom{n + 1}{k} = \binom{n}{k - 1} + \binom{n}{k}.$$

22. Use the formula in Exercise 21 to provide an inductive proof of the Binomial Formula.

23. Let a be a nonzero number and m and n be integers. Prove the following equalities:

a. $a^{m+n} = a^m a^n$.

b. $(ab)^n = a^n b^n$.

24. A natural number n is called *even* if it can be written as $n = 2k$ for some other natural number k, and is called *odd* if either $n = 1$ or $n = 2k + 1$ for some other natural number k.

a. Prove that each natural number n is either odd or even.

b. Prove that if m is a natural number, then $2m > 1$.

c. Prove that a natural number n cannot be both odd and even. (*Hint:* Use part (b).)

d. Suppose that k_1, k_2, ℓ_1, and ℓ_2 are natural numbers such that ℓ_1 and ℓ_2 are odd. Prove that if $2^{k_1}\ell_1 = 2^{k_2}\ell_2$, then $k_1 = k_2$ and $\ell_1 = \ell_2$.

25. **a.** Prove that if n is a natural number, then $2^n > n$.

b. Prove that if n is a natural number, then

$$n = 2^{k_0}\ell_0$$

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for some odd natural number ℓ_0 and some nonnegative integer k_0. (*Hint:* If n is odd, let $k = 0$ and $\ell = n$; if n is even, let $A = \{k \text{ in } \mathbb{N} \mid n = 2^k \ell \text{ for some } \ell \text{ in } \mathbb{N}.\}$ By (a), $A \subseteq \{1, 2, \ldots, n\}$. Choose k_0 to be the maximum of A.)

26. Prove that Exercises 24 and 25 are sufficient to prove the assertions (i) and (ii) about the natural numbers that preceded the proof of the irrationality of $\sqrt{2}$.

27. A real number of the form $m/2^n$, where m and n are integers, is called a *dyadic rational*. Prove that the set of dyadic rationals is dense in $\mathbb{R}$.

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CHAPTER 2

CONVERGENT SEQUENCES

2.1 THE CONVERGENCE OF SEQUENCES

Two of the main topics in the analysis of real-valued functions of a real variable are the differentiation and integration of functions that have as their domain an interval of real numbers. However, sequences of real numbers are also important. The study of sequences is essential for an understanding of infinite series of numbers and functions. Moreover, properties of sequences will play a fundamental role in our study of the continuity, differentiation, and integration of general functions. In this chapter, we study those properties of sequences that will underlie our development of the continuity, differentiation, and integration of general functions.

Definition A *sequence* of real numbers is a real-valued function whose domain is the set of natural numbers.

Since in the first nine chapters we will be considering only sequences of real numbers, we will abbreviate *sequence of real numbers* by writing *sequence*. Also, rather than denoting a sequence with standard functional notation, such as $f : \mathbb{N} \to \mathbb{R}$, it is customary to use subscripts, replacing $f(n)$ with a_n and denoting a sequence by $\{a_n\}$. A natural number n is called an *index* for the sequence, and the number a_n associated with the index n is called the nth *term* of the sequence.

Examples of Sequences

Often sequences are defined by presenting an explicit formula. Thus, for example, $\{1/n\}$ denotes the sequence that has, for each index n, an nth term equal to $1/n$. The sequence $\{1 + (-1)^n\}$ has, for each index n, an nth term equal to $1 + (-1)^n$, so the nth term of this sequence equals 0 if the index n is odd and equals 2 if the index n is even.

Frequently sequences are defined in a less explicit manner, as in the following example.

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Example 2.1 For each index n, define a_n to be the largest natural number that is less than or equal to $\sqrt{n^3}$. In Section 1.2 we proved that any nonempty set of integers that is bounded above has a largest member. Thus, for each natural number n there is a largest natural number that is less than or equal to $\sqrt{n^3}$. Therefore, the sequence $\{a_n\}$ is properly defined. We leave it as an exercise for the reader to find the first four terms of this sequence. ■

We now give an example of a sequence $\{a_n\}$ that is defined *recursively;* that is, the sequence is defined by defining the first term a_1 and then defining a_{n+1} whenever n is a natural number such that the nth term a_n is defined. By the Principle of Mathematical Induction, the nth term a_n is defined for every index n, and thus the sequence $\{a_n\}$ is properly defined.

Example 2.2 Define $a_1 = 1$. If n is an index such that a_n has been defined, then define

$$a_{n+1} = \begin{cases} a_n + 1/n & \text{if } a_n^2 \leq 2 \\ a_n - 1/n & \text{if } a_n^2 > 2. \end{cases}$$

This formula defines the sequence recursively. We leave it as an exercise for the reader to find the first four terms of this sequence. ■

Example 2.3 Consider the sequence whose initial terms are displayed as follows:[1]

$$\{a_n\} \equiv \{1,\ 1/2,\ 2/2,\ 1/3,\ 2/3,\ 3/3,\ 1/4,\ 2/4,\ 3/4,\ 4/4,\ 1/5,\ \ldots\}. \qquad (2.1)$$

The sequence whose first 11 terms are displayed above can be more formally defined as follows. Since for any natural number j,

$$1 + 2 + \cdots + j = \frac{j(j+1)}{2},$$

for an index n written as $n = j(j+1)/2 + k$, where $1 \leq k \leq j+1$,

$$a_n = \frac{k}{j+1}.$$

We leave it as an exercise for the reader to determine a_{20} and a_{30} and also show that there are infinitely many indices n for which $a_n = 1/2$. ■

[1] A sequence cannot be formally determined from explicit knowledge of the initial terms of the sequence, but, as in this example, the construction of the sequence can sometimes be clearly indicated by such a display.

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Example 2.4 Let r be any number. Define the sequence $\{s_n\}$ by

$$s_n = \sum_{k=1}^{n} r^k \qquad \text{for every index } n.$$

If $r = 1$, then $s_n = n$, while if $r \neq 1$, using the Geometric Sum Formula, we can express each term s_n as

$$s_n = \sum_{k=1}^{n} r^k = \frac{r - r^{n+1}}{1 - r}. \qquad \blacksquare$$

Example 2.5 Define the sequence $\{s_n\}$ by

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{for every index } n.$$

It is not clear that each term s_n can be expressed as a simple explicit function of the index n as was done in the preceding example. $\blacksquare$

The two preceding sequences are formed in the following manner: Given a sequence $\{c_n\}$, define a new sequence $\{s_n\}$ by the formula

$$s_n = \sum_{k=1}^{n} c_k \qquad \text{for every index } n.$$

Sequences formed in this manner are called *infinite series*. The nth term s_n is called the nth *partial sum* of the series. We devote a considerable part of Chapters 8 and 9 to the study of infinite series.

The Definition of Convergence

We will be interested in sequences $\{a_n\}$ that have the following property: "As n gets large, the a_n's approach a fixed number a" or, what is supposed to capture the same concept, "As n gets large, the difference between a_n and a becomes arbitrarily small." We place these statements in quotation marks because they are imprecise and vague to the point that they have limited usefulness beyond the intuitively suggestive. After all, what do "approach" and "arbitrarily small" mean in a mathematical sense? Despite the imprecision, the concept is of fundamental importance and the following is a way to make the concept precise.[2]

[2] It takes some reflection to understand why the following definition does capture the concept that "As n gets large, the difference between a_n and a becomes arbitrarily small." This should not be discouraging since, in fact, it took almost two centuries of grappling with the concept before this precise definition of convergence of a sequence was formulated.

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Definition A sequence $\{a_n\}$ is said to *converge* to the number a provided that for every positive number ϵ there is an index N such that

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N.$$

Observe[3] that given a number ℓ and a positive number r, for any number x,

$$|x - \ell| < r \qquad \text{if and only if } \ell - r < x < \ell + r.$$

Thus a sequence $\{a_n\}$ is defined to converge to the number a provided that for every positive number ϵ there is an index N such that

$$a - \epsilon < a_n < a + \epsilon \qquad \text{for all indices } n \geq N,$$

which means that a_n lies in the interval $(a - \epsilon, a + \epsilon)$ for all indices $n \geq N$.

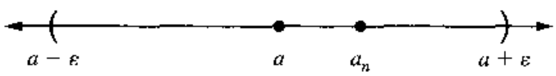

FIGURE 2.1 $|a_n - a| < \epsilon$ for any index $n \geq N$.

A given sequence may or may not converge. But if a sequence does converge, it cannot converge to more than one point. Indeed, suppose the sequence $\{a_n\}$ converges to a and to a' and $a < a'$. Choose a positive ϵ less than half the distance between a and a'. Then $a + \epsilon < a' - \epsilon$, so that the intervals $(a - \epsilon, a + \epsilon)$ and $(a' - \epsilon, a' + \epsilon)$ are disjoint.

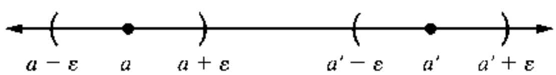

FIGURE 2.2 A sequence cannot converge to a and a' if $a \neq a'$.

However, since the sequence $\{a_n\}$ converges to a there is an index N_1 such that a_n belongs to the interval $(a - \epsilon, a + \epsilon)$ for all indices $n \geq N_1$. On the other hand, since the sequence $\{a_n\}$ converges to a', there is an index N_2 such that a_n belongs to the interval $(a' - \epsilon, a' + \epsilon)$ for all indices $n \geq N_2$. Choose an index n that is greater than both N_1 and N_2. Then a_n belongs to both of the intervals $(a - \epsilon, a + \epsilon)$ and $(a' - \epsilon, a' + \epsilon)$. This contradicts the disjointness of these two intervals. Thus, a convergent sequence cannot converge to two different numbers.

If the sequence $\{a_n\}$ converges to the number a, we call a the *limit of the sequence* $\{a_n\}$ and write

$$\lim_{n \to \infty} a_n = a.$$

[3] Recall the meaning of the construction *if and only if.* Given two mathematical statements A and B, the assertion "A if B" means that B implies A, while the assertion "A only if B" means that A implies B. Thus, "A if and only if B" means that both A implies B and B implies A.

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Proposition 2.6 The sequence $\{1/n\}$ converges to 0; that is,

$$\lim_{n\to\infty} 1/n = 0.$$

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$\left|\frac{1}{n} - 0\right| < \epsilon \qquad \text{for all indices } n \geq N; \tag{2.2}$$

that is, $1/n < \epsilon$ if $n \geq N$. But by the Archimedean Property of $\mathbb{R}$, we can select an index N such that $1/N < \epsilon$, and hence

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon \qquad \text{for all indices } n \geq N.$$

Thus, the required inequality (2.2) holds for this choice of N. ■

Example 2.7 The sequence $\{(-1)^n\}$ does not converge. To see this, we argue by contradiction. Suppose that the sequence $\{(-1)^n\}$ converges to a number a. Take any positive number ϵ such that $\epsilon < 1$. Then the interval $(a - \epsilon, a + \epsilon)$ has length less than 2. Therefore, it is not possible to have both odd and even terms of the sequence $\{(-1)^n\}$ in this interval. Thus, the sequence $\{(-1)^n\}$ cannot converge to a. This contradiction shows that the sequence $\{(-1)^n\}$ does not converge. ■

Example 2.8 The sequence $\{2/n^2 + 4/n + 3\}$ converges to 3; that is,

$$\lim_{n\to\infty}\left[\frac{2}{n^2} + \frac{4}{n} + 3\right] = 3.$$

In order to verify this assertion, we choose $\epsilon > 0$. Then we need to find an index N such that

$$\left|\frac{2}{n^2} + \frac{4}{n} + 3 - 3\right| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.3}$$

Observe that

$$\left|\frac{2}{n^2} + \frac{4}{n} + 3 - 3\right| = \frac{2}{n^2} + \frac{4}{n} \leq \frac{6}{n} \qquad \text{for every index } n.$$

Now, by the Archimedean Property of $\mathbb{R}$, since $\epsilon/6$ is positive, we can select an index N such that $1/N < \epsilon/6$, so that $6/N < \epsilon$ and therefore

$$\left|\frac{2}{n^2} + \frac{4}{n} + 3 - 3\right| \leq \frac{6}{n} \leq \frac{6}{N} < \epsilon \qquad \text{for all indices } n \geq N.$$

Thus, the required inequality (2.3) holds for this choice of N. ■

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A Comparison Criterion for Convergence

We now establish a very useful criterion for showing that a sequence converges. This criterion is so frequently used in the remainder of the book that we name it the Comparison Lemma.

Lemma 2.9 The Comparison Lemma Let the sequence $\{a_n\}$ converge to the number a. Then the sequence $\{b_n\}$ converges to the number b if there is a nonnegative number C and an index N_1 such that

$$|b_n - b| \leq C|a_n - a| \qquad \text{for all indices } n \geq N_1. \tag{2.4}$$

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$|b_n - b| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.5}$$

However, we have the estimate (2.4). If $C = 0$, then the estimate (2.4) implies that the inequality (2.5) holds with $N = N_1$. So suppose that $C > 0$. Since the sequence $\{a_n\}$ converges to a and the number ϵ/C is positive, we can choose an index N_2 such that

$$|a_n - a| < \epsilon/C \qquad \text{for all indices } n \geq N_2. \tag{2.6}$$

Define $N \equiv \max\{N_1, N_2\}$. Then, from the estimates (2.4) and (2.6), we have

$$|b_n - b| \leq C|a_n - a| < C \cdot \epsilon/C = \epsilon \qquad \text{for all indices } n \geq N.$$

Thus, the required inequality (2.5) holds for this choice of N. ■

The Sum, Product, and Quotient of Convergent Sequences

We now turn to proving some general results about the behavior of convergent sequences with respect to addition, multiplication, and division. We prove that the sum of convergent sequences converges to the sum of the limits, the product of convergent sequences converges to the product of the limits, and, when all quotients are defined, the quotient of convergent sequences converges to the quotient of the limits.

Theorem 2.10 The Sum Property Suppose that the sequence $\{a_n\}$ converges to the number a and that the sequence $\{b_n\}$ converges to the number b. Then the sequence of sums $\{a_n + b_n\}$ converges to the sum $a + b$; that is,

$$\lim_{n\to\infty} [a_n + b_n] = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$|(a_n + b_n) - (a + b)| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.7}$$

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In order to do so, we first observe that for every index n,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|,$$

and hence, by the Triangle Inequality,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b|. \tag{2.8}$$

Since the sequence $\{a_n\}$ converges to a and $\epsilon/2$ is positive, we can choose an index N_1 such that

$$|a_n - a| < \frac{\epsilon}{2} \qquad \text{for all indices } n \geq N_1,$$

and since the sequence $\{b_n\}$ converges to b and $\epsilon/2$ is positive, we can choose an index N_2 such that

$$|b_n - b| < \frac{\epsilon}{2} \qquad \text{for all indices } n \geq N_2.$$

Define $N \equiv \max\{N_1, N_2\}$. From inequality (2.8) and the choice of N_1 and N_2, it follows that if $n \geq N$, then

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, the required inequality (2.7) holds for this choice of N. ∎

To simplify the proof that the product of convergent sequences converges to the product of the limits, it is convenient to first prove this for the special case where one of the sequences is constant and then for the special case where both sequences converge to 0.

Lemma 2.11 Suppose that the sequence $\{a_n\}$ converges to the number a. Then, for any number α, the sequence $\{\alpha a_n\}$ converges to αa.

Proof

The proof follows immediately from the Comparison Lemma. Indeed, observe that for every index n,

$$|\alpha a_n - \alpha a| = |\alpha||a_n - a| \leq |\alpha||a_n - a|.$$

Thus, the required comparison inequality (2.4) holds with $N_1 = 1$ and $C = |\alpha|$. The convergence of $\{\alpha a_n\}$ to αa now follows from the Comparison Lemma. ∎

Lemma 2.12 Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ each converge to 0. Then, the product $\{a_n b_n\}$ also converges to 0.

Proof

Let $\epsilon > 0$. We need to find an index N such that

$$|a_n b_n| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.9}$$

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Since $\sqrt{\epsilon} > 0$ and $\lim_{n\to\infty} a_n = 0$, there is an index N_1 such that

$$|a_n| < \sqrt{\epsilon} \qquad \text{for all indices } n \geq N_1.$$

Similarly, there is an index N_2 such that

$$|b_n| < \sqrt{\epsilon} \qquad \text{for all indices } n \geq N_2.$$

Define $N \equiv \max\{N_1,\ N_2\}$. Then, if $n \geq N$,

$$|a_n b_n| = |a_n||b_n| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

Thus, the required inequality (2.9) holds for this choice of index N. ■

Let us explicitly note the following property of convergent sequences that is quite useful; we leave the proof as an exercise. For a sequence $\{c_n\}$ and a number c,

$$\lim_{n\to\infty} c_n = c \qquad \text{if and only if} \quad \lim_{n\to\infty} [c_n - c] = 0.$$

Theorem 2.13 The Product Property Suppose that the sequence $\{a_n\}$ converges to the number a and that the sequence $\{b_n\}$ converges to the number b. Then the sequence of products $\{a_n \cdot b_n\}$ converges to the product ab; that is,

$$\lim_{n\to\infty} [a_n b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

Proof

We need to show that

$$\lim_{n\to\infty} [a_n b_n - ab] = 0. \tag{2.10}$$

To do so, we define, for each index n,

$$\alpha_n \equiv a_n - a \qquad \text{and} \qquad \beta_n \equiv b_n - b.$$

Then

$$\lim_{n\to\infty} \alpha_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n = 0. \tag{2.11}$$

Moreover, for each index n, since $a_n = a + \alpha_n$ and $b_n = b + \beta_n$,

$$\begin{aligned} a_n b_n - ab &= (a + \alpha_n)(b + \beta_n) - ab \\ &= a\beta_n + b\alpha_n + \alpha_n\beta_n. \end{aligned}$$

From the limits (2.11), the sum property of convergent sequences and the preceding two lemmas we conclude that

$$\begin{aligned} \lim_{n\to\infty} [a_n b_n - ab] &= \lim_{n\to\infty} [a\beta_n + b\alpha_n + \alpha_n\beta_n] \\ &= \lim_{n\to\infty} [a\beta_n] + \lim_{n\to\infty} [b\alpha_n] + \lim_{n\to\infty} [\alpha_n\beta_n] \\ &= a \lim_{n\to\infty} \beta_n + b \lim_{n\to\infty} \alpha_n + \lim_{n\to\infty} [\alpha_n\beta_n] \\ &= a \cdot 0 + b \cdot 0 + 0 = 0. \end{aligned}$$

Thus, the required limit (2.10) holds. ■

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Proposition 2.14 Suppose that the sequence $\{b_n\}$ of nonzero numbers converges to the nonzero number b. Then the sequence $\{1/b_n\}$ converges to $1/b$.

Proof

Our strategy is to use the Comparison Lemma. Thus, we will find a nonnegative number C and an index N_1 such that

$$\left|\frac{1}{b_n}-\frac{1}{b}\right| \leq C|b_n-b| \qquad \text{for all indices } n \geq N_1. \tag{2.12}$$

But observe that for every index n,

$$\left|\frac{1}{b_n}-\frac{1}{b}\right| = \left|\frac{b-b_n}{bb_n}\right| = \frac{1}{|b_n||b|}|b_n-b|.$$

Suppose there is an index N_1 such that

$$|b_n| > \frac{|b|}{2} \qquad \text{for all indices } n \geq N_1. \tag{2.13}$$

Then

$$\left|\frac{1}{b_n}-\frac{1}{b}\right| = \frac{1}{|b_n||b|}|b_n-b| \leq \frac{2}{|b|^2}|b_n-b| \qquad \text{for all indices } n \geq N_1. \tag{2.14}$$

Thus, the required comparison inequality (2.12) holds with N_1 as chosen above and $C = 2/|b|^2$. The convergence of $\{1/b_n\}$ to $1/b$ follows from the Comparison Lemma.

To complete the proof we need to show that there is an index N_1 such that the inequality (2.13) holds. Indeed, since the number $|b|/2$ is positive, we can take $\epsilon = |b|/2$ and use the definition of the convergence of a sequence to choose an index N_1 so that

$$b_n \text{ belongs to the interval } (b-\epsilon, b+\epsilon) \text{ for all indices } n \geq N.$$

Observe that if $b > 0$, $\epsilon = b/2$, so that $(b-\epsilon, b+\epsilon) = (b/2, 3b/2)$ and hence $b_n > b/2$ for all indices $n \geq N_1$. Since $b > 0$, the inequality (2.13) holds for this choice of N_1. On the other hand, if $b < 0$, then $\epsilon = -b/2$, so that $(b-\epsilon, b+\epsilon) = (3b/2, b/2)$ and hence $b_n < b/2$ for all indices $n \geq N_1$. Since $b < 0$, the inequality (2.13) holds for this choice of N_1. ■

Theorem 2.15 The Quotient Property Suppose that the sequence $\{a_n\}$ converges to the number a and that the sequence $\{b_n\}$ converges to the number b. Also suppose that $b_n \neq 0$ for all indices n and that $b \neq 0$. Then the sequence of quotients $\{a_n/b_n\}$ converges to the quotient a/b; that is,

$$\lim_{n\to\infty}\left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

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Proof
By the preceding proposition and the product property of convergent sequences, we have

$$\lim_{n\to\infty}\left[\frac{a_n}{b_n}\right] = \lim_{n\to\infty}\left[a_n\cdot\frac{1}{b_n}\right] = a\cdot\frac{1}{b} = \frac{a}{b}. \qquad \blacksquare$$

We have now established the sum, product, and difference properties of convergent sequences $\{a_n\}$ and $\{b_n\}$, and it is useful to label two other properties that follow from these basic three properties.

The Linearity and Polynomial Properties

Directly from the sum property of convergent sequences and Lemma 2.11 we have the following useful property.

Proposition 2.16 The Linearity Property For any two numbers α and β,

$$\lim_{n\to\infty}[\alpha a_n + \beta b_n] = \alpha\lim_{n\to\infty} a_n + \beta\lim_{n\to\infty} b_n.$$

There is a further property of convergent sequences that deserves noting. For a nonnegative integer k and numbers $c_0, c_1, \ldots, c_k$, the function $p:\mathbb{R}\to\mathbb{R}$ defined by

$$p(x) = \sum_{i=0}^{k} c_i x^i \qquad \text{for all } x \text{ in } \mathbb{R}$$

is called a *polynomial*. If $c_k \neq 0$, p is said to have *degree* k.

We leave it to the reader to provide the proof of the following proposition; the proof follows from the product and sum properties of convergent sequences and an induction argument based on the degree of the polynomial.

Proposition 2.17 The Polynomial Property If the sequence $\{a_n\}$ converges to the number a, then for any polynomial $p:\mathbb{R}\to\mathbb{R}$,

$$\lim_{n\to\infty} p(a_n) = p(a).$$

EXERCISES FOR SECTION 2.1

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. If the sequence $\{a_n^2\}$ converges, then the sequence $\{a_n\}$ also converges.
   b. If the sequence $\{a_n + b_n\}$ converges, then the sequences $\{a_n\}$ and $\{b_n\}$ also converge.

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c. If the sequences $\{a_n + b_n\}$ and $\{a_n\}$ converge, then the sequence $\{b_n\}$ also converges.

d. If the sequence $\{|a_n|\}$ converges, then the sequence $\{a_n\}$ also converges.

2. Using only the Archimedean Property of $\mathbb{R}$, give a direct ϵ-N verification of the following limits:

a. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ **b.** $\lim_{n\to\infty} \frac{1}{n+5} = 0$

3. Using only the Archimedean Property of $\mathbb{R}$, give a direct ϵ-N verification of the convergence of the following sequences:

a. $\left\{\frac{2}{\sqrt{n}} + \frac{1}{n} + 3\right\}$ **b.** $\left\{\frac{n^2}{n^2+n}\right\}$

4. For the sequence $\{a_n\}$ defined in Example 2.3:

a. What are the terms a_{10}, a_{20}, a_{30}?

b. Find the second index n for which $a_n = 1/4$ and the fourth index n for which $a_n = 1$.

c. For j an odd natural number, set

$$n = \frac{j(j+1)}{2} + \frac{j+1}{2}$$

and show that $a_n = 1/2$.

d. Show that $\{a_n\}$ does not converge.

5. For the sequence $\{a_n\}$ defined in Example 2.3 and any rational number x in the interval $(0, 1]$, show that there are infinitely many indices n such that $a_n = x$.

6. Suppose that the sequence $\{a_n\}$ converges to a and that $a > 0$. Show that there is an index N such that $a_n > 0$ for all indices $n \geq N$.

7. Suppose that the sequence $\{a_n\}$ converges to ℓ and that the sequence $\{b_n\}$ has the property that there is an index N such that

$$a_n = b_n \qquad \text{for all indices } n \geq N.$$

Show that $\{b_n\}$ also converges to ℓ. (*Suggestion:* Use the Comparison Lemma for a quick proof.)

8. Prove that the sequence $\{c_n\}$ converges to c if and only if the sequence $\{c_n - c\}$ converges to 0.

9. Prove that the Archimedean Property of $\mathbb{R}$ is equivalent to the fact that $\lim_{n\to\infty} 1/n = 0$.

10. Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Hint: Define $\alpha_n = n^{1/n} - 1$ and use the Binomial Formula to show that for each index n,

$$n = (1 + \alpha_n)^n \geq 1 + [n(n-1)/2]\alpha_n^2.$$

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11. We have proven that the sequence $\{1/n\}$ converges to 0 and that it does not converge to any other number. Use this to prove that none of the following assertions is equivalent to the definition of convergence of a sequence $\{a_n\}$ to the number a.

a. For some $\epsilon > 0$ there is an index N such that

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N.$$

b. For each $\epsilon > 0$ and each index N,

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N.$$

c. There is an index N such that for every number $\epsilon > 0$,

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N.$$

12. For the sequence defined in Example 2.2, show that for every index n, $|a_n - \sqrt{2}| < 2/n$. Use this property to show that the sequence converges to $\sqrt{2}$.

13. Prove the Polynomial Property for convergent sequences by using an inductive argument based on the degree of the polynomial.

14. Define the sequence $\{s_n\}$ by

$$s_n = \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{(n+1)(n)} \qquad \text{for every index } n.$$

Prove that

$$\lim_{n \to \infty} s_n = 1.$$

15. Let $\{a_n\}$ be a sequence of real numbers. Suppose that for each positive number c there is an index N such that

$$a_n > c \qquad \text{for all indices } n \geq N.$$

When this is so, the sequence $\{a_n\}$ is said to *converge to infinity*, and we write

$$\lim_{n \to \infty} a_n = \infty.$$

Prove the following:

a. $\lim_{n \to \infty} [n^3 - 4n^2 - 100n] = \infty$ **b.** $\lim_{n \to \infty} \left[\sqrt{n} - \frac{1}{n^2} + 4\right] = \infty$

16. Discuss the convergence to infinity of each of the following sequences:

a. $\{\sqrt{n+1} - \sqrt{n}\}$ **b.** $\{(\sqrt{n+1} - \sqrt{n})\sqrt{n}\}$ **c.** $\{(\sqrt{n+1} - \sqrt{n})n\}$

17. For a sequence $\{a_n\}$ of positive numbers show that

$$\lim_{n \to \infty} a_n = \infty \qquad \text{if and only if } \lim_{n \to \infty} \left[\frac{1}{a_n}\right] = 0.$$

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18. (The Convergence of Cesaro Averages.) Suppose that the sequence $\{a_n\}$ converges to a. Define the sequence $\{\sigma_n\}$ by

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{for every index } n.$$

Prove that the sequence $\{\sigma_n\}$ also converges to a.

2.2 SEQUENCES AND SETS

Boundedness of Convergent Sequences

A set S of numbers has been defined to be *bounded* provided that it is bounded above and below. This is equivalent to the assertion that there is a nonnegative number M such that

$$|x| \leq M \qquad \text{for all points } x \text{ in } S.$$

Definition A sequence $\{a_n\}$ is said to be *bounded* provided that there is a number M such that

$$|a_n| \leq M \qquad \text{for every index } n.$$

Theorem 2.18 Every convergent sequence is bounded.

Proof
Let $\{a_n\}$ be a sequence that converges to the number a. Taking $\epsilon = 1$, it follows from the definition of convergence that we can select an index N such that

$$|a_n - a| < 1 \qquad \text{for all indices } n \geq N.$$

Observe that we have $a_n = (a_n - a) + a$, so that by the Triangle Inequality,

$$|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a|.$$

Thus, by the choice of the index N

$$|a_n| \leq 1 + |a| \qquad \text{for all indices } n \geq N.$$

Define $M \equiv \max\{1 + |a|, |a_1|, \ldots, |a_{N-1}|\}$. Then

$$|a_n| \leq M \qquad \text{for every index } n.$$

Thus, the sequence $\{a_n\}$ is bounded. ∎

Sequential Denseness of the Rationals

A subset S of numbers has been defined to be *dense* in $\mathbb{R}$ provided that every open interval (a, b) contains a point in S. There is a useful characterization of denseness in terms of convergent sequences.

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Definition For a set of numbers S, we say that a sequence $\{x_n\}$ is *in the set S* provided that for each index n, the term x_n belongs to S.

Proposition 2.19 A set S is dense in $\mathbb{R}$ if and only if every number x is the limit of a sequence in S.

Proof

First, assume that the set S is dense in $\mathbb{R}$. Fix a number x. Let n be an index. By the denseness of S in $\mathbb{R}$, there is a member of S in the interval $(x, x + 1/n)$. Choose a member of S that belongs to this interval and label it s_n. This defines a sequence $\{s_n\}$ that has the property that

$$|s_n - x| < 1/n \qquad \text{for every index } n.$$

Since the sequence $\{1/n\}$ converges to 0, it follows from the Comparison Lemma that $\{s_n\}$ converges to x, and, by the above choice, $\{s_n\}$ is a sequence in S.

It remains to prove the converse. Suppose that the set S has the property that every number is the limit of a sequence in S. We will show that S is dense in $\mathbb{R}$. Indeed, consider an interval (a, b). We must show that this interval contains a point of S. Consider the midpoint $x = (a + b)/2$ of the interval. By assumption, there is a sequence $\{s_n\}$ of points in S that converges to s. Define $\epsilon \equiv (b - a)/2$. Then $\epsilon > 0$. By the definition of a convergent sequence, there is an index N such that s_n belongs to $(s - \epsilon, s + \epsilon)$ for each index $n \geq N$. However,

$$(s - \epsilon, s + \epsilon) = (a, b).$$

The point s_N belongs to S and also belongs to (a, b). Thus, S is dense in $\mathbb{R}$. ■

Theorem 2.20 Sequential Density of the Rationals Every number is the limit of a sequence of rational numbers.

Proof

Theorem 1.9 asserts that the set of rational numbers is dense in $\mathbb{R}$. By the preceding proposition, every number is the limit of a sequence of rational numbers. ■

Closed Sets

Lemma 2.21 Suppose that the sequence $\{d_n\}$ converges to the number d and that $d_n \geq 0$ for every natural number n. Then $d \geq 0$.

Proof

We will suppose that the conclusion is false and derive a contradiction. Indeed, suppose that $d < 0$. Let $\epsilon \equiv -d/2$. We see that $\epsilon > 0$ and that $d + \epsilon = d/2 < 0$. Thus, the interval $(d - \epsilon, d + \epsilon)$ consists entirely of negative numbers, so no term of the sequence $\{d_n\}$ belongs to this interval. Therefore, the sequence $\{d_n\}$ cannot converge to d. This is a contradiction. Thus, $d \geq 0$. ■

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Theorem 2.22 Let $\{c_n\}$ be a sequence in the interval $[a, b]$. If $\{c_n\}$ converges to the number c, then c also belongs to the interval $[a, b]$.

Proof
By the linearity property of convergent sequences,

$$\lim_{n\to\infty} [b - c_n] = b - c.$$

However, for each index n, c_n belongs to the interval $[a, b]$, so $b - c_n \geq 0$. The preceding lemma implies that $b - c \geq 0$. A similar argument shows that $c - a \geq 0$. Thus, c belongs to the interval $[a, b]$. ∎

The property of an interval $[a, b]$ described in the above theorem is so important that it has a name.

Definition A subset S of $\mathbb{R}$ is said to be *closed* provided that if $\{a_n\}$ is a sequence in S that converges to a number a, then the limit a also belongs to S.

Using this new name, Lemma 2.21 can be restated by saying that the set of nonnegative numbers is closed. Observe that the set of positive numbers fails to be closed since $\{1/n\}$ is a sequence of positive numbers that converges to a limit that is not positive. Theorem 2.22 can be restated by saying that the interval $[a, b]$ is closed.

Example 2.23 The set $\mathbb{Q}$ of rational numbers is not closed since, by the sequential density of the rationals, Theorem 2.20, there is a sequence $\{r_n\}$ of rational numbers that converges to the number $\sqrt{2}$, and $\sqrt{2}$ is not rational. ∎

Example 2.24 The interval $(0, 1]$ is not closed since $\{1/n\}$ is a sequence in this interval that converges to a point that does not belong to the interval. ∎

EXERCISES FOR SECTION 2.2

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. Every bounded sequence converges.
   b. A convergent sequence of positive numbers has a positive limit.
   c. The sequence $\{n^2 + 1\}$ converges.
   d. A convergent sequence of rational numbers has a rational limit.
   e. The limit of a convergent sequence in the interval (a, b) also belongs to (a, b).

2. Show that the set $(-\infty, 0]$ is closed.

3. Show that every number is the limit of a sequence of irrational numbers.

4. Show that the set of irrational numbers fails to be closed.

5. Show that a sequence $\{a_n\}$ is bounded if and only if there is an interval $[c, d]$ such that $\{a_n\}$ is a sequence in $[c, d]$.

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2.3 THE MONOTONE CONVERGENCE THEOREM

In Section 2.1 we showed that $\lim_{n\to\infty} 1/n = 0$. It is clear that constant sequences converge to their constant values. Thus, using the sum, product, and quotient properties of convergent sequences, we can combine these two examples to obtain further examples of convergent sequences. It is important to analyze much more general sequences. We now turn to the task of providing criteria that are sufficient to determine that a sequence converges but that do not require any explicit knowledge of the limit.

Definition A sequence $\{a_n\}$ is said to be *monotonically increasing* provided that

$$a_{n+1} \geq a_n \qquad \text{for every index } n.$$

A sequence $\{a_n\}$ is said to be *monotonically decreasing* provided that

$$a_{n+1} \leq a_n \qquad \text{for every index } n.$$

A sequence $\{a_n\}$ is called *monotone* if it is either monotonically increasing or monotonically decreasing.

In the preceding section, we proved that if a sequence converges, it must be bounded. Of course, as the sequence $\{(-1)^n\}$ shows, a bounded sequence need not converge. However, in the case of a monotone sequence, there is the following important theorem.

Theorem 2.25 The Monotone Convergence Theorem A monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone sequence $\{a_n\}$ converges to

i. $\sup\{a_n \mid n \text{ in } N\}$ if it is monotonically increasing, and to
ii. $\inf\{a_n \mid n \text{ in } N\}$ if it is monotonically decreasing.

Proof
We have already proven that a convergent sequence is bounded, so it remains to be shown that if the monotone sequence $\{a_n\}$ is bounded, then it converges to limits determined by (i) and (ii). We first suppose that the sequence $\{a_n\}$ is monotonically increasing. Then if we define $S = \{a_n \mid n \in \mathbb{N}\}$, by assumption, the set S is bounded above. According to the Completeness Axiom, S has a least upper bound. Define $\ell \equiv \sup S$. We claim that the sequence $\{a_n\}$ converges to ℓ. Indeed, let $\epsilon > 0$. We need to find an index N such that

$$|a_n - \ell| < \epsilon \qquad \text{for all indices } n \geq N;$$

that is,

$$\ell - \epsilon < a_n < \ell + \epsilon \qquad \text{for all indices } n \geq N. \tag{2.15}$$

Since the number ℓ is an upper bound for the set S, we have

$$a_n \leq \ell < \ell + \epsilon \qquad \text{for every index } n. \tag{2.16}$$

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On the other hand, since ℓ is the least upper bound for S, the number $\ell - \epsilon$ is not an upper bound for S, so there is an index N such that $\ell - \epsilon < a_N$. However, the sequence $\{a_n\}$ is monotonically increasing, so

$$\ell - \epsilon < a_N \le a_n \qquad \text{for all indices } n \ge N. \tag{2.17}$$

From the inequalities (2.16) and (2.17) follows the required inequality (2.15). Thus, the sequence $\{a_n\}$ converges to ℓ.

We leave it to the reader to construct a similar proof when the sequence is monotonically decreasing. ■

Example 2.26 Define

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k} \qquad \text{for every index } n.$$

Then it is clear that the sequence $\{s_n\}$ is monotonically increasing. According to the Monotone Convergence Theorem, $\{s_n\}$ converges if and only if it is bounded. We will show that it is bounded. Indeed, using the Geometric Sum Formula, we see that for every index n,

$$\begin{aligned} s_n &\le \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n \\ &= \frac{1/2 - (1/2)^{n+1}}{1 - 1/2} \le \frac{1/2 - 0}{1 - 1/2} = 1. \end{aligned}$$

Hence the monotonically increasing sequence $\{s_n\}$ is bounded above by 1, and so it converges to a limit ℓ, where $\ell \le 1$. Observe that we have proven that $\{s_n\}$ converges without explicitly identifying the limit. ■

Example 2.27 Define

$$s_n = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{for every index } n.$$

The series $\{s_n\}$ is called the *Harmonic Series*. Again, we see that the sequence $\{s_n\}$ is monotonically increasing. We claim that it is not bounded and hence not convergent. Indeed, to see this, observe that

$$s_2 = 1 + \frac{1}{2} \ge 1 + \frac{1}{2}$$

and that

$$s_4 = s_2 + \frac{1}{3} + \frac{1}{4} \ge s_2 + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

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and

$$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq s_4 + \frac{1}{2} = 1 + \frac{3}{2}.$$

In general, we claim that

$$s_{2^n} \geq 1 + \frac{n}{2} \qquad \text{for every index } n. \tag{2.18}$$

Indeed, observe that for each index $k \geq 1$, there are 2^{k-1} indices i such that

$$2^{k-1} < i \leq 2^k,$$

and for each such index, $1/i \geq 1/2^k$. Therefore,

$$\sum_{2^{k-1} < i \leq 2^k} \frac{1}{i} \geq \frac{1}{2}.$$

Thus,

$$s_{2^n} = 1 + \sum_{2 \leq i \leq 2^n} \frac{1}{i} = 1 + \sum_{k=1}^{n} \left[\sum_{2^{k-1} < i \leq 2^k} \frac{1}{i} \right] \geq 1 + \frac{n}{2}.$$

From this, using the Archimedean Property of $\mathbb{R}$, it follows that the sequence $\{s_n\}$ is not bounded. Thus, by Theorem 2.18, this sequence does not converge. ■

Let us now consider a special but important class of sequences, sequences of the form $\{c^n\}$, where c is a fixed number. This sequence is constant if $c = 1$, so it converges to 1. We have already noted that it does not converge if $c = -1$. We leave it as an exercise to show that if $|c| > 1$, then the sequence is not bounded and therefore, by Theorem 2.18, it fails to converge. For $|c| < 1$ we have the following important result.

Proposition 2.28 Let c be a number such that $|c| < 1$. Then

$$\lim_{n \to \infty} c^n = 0.$$

Proof

The case $c = 0$ is clear since we are then considering the constant sequence all of whose terms are 0. Moreover, since $|c^n| = |(-c)^n|$, the case $c < 0$ follows from the case $c > 0$. So assume $c > 0$. Since $0 < c < 1$, $\{c^n\}$ is a monotonically decreasing sequence bounded below by 0. According to the Monotone Convergence Theorem, the sequence $\{c^n\}$ converges to a number ℓ where

$$\ell = \inf \{c^n \mid n \text{ in } \mathbb{N}\}.$$

We must have $\ell = 0$, since otherwise, because $c > 0$, we have

$$c^n = \frac{c^{n+1}}{c} \geq \frac{\ell}{c} \qquad \text{for every index } n$$

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so that ℓ/c is a lower bound for the sequence. Thus, ℓ/c is a lower bound for the sequence, and it is larger than ℓ since $0 < c < 1$. Hence ℓ is not the greatest lower bound for the sequence. This contradiction shows that $\ell = 0$. ∎

We conclude this section with a geometric consequence of the Monotone Convergence Theorem that can be loosely interpreted as a statement that there are no "holes" in the set of real numbers $\mathbb{R}$.

Theorem 2.29 The Nested Interval Theorem For each natural number n, let a_n and b_n be numbers such that $a_n < b_n$ and consider the interval $I_n \equiv [a_n, b_n]$. Assume that

$$I_{n-1} \subseteq I_n \qquad \text{for every index } n. \tag{2.19}$$

Also assume that

$$\lim_{n\to\infty} [b_n - a_n] = 0. \tag{2.20}$$

Then there is exactly one point x that belongs to the interval I_n for all n, and both of the sequences $\{a_n\}$ and $\{b_n\}$ converge to this point.

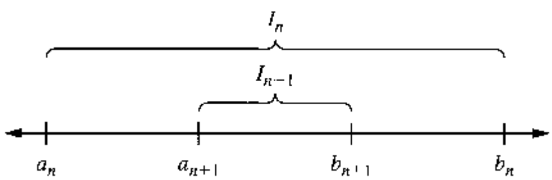

FIGURE 2.3 The sequences of endpoints $\{a_n\}$ and $\{b_n\}$ each converge to c.

Proof
Assumption (2.19) means precisely that for every index n,

$$a_n \le a_{n+1} < b_{n+1} \le b_n.$$

In particular, the sequence $\{a_n\}$ is a monotonically increasing sequence that is bounded above by b_1. The Monotone Convergence Theorem implies that the sequence $\{a_n\}$ converges to a number a and that $a_n \le a$ for every index n. A similar argument shows that the monotonically decreasing sequence $\{b_n\}$ converges to a number that we denote by b such that $b \le b_n$ for every index n. Thus,

$$a_n \le a \text{ and } b \le b_n \text{ for every index } n. \tag{2.21}$$

From assumption (2.20) and the difference property of convergent sequences, we conclude that

$$0 = \lim_{n\to\infty} [b_n - a_n] = b - a.$$

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Thus, $a = b$. Setting $x = a = b$, it follows from (2.21) that the point x belongs to I_n for every natural number n. There can be only one such point since the existence of two such points would contradict the assumption (2.20) that the lengths of the intervals converge to 0. ∎

EXERCISES FOR SECTION 2.3

1. For each of the following statements, determine whether it is true or false and justify your answer.
a. The sum of monotone sequences is monotone.
b. The product of monotone sequences is monotone.
c. Every bounded sequence converges.
d. Every monotone sequence converges.

2. Which of the following sequences is monotone? Justify your conclusions.

a. $\left\{n + \dfrac{(-1)^n}{n}\right\}$ **b.** $\left\{\dfrac{1}{n^2} + \dfrac{(-1)^n}{3^n}\right\}$

3. Suppose that the sequence $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges. Show that this result does not hold without the monotonicity assumption.

4. Suppose that the sequence $\{a_n\}$ converges to a and that $|a| < 1$. Prove that the sequence $\{(a_n)^n\}$ converges to 0.

5. Let c be a number such that $|c| < 1$. Show that $|c|$ can be expressed as $|c| = 1/(1+d)$, where $d > 0$. Then use the Binomial Formula to show that

$$|c^n| \leq \frac{1}{1+nd} \leq \frac{1}{dn} \qquad \text{for every index } n.$$

6. **a.** Use Exercise 5 and the Comparison Lemma (Lemma 2.9) to obtain another proof that if $|c| < 1$, then $\lim_{n\to\infty} c^n = 0$.
b. Use Exercise 5 and the Comparison Lemma to prove that $\lim_{n\to\infty} \sqrt{n}c^n = 0$. Is the sequence $\{\sqrt{n}c^n\}$ necessarily monotone?

7. Prove that if $0 < c < 1$, then

$$\lim_{n\to\infty} nc^n = 0.$$

[*Hint:* Define $a \equiv \sqrt{c}$, observe that $nc^n = (\sqrt{n}a^n)(\sqrt{n}a^n)$, and use part (b) of Exercise 6.]

8. Let $\{b_n\}$ be a bounded sequence of nonnegative numbers and r be any number such that $0 \leq r < 1$. Define

$$s_n = b_1 r + b_2 r^2 + \cdots + b_n r^n \qquad \text{for every index } n.$$

Use the Monotone Convergence Theorem to prove that the series $\{s_n\}$ converges.

9. For each natural number n, let a_n and b_n be numbers such that $a_n < b_n$ and consider the interval $I_n = [a_n, b_n]$. Assume that

$$I_{n+1} \subseteq I_n \qquad \text{for every index } n.$$

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Use the Monotone Convergence Theorem to prove that $\{a_n\}$ converges to a and $\{b_n\}$ converges to b, with $a \leq b$ and the interval $[a, b]$ contained in I_n for every index n.

10. For a pair of positive numbers α and β, the number $\sqrt{\alpha\beta}$ is called the *geometric mean* of α and β, and the number $(\alpha+\beta)/2$ is called the *arithmetic mean* of α and β. By observing that $(\sqrt{\alpha}-\sqrt{\beta})^2 \geq 0$, show that $(\alpha+\beta)/2 \geq \sqrt{\alpha\beta}$.

11. For a pair of positive numbers a and b, define sequences $\{a_n\}$ and $\{b_n\}$ recursively as follows: Define $a_1 = a$ and $b_1 = b$. If n is an index for which a_n and b_n have been defined, define

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \text{and} \qquad b_{n+1} = \sqrt{a_n b_n}.$$

a. Use Exercise 10 to prove that for every index $n \geq 2$,

$$a_n \geq a_{n+1} \geq b_{n+1} \geq b_n.$$

b. Show that the sequences $\{a_n\}$ and $\{b_n\}$ converge. Then show that $\{a_n\}$ and $\{b_n\}$ have the same limit. This common limit is called the *Gauss arithmetic-geometric mean* of a and b; it occurs as the value of an important elliptic integral involving a and b.

2.4 THE SEQUENTIAL COMPACTNESS THEOREM

We will devote this section to a single theorem that is one of the most important in the study of analysis. It is called the Sequential Compactness Theorem and identifies an important property of the set of real numbers that is needed to prove fundamental properties of continuous and differentiable functions. To describe the theorem we first need to introduce the concept of subsequence.

Definition Consider a sequence $\{a_n\}$. Let $\{n_k\}$ be a sequence of natural numbers that is strictly increasing; that is,

$$n_1 < n_2 < n_3 < \cdots.$$

Then the sequence $\{b_k\}$ defined by

$$b_k = a_{n_k} \qquad \text{for every index } k$$

is called a *subsequence* of the sequence $\{a_n\}$.

Often a subsequence of $\{a_n\}$ is simply denoted by $\{a_{n_k}\}$, it being implicitly understood that $\{n_k\}$ is a strictly increasing sequence of natural numbers and that the kth term of the sequence $\{a_{n_k}\}$ is a_{n_k}.[4]

The following result should not be surprising but turns out to be quite useful.

[4] As a rather imprecise intuitive idea, one can think of a subsequence $\{a_{n_k}\}$ as being obtained from a sequence $\{a_n\}$ by deleting terms from the original sequence $\{a_n\}$ and then reindexing the remaining terms so that the kth term in the deleted sequence is the term that was the n_kth term in the original sequence.

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Proposition 2.30 Let the sequence $\{a_n\}$ converge to the limit a. Then every subsequence of $\{a_n\}$ also converges to the same limit a.

Proof
Let $\epsilon > 0$. We need to find an index N such that

$$|a_{n_k} - a| < \epsilon \qquad \text{for all indices } k \geq N. \tag{2.22}$$

Since the whole sequence $\{a_n\}$ converges to a, we can choose an index N such that

$$|a_n - a| < \epsilon \qquad \text{for all indices } n \geq N. \tag{2.23}$$

But observe that since $\{n_k\}$ is a strictly increasing sequence of natural numbers,

$$n_k \geq k \qquad \text{for every index } k.$$

Thus, the required inequality (2.22) follows from inequality (2.23). ■

The sequence $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$. Hence,

$$\text{if } \lim_{n\to\infty} a_n = a, \text{ then } \lim_{n\to\infty} a_{n+1} = a \text{ as well.}$$

This simple observation can be quite useful in analyzing sequences that are defined recursively.

Example 2.31 Define $a_1 = 1$. If n is an index for which a_n has been defined, then define

$$a_{n+1} = \frac{1 + a_n}{2 + a_n}. \tag{2.24}$$

An induction argument shows that $\{a_n\}$ is a sequence of positive numbers. Moreover, directly from the definition of the sequence, it follows that for every index n,

$$a_{n+2} - a_{n+1} = \frac{a_{n+1} - a_n}{(2 + a_n)(2 + a_{n+1})}.$$

Since $a_2 < a_1$, the preceding identity and an induction argument show that $\{a_n\}$ is monotonically decreasing. According to the Monotone Convergence Theorem, the sequence $\{a_n\}$ converges. Denote the limit by a. From the fact that $\lim_{n\to\infty} a_n = a$, and also from the sum, product, and quotient properties of convergent sequences, it follows that the sequence on the right-hand side of (2.24) converges to $(1 + a)/(2 + a)$. On the other hand, since $\lim_{n\to\infty} a_{n+1} = a$, the sequence on the left-hand side of (2.24) converges to a. Thus,

$$a = \frac{1 + a}{2 + a}.$$

It follows from the quadratic formula that

$$\lim_{n\to\infty} a_n = \frac{-1 + \sqrt{5}}{2}.$$ ■

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A given sequence may or may not be monotone. But, in fact, every sequence has a monotone subsequence. This is quite surprising and not at all obvious.[5]

Theorem 2.32 Every sequence has a monotone subsequence.

Proof
Consider a sequence $\{a_n\}$. We call an index m a *peak index* for the sequence $\{a_n\}$ provided that $a_n \leq a_m$ for all indices $n \geq m$. Either there are only finitely many peak indices for the sequence $\{a_n\}$ or there are infinitely many such indices.

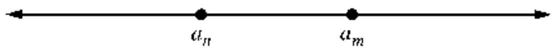

FIGURE 2.4 $a_n \leq a_m$ if $n \geq m$ and m is a peak index.

Case 1: *There are only finitely many peak indices.* Then we can choose an index N such that there are no peak indices greater than N. We will recursively define a monotonically increasing subsequence of $\{a_n\}$. Indeed, define $n_1 = N + 1$. Now suppose that k is an index such that positive integers

$$n_1 < n_2 < \cdots < n_k$$

have been chosen such that

$$a_{n_1} < a_{n_2} < \cdots < a_{n_k}.$$

Since $n_k > N$, the index n_k is not a peak index. Hence there is an index $n_{k+1} > n_k$ such that $a_{n_{k+1}} > a_{n_k}$. Thus, we recursively define a strictly increasing sequence of positive integers $\{n_k\}$ having the property that the subsequence $\{a_{n_k}\}$ is strictly increasing.

Case 2: *There are infinitely many peak indices.* For each natural number k, let n_k be the kth peak index. Directly from the definition of peak index it follows that the subsequence $\{a_{n_k}\}$ is monotonically decreasing. ■

Theorem 2.33 Every bounded sequence has a convergent subsequence.

Proof
Let $\{a_n\}$ be a bounded sequence. According to the preceding theorem, we can choose a monotone subsequence $\{a_{n_k}\}$. Since $\{a_n\}$ is bounded, so is its subsequence $\{a_{n_k}\}$. Hence $\{a_{n_k}\}$ is a bounded monotone sequence. According to the Monotone Convergence Theorem, $\{a_{n_k}\}$ converges. ■

There is a slightly more refined version of Theorem 2.33 that we find very useful.

[5] Knowing that the function $\sin x$ is periodic with irrational period, it is puzzling that the sequence $\{\sin n\}$ has a monotone subsequence.

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Definition A set of real numbers S is said to be *sequentially compact* provided that every sequence $\{a_n\}$ in S has a subsequence that converges to a point that belongs to S.

Example 2.34 Define $S \equiv [0, \infty)$. Then S is not sequentially compact. Indeed, for each index n, set $a_n = n$. Then $\{a_n\}$ is a sequence in S. However, by the Archimedean Property of $\mathbb{R}$, every subsequence of $\{a_n\}$ is unbounded and therefore, by Theorem 2.18, fails to converge. Thus, the set S is not sequentially compact. ■

Example 2.35 Define $S \equiv (0, 2]$. Then S is not sequentially compact. Indeed, $\{1/n\}$ is a sequence in S. This sequence converges to 0, and hence every subsequence also converges to 0. But 0 does not belong to S. Thus, there is no subsequence of $\{1/n\}$ that converges to a point in S. So the set S is not sequentially compact. ■

Theorem 2.36 The Sequential Compactness Theorem Let a and b be numbers such that $a < b$. Then the interval $[a, b]$ is sequentially compact; that is, every sequence in $[a, b]$ has a subsequence that converges to a point in $[a, b]$.

Proof

There are two distinct parts to the proof. First, it is necessary to show that a sequence in $[a, b]$ has a convergent subsequence. Then it must be shown that the limit of this subsequence belongs to the interval $[a, b]$. Let $\{x_n\}$ be a sequence in $[a, b]$. Then $\{x_n\}$ is bounded. Hence, by the preceding theorem, there is a subsequence $\{x_{n_k}\}$ that converges. But the sequence $\{x_{n_k}\}$ is a sequence in $[a, b]$, and hence, according to Theorem 2.22, its limit is also in $[a, b]$. ■

The above Sequential Compactness Theorem is often also referred to as the Bolzano–Weierstrass Theorem.

EXERCISES FOR SECTION 2.4

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. A subsequence of a bounded sequence is bounded.
   b. A subsequence of a monotone sequence is monotone.
   c. A subsequence of a convergent sequence is convergent.
   d. A sequence converges if it has a convergent subsequence.
2. For each of the following statements, determine whether it is true or false and justify your answer.
   a. Every sequence in the interval $(0, 1)$ has a convergent subsequence.
   b. Every sequence in the interval $(0, 1)$ has a subsequence that converges to a point in $(0, 1)$.
   c. Every sequence of rational numbers has a convergent subsequence.
   d. If a sequence of nonnegative numbers converges, its limit also is nonnegative.
   e. Every sequence of nonnegative numbers has a convergent subsequence.

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3. Consider the sequence $\{a_n\}$, where $a_n = 1/n$ for every index n. Write out the first five terms of each of the following subsequences:

a. $\{a_{3k+1}\}$ **b.** $\{a_{k \cdot 5}\}$ **c.** $\{a_{k^2}\}$

4. For each of the following sequences, find the peak indices. Justify your conclusions.

a. $\left\{\dfrac{1}{n}\right\}$ **b.** $\{(-1)^n\}$ **c.** $\{(-1)^n n\}$ **d.** $\left\{\dfrac{(-1)^n}{n}\right\}$

5. Show that a strictly increasing sequence has no peak indices.
6. Show that for a monotonically decreasing sequence every index is a peak index.
7. Show that a monotone sequence is bounded if it has a bounded subsequence.
8. Suppose that the sequence $\{a_n\}$ is monotone and that it has a convergent subsequence. Show that $\{a_n\}$ converges.
9. A sequence $\{a_n\}$ was defined to be bounded provided that there is a number M such that

$$|a_n| \leq M \qquad \text{for every index } n.$$

Show that $\{a_n\}$ is bounded if and only if there are numbers a and b with $a < b$ such that $\{a_n\}$ is a sequence in $[a, b]$.

10. Prove that a sequence $\{a_n\}$ does not converge to the number a if and only if there is some $\epsilon > 0$ and a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \qquad \text{for every index } k.$$

11. For the sequence $\{a_n\}$ defined in Example 2.3:
   a. Show that every rational number x in the interval $(0, 1]$ is the value of a constant subsequence of $\{a_n\}$.
   b. Show that every number x in the interval $[0, 1]$ is the limit of a subsequence of $\{a_n\}$.
12. For $c > 0$, consider the quadratic equation

$$x^2 - x - c = 0, \qquad x > 0.$$

Define the sequence $\{x_n\}$ recursively by fixing $x_1 > 0$ and then, if n is an index for which x_n has been defined, defining

$$x_{n+1} = \sqrt{c + x_n}.$$

Prove that the sequence $\{x_n\}$ converges monotonically to the solution of the above equation.

2.5 COVERING PROPERTIES OF SETS*

This section is a slight detour in our development of analysis. Its purpose is to present a surprisingly different viewpoint of the property of sequential compactness, a viewpoint that is useful in many different areas of mathematics. The subsequent material in the

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book is independent of this section. We begin by showing that a closed bounded set is sequentially compact.

Proposition 2.37 Let S be a subset of $\mathbb{R}$ that is closed and bounded. Then S is sequentially compact.

Proof

We simply retrace the proof of the Sequential Compactness Theorem. Let $\{x_n\}$ be a sequence in S. Then $\{x_n\}$ is bounded since S is bounded. Hence, by Theorem 2.33, there is a subsequence $\{x_{n_k}\}$ that converges to a number x. But the sequence $\{x_{n_k}\}$ is a sequence in S that converges to x, and hence, by the assumption that the set S is closed, the limit x also belongs to S. Therefore, the set S is sequentially compact. ■

We now turn to considering a quite different-looking property possessed by closed intervals $[a, b]$. Suppose that for each natural number n, S_n is a set of real numbers. Then we denote the collection of these sets by $\{S_n\}_{n=1}^{\infty}$. For a set S of real numbers, we say that the collection of sets $\{S_n\}_{n=1}^{\infty}$ is a *cover* for the set S provided that for each point x in S there is an index n such that x belongs to S_n: Using the symbols for unions and set inclusion, this is written

$$S \subseteq \cup_{n=1}^{\infty} S_n.$$

If it is the case that there is an index N such that

$$S \subseteq \cup_{n=1}^{N} S_n,$$

then the finite collection of sets $\{S_1, \ldots, S_N\}$ is called a *finite subcover* of $\{S_n\}_{n=1}^{\infty}$ for the set S. A cover of a set may, or may not, have a finite subcover.

Example 2.38 Let S be the set $[0, \infty)$ of nonnegative real numbers. Define

$$I_n \equiv (-n, n) \qquad \text{for every index } n.$$

It follows from the Archimedean Property that

$$S \subseteq \cup_{n=1}^{\infty} I_n,$$

so the collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covers S. It is not the case that any finite collection covers S since no matter what index N is chosen, $\cup_{n=1}^{N} I_n$ does not contain the number $N+1$. ■

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Example 2.39 Take an interval $[a, b]$ and remove a point c in (a, b) so $S \equiv \{x \mid a \leq x \leq b, x \neq c\}$. Define

$$I_n \equiv (c - n, c - 1/n) \qquad \text{if the index } n \text{ is odd,}$$

$$I_n \equiv (c + 1/n, c + n) \qquad \text{if the index } n \text{ is even.}$$

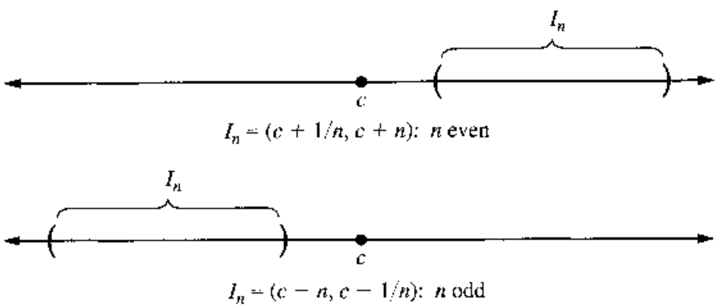

FIGURE 2.5 I_n : n even and n odd.

By the Archimedean Property of $\mathbb{R}$ and the convergence of $\{1/n\}$ to 0, it follows that the collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covers the whole set of real numbers not equal to c, so it certainly covers the set S. It is not the case that any finite subcollection $\{I_n\}_{n=1}^{\infty}$ covers S, since no matter what index N is chosen, $\cup_{n=1}^{N} I_n$ does not contain the points in S whose distance from c is less than $1/N$. ■

So for a general set S it is not the case that a cover of S by a collection of open intervals $\{I_n\}_{n=1}^{\infty}$ necessarily has a finite subcover.

Definition A subset S of $\mathbb{R}$ is said to be *compact* provided that any cover of S by a collection $\{I_n\}_{n=1}^{\infty}$ of open intervals has a finite subcover, that is, if for each index n, I_n is an open interval and

$$S \subseteq \cup_{n=1}^{\infty} I_n,$$

then there is an index N such that

$$S \subseteq \cup_{n=1}^{N} I_n.$$

Motivated by Examples 2.38 and 2.39 we now prove that a compact set must be both bounded and closed.[6]

[6] Sometimes the concept of compactness we define here is referred to as *countable compactness* and compactness is defined slightly differently. In our present context of the real numbers, the two concepts are equivalent.

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Proposition 2.40 Let S be a compact subset of $\mathbb{R}$. Then S is both closed and bounded.

Proof
For each index n, define $I_n \equiv (-n, n)$. By the Archimedean Property of $\mathbb{R}$, $\{I_n\}_{n=1}^{\infty}$ is a collection of open intervals that covers $\mathbb{R}$, so certainly it also covers the set S. Since S is compact, there is an index N such that

$$S \subseteq \cup_{n=1}^{N} A_n,$$

so that

$$|x| < N \qquad \text{for all } x \text{ in } S.$$

Thus, the set S is bounded. It remains to show that S is closed. Let $\{a_n\}$ be a sequence in the set S that converges to the number a. We must show that a also belongs to S. We argue by contradiction. Indeed, suppose that a does not belong to the set S. We proceed as we did in Example 2.38. Define

$$J_n \equiv (a-n, a-1/n) \qquad \text{if the index } n \text{ is odd,}$$

$$J_n \equiv (a+1/n, a+n) \qquad \text{if the index } n \text{ is even.}$$

By the Archimedean Property of $\mathbb{R}$ and the fact that $\{1/n\}$ converges to 0, we see that $\{J_n\}_{n=1}^{\infty}$ is a collection of open intervals that covers the whole set of real numbers not equal to a. But we have supposed that a does not belong to S, and therefore $\{J_n\}_{n=1}^{\infty}$ is a cover of S by a collection of open intervals. It is not the case that any finite subcollection $\{J_n\}_{n=1}^{\infty}$ covers S, since no matter what index N is chosen, because $\{a_n\}$ is a sequence in S that converges to a, there are points in S whose distances from a are less than $1/N$. This contradiction of the compactness property of S shows that in fact a does belong to S. Thus, S is closed. ∎

The following proposition asserts that a sequentially compact set is compact. This is quite surprising since the two concepts seem to be unconnected, apart from our choice of names; the proof of the proposition is rather subtle.

Proposition 2.41 Let S be a sequentially compact subset of $\mathbb{R}$. Then S is compact.

Proof
Suppose that $\{I_n\}_{n=1}^{\infty}$ is a cover of S by a collection of open intervals. We will show there is an index N such that

$$S \subseteq \cup_{n=1}^{N} I_n. \tag{2.25}$$

Since $\{I_n\}_{n=1}^{\infty}$ covers S, for a point x in S we can define its *cover index* to be the smallest index k such that x belongs to I_k and denote this cover index by *cover index* (x). Observe that

$$\text{cover index } (x) \leq k \qquad \text{if and only if } x \text{ belongs to } \cup_{n=1}^{k} I_n$$

and that (2.25) holds if and only if

$$\text{cover index } (x) \leq N \qquad \text{for all } x \text{ in } S. \tag{2.26}$$

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Now observe that given a point x in S, x belongs to I_n, where n is the cover index of x. Since I_n is an open interval, there is an open interval J centered at x such that $J \subseteq I_n$. It follows that every point in $S \cap J$ has a cover index of at most n, that is,

$$\text{cover index } (z) \leq \text{ cover index } (x) \qquad \text{for all } z \text{ in } S \cap J. \tag{2.27}$$

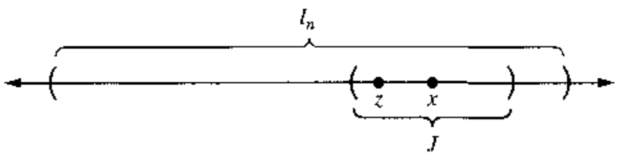

FIGURE 2.6 Cover index $(z) \leq n$ if z belongs to J.

If there is no natural number N such that (2.26) holds, then for each natural number n, there is a point in S whose cover index is greater than n; choose such a point and label it x_n. Thus, $\{x_n\}$ is a sequence in S such that

$$\text{cover index } (x_n) > n \qquad \text{for every index } n. \tag{2.28}$$

But, by assumption, the set S is sequentially compact. Thus, there is a subsequence $\{x_{n_k}\}$ that converges to a point x_0 that also belongs to S. As noted above, we can choose an open interval J centered at x_0 such that (2.27) holds. However, x_0 is the limit of the sequence $\{x_{n_k}\}$, so there is an index K such that

$$x_{n_k} \text{ belongs to } J \text{ for each index } k \geq K.$$

Thus,

$$\text{cover index } (x_{n_k}) \leq \text{ cover index } (x_0) \qquad \text{for each index } k \geq K.$$

This contradicts the property that, by (2.28),

$$\text{cover index } (x_{n_k}) \geq n_k \geq k \qquad \text{for all indices } k.$$

Therefore, the assumption that there was no finite subcover has led to a contradiction and hence we conclude that there is a finite subcover. Thus, the sequentially compact set S is compact. ■

Theorem 2.42 For a subset S of $\mathbb{R}$, the following three assertions are equivalent to each other:

i. S is closed and bounded.
ii. S is sequentially compact.
iii. S is compact.

Proof

Proposition 2.37 is the assertion that (i) implies (ii). Proposition 2.41 is the assertion that (ii) implies (iii). Finally, Proposition 2.40 is the assertion that (iii) implies (i). ■

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The assertion that a closed bounded subset of $\mathbb{R}$ is compact is often referred to as the Heine–Borel Theorem. The assertion that a closed bounded subset of $\mathbb{R}$ is sequentially compact is often referred to as the Bolzano–Weierstrass Theorem.

EXERCISES FOR SECTION 2.5

1. For each of the following statements, determine whether it is true or false and justify your answer.
  a. Every bounded set is closed.
  b. Every closed set is bounded.
  c. Every closed set is compact.
  d. Every bounded set is compact.
  e. A subset of a compact set is also compact.

2. For each of the following statements, determine whether it is true or false and justify your answer.
  a. The set of irrational numbers is closed.
  b. The set of rational numbers in the interval $[0, 1]$ is compact.
  c. The set of negative numbers is closed.

3. Let a and b be numbers with $a < b$. Define $S \equiv [a, b) \equiv \{x \mid a \leq x < b\}$.
  a. Using the definition of sequential compactness, show that S is not sequentially compact.
  b. Using the definition of compactness, show that S is not compact.
  c. Using the definition of closedness, show that S is not closed.

4. Let S be the set of rational numbers in the interval $[0, 2]$.
  a. Using the definition of sequential compactness, show that S is not sequentially compact.
  b. Using the definition of compactness, show that S is not compact.
  c. Using the definition of closedness, show that S is not closed.

5. Let S be a set consisting of a single point. Show that S is compact.

6. Let $S = [0, 1] \cup [3, 4]$. Show that the set S is compact.

7. Let A and B be compact sets. Show that the union $A \cup B$ and the intersection $A \cap B$ are also compact.

8. Let A and B be sets in $\mathbb{R}$. If the union $A \cup B$ is compact, is it true that both A and B must also be compact?

9. At what single point in the proof that sequential compactness implies compactness is the assumption used that the members of the cover are open intervals?

10. For each natural number n, let I_n be a closed bounded interval. Suppose that $\{I_n\}_{n=1}^{\infty}$ covers the compact set consisting of the closed bounded interval $[0, 1]$. Is it true that this cover has a finite subcover?

11. Examine the proof of the theorem that sequential compactness implies compactness and show that the only property of the sets I_n in the cover that we used was that if a point x lies in I_n, then there is an open interval J centered at the point that also lies in I_n. A set having this property is called *open*.

12. Provide a direct proof that a sequentially compact set must be both closed and bounded without taking a detour through the concept of compactness.

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CHAPTER 3

CONTINUOUS FUNCTIONS

3.1 CONTINUITY

In Chapter 2, we considered real-valued functions that have as their domains the set of natural numbers; that is, we considered sequences of real numbers. We now begin the study of real-valued functions having as their domains a general subset of $\mathbb{R}$. There is a standard notation: For a set of real numbers D, by

$$f: D \to \mathbb{R}$$

we denote a function whose domain is D, and for each point x in D we denote by $f(x)$ the value that the function assigns to x. When we write $f: D \to \mathbb{R}$, we will assume without further mention that D is a set of real numbers.

Two of the concepts essential to an analytic description of functions $f: D \to \mathbb{R}$ are *continuity* and *differentiability*. The first five sections of this chapter are devoted to the study of continuity. In the final section we study limits in preparation for the discussion of differentiability, which we will begin in Chapter 4.

Definition A function $f: D \to \mathbb{R}$ is said to be *continuous at the point* x_0 in D provided that whenever $\{x_n\}$ is a sequence in D that converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f: D \to \mathbb{R}$ is said to be *continuous* provided that it is continuous at every point in D.

The definition of continuity of the function $f: D \to \mathbb{R}$ at the point x_0 in D is formulated to make precise the intuitive notion that "if x is a point in D that is close to x_0, then its image $f(x)$ is close to $f(x_0)$," or, what is supposed to describe the same property, "the difference $f(x) - f(x_0)$ becomes arbitrarily small if the point x in D is sufficiently close to x_0." These statements are placed in quotation marks because we are unable to make mathematically precise the concepts of "arbitrarily small" and "close." In Section 4, we will consider a different approach to capturing the concept of continuity.

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Three Examples

Example 3.1 For each number x, define $f(x) = x^2 - 2x + 4$. Then the function $f: \mathbb{R} \to \mathbb{R}$ is continuous. To verify this, we select a point x_0 in $\mathbb{R}$, and we will show that the function is continuous at x_0. Let $\{x_n\}$ be a sequence that converges to x_0. By the sum and product properties of convergent sequences,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left[x_n^2 - 2x_n + 4\right] = x_0^2 - 2x_0 + 4 = f(x_0).$$

Thus, f is continuous at x_0. ∎

The above example is a special case of the continuity of polynomials. The Polynomial Property of convergent sequences stated in Section 2.1 is a statement of the continuity of polynomials.

Example 3.2 Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0. \end{cases}$$

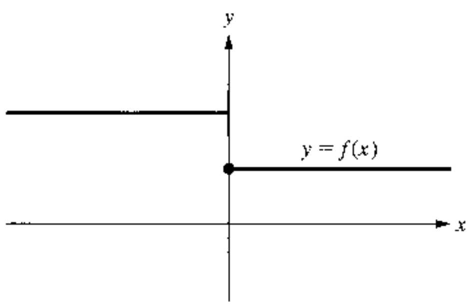

FIGURE 3.1 The function $f: \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$.

The function $f: \mathbb{R} \to \mathbb{R}$ is continuous at each point x_0 except for $x_0 = 0$. First consider $x_0 = 0$. The sequence ($\{-1/n\}$ converges to 0. But $\{f(-1/n)\}$ is a constant sequence having all terms equal to 2. Thus,

$$\lim_{n\to\infty} f(-1/n) = 2 \neq 1 = f(0),$$

and so f is not continuous at $x_0 = 0$. Now consider $x_0 \neq 0$. If a sequence $\{x_n\}$ converges to x_0, then there is an index N such that

$$f(x_n) = f(x_0) \qquad \text{for all indices } n \geq N.$$

Thus,

$$\lim_{n\to\infty} f(x_n) = f(x_0),$$

and so f is continuous at the point x_0. ∎

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Example 3.3 Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is called *Dirichlet's function*. There is no point x_0 in $\mathbb{R}$ at which Dirichlet's function is continuous. Indeed, given a point x_0 in $\mathbb{R}$, by the sequential density of the rationals and irrationals (recall Theorem 2.20), there is a sequence $\{u_n\}$ of rational numbers that converges to x_0 and also a sequence $\{v_n\}$ of irrational numbers that converges to x_0. But $\{f(u_n)\}$ is a constant sequence all of whose terms equal 1 while $\{f(v_n)\}$ is a constant sequence all of whose terms equal 0. Thus,

$$\lim_{n\to\infty} f(u_n) = 1 \neq 0 = \lim_{n\to\infty} f(v_n).$$

Since both of the sequences $\{u_n\}$ and $\{v_n\}$ converge to x_0, it is not possible for f to be continuous at x_0. Observe that one expression of the discontinuous nature of Dirichlet's function is that there is no way to graph it. ■

Sums, Products, and Quotients of Continuous Functions

Given two functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$, we define the *sum* $f + g: D \to \mathbb{R}$ and the *product* $fg: D \to \mathbb{R}$ by

$$(f+g)(x) \equiv f(x) + g(x) \qquad \text{and} \qquad (fg)(x) \equiv f(x)g(x) \qquad \text{for all } x \text{ in } D.$$

Moreover, if $g(x) \neq 0$ for all x in D, the *quotient* $f/g: D \to \mathbb{R}$ is defined by

$$(f/g)(x) \equiv \frac{f(x)}{g(x)} \qquad \text{for all } x \text{ in } D.$$

The following theorem is an analog, and also a consequence, of the sum, product, and quotient properties of convergent sequences.

Theorem 3.4 Suppose that the functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are continuous at the point x_0 in D. Then the sum

$$f + g: D \to \mathbb{R} \text{ is continuous at } x_0, \tag{3.1}$$

the product

$$fg: D \to \mathbb{R} \text{ is continuous at } x_0, \tag{3.2}$$

and, if $g(x) \neq 0$ for all x in D, the quotient

$$f/g: D \to \mathbb{R} \text{ is continuous at } x_0. \tag{3.3}$$

Proof

Let $\{x_n\}$ be a sequence in D that converges to x_0. By the definition of continuity,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = g(x_0).$$

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The sum property of convergent sequences implies that

$$\lim_{n\to\infty} [f(x_n) + g(x_n)] = f(x_0) + g(x_0), \tag{3.4}$$

and the product property of convergent sequences implies that

$$\lim_{n\to\infty} [f(x_n)g(x_n)] = f(x_0)g(x_0). \tag{3.5}$$

If $g(x) \neq 0$ for all x in D, the quotient property of convergent sequences implies that

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}. \tag{3.6}$$

By the definition of continuity, (3.1), (3.2), and (3.3) follow from (3.4), (3.5), and (3.6), respectively. ■

The Polynomial Property for convergent sequences stated in Section 2.1 is precisely the assertion that a polynomial is continuous. Thus, by the quotient property for continuous functions, we have the following corollary describing a general class of continuous functions.

Corollary 3.5 Let $p:\mathbb{R} \to \mathbb{R}$ and $q:\mathbb{R} \to \mathbb{R}$ be polynomials. Then the quotient $p/q : D \to \mathbb{R}$ is continuous, where $D = \{x \text{ in } \mathbb{R} \mid q(x) \neq 0\}$.

Compositions of Continuous Functions

In addition to forming the sum, product, and quotient of functions, there is another useful way to combine functions: They can be *composed*.

Definition For functions $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$ such that $f(D)$ is contained in U, we define the composition of $f: D \to \mathbb{R}$ with $g: U \to \mathbb{R}$, denoted by $g \circ f: D \to \mathbb{R}$, by

$$(g \circ f)(x) \equiv g(f(x)) \qquad \text{for all } x \text{ in } D.$$

We have the following composition property for continuous functions.

Theorem 3.6 For functions $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$ such that $f(D)$ is contained in U, suppose that $f: D \to \mathbb{R}$ is continuous at the point x_0 in D and $g: U \to \mathbb{R}$ is continuous at the point $f(x_0)$. Then the composition

$$g \circ f: D \to \mathbb{R}$$

is continuous at x_0.

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Proof

Let $\{x_n\}$ be a sequence in D that converges to x_0. By the continuity of the function $f: D \to \mathbb{R}$ at the point x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. But then $\{f(x_n)\}$ is a sequence in U that converges to $f(x_0)$, so by the continuity of $g: U \to \mathbb{R}$ at the point $f(x_0)$, the sequence $\{g(f(x_n))\}$ converges to $g(f(x_0))$; that is,

$$\lim_{n\to\infty} (g \circ f)(x_n) = (g \circ f)(x_0).$$

Thus, the composition $g \circ f$ is continuous at x_0. ∎

EXERCISES FOR SECTION 3.1

1. For each of the following statements, determine whether it is true or false and justify your answer.

a. If the function $f + g: \mathbb{R} \to \mathbb{R}$ is continuous, then the functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ also are continuous.

b. If the function $f^2: \mathbb{R} \to \mathbb{R}$ is continuous, then so is the function $f: \mathbb{R} \to \mathbb{R}$.

c. If the functions $f + g: \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are continuous, then so is the function $f: \mathbb{R} \to \mathbb{R}$.

d. Every function $f: \mathbb{N} \to \mathbb{R}$ is continuous, where $\mathbb{N}$ denotes the set of natural numbers.

2. Define

$$f(x) = \begin{cases} 11 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2. \end{cases}$$

At what points is the function $f: [0, 2] \to \mathbb{R}$ continuous? Justify your answer.

3. Define

$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0. \end{cases}$$

At what points is the function $f: \mathbb{R} \to \mathbb{R}$ continuous? Justify your answer.

4. For a function $f: D \to \mathbb{R}$ and a point x_0 in D, define $A = \{x \text{ in } D \mid x \ge x_0\}$ and $B = \{x \text{ in } D \mid x \le x_0\}$. Prove that $f: D \to \mathbb{R}$ is continuous at x_0 if and only if $f: A \to \mathbb{R}$ and $f: B \to \mathbb{R}$ are continuous at x_0.

5. Define

$$f(x) = \begin{cases} x^2 & \text{if } x \ge 0 \\ x & \text{if } x < 0. \end{cases}$$

Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is continuous. (*Hint:* Use Exercise 4.)

6. Define the function $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

At what points is the function continuous? Justify your answer.

7. Suppose that the function $f: [0, 1] \to \mathbb{R}$ is continuous and that

$$f(x) \ge 2 \qquad \text{if } 0 \le x < 1.$$

Show that $f(1) \ge 2$.

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8. Suppose that the function $f:[0,\ 1] \to \mathbb{R}$ is continuous and that

$$f(x) > 2 \qquad \text{if } 0 \leq x < 1.$$

Is it necessarily the case that $f(1) > 2$?

9. Suppose that the function $f:\mathbb{R} \to \mathbb{R}$ is continuous at the point x_0 and that $f(x_0) > 0$. Prove that there is an interval $I \equiv (x_0 - 1/n,\ x_0 + 1/n)$, where n is a natural number, such that $f(x) > 0$ for all x in I. (*Hint:* Argue by contradiction.)

10. Suppose that the function $f:\mathbb{R} \to \mathbb{R}$ is continuous at the point x_0. Prove that there is an interval $I \equiv (x_0 - 1/n, x_0 + 1/n)$, where n is a natural number, such that $f(x) < n$ for all x in I. (*Hint:* Argue by contradiction.)

11. Suppose that the function $g:\mathbb{R} \to \mathbb{R}$ is continuous and that $g(x) = 0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$.

12. Let the function $f: D \to \mathbb{R}$ be continuous. Then define the function $|f|: D \to \mathbb{R}$ by $|f|(x) = |f(x)|$ for x in D. Prove that the function $|f|: D \to \mathbb{R}$ also is continuous.

13. A function $f: D \to \mathbb{R}$ is said to be a *Lipschitz function* provided that there is a nonnegative number C such that

$$|f(u) - f(v)| \leq C|u - v| \qquad \text{for all } u \text{ and } v \text{ in } D.$$

Use the Comparison Lemma of Section 2.1 to show that a Lipschitz function is continuous.

14. Suppose that the function $f:\mathbb{R} \to \mathbb{R}$ has the property that

$$f(u + v) = f(u) + f(v) \qquad \text{for all } u \text{ and } v.$$

a. Define $m \equiv f(1)$. Prove that

$$f(x) = mx \qquad \text{for all rational numbers } x.$$

b. Use (a) to prove that if $f:\mathbb{R} \to \mathbb{R}$ is continuous, then

$$f(x) = mx \qquad \text{for all } x.$$

3.2 THE EXTREME VALUE THEOREM

For a function $f: D \to \mathbb{R}$, we define

$$f(D) \equiv \{y \mid y = f(x) \qquad \text{for some } x \text{ in } D\}$$

and call the set $f(D)$ the *image* of the function $f: D \to \mathbb{R}$. We say that the function $f: D \to \mathbb{R}$ attains a *maximum value* provided that its image $f(D)$ has a maximum; that is, there is a point x_0 in D such that

$$f(x) \leq f(x_0) \qquad \text{for all } x \text{ in } D.$$

We will call such a point x_0 in D a *maximizer* of the function $f: D \to \mathbb{R}$. Similarly, the function $f: D \to \mathbb{R}$ is said to attain a *minimum value* provided that its image $f(D)$ has a minimum; a point in D at which this minimum value is attained is called a *minimizer* of the function $f: D \to \mathbb{R}$.

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In general, a nonempty set has a maximum provided that the set is bounded above and contains its supremum. Thus, a function $f : D \to \mathbb{R}$ has a maximum precisely when the image $f(D)$ is bounded above and the supremum of the image is a functional value.

In general, no assertion can be made concerning the existence of a minimum or maximum value for a function $f : D \to \mathbb{R}$.

Example 3.7 Define the function $f : (0, 1) \to \mathbb{R}$ by $f(x) = 2x$ for all x in $(0, 1)$. This function does not have a maximum value since no matter what x_0 in $(0, 1)$ is chosen, all the points in the interval $(x_0, 1)$ have functional values greater than $f(x_0)$. Observe that the image is bounded above with supremum 2 but that 2 is not attained as a functional value.

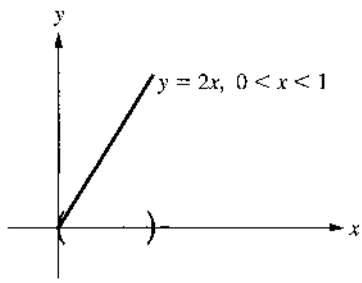

FIGURE 3.2 The supremun of the image is not a functional value. ■

Example 3.8 Define the function $f : (0, 1) \to \mathbb{R}$ by $f(x) = 1/x$ for all x in $(0, 1)$. For each natural number n, $f(1/n) = n$, so the image is not even bounded above. Thus, the function certainly cannot attain a maximum value.

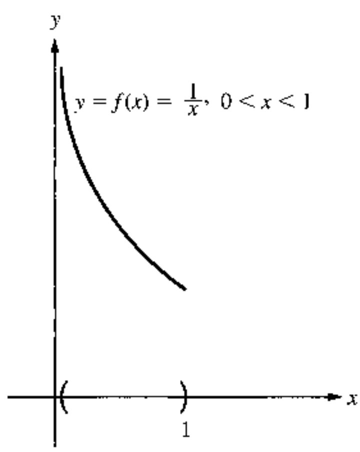

FIGURE 3.3 The image of the function $f(x) = 1/x$ on $(0, 1]$ is not bounded above. ■

However, in the case that the domain D is a closed bounded interval $[a, b]$ and the function $f: [a, b] \to \mathbb{R}$ is continuous, we have the following important result.

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Theorem 3.9 The Extreme Value Theorem A continuous function on a closed bounded interval,

$$f: [a, b] \to \mathbb{R},$$

attains both a minimum and a maximum value.

In order to prove that a function $f: D \to \mathbb{R}$ has a maximum, a reasonable strategy is to show that

- The image $f(D)$ is bounded above, and then that
- The number $\sup f(D)$ is a functional value.

The following lemma achieves the first goal of this strategy to prove the Extreme Value Theorem.

Lemma 3.10 The image of a continuous function on a closed bounded interval,

$$f: [a, b] \to \mathbb{R},$$

is bounded above; that is, there is a number M such that

$$f(x) \leq M \qquad \text{for all } x \text{ in } [a, b].$$

Proof of Lemma 3.10

We will argue by contradiction. Assume that there is no such number M. Let n be a natural number. Then it is not true that

$$f(x) \leq n \qquad \text{for all } x \text{ in } [a, b].$$

Thus, there is a point x in $[a, b]$ at which $f(x) > n$. Choose such a point and label it x_n. This defines a sequence $\{x_n\}$ in $[a, b]$ with the property that $f(x_n) > n$ for every index n. We can employ the Sequential Compactness Theorem (Theorem 2.36) to choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point x_0 in $[a, b]$. Since the function $f: [a, b] \to \mathbb{R}$ is continuous at x_0, the image sequence $\{f(x_{n_k})\}$ converges to $f(x_0)$. But a convergent sequence is bounded (Theorem 2.18), so the sequence $\{f(x_{n_k})\}$ is bounded. This contradicts the property that

$$f(x_{n_k}) > n_k \geq k \qquad \text{for all indices } k.$$

This contradiction proves that the image of $f: [a, b] \to \mathbb{R}$ is bounded above. ∎

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Proof of the Extreme Value Theorem
Define $S \equiv f([a, b])$. Then S is a nonempty set of real numbers that, by the preceding lemma, is bounded above. According to the Completeness Axiom, S has a supremum. Define $c \equiv \sup S$. It is necessary to find a point x_0 in $[a, b]$ at which $c = f(x_0)$.

Let n be a natural number. Then the number $c - 1/n$ is smaller than c and is therefore not an upper bound for the set S. Thus, there is a point x in $[a, b]$ at which $f(x) > c - 1/n$. Choose such a point and label it x_n. From this choice and from the fact that c is an upper bound for S, we see that $c - 1/n < f(x_n) \leq c$ for every index n. Hence the sequence $\{f(x_n)\}$ converges to c.

The Sequential Compactness Theorem (Theorem 2.36) asserts that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point x_0 in $[a, b]$. Since $f\colon [a, b] \to \mathbb{R}$ is continuous at x_0, $\{f(x_{n_k})\}$ converges to $f(x_0)$. But $\{f(x_{n_k})\}$ is a subsequence of the sequence $\{f(x_n)\}$ that converges to c, so $c = f(x_0)$. The point x_0 is a maximizer of the function $f\colon [a, b] \to \mathbb{R}$.

To complete the proof, we observe that the function $-f\colon [a, b] \to \mathbb{R}$ is also continuous. Consequently, using what we have just proven, we can select a point in $[a, b]$ at which $-f\colon [a, b] \to \mathbb{R}$ attains a maximum value, and at this point the function $f\colon [a, b] \to \mathbb{R}$ attains a minimum value. ∎

EXERCISES FOR SECTION 3.2

1. For each of the following statements, determine whether it is true or false and justify your answer.
 a. Every function $f : [0, 1] \to \mathbb{R}$ has a maximum.
 b. Every continuous function $f : [a, b] \to \mathbb{R}$ has a minimum.
 c. Every continuous function $f : (0, 1) \to \mathbb{R}$ has a maximum.
 d. Every continuous function $f : (0, 1) \to \mathbb{R}$ has a bounded image.
 e. If the image of the continuous function $f : (0, 1) \to \mathbb{R}$ is bounded below, then the function has a minimum.

2. Find a maximizer for each of the following functions.
 a. $f\colon [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x} + x^{10} + 4$ for $0 \leq x \leq 1$
 b. $g\colon [-1, 1] \to \mathbb{R}$ defined by $g(x) = -x^{10}(x - 1/4)^{24}$ for $-1 \leq x \leq 1$
 c. $h\colon [-1, 1] \to \mathbb{R}$ defined by $h(x) = 4 - 2x^3$ for $-1 \leq x \leq 1$

3. Let a and b be real numbers with $a < b$. Find a continuous function $f\colon (a, b) \to \mathbb{R}$ having an image that is unbounded above. Also, find a continuous function $f\colon (a, b) \to \mathbb{R}$ having an image that is bounded above but does not attain a maximum value.

4. Suppose that S is a nonempty set of real numbers that is not sequentially compact. Prove that either (i) there is an unbounded sequence in S or (ii) there is a sequence in S that converges to a point x_0 that is not in S.

5. If a set S contains an unbounded sequence, show that the function $f : S \to \mathbb{R}$, defined by $f(x) = x$ for all x in S, is continuous but unbounded. If a set S contains a sequence that converges to a point x_0 not in S, show that the function $f : S \to \mathbb{R}$, defined by $f(x) = 1/|x - x_0|$ for all x in S, is continuous but unbounded.

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6. Use Exercises 4 and 5 to show that if S is a nonempty subset of $\mathbb{R}$ that fails to be sequentially compact, then there is a function $f : S \to \mathbb{R}$ that is continuous but unbounded.

7. Suppose that the function $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) > 0$, and $f(1) = 0$. Prove that there is a number x_0 in $(0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for $0 \le x < x_0$; that is, there is a smallest point in the interval $[0, 1]$ at which the function f attains the value 0.

3.3 THE INTERMEDIATE VALUE THEOREM

The second important geometric property of the graph of a continuous function that we will establish is that if a continuous function has a domain consisting of an interval, and if its graph contains points that are both above and below a line $y = c$, then, in fact, the graph intersects the line $y = c$.

Theorem 3.11 The Intermediate Value Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Let c be a number strictly between $f(a)$ and $f(b)$; that is,

$$f(a) < c < f(b) \qquad \text{or} \qquad f(b) < c < f(a).$$

Then there is a point x_0 in the open interval (a, b) at which $f(x_0) = c$.

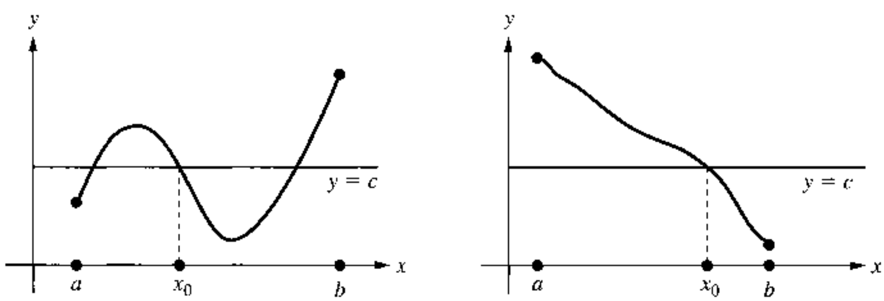

FIGURE 3.4 A number between two functional values is also a functional value.

Proof The Bisection Method
We will consider only the case that $f(a) < c$ and $f(b) > c$. The other case follows from this case by replacing f by $-f$ and c by $-c$. We will recursively define a sequence of nested, closed subintervals of $[a, b]$ whose endpoints converge to a point in $[a, b]$ at which $f(x) = c$.

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Let $a_1 = a$ and $b_1 = b$. For a natural number n, suppose that the interval $[a_n, b_n]$ contained in $[a, b]$ has been defined such that $f(a_n) \leq c$ and $f(b_n) > c$. Consider the midpoint $m_n = (a_n + b_n)/2$.

- If $f(m_n) \leq c$, define $a_{n+1} = m_n$ and $b_{n+1} = b_n$.
- If $f(m_n) > c$, define $a_{n+1} = a_n$ and $b_{n+1} = m_n$.

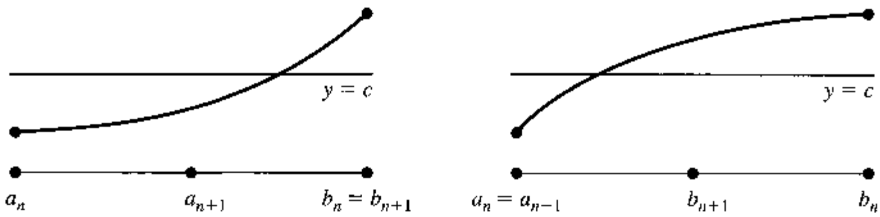

FIGURE 3.5 The bisection method.

Observe that for each natural number n,

$$a \leq a_n \leq a_{n+1} < b_{n+1} \leq b_n \leq b,$$

$$f(a_{n+1}) \leq c \qquad \text{and} \qquad f(b_{n+1}) > c,$$

and

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}.$$

It follows that $(b_n - a_n) = (b - a)/2^{n-1}$ for all n. Thus, the sequences $\{a_n\}$ and $\{b_n\}$ satisfy the assumptions of the Nested Interval Theorem (Theorem 2.29), so there is a point x_0 in $[a, b]$ to which both $\{a_n\}$ and $\{b_n\}$ converge. Since $f: [a, b] \to \mathbb{R}$ is continuous at x_0, the image sequences $\{f(a_n)\}$ and $\{f(b_n)\}$ converge to $f(x_0)$. It follows that $f(x_0) \leq c$ since $f(a_n) \leq c$ for each index n, and that $f(x_0) \geq c$ since $f(b_n) \geq c$ for each index n. Consequently, $f(x_0) = c$. ■

Example 3.12 For any positive number c, there is a solution to the equation

$$x^2 = c, \qquad x > 0. \tag{3.7}$$

Indeed, consider the function $f : [0, c + 1] \to \mathbb{R}$ defined by

$$f(x) = x^2 \qquad \text{for } 0 \leq x \leq c + 1.$$

The function is continuous since it is a polynomial and

$$f(0) = 0 < c, \text{ while } f(c + 1) = c^2 + 2c + 1 > c.$$

The Intermediate Value Theorem implies that equation (3.7) has a solution. ■

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For a natural number k and real numbers $a_0, a_1, \ldots, a_k$, consider the equation

$$a_0 + a_1 x + \cdots + a_k x^k = 0, \qquad x \text{ in } \mathbb{R}. \tag{3.8}$$

In general, of course, this equation might not have any solution. For instance, the equation

$$1 + x^2 = 0, \qquad x \text{ in } \mathbb{R},$$

has no solution since $1 > 0$ and $x^2 \geq 0$. For $k = 1$, we can easily analyze (3.8). For $k = 2$, Example 3.12, which guarantees that the square root of a positive number is properly defined, and the quadratic formula permit us to analyze equation (3.8). For $k = 3$ and $k = 4$, there are explicit formulas similar to, but more complicated than, the quadratic formula for determining the solutions of this equation. *However, for $k \geq 5$ there cannot be a formula for determining the solutions of equation* (3.8) *for arbitrary choices of the coefficients $a_0, a_1, \ldots, a_k$; this follows from a beautiful theorem of Galois, which, unfortunately, lies outside the scope of this book.*[1] Hence, even when the function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial, if its degree is greater than 4, it is usually not possible to explicitly determine the solutions of the equation

$$f(x) = 0, \qquad x \text{ in } \mathbb{R}. \tag{3.9}$$

So one can imagine how difficult it is to determine the solutions of equation (3.9) when $f: \mathbb{R} \to \mathbb{R}$ is defined in terms of, say, trigonometric and exponential functions.

However, the Intermediate Value Theorem is useful in the study of equation (3.9). If $f: \mathbb{R} \to \mathbb{R}$ is continuous and we can find numbers a and b with $a < b$ and $f(a) \cdot f(b) < 0$, then equation (3.9) has a solution in the open interval (a, b). Moreover, the method of proof we gave for Theorem 3.11, which is called the *bisection method,* provides a recursive method that, after n steps, determines a subinterval of $[a, b]$ of length $(b - a)/2^{n-1}$ that contains a solution of equation (3.9).

Example 3.13 Consider the equation

$$x^5 + x + 1 = 0, \qquad x \text{ in } \mathbb{R}.$$

We claim that there is a solution of the above equation. Indeed, define $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = x^5 + x + 1$ for all x. Observe that $h(-2) < 0$ and $h(0) > 0$. Thus, we can apply the Intermediate Value Theorem to the restriction $h: [-2, 0] \to \mathbb{R}$ to conclude that there is a point x_0 in the open interval $(-2, 0)$ that is a solution of this equation. ■

There is a slightly more general form of the Intermediate Value Theorem that is of interest because it can be generalized to the situation in which one considers real-valued functions of several real variables.

Definition A subset D of $\mathbb{R}$ is said to be *convex* provided that whenever the points u and v are in D and $u < v$, then the whole interval $[u, v]$ is contained in D.

[1] See I. N. Herstein, *Topics in Algebra.*

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In Exercise 11 we outline the proof that the convex subsets of $\mathbb{R}$ are just the intervals. It is useful to isolate the property of convexity since this is precisely the ingredient used in the proof of the following slight generalization of the Intermediate Value Theorem.

Theorem 3.14 Let I be an interval and suppose that the function $f: I \to \mathbb{R}$ is continuous. Then its image $f(I)$ also is an interval.

Proof

We show that the image $f(I)$ is a convex set. Let y_1 and y_2 be points in $f(I)$, with $y_1 < y_2$. We must show that the closed interval $[y_1, y_2]$ is also contained in $f(I)$. Indeed, let $y_1 < c < y_2$. Since y_1 and y_2 are in $f(I)$, there are points x_1 and x_2 in I with $f(x_1) = y_1$ and $f(x_2) = y_2$. If we let J be the closed interval having x_1 and x_2 as endpoints, then J is contained in I since, by assumption, the set I is an interval and therefore is convex. Thus, we can apply the Intermediate Value Theorem to the function $f: J \to \mathbb{R}$ in order to conclude that there is a point x_0 in J at which $f(x_0) = c$. Thus, x_0 belongs to I and $f(x_0) = c$. It follows that $[y_1, y_2]$ is contained in $f(I)$. ∎

EXERCISES FOR SECTION 3.3

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. If the function $f: \mathbb{R} \to \mathbb{R}$ is continuous, then $f(\mathbb{R}) = \mathbb{R}$.
   b. For any function $f: [0, 1] \to \mathbb{R}$, its image $f([0, 1])$ is an interval.
   c. For any continuous function $f: D \to \mathbb{R}$, its image $f(D)$ is an interval.
   d. For a continuous strictly increasing function $f: [0, 1] \to \mathbb{R}$, its image is the interval $[f(0), f(1)]$.
2. Prove that there is a solution of the equation
$$x^9 + x^2 + 4 = 0, \qquad x \text{ in } \mathbb{R}.$$
3. Prove that there is a solution of the equation
$$\frac{1}{\sqrt{x + x^2}} + x^2 - 2x = 0, \qquad x > 0.$$
4. For a function $f: D \to \mathbb{R}$, a solution of the equation
$$f(x) = x, \qquad x \text{ in } D$$
is called a *fixed point* of f. A fixed point corresponds to a point at which the graph of the function f intersects the line $y = x$. If $f: [-1, 1] \to \mathbb{R}$ is continuous, $f(-1) > -1$, and $f(1) < 1$, show that $f: [-1, 1] \to \mathbb{R}$ has a fixed point.
5. Suppose that the functions $h: [a, b] \to \mathbb{R}$ and $g: [a, b] \to \mathbb{R}$ are continuous. Observe that a solution of the equation
$$h(x) = g(x), \qquad x \text{ in } [a, b],$$
corresponds to a point where the graphs intersect. Show that if $h(a) \leq g(a)$ and $h(b) \geq g(b)$, then this equation has a solution.

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6. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and that its image $f(\mathbb{R})$ is bounded. Prove that there is a solution of the equation

$$f(x) = x, \qquad x \text{ in } \mathbb{R}.$$

7. Suppose that the function $f: [a, b] \to \mathbb{R}$ is continuous. For a natural number k, let $x_1, \ldots, x_k$ be points in $[a, b]$. Prove that there is a point z in $[a, b]$ at which

$$f(z) = \frac{f(x_1) + \cdots + f(x_k)}{k}.$$

8. The proof of Theorem 3.11 has a constructive aspect. At the nth stage, an interval $[a_n, b_n]$ of length $(b - a)/2^{n-1}$ has been determined that contains a solution of the equation

$$f(x) = c, \qquad x \text{ in } [a, b].$$

Find an interval of length smaller than $1/8$ that contains a solution of the equation

$$x^7 + 2x^3 = 2, \qquad x > 0.$$

9. Let $p: \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. Prove that there is a solution of the equation

$$p(x) = 0, \qquad x \text{ in } \mathbb{R}.$$

10. Suppose that the function $f: [0, 1] \to \mathbb{R}$ is continuous and that its image consists entirely of rational numbers. Prove that $f: [0, 1] \to \mathbb{R}$ is a constant function.

11. Let I be a nonempty convex subset of $\mathbb{R}$. If I is bounded above, define $b = \sup I$; if I is bounded below, define $a = \inf I$. Prove the following:
- **a.** If I is unbounded above and below, then $I = \mathbb{R}$.
- **b.** If I is bounded below but not above, then $I = (a, \infty)$ or $I = [a, \infty)$.
- **c.** If I is bounded above but not below, then $I = (-\infty, b]$ or $I = (-\infty, b)$.
- **d.** If I is bounded, then I is one of the sets $[a, b]$, (a, b), $[a, b)$, $(a, b]$.

3.4 UNIFORM CONTINUITY

For a function $f: D \to \mathbb{R}$ and a point x_0 in its domain D, we have defined f to be continuous at x_0 provided that whenever a sequence $\{x_n\}$ in D converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f: D \to \mathbb{R}$ has been defined to be continuous provided that it is continuous at each point in its domain D. There is a concept of uniform continuity of a function $f: D \to \mathbb{R}$ that, in general, requires more than continuity at each point in the domain. Uniform continuity plays an important role in the development of the integral in Chapter 6.

Definition A function $f: D \to \mathbb{R}$ is said to be *uniformly continuous* provided that whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D such that

$$\lim_{n\to\infty} [u_n - v_n] = 0,$$

then

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0.$$

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Note carefully that in the above definition of uniform continuity there is no information regarding convergence of the sequence $\{u_n\}$ to a point in the domain of f. The concept of uniform continuity of f on D is formulated to capture the intuitive notion that "the difference $f(u) - f(v)$ becomes arbitrarily small for any two points u and v in D that are sufficiently close to each other, no matter where the two points are located in the domain."

If a function $f: D \to \mathbb{R}$ is uniformly continuous, then it is continuous. Indeed, for a point x_0 in D, let $\{x_n\}$ be a sequence in D that converges to x_0. Define $u_n = x_n$ and $v_n = x_0$ for each index n. We have

$$\lim_{n\to\infty} [u_n - v_n] = \lim_{n\to\infty} [x_n - x_0] = 0,$$

so that, by the uniform continuity of $f: D \to \mathbb{R}$,

$$\lim_{n\to\infty} [f(x_n) - f(x_0)] = \lim_{n\to\infty} [f(u_n) - f(v_n)] = 0;$$

that is, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. Thus, f is continuous at x_0.

In general, it is not the case that continuity implies uniform continuity, as the following two examples demonstrate.

Example 3.15 Define $f(x) = x^2$ for x in $\mathbb{R}$. Then the function $f: \mathbb{R} \to \mathbb{R}$ is continuous since it is a polynomial. But this function is not uniformly continuous. Indeed, for each index n, set

$$u_n = n \qquad \text{and} \qquad v_n = n + 1/n.$$

Clearly,

$$\lim_{n\to\infty} [u_n - v_n] = \lim_{n\to\infty} 1/n = 0.$$

However, for each index n, $f(u_n) - f(v_n) = 2 + 1/n^2$, so that

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] \neq 0.$$

■

Example 3.16 Define $f(x) = 1/x$ for x in $(0, 1)$. The function $f: (0, 1) \to \mathbb{R}$ is continuous since it is the reciprocal of a polynomial that is never zero on the domain. But this function is not uniformly continuous. Indeed, for each index n, set

$$u_n = 1/n \qquad \text{and} \qquad v_n = 1/2n.$$

Clearly,

$$\lim_{n\to\infty} [u_n - v_n] = \lim_{n\to\infty} 1/2n = 0.$$

However, for each index n, $f(u_n) - f(v_n) = -n$, so that

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] \neq 0.$$

■

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The domains of the continuous functions in the preceding two examples were not closed bounded intervals. The following theorem asserts that if the domain of a continuous function is a closed bounded interval, then the function is uniformly continuous.

Theorem 3.17 A continuous function on a closed bounded interval,

$$f: [a, b] \to \mathbb{R},$$

is uniformly continuous.

Proof
Let $\{u_n\}$ and $\{v_n\}$ be sequences in $[a, b]$ such that

$$\lim_{n\to\infty} [u_n - v_n] = 0. \tag{3.10}$$

We need to show that

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0. \tag{3.11}$$

We will do so by arguing by contradiction. Suppose that the sequence $\{f(u_n) - f(v_n)\}$ does not converge to 0. Then (Exercise 12) by possible passing to a subsequence, there is some $\epsilon > 0$ such that

$$|f(u_n) - f(v_n)| \geq \epsilon \qquad \text{for every index } n. \tag{3.12}$$

We now use the assumption that the domain of f is a closed bounded interval. According to the Sequential Compactness Theorem (Theorem 2.36) there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a point x_0 in $[a, b]$ such that

$$\lim_{k\to\infty} u_{n_k} = x_0.$$

From this convergence and (3.10), we also conclude that

$$\lim_{k\to\infty} v_{n_k} = x_0.$$

However, f is continuous at the point x_0. Thus,

$$\lim_{k\to\infty} f(u_{n_k}) = f(x_0) \qquad \text{and} \qquad \lim_{k\to\infty} f(v_{n_k}) = f(x_0),$$

and therefore,

$$\lim_{k\to\infty} [f(u_{n_k}) - f(v_{n_k})] = 0.$$

However, by (3.12),

$$|f(u_{n_k}) - f(v_{n_k})| \geq \epsilon \qquad \text{for every index } k.$$

From this contradiction we conclude that f is uniformly continuous on $[a, b]$. ■

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EXERCISES FOR SECTION 3.4

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.
   b. Every continuous function $f : [0, 1) \to \mathbb{R}$ is uniformly continuous.
   c. Every continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous.
   d. Every uniformly continuous function $f : D \to \mathbb{R}$ is continuous.
2. If the function $f : D \to \mathbb{R}$ is uniformly continuous and α is any number, show that the function $\alpha f : D \to \mathbb{R}$ also is uniformly continuous.
3. Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous, then so is the sum $f + g : D \to \mathbb{R}$.
4. Define $f(x) = mx + b$ for all x. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous.
5. Define $f(x) = x^3$ for all x. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.
6. Show that it is not necessarily the case that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are each uniformly continuous, then so is the product $fg : D \to \mathbb{R}$.
7. Suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous and bounded. Prove that the product $fg : D \to \mathbb{R}$ also is uniformly continuous. *Hint:* Write
$$f(u)g(u) - f(v)g(v) = f(u)[g(u) - g(v)] + g(v)[f(u) - f(v)].$$
8. For any open interval $I = (a, b)$, find a continuous function $f : I \to \mathbb{R}$ that is not uniformly continuous.
9. For an unbounded nonempty set of real numbers D, does there necessarily exist a continuous function $f : D \to \mathbb{R}$ that is not uniformly continuous?
10. Suppose that the function $f: (a, b) \to \mathbb{R}$ is uniformly continuous. Prove that $f: (a, b) \to \mathbb{R}$ is bounded.
11. A function $f: D \to \mathbb{R}$ is called a Lipschitz function if there is some nonnegative number C such that
$$|f(u) - f(v)| \leq C|u - v| \qquad \text{for all points } u \text{ and } v \text{ in } D.$$
Prove that if $f: D \to \mathbb{R}$ is a Lipschitz function, then it is uniformly continuous.
12. Suppose that the function $f : D \to \mathbb{R}$ is not uniformly continuous. Then, by definition, there are sequences $\{s_n\}$ and $\{t_n\}$ in D such that
$$\lim_{n\to\infty} [s_n - t_n] = 0, \text{ but } \lim_{n\to\infty} [f(s_n) - f(t_n)] \neq 0.$$
   a. Show that there is an $\epsilon > 0$ and a strictly increasing sequence of indices $\{n_k\}$ such that for each index k, $|f(s_{n_k}) - f(t_{n_k})| \geq \epsilon$.
   b. Define $u_k = s_{n_k}$ and $v_k = t_{n_k}$ for each index k. Show that
$$\lim_{n\to\infty} [u_n - v_n] = 0, \text{ but } |f(u_n) - f(v_n)| \geq \epsilon \text{ for each index } n.$$

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3.5 THE ϵ-δ CRITERION FOR CONTINUITY

We defined the *continuity of the function* $f : D \to \mathbb{R}$ *at the point* x_0 *in* D in terms of convergent sequences in order to make precise the intuitive notion that "if x is a point in D that is close to x_0, then its image $f(x)$ is close to $f(x_0)$." There is another way to capture this idea that is equivalent to the sequential convergence criterion but which appears to be rather different. From the alternative perspective, certain properties of continuous functions can be seen more clearly. The alternate criterion for continuity is the following.

Definition The ϵ-δ Criterion at a Point A function $f : D \to \mathbb{R}$ is said to satisfy the ϵ-δ criterion at a point x_0 in the domain D provided that for each positive number ϵ there is a positive number δ such that for x in D,

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta. \tag{3.13}$$

In terms of the graph of the function $f : D \to \mathbb{R}$, the ϵ-δ criterion at a point x_0 in D can be reworded as follows: *For each symmetric band of width* 2ϵ *about the line* $y = f(x_0)$ *(no matter how small this width is)*, there is an interval $(x_0 - \delta, x_0 + \delta)$, centered at x_0 and of diameter $2\delta > 0$, such that the graph of the restriction of f to this interval lies in the given band.

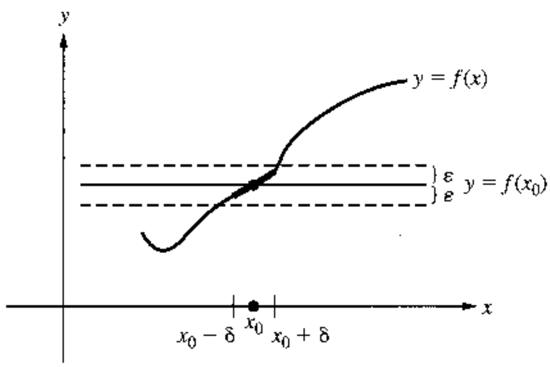

FIGURE 3.6 The ϵ-δ criterion for continuity at x_0.

Example 3.18 Define $f(x) = x^3$ for all x. We claim that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the ϵ-δ criterion at the point $x_0 = 2$. Let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$|x^3 - 8| < \epsilon \qquad \text{if } |x - 2| < \delta. \tag{3.14}$$

But observe that the difference of cubes formula and the Triangle Inequality imply that for all x,

$$|x^3 - 8| = |(x^2 + 2x + 4)(x - 2)| \leq [|x|^2 + 2|x| + 4]|x - 2|.$$

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However,

$$|x|^2 + 2|x| + 4 \leq 19 \qquad \text{if } 1 < x < 3$$

so that

$$|x^3 - 8| \leq 19|x - 2| \qquad \text{if } 1 < x < 3. \tag{3.15}$$

Define

$$\delta = \min\{1, \epsilon/19\}. \tag{3.16}$$

If $|x - 2| < \delta$, then $1 < x < 3$ and $19|x - 2| < \epsilon$, so from (3.15) it follows that $|x^3 - 8| < \epsilon$. Therefore, for $\epsilon > 0$, if we define $\delta > 0$ by (3.16), then condition (3.14) holds. ■

Example 3.19 Define

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 0. \end{cases}$$

The function $f : \mathbb{R} \to \mathbb{R}$ does not satisfy the ϵ-δ criterion at the point $x_0 = 0$. Indeed, take $\epsilon \equiv 1/2$. Then there is no positive number δ having the property that

$$-1/2 < f(x) < 1/2 \qquad \text{if } -\delta < x < \delta$$

since no matter what positive number δ is selected, there are positive numbers x in the interval $(-\delta, \delta)$ such that $f(x) < -1/2$.

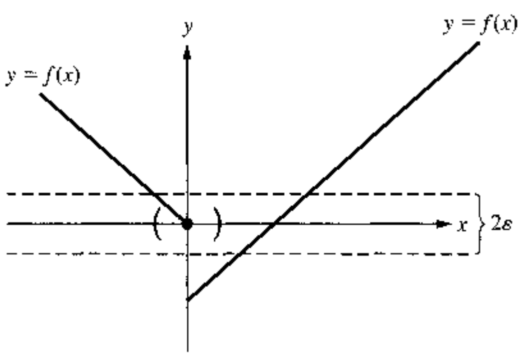

FIGURE 3.7 The ϵ-δ criterion for continuity is not satisfied at the point $x_0 = 0$. ■

The following theorem reconciles the ϵ-δ criterion at a point with the sequential criterion for continuity at a point.

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Theorem 3.20 For a function $f: D \to \mathbb{R}$ and a point x_0 in its domain D, the following two assertions are equivalent:

i. The function $f: D \to \mathbb{R}$ is continuous at x_0; that is, for a sequence $\{x_n\}$ in D,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \qquad \text{if } \lim_{n\to\infty} x_n = x_0.$$

ii. The ϵ-δ criterion at the point x_0 holds; that is, for each positive number ϵ there is a positive number δ such that for x in D,

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta. \tag{3.17}$$

Proof

First, suppose that $f: D \to \mathbb{R}$ is continuous at x_0. We will argue by contradiction to verify the ϵ-δ criterion at the point x_0. Suppose that this criterion does not hold. Then there is some $\epsilon_0 > 0$ such that for $\epsilon = \epsilon_0$ there is no $\delta > 0$ for which (3.17) holds. Let n be a natural number. Then (3.17) does not hold for $\epsilon = \epsilon_0$ and $\delta = 1/n$. This means precisely that for each natural number n, there is a point x in D such that $|x - x_0| < 1/n$ but $|f(x) - f(x_0)| \geq \epsilon_0$. Choose such a point and label it x_n. This defines a sequence $\{x_n\}$ in D that converges to x_0. But by the continuity of $f: D \to \mathbb{R}$ at x_0, $\{f(x_n)\}$ converges to $f(x_0)$. This clearly contradicts the assertion that $|f(x_n) - f(x_0)| \geq \epsilon_0$ for every index n. Thus, the ϵ-δ criterion at the point x_0 holds.

Now to prove the converse. Assume that the ϵ-δ criterion at the point x_0 holds. We will show that $f : D \to \mathbb{R}$ is continuous at x_0. Indeed, let $\{x_n\}$ be a sequence in D that converges to x_0. To show that $\{f(x_n)\}$ converges to $f(x_0)$, we let $\epsilon > 0$ and seek a natural number N such that

$$|f(x_n) - f(x_0)| < \epsilon \qquad \text{for all indices } n \geq N. \tag{3.18}$$

But the ϵ-δ criterion at the point x_0 asserts that we can select a positive number δ such that for x in D,

$$|f(x) - f(x_0)| < \epsilon \qquad \text{if } |x - x_0| < \delta. \tag{3.19}$$

Moreover, since $\{x_n\}$ converges to x_0, we can choose a natural number N such that

$$|x_n - x_0| < \delta \qquad \text{for all indices } n \geq N. \tag{3.20}$$

Clearly (3.20) and (3.19) imply (3.18). ∎

Frequently, for a continuous function $f : D \to \mathbb{R}$, it happens that the choice of $\delta > 0$ depends only on $\epsilon > 0$ and is independent of the choice of point in D. We formulate this criterion as follows.

Definition The ϵ-δ Criterion on the Domain A function $f : D \to \mathbb{R}$ is said to satisfy the ϵ-δ criterion on the domain D provided that for each positive number ϵ there is a positive number δ such that for all u, v in D,

$$|f(u) - f(v)| < \epsilon \qquad \text{if } |u - v| < \delta. \tag{3.21}$$

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Example 3.21 Define $f(x) = x^3$ for x in $[0, 20]$. Then the function $f: [0, 20] \to \mathbb{R}$ is uniformly continuous. To see this, observe that for all u and v in $[0, 20]$,

$$|f(u) - f(v)| = |u^2 + uv + v^2||u - v| \leq 1200|u - v|.$$

Hence, for $\epsilon > 0$, if we define $\delta = \epsilon/1200$, then (3.21) holds. ■

We just showed in Theorem 3.20 that the sequential criterion for continuity at a point is equivalent to the ϵ-δ criterion at a point. Following an entirely similar proof, which we leave as an exercise, we have the following theorem regarding the equivalence of the sequential uniform continuity criterion studied in the preceding section with the ϵ-δ criterion on the domain.

Theorem 3.22 For a function $f: D \to \mathbb{R}$, the following two assertions are equivalent:

i. The function $f: D \to \mathbb{R}$ is uniformly continuous; that is, for two sequences $\{u_n\}$ and $\{v_n\}$ in D,

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0 \qquad \text{if } \lim_{n\to\infty} [u_n - v_n] = 0.$$

ii. The function $f: D \to \mathbb{R}$ satisfies the ϵ-δ criterion at the domain D; that is, for each positive number ϵ there is a positive number δ such that for u, v in D,

$$|f(u) - f(v)| < \epsilon \qquad \text{if } |u - v| < \delta. \tag{3.22}$$

EXERCISES FOR SECTION 3.5

1. Define $f(x) = x^2$ for all x. Verify the ϵ-δ criterion for continuity at $x = 2$ and at $x = 50$.

2. Define $f(x) = \sqrt{x}$ for all $x \geq 0$. Verify the ϵ-δ criterion for continuity at $x = 4$ and at $x = 100$. *Hint:* First show that for $x \geq 0$, $x_0 > 0$,

$$|\sqrt{x} - \sqrt{x_0}| \leq |x - x_0|/\sqrt{x_0}.$$

3. Define $f(x) = x^3$ for all x. Verify the ϵ-δ criterion for continuity at each point x_0.

4. Define

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 3/4 \\ 2 & \text{if } x > 3/4. \end{cases}$$

Use the ϵ-δ criterion for continuity at a point to show that $f: \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 3/4$.

5. Define $h(x) = 1/(1 + x^2)$ for all x. Prove that the function $h: \mathbb{R} \to \mathbb{R}$ satisfies the ϵ-δ criterion on $\mathbb{R}$.

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6. A function $f : D \to \mathbb{R}$ is said to be a Lipschitz function provided that there is a nonnegative number C such that

$$|f(u) - f(v)| \leq C|u - v| \qquad \text{for all } u \text{ and } v \text{ in } D.$$

Show that a Lipschitz function satisfies the ϵ-δ criterion on D.

7. Define $f(x) = \sqrt{x}$ for $0 \leq x \leq 1$.
   a. Prove that the function $f : [0, 1] \to \mathbb{R}$ is continuous.
   b. Use part (a) to show that $f : [0, 1] \to \mathbb{R}$ is uniformly continuous.
   c. Show that $f : [0, 1] \to \mathbb{R}$ is not a Lipschitz function.
8. Suppose that the continuous function $f : \mathbb{R} \to \mathbb{R}$ is periodic, that is, there is a number $p > 0$ such that $f(x + p) = f(x)$ for all x. Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.
9. Define the function $h : [1, 2] \to \mathbb{R}$ as follows: $h(x) = 0$ if the point x in $[1, 2]$ is irrational; $h(x) = 1/n$ if the point x in $[1, 2]$ is rational and $x = m/n$, where m and n are natural numbers having no common positive integer factor other than 1.
   a. Prove that $h : [1, 2] \to \mathbb{R}$ fails to be continuous at each rational number in $[1, 2]$.
   b. Prove that if $\epsilon > 0$, then the set $\{x \text{ in } [1, 2] \mid h(x) > \epsilon\}$ has only a finite number of points.
   c. Use part (b) to prove that $h : [1, 2] \to \mathbb{R}$ is continuous at each irrational number in $[1, 2]$.
10. Prove Theorem 3.22 following the strategy of the proof of Theorem 3.20.

3.6 IMAGES AND INVERSES; MONOTONE FUNCTIONS

In Chapter 2, we proved the Monotone Convergence Theorem: A monotone sequence converges if and only if it is bounded. We shall now introduce the natural concept of a monotone function and show that such functions also have quite special properties not possessed by general functions. Functions that are monotone occur naturally in considering the *inverses* of functions.

A Continuity Criterion for Monotone Functions

Definition The function $f : D \to \mathbb{R}$ is called *monotonically increasing* provided that

$$f(v) \geq f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

The function $f : D \to \mathbb{R}$ is called *monotonically decreasing* provided that

$$f(v) \leq f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

A function that is either monotonically increasing or monotonically decreasing is said to be *monotone*.

Observe that in the definition of a monotone function the inequalities between the functional values are not strict, so, for instance, constant functions are considered to be monotone.

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We will now show that a monotone function has the remarkable property that it is continuous if its image is an interval. This result stands in sharp contrast to many results in this chapter in which continuity of the function has been the *assumption*, not the *conclusion*.

Theorem 3.23 Suppose that the function $f: D \to \mathbb{R}$ is monotone. If its image $f(D)$ is an interval, then the function f is continuous.

Proof

We will consider the case that the function $f: D \to \mathbb{R}$ is monotonically increasing. The case of a monotonically decreasing function follows from the monotonically increasing case by considering the function $-f: D \to \mathbb{R}$.

Let x_0 be a point in D. Let $\{x_n\}$ be a sequence in D that converges to x_0. For each index n, set $y_n = f(x_n)$ and set $y_0 = f(x_0)$. We must show that $\{y_n\}$ converges to y_0. To do so, select $\epsilon > 0$. We must find an index N such that

$$|y_n - y_0| < \epsilon \qquad \text{for all indices } n \geq N;$$

that is,

$$y_n < y_0 + \epsilon \qquad \text{for all indices } n \geq N \tag{3.23}$$

and

$$y_0 - \epsilon < y_n \qquad \text{for all indices } n \geq N. \tag{3.24}$$

We will just find an index N such that (3.23) holds and leave the entirely similar verification of (3.24) as an exercise.

Of course, if $y_n \leq y_0$ for all indices n, (3.23) holds for $N = 1$. Otherwise, there is some term of the sequence $\{y_n\}$ that is greater than y_0. In particular, there is a point that we denote by y^* such that

$$y^* > y_0 \text{ and } y^* \text{ belongs to the image } f(D).$$

But, by assumption, the image $f(D)$ is an interval. Thus, since y_0 and y^* belong to $f(D)$,

$$\text{the interval } [y_0, \ y^*] \text{ is contained in the image } f(D). \tag{3.25}$$

Choose x^* in D such that $f(x^*) = y^*$. Observe that $x_0 < x^*$ since the function f is monotone and $f(x_0) < f(x^*)$.

Define $y_\epsilon = \min\{y_0 + \epsilon/2, \ y^*\}$. Then $y_0 < y_\epsilon \leq y^*$, so that by (3.25), there is a point x_ϵ in D such that $f(x_\epsilon) = y_\epsilon$ and $x_\epsilon > x_0$ since f is monotonically increasing.

However, the sequence $\{x_n\}$ converges to x_0 and $x_\epsilon > x_0$. Thus, there is an index N such that $x_n < x_\epsilon$ if $n \geq N$. Thus, since f is monotonically increasing,

$$f(x_n) \leq f(x_\epsilon) = y_\epsilon < y_0 + \epsilon \qquad \text{for all indices } n \geq N.$$

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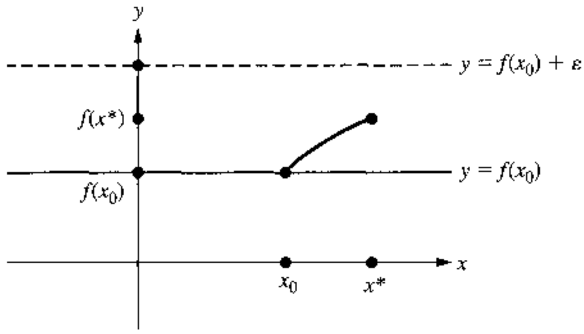

FIGURE 3.8 $f(x) \leq y^*$ if $x < x^*$.

Thus, (3.23) holds for this choice of index N. As already stated, the verification of (3.24) is similar. Therefore, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. ∎

It is not true that a general function whose image is an interval is necessarily continuous, as the following example shows.

Example 3.24 Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 0. \end{cases}$$

It is clear that the image $f(\mathbb{R})$ equals all of $\mathbb{R}$; so it is an interval. But $f : \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$ since

$$\lim_{n\to\infty} 1/n = 0 \text{ while } \lim_{n\to\infty} f(1/n) = -1 \neq f(0).$$ ∎

Corollary 3.25 Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is monotone. Then the function f is continuous if and only if its image $f(I)$ is an interval.

Proof
If the image $f(I)$ is an interval, then the preceding theorem asserts that f is continuous. Conversely, the Intermediate Value Theorem, as expressed in Theorem 3.14, asserts that a continuous function whose domain is an interval has an interval as its image. ∎

Continuity of Inverse Functions

Definition The function $f : D \to \mathbb{R}$ is called *strictly increasing* provided that

$$f(v) > f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

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The function $f: D \to \mathbb{R}$ is called *strictly decreasing* provided that

$$f(v) < f(u) \qquad \text{for all points } u \text{ and } v \text{ in } D \text{ such that } v > u.$$

A function that is either strictly increasing or strictly decreasing is said to be *strictly monotone*.

Example 3.26 Define $f(x) = x^2$ for $x \geq 0$. Then the function $f:[0, \infty) \to \mathbb{R}$ is strictly increasing since by the difference of squares formula

$$u^2 - v^2 = (u - v)(u + v) > 0 \qquad \text{if } u \geq 0, v \geq 0, u > v. \quad \blacksquare$$

Example 3.27 Define $f(x) = x^3$ for all x. Then the function $f:\mathbb{R} \to \mathbb{R}$ is strictly increasing. Indeed, the difference of cubes formula states:

$$u^3 - v^3 = (u - v)(u^2 + uv + v^2) \qquad \text{for all } u, v.$$

If u and v have the same sign, then $uv > 0$; thus $u^2 + uv + v^2 > 0$, so from the difference of cubes formula $u^3 > v^3$ if $u > v$. On the other hand, if $u > 0 > v$, then $u^3 > 0 > v^3$. Finally, if $u > v$ and either u or v equals 0, then clearly $u^3 > v^3$. ■

Strictly monotone functions $f: D \to \mathbb{R}$ have the property that for a point y in the image $f(D)$ there is *exactly one point* x in its domain D such that $f(x) = y$. This property is important in general and therefore it has a name.

Definition A function $f: D \to \mathbb{R}$ is said to be *one-to-one* provided that for each point y in its image $f(D)$, there is exactly one point x in its domain D such that $f(x) = y$.

It is not difficult to see that $f: D \to \mathbb{R}$ is one-to-one provided that for u and v in D, if $f(u) = f(v)$, then $u = v$.

For a function $f: D \to \mathbb{R}$ that is one-to-one, by definition, if y is a point in $f(D)$, there is exactly one point x in D such that $f(x) = y$. We will denote this point x by $f^{-1}(y)$, so we have defined the function

$$f^{-1}: f(D) \to \mathbb{R},$$

which we call the *inverse* of the function $f: D \to \mathbb{R}$. To clarify matters in considering functions and their inverses, it sometimes is useful to denote the variable in the domain by x and the variable in the image by y. The inverse function is characterized by the following relations:

$$f^{-1}(f(x)) = x \qquad \text{for all } x \text{ in } D;$$
$$f(f^{-1}(y)) = y \qquad \text{for all } y \text{ in } f(D).$$

As noted above, strictly monotone functions are one-to-one and therefore have inverse functions.

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Example 3.28 Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 \qquad \text{for all } x.$$

Since $f : \mathbb{R} \to \mathbb{R}$ is a polynomial, it is continuous. We verified in Example 3.27 that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Now the Intermediate Value Theorem, as expressed in Theorem 3.14, implies that the image $f(\mathbb{R})$ is an interval. However,

$$f(n) = n^3 > n \qquad \text{and} \qquad f(-n) = -n^3 < -n \qquad \text{for every natural number } n.$$

So $f(\mathbb{R})$ is an interval that is unbounded above and below, and hence $f(\mathbb{R}) = \mathbb{R}$. The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ has the following two characteristic properties of inverse functions:

$$f^{-1}(x^3) = x \qquad \text{for all } x \text{ in } \mathbb{R};$$
$$(f^{-1}(y))^3 = y \qquad \text{for all } y \text{ in } f(\mathbb{R}) = \mathbb{R}.$$

The standard exponential notation, of course, is that

$$y^{1/3} \equiv f^{-1}(y) \qquad \text{for all } y \text{ in } \mathbb{R}.$$

■

Many functions occur as the *inverse* of another function $f : D \to \mathbb{R}$, and properties of the inverse function can be deduced from properties of the original function $f : D \to \mathbb{R}$. When we study differentiation, we will see examples of this. Observe that the inverse of a strictly monotone function is strictly monotone. The following is also an interesting example of the way inverse functions inherit properties from their origins.

Theorem 3.29 Let I be an interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

Proof

Define $D \equiv f(I)$. Then the function $f^{-1} : D \to \mathbb{R}$ is a monotonically increasing function that has the interval I as its image. Theorem 3.23 implies that the function $f^{-1} : D \to \mathbb{R}$ is continuous. ■

The preceding theorem provides us with a new class of continuous functions. For n a natural number define $f : [0, \infty) \to \mathbb{R}$ by

$$f(x) = x^n \qquad \text{for all } x \geq 0.$$

Arguing just as in Example 3.28, by the difference of powers formula and the Intermediate Value Theorem, we see that $f : [0, \infty) \to \mathbb{R}$ is strictly increasing and $f([0, \infty)) = [0, \infty)$. The preceding theorem implies that the inverse function $f^{-1} : [0, \infty) \to \mathbb{R}$ is continuous and, of course, it has the following two characteristic properties of an inverse function:

$$f^{-1}(x^n) = x \qquad \text{for all } x \geq 0;$$
$$(f^{-1}(y))^n = y \qquad \text{for all } y \text{ in } f([0, \infty)) = [0, \infty).$$

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The standard exponential notation, of course, is that

$$y^{1/n} \equiv f^{-1}(y) \qquad \text{for all } y \geq 0,$$

and $y^{1/n}$ is called the *nth root of* y.

For a negative integer n and any number $x \neq 0$, we define

$$x^n \equiv 1/x^{-n}.$$

By using an induction argument to first prove it for natural numbers, we have the following familiar algebraic formulas for integer powers, whose proof we leave as an exercise; for integers m and n and any number $x \neq 0$,

$$x^n \cdot x^m = x^{m+n} \qquad \text{and} \qquad (x^n)^m = x^{nm}. \tag{3.26}$$

We will now define *rational powers* so that these two algebraic properties continue to hold.

Definition For $x > 0$ and rational number $r = m/n$, where m and n are integers with n positive, we define

$$x^r \equiv (x^m)^{1/n}.$$

But there is an obstacle here. Since rational numbers can be expressed in different ways as quotients of integers, we need to establish that if m is an integer and n and k are natural numbers, then

$$(x^m)^{1/n} = (x^{km})^{1/kn}. \tag{3.27}$$

Indeed, since, for $u > 0$,

$$u^k = [(u^{1/n})^n]^k = (u^{1/n})^{kn},$$

setting $u = x^m$, we have

$$x^{km} = (x^m)^k = [(x^m)^{1/n}]^{kn},$$

from which (3.27) follows and so rational powers are properly defined.

We leave it as an exercise to first verify for each positive number x and integers m and n with n positive,

$$(x^m)^{1/n} = (x^{1/n})^m, \tag{3.28}$$

and to then use this to verify the extension of formulas (3.26) to rational numbers r and s: for $x > 0$,

$$x^r \cdot x^s = x^{r+s} \qquad \text{and} \qquad (x^r)^s = x^{rs}. \tag{3.29}$$

Proposition 3.30 For r a rational number, define

$$f(x) = x^r \qquad \text{for } x \geq 0.$$

The function $f : [0, \infty) \to \mathbb{R}$ is continuous.

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Proof

We will express the function, $f : [0, \infty) \to \mathbb{R}$ as the composition of continuous functions, and hence, by Theorem 3.6, $f : [0, \infty) \to \mathbb{R}$ is continuous. Indeed, define

$$g(x) = x^{1/n} \qquad \text{and} \qquad h(x) = x^m \qquad \text{for } x \geq 0.$$

By definition,

$$f(x) = g(h(x)) = (g \circ h)(x) \qquad \text{for } x \geq 0.$$

The function $h : [0, \infty) \to \mathbb{R}$ is continuous since it is a polynomial and, by Theorem 3.29, the function $g : [0, \infty) \to \mathbb{R}$ is continuous since it is the inverse of a strictly increasing function defined on an interval. ■

EXERCISES FOR SECTION 3.6

1. For each of the following statements, determine whether it is true or false and justify your answer.
 a. A monotone function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.
 b. A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.
 c. A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
 d. A one-to-one function $f : \mathbb{R} \to \mathbb{R}$ is monotone.

2. **a.** Find a continuous function $f: (0, 1) \to \mathbb{R}$ with an image equal to $\mathbb{R}$.
 b. Find a continuous function $f: (0, 1) \to \mathbb{R}$ with an image equal to $[0, 1]$.
 c. Find a continuous function $f: \mathbb{R} \to \mathbb{R}$ that is strictly increasing and has an image equal to $(-1, 1)$.

3. Find the images of each of the following functions:
 a. $f: [0, \infty) \to \mathbb{R}$ defined by $f(x) = 1/(1 + x^2)$ for $x \geq 0$.
 b. $h: (0, 1) \to \mathbb{R}$ defined by $h(x) = 1/(x^2 + 8x)$ for $0 < x < 1$.

4. Define

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0. \end{cases}$$

Show that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and that $f^{-1}: f(\mathbb{R}) \to \mathbb{R}$ is continuous at 1.

5. Let $D = [0, 1] \cup (2, 3]$ and define $f: D \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ x - 1 & \text{if } 2 < x \leq 3. \end{cases}$$

Prove that $f: D \to \mathbb{R}$ is continuous. Determine $f^{-1}: f(D) \to \mathbb{R}$ and prove that $f^{-1}: f(D) \to \mathbb{R}$ is not continuous. Does this contradict Theorem 3.29?

6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be odd provided that

$$f(-x) = -f(x) \qquad \text{for all } x.$$

Show that if $f : \mathbb{R} \to \mathbb{R}$ is odd and the restriction of this function to the interval $[0, \infty)$ is strictly increasing, then $f : \mathbb{R} \to \mathbb{R}$ itself is strictly increasing.

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7. For an odd natural number n, define $f(x) = x^n$ for all x. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and $f(\mathbb{R}) = \mathbb{R}$.

8. Recall that $\mathbb{Q}$ denotes the set of rational numbers. Show that there does not exist a strictly increasing function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(\mathbb{Q}) = \mathbb{R}$.

9. Prove the algebraic identities (3.26) for any integers m and n and any number $x \neq 0$.

10. For positive numbers a and b and natural numbers n and m, show that

$$a = b \text{ if and only if } a^n = b^n \text{ if and only if } a^{1/m} = b^{1/m}.$$

11. Show that for a positive number x and integers m and n, with n positive,

$$(x^{1/n})^m = (x^m)^{1/n}.$$

12. Use Exercises 10 and 11 to prove formulas (3.29).

13. Let the function $f : [a, b] \to \mathbb{R}$ be continuous and one-to-one and such that $f(a) < f(b)$. Let c be a point in the open interval (a, b). Prove that $f(a) < f(c) < f(b)$.

14. Let the function $f : [a, b] \to \mathbb{R}$ be continuous and one-to-one and such that $f(a) < f(b)$. Prove that $f : [a, b] \to \mathbb{R}$ is strictly increasing. (*Hint:* Use Exercise 13.)

15. Let the function $f : [a, b] \to \mathbb{R}$ be continuous and one-to-one. Prove that $f : [a, b] \to \mathbb{R}$ is strictly monotone. (*Hint:* Use Exercise 14.)

3.7 LIMITS

In the preceding sections of this chapter we have studied the properties of continuous functions. We now turn to the study of the behavior of functions near points that are not necessarily in the domain of the given function. For a set D of numbers and a number x_0, we use the notation $D\backslash\{x_0\}$ to denote the set $\{x \text{ in } D \mid x \neq x_0\}$. It is convenient to say that a sequence $\{x_n\}$ is *distinct from* the point x_0, if $x_n \neq x_0$, for every index n.

Definition For a set D of real numbers, the number x_0 is called a *limit point of* $\mathbb{D}$ provided that there is a sequence of points in $D\backslash\{x_0\}$ that converges to x_0.

Example 3.31 For numbers a and b such that $a < b$, both a and b are limit points of the open interval (a, b), although neither point belongs to (a, b). To see that a is a limit point of (a, b), observe that $\{a + (b - a)/2n\}$ is a sequence in (a, b), distinct from a, that converges to a. ∎

Example 3.32 Every real number is a limit point of $\mathbb{Q}$, the set of rational numbers. Indeed, let x_0 be any real number. Then, by the density of the rational numbers (Theorem 1.9), for each natural number n we can select a rational number q_n in the interval $(x_0, x_0 + 1/n)$. Then $\{q_n\}$ is a sequence of rational numbers, distinct from x_0, that converges to x_0. A similar argument shows that every real number is also a limit point of the set of irrational numbers. ∎

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Definition Given a function $f: D \to \mathbb{R}$ and a limit point x_0 of its domain D, for a number ℓ, we write

$$\lim_{x \to x_0} f(x) = \ell \tag{3.30}$$

provided that whenever $\{x_n\}$ is a sequence in $D \backslash \{x_0\}$ that converges to x_0,

$$\lim_{n \to \infty} f(x_n) = \ell.$$

We read (3.30) as "The limit of $f(x)$ as x approaches x_0, with x in D, equals ℓ."

For a function $f: D \to \mathbb{R}$ and a limit point x_0 of its domain D, if there is a number ℓ such that $\lim_{x \to x_0} f(x) = \ell$, we write "$\lim_{x \to x_0} f(x)$ exists," and if there is no such number ℓ, we write "$\lim_{x \to x_0} f(x)$ does not exist."

Comparing the definition of *limit* with the definition of *continuity* of a function at a point in its domain, it is not difficult (Exercise 8) to see that if the number x_0 is a limit point of the set of numbers D and also belongs to D, then a function $f: D \to \mathbb{R}$ is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Therefore, since we have already provided many examples of continuous functions, we have already computed many limits.

Example 3.33 We have proven that the quotient of polynomials is continuous at points where the denominator is nonzero, that the square root function is continuous, and that composition of continuous functions is continuous. From this it follows that

$$\lim_{x \to 2} \sqrt{\frac{3x+3}{x^3-4}} = \frac{3}{2}.$$

■

Example 3.34

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2. \tag{3.31}$$

To verify this, we let the sequence $\{x_n\}$ converge to 1, with $x_n \neq 1$ for all n. Then, by the difference of squares formula, $(x_n^2 - 1)/(x_n - 1) = x_n + 1$ for all n. Thus,

$$\lim_{n \to \infty} \frac{x_n^2 - 1}{x_n - 1} = \lim_{n \to \infty} [x_n + 1] = 2,$$

and this proves (3.31). ■

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Example 3.35

$$\lim_{x \to 8} \frac{x-8}{x^{1/3}-2} = 12. \tag{3.32}$$

To verify this, we let the sequence $\{x_n\}$ converge to 8, with $x_n \neq 8$ for all n. Then, by the difference of cubes formula, for each n,

$$x - 8 = (x^{1/3})^3 - 2^3 = (x^{1/3} - 2)(x^{2/3} + 2x^{1/3} + 4),$$

so that by the continuity of the nth root functions,

$$\lim_{n \to \infty} \frac{x_n - 8}{x_n^{1/3} - 2} = \lim_{n \to \infty} \left[x_n^{2/3} + 2x_n^{1/3} + 4\right] = 12. \qquad \blacksquare$$

The following theorem is an analog, and also a consequence, of the sum, product, and quotient properties of convergent sequences. A completely similar result was established for continuous functions in Section 3.1.

Theorem 3.36 For functions $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$, and a limit point x_0 of their domains D, suppose that

$$\lim_{x \to x_0} f(x) = A \qquad \text{and} \qquad \lim_{x \to x_0} g(x) = B.$$

Then

$$\lim_{x \to x_0} [f(x) + g(x)] = A + B, \tag{3.33}$$

$$\lim_{x \to x_0} [f(x)g(x)] = AB, \tag{3.34}$$

and, if $B \neq 0$ and $g(x) \neq 0$ for all x in D,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}. \tag{3.35}$$

Proof

Let $\{x_n\}$ be a sequence in $D\backslash\{x_0\}$ that converges to x_0. From the definition of limit, it follows that

$$\lim_{n \to \infty} f(x_n) = A \qquad \text{and} \qquad \lim_{n \to \infty} g(x_n) = B.$$

The sum property of convergent sequences implies that

$$\lim_{n \to \infty} [f(x_n) + g(x_n)] = A + B, \tag{3.36}$$

and the product property of convergent sequences implies that

$$\lim_{n \to \infty} [f(x_n)g(x_n)] = AB. \tag{3.37}$$

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If $g(x) \neq 0$ for all x in D, the quotient property of convergent sequences implies that

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{A}{B}. \tag{3.38}$$

From the definition of limit, (3.33), (3.34), and (3.35) follow from (3.36), (3.37), and (3.38), respectively. ■

We have the following composition property for limits.

Theorem 3.37 For functions $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$, suppose that x_0 is a limit point of D such that

$$\lim_{x\to x_0} f(x) = y_0 \tag{3.39}$$

and that y_0 is a limit point of U such that

$$\lim_{y\to y_0} g(y) = \ell. \tag{3.40}$$

Moreover, suppose that

$$f(D\backslash\{x_0\}) \text{ is contained in } U\backslash\{y_0\}. \tag{3.41}$$

Then

$$\lim_{x\to x_0} (g \circ f)(x) = \ell.$$

Proof

Let $\{x_n\}$ be a sequence in $D\backslash\{x_0\}$ that converges to x_0. From (3.39) it follows that $\{f(x_n)\}$ converges to y_0. Set $y_n = f(x_n)$ for each natural number n. Then the sequence $\{y_n\}$ converges to y_0. Assumption (3.41) implies that $\{y_n\}$ is a sequence in $U\backslash\{y_0\}$. From (3.40) it follows that $\{g(y_n)\}$ converges to ℓ. Thus,

$$\lim_{n\to\infty} (g \circ f)(x_n) = \ell.$$ ■

Example 3.38 Suppose that the function $f: \mathbb{R} \to \mathbb{R}$ has the property that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \ell.$$

From the composition property of limits it follows that if k is any natural number, then

$$\lim_{x\to 0} \frac{f(x^k) - f(0)}{x^k} = \ell.$$

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Moreover, for any $c \neq 0$, the composition property also implies that

$$\lim_{x\to 0} \frac{f(cx) - f(0)}{cx} = \ell,$$

and so

$$\lim_{x\to 0} \frac{f(cx) - f(0)}{x} = c \lim_{x\to 0} \frac{f(cx) - f(0)}{cx} = c\ell.$$

■

EXERCISES FOR SECTION 3.7

1. Find the following limits or determine that they do not exist:

a. $\lim_{x\to 0} |x|$

b. $\lim_{x\to 0, x>0} \dfrac{x+\sqrt{x}}{2+\sqrt{x}}$

c. $\lim_{x\to 0} \dfrac{|x|^2}{x}$

d. $\lim_{x\to 0} \dfrac{1}{x}$

2. Prove that

a. $\lim_{x\to 1} \dfrac{x^4-1}{x-1} = 4$

b. $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} = \dfrac{1}{2}$

3. Define the function $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$ if $x \neq 0$ and $f(0) = 4$. Show that $\lim_{x\to 0} f(x) = 1$ and f is not continuous at $x = 0$.

4. Find the following limits or determine that they do not exist:

a. $\lim_{x\to 0} \dfrac{1+1/x}{1+1/x^2}$

b. $\lim_{x\to 0} \dfrac{1+1/x^2}{1+1/x}$

c. $\lim_{x\to 1} \dfrac{1+1/(x-1)}{2+1/(x-1)^2}$

5. Let D be the set of real numbers consisting of the single number x_0. Show that the set D has no limit points. Also show that the set $\mathbb{N}$ of natural numbers has no limit points.

6. Let D be a nonempty subset of $\mathbb{R}$ that is bounded above. Is the supremum of D a limit point of D?

7. Explain why, in the definition of $\lim_{x\to x_0} f(x)$, it is necessary to require that x_0 be a limit point of D.

8. **a.** A point x_0 in D is said to be an *isolated point* of D provided that there is an $r > 0$ such that the only point of D in the interval $(x_0 - r, x_0 + r)$ is x_0 itself. Prove that a point x_0 in D is either an isolated point or a limit point of D.

b. Suppose that x_0 is an isolated point of D. Prove that every function $f: D \to \mathbb{R}$ is continuous at x_0.

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c. Prove that if the point x_0 in D is a limit point of D, then a function $f: D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

9. Suppose the function $f: \mathbb{R} \to \mathbb{R}$ has the property that there is some $M > 0$ such that

$$|f(x)| \leq M|x|^2 \qquad \text{for all } x.$$

Prove that

$$\lim_{x \to 0} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

10. For each number x, define $f(x)$ to be the largest integer that is less than or equal to x. Graph the function $f: \mathbb{R} \to \mathbb{R}$. Given a number x_0, examine

$$\lim_{x \to x_0} f(x).$$

11. Let k be a natural number. Prove that

$$\lim_{x \to 1} \frac{x^k - 1}{x - 1} = k.$$

12. (A General Monotone Convergence Principle.) Let a and b be numbers with $a < b$ and set $I = (a, b)$. Suppose that the function $f: I \to \mathbb{R}$ is monotonically increasing and bounded. Prove that $\lim_{x \to a} f(x)$ exists.

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CHAPTER 4

DIFFERENTIATION

4.1 THE ALGEBRA OF DERIVATIVES

The simplest type of function $f:\mathbb{R} \to \mathbb{R}$ is one whose graph is a line. For such a function, the ratio

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

where $x_1 \neq x_2$, does not depend on the choice of points x_1 and x_2. We denote this ratio by m and call m the *slope* of the graph of f. So a function f whose graph is a line is completely determined by prescribing its functional value at one point, say at x_0, and then prescribing its slope m; it is then defined by the formula

$$f(x) = f(x_0) + m(x - x_0) \qquad \text{for all } x. \tag{4.1}$$

For a function whose graph is not a line, it makes no sense to speak of "the slope of the graph." However, many functions have the property that at certain points on their graph, the graph can be approximated, in a sense that we will soon make precise, by a tangent line. One then defines the slope of the graph at that point to be the slope of the tangent line. The slope will vary from point to point, and when we can determine the slope at each point we have very useful information for analyzing the function. This is the basic geometric idea behind differentiation.[1]

An open interval $I = (a, b)$ that contains the point x_0 is called a *neighborhood* of x_0.

[1] We will prove a version of formula (4.1) for differentiable functions whose graphs are not lines; it is called the First Fundamental Theorem (Integrating Derivatives): For a differentiable function $f:\mathbb{R} \to \mathbb{R}$ whose derivative is continuous, formula (4.1) becomes

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt \qquad \text{for all } x.$$

The symbols and the formula will be explained in Chapter 6.

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Tangent Lines and Derivatives

To make the above precise, we need to define the *tangent line*. For a function $f : I \to \mathbb{R}$, where I is a neighborhood of the point x_0, observe that for a point x in I, with $x \neq x_0$, the slope of the line joining the points $(x_0, f(x_0))$ and $(x, f(x))$ is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

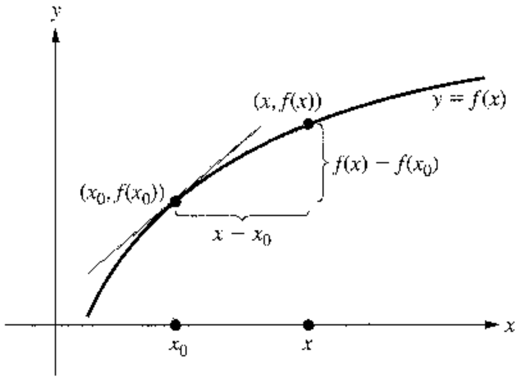

FIGURE 4.1 Approximation of the slope of the tangent line at the point $(x_0, f(x_0))$.

It is reasonable to expect that if there is a tangent line to the graph of $f : I \to \mathbb{R}$ at $(x_0, f(x_0))$, which has a slope m_0, then one should have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0.$$

For a number x_0, an open interval $I = (a, b)$ that contains x_0 is called a *neighborhood* of x_0.

Definition Let I be a neighborhood of x_0. Then the function $f : I \to \mathbb{R}$ is said to be differentiable at x_0 provided that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{4.2}$$

exists, in which case we denote this limit by $f'(x_0)$ and call it the derivative of f at x_0; that is,

$$f'(x_0) \equiv \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{4.3}$$

If the function $f : I \to \mathbb{R}$ is differentiable at every point in I, we say that f is *differentiable* and call the function $f' : I \to \mathbb{R}$ the *derivative* of f.

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For a function $f : I \to \mathbb{R}$ that is differentiable at x_0, we call the line determined by the equation

$$y = f(x_0) + f'(x_0)(x - x_0), \qquad \text{for all } x,$$

the *tangent line to the graph of f at the point* $(x_0, f(x_0))$.

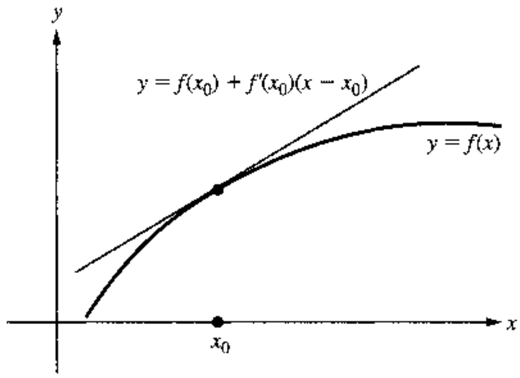

FIGURE 4.2 The tangent line to the graph of f at the point $(x_0, f(x_0))$.

Observe that since $\lim_{x \to x_0}[x - x_0] = 0$, we cannot use the quotient formula for limits in the determination of differentiability. To overcome this obstacle, in this and the next section we will develop techniques for evaluating limits of the type (4.2), which are referred to as *differentiation rules*. Before turning to these, we will consider some specific examples.

Three Examples

Example 4.1 Define $f(x) = mx + b$ for all x. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = m \qquad \text{for all } x.$$

Indeed, for x_0 in $\mathbb{R}$,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{m(x - x_0)}{x - x_0} = m \qquad \text{if } x \neq x_0.$$

Thus,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} m = m. \qquad \blacksquare$$

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Example 4.2 Consider the simplest polynomial whose graph is not a line. Define $f(x) = x^2$ for all x. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = 2x \qquad \text{for all } x.$$

Indeed, for x_0 in $\mathbb{R}$, by the difference of squares formula,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0 \qquad \text{if } x \neq x_0.$$

Thus,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} [x + x_0] = 2x_0.$$

■

Example 4.3 Define $f(x) = |x|$ for all x. Then $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at $x = 0$. To see this, observe that

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = 1 \qquad \text{if } x > 0,$$

while

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x} = -1 \qquad \text{if } x < 0.$$

It follows that

$$\lim_{x \to 0, x > 0} \frac{f(x) - f(0)}{x - 0} = 1 \qquad \text{and} \qquad \lim_{x \to 0, x > 0} \frac{f(x) - f(0)}{x - 0} = -1.$$

Thus,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \text{ does not exist.}$$

It is easy to see that if $x \neq 0$, then f is differentiable at x, and $f'(x) = 1$ if $x > 0$, while $f'(x) = -1$ if $x < 0$. ■

Differentiating Positive Integral Powers

Proposition 4.4 For a natural number n, define $f(x) = x^n$ for all x. Then the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = nx^{n-1} \qquad \text{for all } x.$$

Proof

Fix a number x_0. Observe that by the difference of powers formula,

$$x^n - x_0^n = (x - x_0)\left(x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-2} + x_0^{n-1}\right) \qquad \text{for all } x,$$

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and hence

$$\frac{f(x)-f(x_0)}{x-x_0}=x^{n-1}+x^{n-2}x_0+\cdots+x_0^{n-2}+x_0^{n-1} \qquad \text{if } x\neq x_0.$$

Observe that there are n terms on the right-hand side and that each has x_0^{n-1} as its limit as x approaches x_0. Thus, by the sum property of limits,

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=nx_0^{n-1}.$$

■

Differentiable Functions are Continuous

Proposition 4.5 Let I be a neighborhood of x_0 and suppose that the function $f: I\to\mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.

Proof
Since

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0) \qquad \text{and} \qquad \lim_{x\to x_0}[x-x_0]=0,$$

it follows from the product property of limits that

$$\lim_{x\to x_0}[f(x)-f(x_0)]=\lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right]=f'(x_0)\cdot 0=0.$$

Thus, $\lim_{x\to x_0} f(x)=f(x_0)$, which means that f is continuous at x_0. ■

As Example 4.3 shows, it is not true that continuity of a function at a point implies the differentiability of the function at that point.

Differentiating Sums, Products, and Quotients

Theorem 4.6 Let I be a neighborhood of x_0 and suppose that the functions $f: I\to\mathbb{R}$ and $g: I\to\mathbb{R}$ are differentiable at x_0. Then

i. the sum $f+g: I\to\mathbb{R}$ is differentiable at x_0 and

$$(f+g)'(x_0)=f'(x_0)+g'(x_0);$$

ii. the product $fg: I\to\mathbb{R}$ is differentiable at x_0 and

$$(fg)'(x_0)=f(x_0)g'(x_0)+f'(x_0)g(x_0);$$

iii. if $g(x)\neq 0$ for all x in I, then the reciprocal $1/g: I\to\mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{1}{g}\right)'(x_0)=\frac{-g'(x_0)}{(g(x_0))^2};$$

and

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iv. if $g(x) \neq 0$ for all x in I, then the quotient $f/g : I \to \mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proof of (i)
For x in I, with $x \neq x_0$,

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}.$$

Hence, by the definition of derivative and the sum property of limits,

$$\lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = f'(x_0) + g'(x_0). \quad \blacksquare$$

Proof of (ii)
In this proof, observe that in order to facilitate factorization, in the numerator we subtract and add the term $f(x)g(x_0)$. For x in I, with $x \neq x_0$,

$$\begin{aligned}\frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\ &= f(x)\left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\left[\frac{f(x) - f(x_0)}{x - x_0}\right].\end{aligned}$$

Since differentiability implies continuity, $\lim_{x \to x_0} f(x) = f(x_0)$. Consequently, using the definition of derivative and the sum and product properties for limits,

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f(x_0)g'(x_0) + g(x_0)f'(x_0). \quad \blacksquare$$

Proof of (iii)
For x in I, with $x \neq x_0$,

$$\begin{aligned}\frac{(1/g)(x) - (1/g)(x_0)}{x - x_0} &= \frac{1/g(x) - 1/g(x_0)}{x - x_0} \\ &= \frac{1}{g(x)g(x_0)}\left[\frac{g(x_0) - g(x)}{x - x_0}\right] \\ &= \frac{-1}{g(x)g(x_0)}\left[\frac{g(x) - g(x_0)}{x - x_0}\right].\end{aligned}$$

Since differentiability implies continuity, $\lim_{x \to x_0} g(x) = g(x_0)$. Hence we can use the definition of derivative, together with the product and quotient properties of limits, to conclude from the preceding identity that

$$\lim_{x \to x_0} \left[\frac{(1/g)(x) - (1/g)(x_0)}{x - x_0}\right] = \frac{-g'(x_0)}{(g(x_0))^2}. \quad \blacksquare$$

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Proof of (iv)
For x in I, with $x \neq x_0$, observe that

$$\frac{f(x)}{g(x)} = \frac{1}{g(x)} \cdot f(x).$$

The quotient formula for derivatives now follows from parts (ii) and (iii). ■

Proposition 4.7 For an integer n, define the set $\mathcal{O}$ to be $\mathbb{R}$ if $n \geq 0$ and to be $\{x \text{ in } \mathbb{R} \mid x \neq 0\}$ if $n < 0$. Then define

$$f(x) = x^n \qquad \text{for all } x \text{ in } \mathcal{O}.$$

The function $f : \mathcal{O} \to \mathbb{R}$ is differentiable, and

$$f'(x) = nx^{n-1} \qquad \text{for all } x \text{ in } \mathcal{O}.$$

Proof
The case in which $n > 0$ is precisely Proposition 4.4, so we need only consider the case $n < 0$. But if $n < 0$, then

$$f(x) = \frac{1}{x^{-n}} \qquad \text{for all } x \text{ in } \mathcal{O},$$

where $-n$ is a natural number. Then from Proposition 4.4 and the formula for differentiating the reciprocal of a differentiable function [part (iii) of Theorem 4.6], it follows that $f : \mathcal{O} \to \mathbb{R}$ is differentiable and

$$\begin{aligned} f'(x) &= \frac{-[(-n)x^{-n-1}]}{(x^{-n})^2} \\ &= nx^{n-1} \qquad \text{for all } x \text{ in } \mathcal{O}. \end{aligned}$$ ■

Corollary 4.8 For polynomials $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$, define the set $\mathcal{O}$ to be $\{x \text{ in } \mathbb{R} \mid q(x) \neq 0\}$. Then the quotient $p/q : \mathcal{O} \to \mathbb{R}$ is differentiable.

Proof
From Proposition 4.4 and parts (i) and (ii) of Theorem 4.6 it follows that both $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are differentiable. Then part (iv) of Theorem 4.6 implies that $p/q : \mathcal{O} \to \mathbb{R}$ is differentiable. ■

EXERCISES FOR SECTION 4.1

1. For each of the following statements, determine whether it is true or false and justify your answer.

a. If the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0, then it is differentiable at x_0.

b. If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.

c. The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if the function $f^2 : \mathbb{R} \to \mathbb{R}$ is differentiable.

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2. Define $f(x) = x^3 + 2x + 1$ for all x. Find the equation of the tangent line to the graph of $f : \mathbb{R} \to \mathbb{R}$ at the point $(2, 13)$.

3. For m_1 and m_2 numbers, with $m_1 \neq m_2$, define

$$f(x) = \begin{cases} m_1 x + 4 & \text{if } x \leq 0 \\ m_2 x + 4 & \text{if } x \geq 0. \end{cases}$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous but not differentiable at $x = 0$.

4. Use the definition of derivative to compute the derivative of the following functions at $x = 1$:

a. $f(x) = \sqrt{x+1}$ for all $x > 0$.
b. $f(x) = x^3 + 2x$ for all x.
c. $f(x) = 1/(1 + x^2)$ for all x.

5. Evaluate the following limits or determine that they do not exist:

a. $\lim_{x \to 0} \dfrac{x^2}{x}$

b. $\lim_{x \to 1} \dfrac{x^2 - 1}{\sqrt{x} - 1}$

c. $\lim_{x \to 0} \dfrac{x - 1}{\sqrt{x} - 1}$

d. $\lim_{x \to 2} \dfrac{x^4 - 16}{x - 2}$

6. Let I and J be open intervals, and the functions $f : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ have the property that $h(J) \subseteq I$, so the composition $f \circ h : J \to \mathbb{R}$ is defined. Show that if x_0 is in J, $h : J \to \mathbb{R}$ is continuous at x_0, $h(x) \neq h(x_0)$ if $x \neq x_0$, and $f : I \to \mathbb{R}$ is differentiable at $h(x_0)$, then

$$\lim_{x \to x_0} \frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)} = f'(h(x_0)).$$

7. Use Exercise 6 to show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 = 1$, then:

a. $\lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = f'(1)$

b. $\lim_{t \to 1} \dfrac{f(\sqrt{t}) - f(1)}{\sqrt{t} - 1} = f'(1)$

c. $\lim_{x \to 1} \dfrac{f(x^2) - f(1)}{x^2 - 1} = f'(1)$

d. $\lim_{x \to 1} \dfrac{f(x^2) - f(1)}{x - 1} = 2f'(1)$

e. $\lim_{x \to 1} \dfrac{f(x^3) - f(1)}{x - 1} = 3f'(1)$.

(*Hint:* For the last two limits, first make use of the difference of powers formula.)

8. For a natural number $n \geq 2$, define

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^n & \text{if } x > 0. \end{cases}$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

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9. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$-x^2 \leq f(x) \leq x^2 \qquad \text{for all } x.$$

Prove that f is differentiable at $x = 0$ and that $f'(0) = 0$.

10. For real numbers a and b, define

$$g(x) = \begin{cases} 3x^2 & \text{if } x \leq 1 \\ a + bx & \text{if } x > 1. \end{cases}$$

For what values of a and b is the function $g : \mathbb{R} \to \mathbb{R}$ differentiable at $x = 1$?

11. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$. Also, suppose that for each natural number n, $g(1/n) = 0$. Prove that $g(0) = 0$ and $g'(0) = 0$.

12. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and monotonically increasing. Show that $f'(x) \geq 0$ for all x.

13. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a bounded sequence $\{x_n\}$ with $x_n \neq x_m$, if $n \neq m$, such that $f(x_n) = 0$ for every index n. Show that there is a point x_0 at which $f(x_0) = 0$ and $f'(x_0) = 0$. (*Hint:* Use the Sequential Compactness Theorem.)

14. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Analyze the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h}.$$

[*Hint:* Subtract and add $f(x_0)$ to the numerator.]

15. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Prove that

$$\lim_{x \to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0).$$

16. Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 0$. Prove that

$$\lim_{x \to 0} \frac{f(x^2) - f(0)}{x} = 0.$$

17. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0. For real numbers a, b, and c, with $c \neq 0$, prove that

$$\lim_{x \to 0} \frac{f(ax) - f(bx)}{cx} = \left[\frac{a - b}{c}\right] f'(0).$$

18. Let the function $h : \mathbb{R} \to \mathbb{R}$ be bounded. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 1 + 4x + x^2 h(x) \qquad \text{for all } x.$$

Prove that $f(0) = 1$ and $f'(0) = 4$. (*Note:* There is no assumption about the differentiability of the function h.)

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19. For a natural number n, the Geometric Sum Formula asserts that

$$1 + x + \cdots + x^n = \frac{1 - x^{n-1}}{1 - x} \qquad \text{if } x \neq 1.$$

By differentiating, find a formula for

$$1 + x + 2x^2 + \cdots + nx^n$$

and then for

$$1^2 + 2^2 x + \cdots + n^2 x^{n-1}.$$

4.2 DIFFERENTIATING INVERSES AND COMPOSITIONS

Theorem 3.29 asserts that if I is an interval and the function $f : I \to \mathbb{R}$ is strictly monotone with image $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is continuous. It is natural to consider the question of the differentiability of the inverse function f^{-1} at the point $y_0 \equiv f(x_0)$ in J if f is differentiable at x_0. We will show that if f is differentiable at x_0 with $m \equiv f'(x_0)$, then if $m \neq 0$, the inverse function f^{-1} is differentiable at y_0 and its derivative at y_0 equals $1/m$. Before proving this, we explain geometrically why this formula is natural.

Indeed, suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0 and its tangent line ℓ at the point $p = (x_0, y_0)$ is not horizontal. This means that

$$m \equiv f'(x_0) \neq 0.$$

Then it appears that the tangent line to the inverse function at the point p is also the same line ℓ. From the viewpoint of the inverse function, the line ℓ is now defined as the graph of a function with domain on the vertical axis, and therefore its slope is the reciprocal of $m \equiv f'(x_0)$.

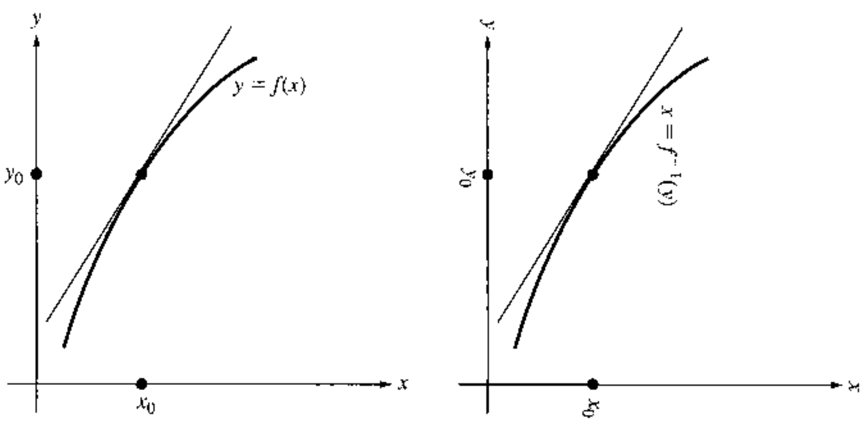

FIGURE 4.3 $(f^{-1})'(y_0) = 1/f'(x_0)$.

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Example 4.9 Define

$$f(x) = 2x + 1 \qquad \text{for all } x.$$

The graph of the function $f : \mathbb{R} \to \mathbb{R}$ is a line of slope 2, and the derivative of the function is constant, with constant value 2. The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is explicitly given by

$$f^{-1}(y) = y/2 - 1/2 \qquad \text{for all } y.$$

The graph of the inverse function is a line of slope $1/2$, and the derivative of the inverse function is constant, with constant value $1/2$. ■

Example 4.10 Define

$$f(x) = x^2 \qquad \text{for } x > 0.$$

The function $f : (0, \infty) \to \mathbb{R}$ is strictly increasing, and $f'(3) = 6$. Thus, the slope of the tangent line to the graph of f at the point $(3, 9)$ equals 6. The inverse function $f^{-1} : (0, \infty) \to \mathbb{R}$ is explicitly given by

$$f^{-1}(y) = \sqrt{y} \qquad \text{for all } y > 0.$$

We expect that the tangent line to the graph of the inverse at the point on the graph at which $y_0 = 9$ will be $1/6$. Indeed, if $y > 0$ and $y \neq 9$, then since $y - 9 = (\sqrt{y} + 3)(\sqrt{y} - 3)$,

$$\begin{aligned}\frac{f^{-1}(y) - f^{-1}(9)}{y - 9} &= \frac{\sqrt{y} - 3}{y - 9} \\ &= \frac{1}{\sqrt{y} + 3} \cdot \frac{\sqrt{y} - 3}{\sqrt{y} - 3} \\ &= \frac{1}{\sqrt{y} + 3}.\end{aligned}$$

Thus, by the continuity of the square root function,

$$\lim_{y \to 9} \frac{f^{-1}(y) - f^{-1}(9)}{y - 9} = \lim_{y \to 9} \frac{1}{\sqrt{y} + 3} = \frac{1}{6}. \qquad ■$$

The Derivative of the Inverse Function

Theorem 4.11 Let I be a neighborhood of x_0 and let the function $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose that f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $J = f(I)$. Then the inverse $f^{-1} : J \to \mathbb{R}$ is differentiable at the point $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}. \tag{4.4}$$

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Proof
It follows from the Intermediate Value Theorem that J is a neighborhood of $y_0 = f(x_0)$. For a point y in J, with $y \neq y_0$, define

$$x \equiv f^{-1}(y),$$

so that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = 1 \Big/ \frac{f(x) - f(x_0)}{x - x_0}. \tag{4.5}$$

Since the inverse function $f^{-1} : J \to \mathbb{R}$ is continuous,

$$\lim_{y \to y_0} x \equiv \lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) \equiv x_0.$$

By the composition property for limits, the quotient property of limits, and the definition of the differentiability of $f : I \to \mathbb{R}$ at x_0, it follows that

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} 1 \Big/ \frac{f(x) - f(x_0)}{x - x_0} = \frac{1}{f'(x_0)}.$$

Thus, $f^{-1} : J \to \mathbb{R}$ is differentiable at y_0, and its derivative at y_0 is given by (4.4). ∎

Frequently, the inverse function is considered the primary object, in which case we denote the variable that it depends on by x, and because of this we present the following corollary.

Corollary 4.12 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is strictly monotone and differentiable with a nonzero derivative at each point in I. Define $J = f(I)$. Then the inverse function $f^{-1} : J \to \mathbb{R}$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)} \qquad \text{for all } x \text{ in } J. \tag{4.6}$$

Proof
Since differentiability implies continuity, the function $f : I \to \mathbb{R}$ is continuous. Hence we can apply the previous theorem at x in J, where we have $x = f(f^{-1}(x))$ and $f^{-1}(x)$ plays the role of x_0 in the preceding theorem. ∎

Proposition 4.13 For a natural number n, define $g(x) = x^{1/n}$ for all $x > 0$. Then the function $g : (0, \infty) \to \mathbb{R}$ is differentiable and

$$g'(x) = \frac{1}{n} x^{1/n - 1} \qquad \text{for all } x > 0.$$

Proof
If $f : (0, \infty) \to \mathbb{R}$ is defined by $f(x) = x^n$ for $x > 0$, then by definition $g : (0, \infty) \to \mathbb{R}$ is the inverse of $f : (0, \infty) \to \mathbb{R}$. According to Proposition 4.7,

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$f'(x) = nx^{n-1}$ if $x > 0$. Using Corollary 4.12, we conclude that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n\left(x^{1/n}\right)^{n-1}} = \frac{1}{n}x^{1/n-1} \qquad \text{if } x > 0. \qquad \blacksquare$$

The Derivative of the Composition

We have shown that the composition of continuous functions is continuous. The composition of differentiable functions is differentiable, and there is a formula for the derivative of the composition. This is the content of the following theorem.

Theorem 4.14 The Chain Rule Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. Let J be an open interval such that $f(I) \subseteq J$ and suppose that the function $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then the composition $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0). \tag{4.7}$$

Proof

Define $y_0 = f(x_0)$. For each x in I with $x \neq x_0$, if we let $y = f(x)$, then since

$$\frac{f(x) - f(x_0)}{y - y_0} = 1,$$

we have

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \frac{g(y) - g(y_0)}{x - x} = \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{f(x) - f(x_0)}{x - x}, \tag{4.8}$$

provided that $y - y_0 = f(x) - f(x_0) \neq 0$. If there is an open interval containing x_0 in which

$$f(x) - f(x_0) \neq 0 \qquad \text{if } x \neq x_0,$$

then the result follows by taking limits in the above identity and using the composition and product properties of limits. To account for the possibility that there is no such interval, we introduce an auxiliary function $h : J \to \mathbb{R}$ by defining

$$h(y) = \begin{cases} [g(y) - g(y_0)]/[y - y_0] & \text{for } y \text{ in } J \text{ with } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0. \end{cases}$$

Observe that

$$g(y) - g(y_0) = h(y)[y - y_0] \qquad \text{for all } y \text{ in } J,$$

so the preceding identity (4.8) can be rewritten as

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = h(f(x))\left[\frac{f(x) - f(x_0)}{x - x_0}\right] \qquad \text{for } x \neq x_0 \text{ in } I. \tag{4.9}$$

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Indeed, if $f(x) \neq f(x_0)$, then (4.9) agrees with (4.8), while if $f(x) = f(x_0)$, then each side of (4.9) is equal to 0.

From the very definition of $g'(y_0)$ it follows that h is continuous at y_0. Furthermore, the differentiability of f at x_0 implies the continuity of f at x_0, and hence, since the composition of continuous functions is continuous, the composition $h \circ f : I \to \mathbb{R}$ is also continuous at x_0. From this, using the product theorem for limits and the identity (4.9), we see that

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x_0))f'(x_0) = g'(f(x_0))f'(x_0). \qquad \blacksquare$$

Recall that, in Section 3.6, for a rational number $r = m/n$, where m and $n > 0$ are integers, we defined the rth power function by setting $x^r \equiv (x^m)^{1/n}$ for $x > 0$ and showed that this defines a continuous function. We now show that the rth power function is differentiable.

Proposition 4.15 For a rational number r, define $h(x) = x^r$ for $x > 0$. Then the function $h : (0, \infty) \to \mathbb{R}$ is differentiable and

$$h'(x) = rx^{r-1} \qquad \text{for } x > 0.$$

Proof

Since r is a rational number, we can choose integers m and n with $n > 0$ such that $r = m/n$. For $x > 0$, define $f(x) = x^m$ and $g(x) = x^{1/n}$, so that by definition $h(x) = g(f(x))$. According to Proposition 4.7, the function $f : (0, \infty) \to \mathbb{R}$ is differentiable and $f'(x) = mx^{m-1}$ if $x > 0$. On the other hand, according to Proposition 4.13, the function $g : (0, \infty) \to \mathbb{R}$ is differentiable and $g'(x) = (1/n)x^{1/n-1}$ if $x > 0$. From the Chain Rule, we conclude that for $x > 0$,

$$\begin{aligned} h'(x) &= (g \circ f)'(x) \\ &= g'(f(x))f'(x) \\ &= (1/n)(x^m)^{1/n-1}(mx^{m-1}) \\ &= rx^{r-1}. \end{aligned} \qquad \blacksquare$$

EXERCISES FOR SECTION 4.2

1. Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable and define $h \equiv f \circ g : \mathbb{R} \to \mathbb{R}$. Find $h'(1)$ and $h'(2)$ if

$$g(1) = 2,\ g(2) = 1,\ f'(1) = -1,\ f'(2) = 2,\ g'(1) = 3,\ g'(2) = 4.$$

2. Define

$$f(x) = \frac{1}{\sqrt{1 + x^2}} \qquad \text{for all } x > 0.$$

Find $(f^{-1})'(\sqrt{1/5})$.

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3. Define $f(x) = 1/x^2$ for $x > 0$. Show that $f^{-1}(y) = 1/\sqrt{y}$ for $y > 0$. Calculate the derivative of the inverse directly and then check that this calculation agrees with formula (4.6).

4. Define $f(x) = 1/(1+x)$ for x in $I \equiv (0, 1)$. Show that $f : I \to \mathbb{R}$ is strictly decreasing and differentiable and that $f(I) = (1/2, 1) \equiv J$. Show that $f^{-1}(y) = (1-y)/y$ for y in J. Calculate the derivative of the inverse directly and then check that this calculation agrees with formula (4.6).

5. Let I be a neighborhood of x_0 and let $f : I \to \mathbb{R}$ be continuous, strictly monotone, and differentiable at x_0. Assume that $f'(x_0) = 0$. Use the characteristic property of inverses,

$$f^{-1}(f(x)) = x \qquad \text{for } x \text{ in } I,$$

and the Chain Rule to prove that the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is not differentiable at $f(x_0)$. Thus, the assumption in Theorem 4.11 that $f'(x_0) \neq 0$ is necessary.

6. Suppose that the function $f : (0, \infty) \to \mathbb{R}$ is differentiable and let $c > 0$. Now define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = f(cx)$ for $x > 0$. Just using the definition of derivative, show that $g'(x) = cf'(cx)$ for $x > 0$.

7. Suppose that the function $h : \mathbb{R} \to \mathbb{R}$ is strictly monotone differentiable, $h'(x) > 0$ for all x, and $h(\mathbb{R}) = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and define $g(x) = f(h^{-1}(x))$ for all x. Find $g'(x)$.

8. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $\{x_n\}$ is a strictly increasing bounded sequence with $f(x_n) \leq f(x_{n+1})$ for all n in $\mathbb{N}$. Prove that there is a number x_0 at which $f'(x_0) \geq 0$. (*Hint:* Apply the Monotone Convergence Theorem.)

9. A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(x) = f(-x) \qquad \text{for all } x,$$

and $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(x) = -f(-x) \qquad \text{for all } x.$$

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and odd, $f' : \mathbb{R} \to \mathbb{R}$ is even.

4.3 THE MEAN VALUE THEOREM AND ITS GEOMETRIC CONSEQUENCES

We will now prove one of the most useful and geometrically attractive results in calculus, the Mean Value Theorem[2]. It asserts that if the function $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to the open interval (a, b) is differentiable, then there is a point x_0 in the open interval (a, b) with the property that the tangent line to the graph at the point $(x_0, f(x_0))$ is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$.

[2] This theorem is often called the Lagrange Mean Value Theorem in order to distinguish it from the Cauchy Mean Value Theorem that we will prove in the next section.

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To prove the Mean Value Theorem, it is convenient first to prove some preliminary results.

Lemma 4.16 Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ is differentiable at x_0. If the point x_0 is either a maximizer or a minimizer of the function $f : I \to \mathbb{R}$, then $f'(x_0) = 0$.

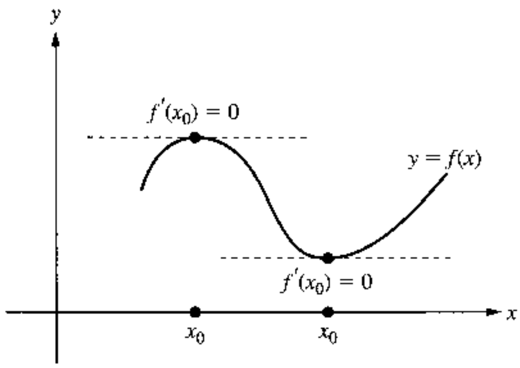

FIGURE 4.4 $f'(x_0) = 0$ if the point x_0 is a minimizer or a maximizer for the function f.

Proof

Observe that by the very definition of a derivative,

$$\lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

First suppose that x_0 is a maximizer. Then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \qquad \text{for } x \text{ in } I \text{ with } x < x_0,$$

and hence

$$f'(x_0) = \lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

On the other hand,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \qquad \text{for } x \text{ in } I \text{ with } x > x_0,$$

and hence

$$f'(x_0) = \lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Thus, $f'(x_0) = 0$.

In the case where x_0 is a minimizer, the same proof applies, with inequalities reversed. ■

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Theorem 4.17 Rolle's Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Assume, moreover, that

$$f(a) = f(b).$$

Then there is a point x_0 in the open interval (a, b) at which

$$f'(x_0) = 0.$$

Proof

Since $f : [a, b] \to \mathbb{R}$ is continuous, according to the Extreme Value Theorem, it attains both a minimum value and a maximum value on $[a, b]$. Since $f(a) = f(b)$, if both the maximizers and the minimizers occur at the endpoints, then the function $f : [a, b] \to \mathbb{R}$ is constant, so $f'(x) = 0$ at every point x in (a, b). Otherwise, the function has either a maximizer or a minimizer at some point x_0 in the open interval $I = (a, b)$, and hence, by the preceding lemma, at this point $f'(x_0) = 0$. ■

Rolle's Theorem is a special case of the Mean Value Theorem, but in fact the general result follows immediately from Rolle's Theorem.

Theorem 4.18 The Mean Value Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the restriction of f to the open interval (a, b) is differentiable. Then there is a point x_0 in the open interval (a, b) at which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

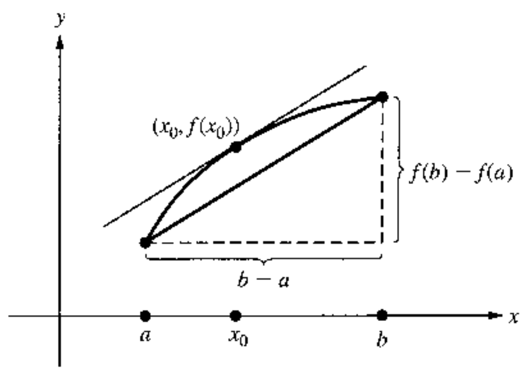

FIGURE 4.5 The tangent line is parallel to the segment joining the endpoints.

Proof

For a number m, we wish to apply Rolle's Theorem to the function $h : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - mx$ for x in $[a, b]$. To do so, we must have $h(a) = h(b)$,

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and this occurs precisely when we choose

$$m \equiv \frac{f(b) - f(a)}{b - a}.$$

For this choice of m, we apply Rolle's Theorem to choose a point x_0 in the open interval (a, b) at which $h'(x_0) = 0$. Since $h'(x) = f'(x) - m$ at this point x_0,

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}. \qquad \blacksquare$$

The preceding Mean Value Theorem is one of the essential tools for analyzing functions. Observe that its proof depends merely on the definition of the derivative and the Extreme Value Theorem.

As a general principle, if we have information about the derivative of a function that we wish to use in order to analyze the function, we should first try to apply the Mean Value Theorem. The remainder of this section comprises various applications of this strategy.

The Identity Criterion

A function $f : D \to \mathbb{R}$ is said to be *constant* provided that there is some number c such that $f(x) = c$ for all x in D.

Lemma 4.19 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is differentiable. Then $f : I \to \mathbb{R}$ is constant if and only if

$$f'(x) = 0 \qquad \text{for all } x \text{ in } I.$$

Proof

First assume that $f : I \to \mathbb{R}$ is constant. Then clearly $f'(x) = 0$ for all x in I. To prove the converse, assume that $f'(x) = 0$ for all x in I. Choose a point x_0 in I and define $c \equiv f(x_0)$. We will show that

$$f(x) = c \qquad \text{for all } x \text{ in } I.$$

Let x be a point in I and suppose that $x < x_0$. Since differentiability implies continuity, the restriction of f to the interval $[x, x_0]$ is continuous, and, of course, its restriction to the open interval (x, x_0) is differentiable. According to the Mean Value Theorem, there is a point z in the open interval (x, x_0) at which

$$f'(z) = \frac{f(x_0) - f(x)}{x_0 - x}.$$

But $f'(z) = 0$, so that $f(x) = f(x_0) = c$. For $x > x_0$, the same argument can be used for the restriction of f to the interval $[x_0, x]$ to conclude that $f(x) = c$. Consequently, the function $f : I \to \mathbb{R}$ has a constant value c. $\blacksquare$

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Two functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are said to *differ by a constant* if there is some number c such that

$$g(x) = h(x) + c \qquad \text{for all } x \text{ in } I.$$

Of course, two functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are equal provided that $g(x) = h(x)$ for all x in I. Sometimes we say that equal functions are *identically equal* in order to emphasize that the functions have the same value at *all points of their domain*. The following result is quite clear from a geometric viewpoint. We will refer to it very frequently, and so we name it.

Proposition 4.20 The Identity Criterion Let I be an open interval and let the functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ be differentiable. Then these functions differ by a constant if and only if

$$g'(x) = h'(x) \qquad \text{for all } x \text{ in } I. \tag{4.10}$$

In particular, these functions are identically equal if and only if (4.10) holds and there is some point x_0 in I at which

$$g(x_0) = h(x_0).$$

Proof
Define $f = g - h : I \to \mathbb{R}$. According to the differentiation rule for sums, $f : I \to \mathbb{R}$ is differentiable and

$$f'(x) = g'(x) - h'(x) \qquad \text{for all } x \text{ in } I.$$

Also, observe that $f : I \to \mathbb{R}$ is constant if and only if the functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ differ by a constant. The result now follows from the preceding lemma. ∎

A Criterion for Strict Monotonicity

Corollary 4.21 Let I be an open interval and the function $f : I \to \mathbb{R}$ be differentiable. Suppose that $f'(x) > 0$ for all x in I. Then $f : I \to \mathbb{R}$ is strictly increasing.

Proof
Let u and v be points in I with $u < v$. Then we can apply the Mean Value Theorem to the restriction of f to the closed bounded interval $[u, v]$ and choose a point x_0 in the open interval (u, v) at which

$$f'(x_0) = \frac{f(v) - f(u)}{v - u}.$$

Since $f'(x_0) > 0$ and $v - u > 0$, it follows that $f(u) < f(v)$. ∎

By replacing $f : I \to \mathbb{R}$ with $-f : I \to \mathbb{R}$, the above corollary implies that if $f : I \to \mathbb{R}$ has a negative derivative at each point x in I, then $f : I \to \mathbb{R}$ is strictly decreasing.

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The above results give a method for finding intervals on which a differentiable function $f : I \to \mathbb{R}$ is strictly monotonic. The effectiveness of the method depends on being able to find the points at which $f'(x) = 0$. In fact, unless the function $f : I \to \mathbb{R}$ is quite simple, it is usually very difficult to find these points.

A Criterion for Selecting Maximizers and Minimizers

Definition A point x_0 in the domain of a function $f : D \to \mathbb{R}$ is said to be a *local maximizer* for f provided that there is some $\delta > 0$ such that

$$f(x) \leq f(x_0) \qquad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

We call x_0 a *local minimizer* for f provided that there is some $\delta > 0$ such that

$$f(x) \geq f(x_0) \qquad \text{for all } x \text{ in } D \text{ such that } |x - x_0| < \delta.$$

Lemma 4.16 asserts that if I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable at x_0, then for x_0 to be either a local minimizer or a local maximizer for f, it is necessary that

$$f'(x_0) = 0.$$

However, knowing that $f'(x_0) = 0$ does not guarantee that x_0 is either a local maximizer or a local minimizer. For instance, if $f(x) = x^3$ for all x, then $f'(0) = 0$, but the point 0 is neither a local maximizer nor a local minimizer for the function f. In order to establish criteria that are sufficient for the existence of local maximizers and local minimizers, it is necessary to introduce higher derivatives.

For a differentiable function $f : I \to \mathbb{R}$ that has as its domain an open interval I, we say that $f : I \to \mathbb{R}$ has *one derivative* if $f : I \to \mathbb{R}$ is differentiable and define $f^{(1)}(x) = f'(x)$ for all x in I. If the function $f' : I \to \mathbb{R}$ itself has a derivative, we say that $f : I \to \mathbb{R}$ has *two derivatives,* or has a *second derivative,* and denote the derivative of $f' : I \to \mathbb{R}$ by $f'' : I \to \mathbb{R}$ or by $f^{(2)} : I \to \mathbb{R}$. Now let k be a natural number for which we have defined what it means for $f : I \to \mathbb{R}$ to *have k derivatives* and have defined $f^{(k)} : I \to \mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to *have $k + 1$ derivatives* if $f^{(k)} : I \to \mathbb{R}$ is differentiable, and we define $f^{(k+1)} : I \to \mathbb{R}$ to be the derivative of $f^{(k)} : I \to \mathbb{R}$. In this context, is it useful to denote $f(x)$ by $f^{(0)}(x)$.

In general, if a function has k derivatives, it does not necessarily have $k + 1$ derivatives. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|x$ for all x is differentiable but does not have a second derivative.

Theorem 4.22 Let I be an open interval containing the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has a second derivative. Suppose that

$$f'(x_0) = 0.$$

If $f''(x_0) > 0$, then x_0 is a local minimizer of f.

If $f''(x_0) < 0$, then x_0 is a local maximizer of f.

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Proof
First suppose that $f''(x_0) > 0$. Since

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} > 0,$$

it follows (Exercise 18) that there is a $\delta > 0$ such that the open interval $(x_0 - \delta, x_0 + \delta)$ is contained in I and

$$\frac{f'(x) - f'(x_0)}{x - x_0} > 0 \qquad \text{if } x \text{ belongs to } (x_0 - \delta,\ x_0 + \delta). \tag{4.11}$$

But $f'(x_0) = 0$, so (4.11) amounts to the assertion that

$$f'(x) > 0 \text{ if } x_0 < x_0 + \delta \text{ and } f'(x) < 0 \text{ if } x_0 - \delta < x < x_0.$$

Using these two inequalities and the Mean Value Theorem, it follows that

$$f(x) > f(x_0) \qquad \text{if } 0 < |x - x_0| < \delta.$$

A similar argument applies when $f''(x_0) < 0$. ■

The preceding theorem provides no information about $f(x_0)$ as a local extreme point if both $f'(x_0) = 0$ and $f''(x_0) = 0$. As we see from examining functions of the form $f(x) = cx^n$ for all x at $x_0 = 0$, if $f'(x_0) = 0$ and $f''(x_0) = 0$, then x_0 may be a local maximizer, a local minimizer, or neither.

The geometric consequences of the Mean Value Theorem that we have presented so far certainly conform to one's geometric intuition. However, it is always necessary to be careful to confirm what appears to be clear from a geometric viewpoint. In Section 9.5, we will describe a function $f : \mathbb{R} \to \mathbb{R}$ possessing the following three properties:

i. The function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
ii. There is no point at which f is differentiable.
iii. There is no interval I such that $f : I \to \mathbb{R}$ is monotonic.

It is the existence of such functions that makes it absolutely necessary to root intuitive, geometric arguments on firm analytic ground. A less startling, but still somewhat surprising, phenomenon is the existence of a function $f : \mathbb{R} \to \mathbb{R}$ that has $f'(0) > 0$ but for which there is no neighborhood I of 0 on which $f : I \to \mathbb{R}$ is increasing. We describe such a function in Exercise 20.[3]

[3] Assuming that the periodicity and differentiability properties of the sine function are familiar, the following is an example of a differentiable function having a positive derivative at $x = 0$ but such that there is no neighborhood of 0 on which it is monotonically increasing:

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The source of this counterintuitive behavior is that the derivative f' is not continuous at $x = 0$.

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EXERCISES FOR SECTION 4.3

1. For each of the following statements, determine whether it is true or false and justify your answer.

a. If the differentiable function $f:\mathbb{R}\to\mathbb{R}$ is strictly increasing, then $f'(x) > 0$ for all x.

b. If the differentiable function $f:\mathbb{R}\to\mathbb{R}$ is monotonically increasing, then $f'(x)\geq 0$ for all x.

c. If the function $f:\mathbb{R}\to\mathbb{R}$ is differentiable and

$$f(x)\leq f(0) \qquad \text{for all } x \text{ in } [-1,1],$$

then $f'(0)=0$.

d. If the function $f:\mathbb{R}\to\mathbb{R}$ is differentiable and

$$f(x)\leq f(1) \qquad \text{for all } x \text{ in } [-1,1],$$

then $f'(1)=0$.

2. Sketch the graphs of the following functions. Find the intervals on which they are increasing or decreasing.

a. $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3+ax^2+bx+c$ for all x.

b. $h:(0,\infty)\to\mathbb{R}$ defined by $h(x)=a+b/x$ for $x>0$, where $a>0$, $b>0$.

3. For real numbers $a, b, c,$ and d, define $\mathcal{O}=\{x \mid cx+d\neq 0\}$. Then define

$$f(x)=\frac{ax+b}{cx+d} \qquad \text{for all } x \text{ in } \mathcal{O}.$$

Show that if the function $f:\mathcal{O}\to\mathbb{R}$ is not constant, then it fails to have any local maximizers or minimizers. Sketch the graph.

4. For $c>0$, prove that the following equation does not have two solutions:

$$x^3-3x+c=0, \qquad 0<x<1.$$

5. Prove that the following equation has exactly one solution:

$$x^5+5x+1=0, \qquad -1<x<0.$$

6. Prove that the following equation has exactly two solutions:

$$x^4+2x^2-6x+2=0, \qquad x \text{ in } \mathbb{R}.$$

7. For any numbers a and b and an even natural number n, show that the following equation has at most two solutions:

$$x^n+ax+b=0, \qquad x \text{ in } \mathbb{R}.$$

Is this true if n is odd?

8. For numbers a and b, prove that the following equation has exactly three solutions if and only if $4a^3+27b^2<0$:

$$x^3+ax+b=0, \qquad x \text{ in } \mathbb{R}.$$

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9. Let D be the set of nonzero real numbers. Suppose that the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ are differentiable and that

$$g'(x) = h'(x) \qquad \text{for all } x \text{ in } D.$$

Do the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ differ by a constant? (*Hint:* Is D an interval?)

10. Show that there does not exist a differentiable function $F : \mathbb{R} \to \mathbb{R}$, with $F'(x) = 0$ if $x < 0$ and $F'(x) = 1$ if $x \geq 0$, by arguing that such a function would necessarily be (i) continuous, (ii) constant on $(-\infty, 0)$, and (iii) of the form $F(x) = A + Bx$ on $(0, \infty)$, and then deriving a contradiction.

11. Let n be a natural number. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that the following equation has at most $n - 1$ solutions:

$$f'(x) = 0, \qquad x \text{ in } \mathbb{R}.$$

Prove that the following equation has at most n solutions:

$$f(x) = 0, \qquad x \text{ in } \mathbb{R}.$$

12. Use an induction argument together with Exercise 11 to prove that if $p : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree n, then there are at most n solutions of the equation

$$p(x) = 0, \qquad x \text{ in } \mathbb{R}.$$

13. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that

$$\begin{cases} f'(x) = x + x^3 + 2 & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 5. \end{cases}$$

What is the function $f : \mathbb{R} \to \mathbb{R}$?

14. Suppose that the function $g : (-1, 1) \to \mathbb{R}$ is differentiable and that

$$\begin{cases} g'(x) = x/\sqrt{1 - x^2} & \text{for } -1 < x < 1 \\ g(0) = 25. \end{cases}$$

What is the function $g : (-1, 1) \to \mathbb{R}$?

15. Let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable functions and suppose that

$$g(x)f'(x) = f(x)g'(x) \qquad \text{for all } x.$$

If $g(x) \neq 0$ for all x in $\mathbb{R}$, show that there is some c in $\mathbb{R}$ such that $f(x) = cg(x)$ for all x in $\mathbb{R}$.

16. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are each differentiable and that

$$\begin{cases} f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x) & \text{for all } x \\ f(0) = 0 \quad \text{and} \quad g(0) = 1. \end{cases}$$

Prove that

$$[f(x)]^2 + [g(x)]^2 = 1 \qquad \text{for all } x.$$

(*Hint:* Define $h(x) \equiv [f(x)]^2 + [g(x)]^2$ for all x. Show that $h : \mathbb{R} \to \mathbb{R}$ is a constant function.)

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17. Let I be an open interval. Suppose that the function $f : I \to \mathbb{R}$ is continuous and that at the point x_0 in I, $f(x_0) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ if $|x - x_0| < \delta$.

18. Let I be a neighborhood of x_0 and suppose that the function $g : I \to \mathbb{R}$ is differentiable. Define

$$h(x) = \begin{cases} [g(x) - g(x_0)]/[x - x_0] & \text{if } x \neq x_0 \\ g'(x_0) & \text{if } x = x_0. \end{cases}$$

Show that $h : I \to \mathbb{R}$ is continuous. If $g'(x_0) > 0$, use Exercise 17 to show that there is a $\delta > 0$ such that $[g(x) - g(x_0)]/[x - x_0] > 0$ if $0 < |x - x_0| < \delta$.

19. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, that $f' : \mathbb{R} \to \mathbb{R}$ is continuous at 0, and that $f'(0) > 0$. Prove that there is an open interval I containing 0 such that $f : I \to \mathbb{R}$ is strictly monotonic.

20. Define

$$f(x) = \begin{cases} x - x^2 & \text{if } x \text{ is rational} \\ x + x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that $f'(0) = 1$ and yet there is no neighborhood I of the point 0 on which this function is monotonically increasing.

21. Let the function $f : \mathbb{R} \to \mathbb{R}$ have the property that there is a positive number c such that $|f(u) - f(v)| \leq c(u - v)^2$ for all u, v in $\mathbb{R}$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is constant.

22. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a positive number c such that

$$f'(x) \geq c \qquad \text{for all } x.$$

Prove that

$$f(x) \geq f(0) + cx \text{ if } x \geq 0 \qquad \text{and} \qquad f(x) \leq f(0) + cx \text{ if } x \leq 0.$$

Use these inequalities to prove that $f(\mathbb{R}) = \mathbb{R}$.

23. Let the function $f : \mathbb{R} \to \mathbb{R}$ have two derivatives and suppose that

$$f(x) \leq 0 \qquad \text{and} \qquad f''(x) \geq 0 \qquad \text{for all } x.$$

Prove that $f : \mathbb{R} \to \mathbb{R}$ is constant. (*Hint:* Observe that $f' : \mathbb{R} \to \mathbb{R}$ is increasing.)

24. Let the function $f : \mathbb{R} \to \mathbb{R}$ have two derivatives with $f(0) = 0$ and

$$f'(x) \leq f(x) \qquad \text{for all } x.$$

Is $f(x) = 0$ for all x?

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4.4 THE CAUCHY MEAN VALUE THEOREM AND ITS ANALYTIC CONSEQUENCES

The following is a useful extension of the Mean Value Theorem.

Theorem 4.23 The Cauchy Mean Value Theorem Suppose that the functions $f:[a, b] \to \mathbb{R}$ and $g:[a, b] \to \mathbb{R}$ are continuous and that their restrictions to the open interval (a, b) are differentiable. Moreover, assume that

$$g'(x) \neq 0 \qquad \text{for all } x \text{ in } (a, b).$$

Then there is a point x_0 in the open interval (a, b) at which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}. \tag{4.12}$$

Proof

To prove this theorem we will use Rolle's Theorem in a manner similar to the way we proved the Mean Value Theorem. For a number m, we wish to apply Rolle's Theorem to the function $h:[a, b] \to \mathbb{R}$ defined by

$$h(x) \equiv f(x) - mg(x) \qquad \text{for } x \text{ in } [a, b].$$

To do so, we must have $h(a) = h(b)$, and this occurs precisely when we choose

$$m = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

For this choice of m, we apply Rolle's Theorem to choose a point x_0 in the open interval (a, b) at which $h'(x_0) = 0$. Since $h'(x) = f'(x) - mg'(x)$, at this point x_0,

$$\frac{f'(x_0)}{g'(x_0)} = m = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

Observe that if $g(x) = x$ for $a \le x \le b$, then the Cauchy Mean Value Theorem reduces to the Mean Value Theorem.

The following consequence of the Cauchy Mean Value Theorem will be an important tool in estimating the errors that occur when, in Chapter 8, we approximate functions by polynomials.

Theorem 4.24 Let I be an open interval and n be a natural number and suppose that the function $f: I \to \mathbb{R}$ has n derivatives. Suppose also that at the point x_0 in I,

$$f^{(k)}(x_0) = 0 \qquad \text{for } 0 \le k \le n - 1.$$

Then, for each point $x \neq x_0$ in I, there is a point z strictly between x and x_0 at which

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n. \tag{4.13}$$

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Proof

Define $g(x) = (x - x_0)^n$ for all x in I. Then $g^{(k)}(x_0) = 0$ for $0 \le k \le n-1$ and $g^{(n)}(x_0) = n!$ Let x be a point in I, with $x \neq x_0$. We can suppose that $x > x_0$. By applying the Cauchy Mean Value Theorem to the functions $f : [x_0, x] \to \mathbb{R}$ and $g : [x_0, x] \to \mathbb{R}$, we can select a point x_1 in (x_0, x) at which

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(x_1)}{g'(x_1)}. \tag{4.14}$$

Now apply the Cauchy Mean Value Theorem to $f' : [x_0, x_1] \to \mathbb{R}$ and $g' : [x_0, x_1] \to \mathbb{R}$ in order to select x_2 in (x_0, x_1), at which

$$\frac{f'(x_1)}{g'(x_1)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} = \frac{f''(x_2)}{g''(x_2)}, \tag{4.15}$$

so that by (4.14),

$$\frac{f(x)}{g(x)} = \frac{f''(x_2)}{g''(x_2)}.$$

Continuing with successively higher derivatives, we obtain a point x_n in (x_0, x) such that

$$\frac{f(x)}{g(x)} = \frac{f^{(n)}(x_n)}{g^{(n)}(x_0)} = \frac{f^{(n)}(x_n)}{n!},$$

and setting $z = x_n$, we obtain (4.13). ∎

EXERCISES FOR SECTION 4.4

1. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has two derivatives, with $f(0) = f'(0) = 0$ and $|f''(x)| \le 1$ if $|x| \le 1$. Prove that $f(x) \le 1/2$ if $|x| \le 1$.

2. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree no greater than 5. Suppose that at some point x_0 in $\mathbb{R}$,

$$p(x_0) = p'(x_0) = \cdots = p^{(5)}(x_0) = 0.$$

Prove that $p(x) = 0$ for all x in $\mathbb{R}$.

3. Define $f(t) = t^2$ for $0 \le t \le 1$ and $g(t) = t^3$ for $0 \le t \le 1$.

a. Find the number c with $0 < c < 1$ at which

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}.$$

b. Show that there does not exist a number c with $0 < c < 1$ at which

$$\begin{cases} f(1) - f(0) = f'(c)(1-0) \\ \qquad\qquad \text{and} \\ g(1) - g(0) = g'(c)(1-0). \end{cases}$$

4. Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that their restrictions to the open interval (a, b) are differentiable. Also suppose that

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$|f'(x)| \ge |g'(x)| > 0$ for all x in (a, b). Prove that

$$|f(u) - f(v)| \ge |g(u) - g(v)| \qquad \text{for all } u, v \text{ in } [a, b].$$

5. Suppose that the function $f : (-1, 1) \to \mathbb{R}$ has n derivatives and that its nth derivative $f^{(n)} : (-1, 1) \to \mathbb{R}$ is bounded. Assume also that

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0.$$

Prove that there is a positive number M such that

$$|f(x)| \le M|x|^n \qquad \text{for all } x \text{ in } (-1, 1).$$

6. Suppose that the function $f : (-1, 1) \to \mathbb{R}$ has n derivatives. Assume that there is a positive number M such that

$$|f(x)| \le M|x|^n \qquad \text{for all } x \text{ in } (-1, 1).$$

Prove that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.

7. Let I be a neighborhood of x_0 and suppose that the function $f : I \to \mathbb{R}$ has two continuous derivatives. Prove that

$$\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).$$

8. Let I be an open interval and n be a natural number. Suppose that both $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ have n derivatives. Prove that $fg : I \to \mathbb{R}$ has n derivatives, obtaining the following formula called *Leibnitz's formula:*

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \qquad \text{for all } x \text{ in } I.$$

Write the formula out explicitly for $n = 2$ and $n = 3$.

4.5 THE NOTATION OF LEIBNITZ

So far, given an open interval I and a differentiable function $f : I \to \mathbb{R}$, we have denoted the function's derivative by

$$f' : I \to \mathbb{R},$$

so that $f'(x)$ is the derivative of $f : I \to \mathbb{R}$ at x in I. This notation has been completely adequate. However, as we introduce new classes of functions and when we study integration, certain formulas and algorithmic techniques become easier to assimilate using an alternate notation due to Leibnitz. Moreover, this Leibnitz notation is widely used in texts on science and engineering, so acquaintance with it is necessary.

For a differentiable function $f : I \to \mathbb{R}$, we denote $f'(x)$ by

$$\frac{d}{dx}(f(x)) \qquad \text{or} \qquad \frac{df}{dx}.$$

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If points on the graph of $f : I \to \mathbb{R}$ are denoted by (x, y), we also denote $f'(x)$ by

$$\frac{dy}{dx} \qquad \text{or} \qquad y'.$$

The great advantage of Leibnitz symbols is that, when they are properly interpreted, we can treat the symbols df, dy, dx, and so on, as if they represent members of $\mathbb{R}$ and carry out various algebraic operations, and the resulting formulas will have meaning. This notation of Leibnitz, which has been used in mathematics and science for more than 300 years, is very useful. However, the Leibnitz symbolism has certain ambiguities, so caution is needed when interpreting the formulas. This is well illustrated in the following formulation of the Chain Rule in terms of Leibnitz symbols.

The Chain Rule and Leibnitz Symbols

For numbers $a, b \neq 0$, and $c \neq 0$,

$$\frac{a}{b} = \frac{a}{c} \cdot \frac{c}{b}.$$

A corresponding cancelation formula for the Leibnitz symbols is

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}. \tag{4.16}$$

We seek a suitable interpretation of (4.16). The following is reasonable: Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$ are differentiable. Consider the composition $f \circ u : \mathbb{R} \to \mathbb{R}$. According to the Chain Rule, $f \circ u : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$(f \circ u)'(x) = f'(u(x))u'(x) \qquad \text{for all } x. \tag{4.17}$$

If we substitute

$$\frac{df}{dx} \text{ for } (f \circ u)'(x), \ \frac{df}{du} \text{ for } f'(u(x)), \text{ and } \frac{du}{dx} \text{ for } u'(x),$$

then (4.17) becomes (4.16).

Formulas such as (4.16) occur frequently, and they are very useful. In fact, similar formulas are even more useful in the calculus of functions of several variables. But, in formula (4.16), the symbol df/dt has quite a different meaning when $t = x$ and when $t = u$. While we introduced the symbol df/dx as a substitute for $f'(x)$, in formula (4.16) it means $(f \circ u)'(x)$. The whole formula (4.16) is a reasonable symbolic encapsulation of the Chain Rule, but there is ambiguity in the individual symbols. When using Leibnitz symbols, it is always necessary to interpret the meaning of the individual symbols in terms of the context in which they are being used and justify the interpretation using precise proven results.

The Derivatives of the Inverse and Leibnitz Symbols

For any two real nonzero numbers,

$$\frac{a}{b} = \frac{1}{b/a}.$$

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A corresponding inversion formula for the Leibnitz symbols is

$$\frac{dx}{dy} = \frac{1}{dy/dx}. \tag{4.18}$$

What significance can be attached to (4.18)? Well, suppose that $f : I \to \mathbb{R}$ is differentiable and strictly monotonic, with $f'(x) \neq 0$ for all x in I. Define $J = f(I)$ and set $y = f(x)$ for x in I. According to the formula for differentiating inverse functions, as expressed in Corollary 4.12, the inverse function $f^{-1} : J \to \mathbb{R}$ is differentiable and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \qquad \text{for all } x \text{ in } I. \tag{4.19}$$

If we now let

$$x = f^{-1}(y) \qquad \text{for all } y \text{ in } J,$$

then, according to the symbolism introduced above,

$$\frac{dx}{dy} = (f^{-1})'(y) = (f^{-1})'(f(x)),$$

and so, since

$$\frac{dy}{dx} = f'(x),$$

we see that (4.18) can be interpreted as a compact rewriting of (4.19).

One final notational convention: When $f : I \to \mathbb{R}$ has a second derivative, we denote $f''(x)$ by

$$\frac{d^2}{dx^2}(f(x)), \quad \frac{d^2 f}{dx^2}, \quad \frac{d^2 y}{dx^2}, \text{ and } y''.$$

EXERCISES FOR SECTION 4.5

1. Give a reasonable interpretation of the formula

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}.$$

2. Give a reasonable interpretation of the formula

$$\frac{df}{dr} = \frac{df}{du} \cdot \frac{du}{ds} \cdot \frac{ds}{dr}.$$

3. Give a reasonable interpretation of the formula

$$\frac{ds}{ds} = 1.$$

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CHAPTER 5*

ELEMENTARY FUNCTIONS AS SOLUTIONS OF DIFFERENTIAL EQUATIONS

5.1 SOLUTIONS OF DIFFERENTIAL EQUATIONS

So far, our available stock of differentiable functions is very limited. The functions we have shown to be differentiable are constant functions, rational powers, and functions formed from these by addition, multiplication, division, and composition of functions and also by considering inverse functions. But in the study of the most basic problems in mathematics and science there arise much broader classes of functions. For example, there are trigonometric and exponential functions and their inverses.

In this chapter, we will introduce exponential and trigonometric functions as solutions of differential equations. In addition to expanding the stock of available functions, this also provides an opportunity to demonstrate the power of the tools we have developed in the first four chapters.

It is important to be aware of a *provisional* aspect of our development. We are going to provisionally *assume* that two differential equations have solutions.

First, we provisionally assume that there is a solution of the following differential equation.

The Logarithmic Differential Equation

$$\begin{cases} F'(x) = 1/x & \text{for all } x > 0 \\ F(1) = 0. \end{cases} \tag{5.1}$$

The provisional assumption will be removed after we have studied integration. In Chapter 6, as a consequence of the Second Fundamental Theorem (Differentiating Integrals), we will show that there is a solution of this differential equation that is explicitly

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expressed as an integral.[1] But assuming that there is a solution of equation (5.1), we prove that the solution is unique, the solution possesses all the familiar properties of the natural logarithm, and, moreover, it has an inverse that possesses all the familiar properties of the exponential functions.

Second, we provisionally assume that there is a solution of the following differential equation.

The Trigonometric Differential Equation

$$\begin{cases} f''(x) + f(x) = 0 & \text{for all } x \\ f(0) = 1 \quad \text{and} & f'(0) = 0. \end{cases} \tag{5.2}$$

Again, this provisional assumption will be removed when we study the properties of convergent power series. In Chapter 9, as a consequence of the validity of term-by-term differentiation for a power series, we will show that there is a solution of this differential equation that is explicitly expressed as a power series.[2] But assuming that there is a solution of the differential equation (5.2), we prove that the solution is unique, the solution possesses all the familiar properties of the cosine function, and the negative of its derivative possesses all the familiar properties of the sine function. In particular, we show that the solution of the differential equation (5.2) is a periodic function.

In our analysis of the solutions of the differential equations we study in this chapter, we will frequently make use of the Identity Criterion established in Section 4.3, which we reformulate here as follows.

The Identity Criterion

A differentiable function $g : I \to \mathbb{R}$, where I is an open interval, is identically equal to 0 if and only if

i. its derivative $g' : I \to \mathbb{R}$ is identically equal to 0, and
ii. there is some point x_0 in I at which $g(x_0) = 0$.

As a preview of the techniques we will use in this chapter, we here provide a representative example of how we will use the Identity Criterion. We wish to show that the solution of the differential equation (5.1) has the following familiar property of the natural logarithm: For any positive numbers a and b,

$$F(ab) = F(a) + F(b). \tag{5.3}$$

---

[1] The solution of the differential equation (5.1) is defined by

$$F(x) = \int_1^x \frac{1}{t}\, dt \qquad \text{for all } x > 0.$$

[2] The solution of the differential equation (5.2) is defined by

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \qquad \text{for all } x.$$

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But if we treat b as a variable that we denote by x, this is equivalent to showing that if we define the function $g : (0, \infty) \to \mathbb{R}$ by

$$g(x) \equiv F(ax) - F(a) - F(x) \qquad \text{for } x > 0,$$

then g is identically equal to 0. The Chain Rule and the fact that F is a solution of the differential equation (5.1) allows us to invoke the Identity Criteria to show that the function g is identically equal to 0. Indeed, by the Chain Rule, we see that

$$g'(x) = \frac{d}{dx}[F(ax) - F(a) - F(x)] = \frac{a}{ax} - \frac{1}{x} = 0 \qquad \text{for } x > 0$$

and

$$g(1) = F(a) - F(a) - F(1) = -F(1) = 0.$$

Thus, by the Identity Criterion, g is identically equal to 0 and therefore the identity (5.3) holds.

5.2 THE NATURAL LOGARITHM AND EXPONENTIAL FUNCTIONS

Given a rational number r, we seek a differentiable function $F: (0, \infty) \to \mathbb{R}$ such that

$$F'(x) = x^r \qquad \text{for all } x > 0.$$

We immediately recognize that if $r \neq -1$ and c is any real number, then the function $F: (0, \infty) \to \mathbb{R}$ defined by

$$F(x) = \frac{x^{r+1}}{r+1} + c \qquad \text{for all } x > 0$$

is a solution of the above differential equation. By the Identity Criterion, all the solutions are of this form. The exceptional case, $r = -1$, is more interesting.

The Natural Logarithm

The exceptional problem is to find a differentiable function $F: (0, \infty) \to \mathbb{R}$ such that

$$F'(x) = \frac{1}{x} \qquad \text{for all } x > 0.$$

Based on what we have so far established, we cannot determine whether there is such a function.

In order to extend our stock of differentiable functions, let us provisionally *assume* that there is a differentiable function $F: (0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} F'(x) = 1/x & \text{for all } x > 0 \\ F(1) = 0. \end{cases} \tag{5.4}$$

In Chapter 6, we will prove that there is indeed such a function. There is only one solution to the above differential equation since if the functions F_1 and F_2 were both solutions, then their difference $g \equiv F_1 - F_2$ would have a derivative identically equal to 0 and have

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$g(1) = 0$ also. It follows from the Identity Criterion that g is identically equal to 0, and therefore $F_1 = F_2$.

Theorem 5.1 Let the function $F\colon (0, \infty) \to \mathbb{R}$ satisfy the differential equation (5.4). Then

i. $F(ab) = F(a) + F(b)$ for all $a, b > 0$.
ii. $F(a^r) = rF(a)$ if $a > 0$, and r is rational.
iii. For each number c there is a unique positive number x such that $F(x) = c$.

Proof of (i)
This property was proved at the end of the preceding section. ■

Proof of (ii)
Define

$$h(x) \equiv F(x^r) - rF(x) \qquad \text{for all } x > 0.$$

To verify the identity (ii) is to verify that the function $h : (0, \infty) \to \mathbb{R}$ is identically equal to 0. Since $F'(x) = 1/x$ for all $x > 0$, using the Chain Rule, we see that the function h is differentiable and that

$$h'(x) = \frac{d}{dx}[F(x^r) - rF(x)] = \frac{rx^{r-1}}{x^r} - \frac{r}{x} = 0.$$

Moreover, since $F(1) = 0$,

$$h(1) = F(1) - rF(1) = 0.$$

Thus, by the Identity Criterion, the function h is identically equal to 0, and therefore the identity (ii) holds. ■

Proof of (iii)
Since $F'(x) = 1/x > 0$ for all $x > 0$, the function $F\colon (0, \infty) \to \mathbb{R}$ is strictly increasing. Also observe, using (i), that

$$0 = F(1) = F\left(x \cdot \frac{1}{x}\right) = F(x) + F\left(\frac{1}{x}\right) \qquad \text{for all } x > 0,$$

so that $F(1/x) = -F(x)$ for all $x > 0$. Thus, to verify (iii), we can assume $c > 0$, and it suffices to show that there is a solution of the equation

$$F(x) = c, \qquad x > 1. \tag{5.5}$$

Since differentiability implies continuity, the function $F\colon (0, \infty) \to \mathbb{R}$ is continuous. Consequently, since $F(1) = 0 < c$, according to the Intermediate Value Theorem, to show that equation (5.5) has a solution it will suffice to show that there is a number $x_0 > 1$ such that $F(x_0) > c$. However, property (ii) implies that

$$F(2^n) = nF(2) \qquad \text{for every natural number } n.$$

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According to the Archimedean Property, we can choose a natural number n such that

$$n > \frac{c}{F(2)}$$

and so, since $F(2) > F(1) = 0$, if we define $x_0 = 2^n$, then $F(x_0) = nF(2) > c$.

■

The unique function $F: (0, \infty) \to \mathbb{R}$ that is a solution of the differential equation (5.4), and therefore possesses the properties described in the above theorem, occurs so frequently in science that it has a special name—it is called the *natural logarithm*—and $F(x)$ is denoted by $\ln x$ for $x > 0$.

From the definition of the natural logarithm and the Chain Rule, it follows that *if I is an open interval and the function $h: I \to \mathbb{R}$ is differentiable and is such that $h(x) > 0$ for all x in I, then*

$$\frac{d}{dx}(\ln h(x)) = \frac{h'(x)}{h(x)} \qquad \text{for all } x \text{ in } I. \tag{5.6}$$

Exponential Functions

Since at each point in its domain the derivative of the natural logarithm is positive, this function is strictly increasing and therefore has an inverse function $\mathbb{R}$. Denote the inverse function by $g: \mathbb{R} \to \mathbb{R}$. Then, by the two characteristic properties of the inverse function,

$$\begin{cases} g(\ln x) = x & \text{for all } x > 0 \\ & \text{and} \\ \ln g(x) = x & \text{for all } x. \end{cases}$$

Moreover, according to Theorem 4.11, the inverse function $g: \mathbb{R} \to \mathbb{R}$ is differentiable. Since

$$\ln g(x) = x \qquad \text{for all } x,$$

$$\frac{d}{dx}[\ln g(x)] = \frac{d}{dx}[x] \qquad \text{for all } x,$$

and therefore, by the differentiation formula (5.6),

$$\frac{g'(x)}{g(x)} = 1 \qquad \text{for all } x.$$

Moreover, since $\ln 1 = 0$,

$$g(0) = g(\ln 1) = 1.$$

Thus, $g: \mathbb{R} \to \mathbb{R}$, the inverse of the natural logarithm, is a solution of the following differential equation:

$$\begin{cases} g'(x) = g(x) & \text{for all } x \\ g(0) = 1. \end{cases} \tag{5.7}$$

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Since the natural logarithm is strictly increasing and its image is all of $\mathbb{R}$, there is a unique solution of the equation

$$\ln x = 1, \qquad x > 0. \tag{5.8}$$

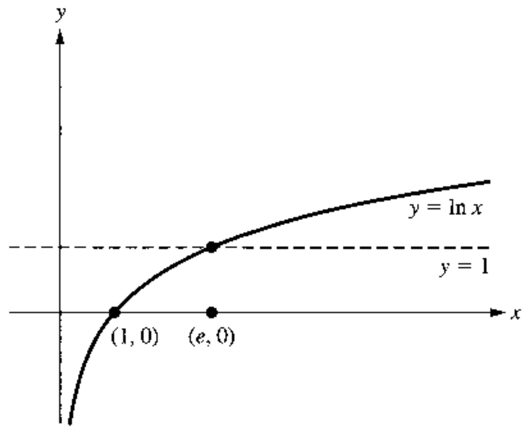

FIGURE 5.1 e is the unique number x such that $\ln x = 1$.

Definition The unique positive number x such that $\ln x = 1$ is denoted by e.

Recall that if a is a positive number and $x = m/n$ is a rational number, where m and n are integers with $n > 0$, we have defined a^x by

$$a^x \equiv (a^m)^{1/n}. \tag{5.9}$$

Up until now, we have not defined the symbol a^x if x is irrational. For instance, we have not defined the symbol $3^{\sqrt{2}}$. However, the natural logarithm and its inverse function now permit us to define irrational powers of positive numbers. Indeed, by conclusion (ii) of Theorem 5.1, for a rational number x and positive number a,

$$\ln a^x = x \ln a,$$

so that

$$g(\ln a^x) = g(x \ln a).$$

On the other hand, since g is the inverse function of the natural logarithm,

$$g(\ln a^x) = a^x.$$

Thus,

$$a^x = g(x \ln a) \qquad \text{for } a > 0 \text{ and } x \text{ rational}. \tag{5.10}$$

In particular,

$$e^x = g(x) \qquad \text{for all rational numbers } x. \tag{5.11}$$

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However, since the right-hand side of formula (5.10) is defined for any number x, rational or irrational, we now have a natural way to *define irrational powers of positive numbers.*

Definition For any positive number a and any number x we define

$$a^x \equiv g(x \ln a),$$

where $g : \mathbb{R} \to \mathbb{R}$ is the inverse of the natural logarithm.

Formula (5.10) shows that the above definition of a^x coincided with the preceding definition of a^x when x is rational. Moreover, when $a = e$, we have

$$e^x = g(x) \qquad \text{for all } x.$$

Since $g'(x) = g(x)$ for all x, using the Chain Rule we conclude that if I is an open interval and the function $h: I \to \mathbb{R}$ is differentiable, then, for all x in I,

$$\frac{d}{dx}[g(h(x))] = g'(h(x))h'(x) = g(h(x))h'(x).$$

But $g(h(x)) = e^{h(x)}$ for all x in I, so the preceding differentiation formula can be rewritten as

$$\frac{d}{dx}\left[e^{h(x)}\right] = e^{h(x)}h'(x). \tag{5.12}$$

We now have two new classes of differentiable functions.

Proposition 5.2 Let $a > 0$. Then

$$\frac{d}{dx}[a^x] = a^x \ln a \qquad \text{for all } x. \tag{5.13}$$

Proof
Using formula (5.12), for all x we have

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[g(x \ln a)] = \frac{d}{dx}[e^{x \ln a}] = e^{x \ln a} \ln a = a^x \ln a. \qquad \blacksquare$$

Proposition 5.3 Let r be any number. Then

$$\frac{d}{dx}[x^r] = rx^{r-1} \qquad \text{for all } x > 0. \tag{5.14}$$

Proof
Again using formula (5.12), for $x > 0$, since $x^r = g(r \ln x) = e^{r \ln x}$, we have

$$\frac{d}{dx}[x^r] = \frac{d}{dx}[e^{r \ln x}] = [e^{r \ln x}]\frac{d}{dx}(r \ln x) = x^r \cdot \frac{r}{x} = rx^{r-1}. \qquad \blacksquare$$

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The Exponential Differential Equation

Theorem 5.4 Let c and k be any real numbers. Then the differential equation

$$\begin{cases} F'(x) = kF(x) & \text{for all } x \\ F(0) = c \end{cases} \tag{5.15}$$

has exactly one solution. It is given by the formula

$$F(x) = ce^{kx} \qquad \text{for all } x. \tag{5.16}$$

Proof

From the differentiation formula (5.12), we see that the function defined by (5.16) defines a solution of (5.15). It remains to prove uniqueness. Let the function $F: \mathbb{R} \to \mathbb{R}$ be a solution of (5.15). Define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) \equiv \frac{F(x)}{e^{kx}} - c \qquad \text{for all } x.$$

Using the quotient rule for derivatives, we have

$$h'(x) = \frac{ke^{kx}F(x) - ke^{kx}F(x)}{(e^{kx})^2} = 0 \qquad \text{for all } x.$$

Moreover,

$$h(0) = F(0) - c = 0.$$

The Identity Criterion implies that the function h is identically equal to 0. Thus,

$$F(x) = ce^{kx} \qquad \text{for all } x.$$

So there is exactly one solution of the differential equation (5.15). ■

EXERCISES FOR SECTION 5.2

1. Let $a > 0$. Prove that for any numbers x_1 and x_2,
a. $a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$
b. $(a^{x_1})^{x_2} = a^{x_1 x_2}$

2. For $a > 0$, show that

$$\lim_{n\to\infty} n[a^{1/n} - 1] = \ln a.$$

3. Let $0 < a \leq b$. Prove that

$$\frac{b-a}{b} \leq \ln\left[\frac{b}{a}\right] \leq \frac{b-a}{a}.$$

4. Let $a > 0$. Prove that there is a number k such that

$$a^x = e^{kx} \qquad \text{for all } x.$$

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5. Use the Mean Value Theorem to show that $e^x > 1 + x$ if $x \neq 0$. Then show that the following equation has exactly one solution:

$$2e^x = (1+x)^2, \qquad x \text{ in } \mathbb{R}.$$

6. Show that there is a number c in the open interval $(1, e)$ such that

$$1 = \ln e - \ln 1 = \frac{1}{c}(e-1).$$

From this conclude that $e > 2$.

7. Use Exercise 6 to prove that the following equation has exactly one solution:

$$xe^x = 2, \qquad 0 < x < 1.$$

8. For a fixed number a, how many solutions does the following equation have?

$$x \ln x = a, \qquad x > 0.$$

9. Suppose that $h: \mathbb{R} \to \mathbb{R}$ is a differentiable function having the property that

$$h(a+b) = h(a)h(b) \qquad \text{for all } a \text{ and } b$$

and that the function is not identically equal to 0.

a. Using the definition of a derivative, prove that

$$h'(x) = h'(0)h(x) \qquad \text{for all } x.$$

b. Show that if $k = h'(0)$, then $h(x) = e^{kx}$ for all x.

10. The *hyperbolic cosine* of x, denoted by $\cosh x$, and the *hyperbolic sine* of x, denoted by $\sinh x$, are defined by

$$\cosh x \equiv \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh x \equiv \frac{e^x - e^{-x}}{2} \qquad \text{for all } x.$$

Given numbers a, α, and β, find a solution of the equation

$$\begin{cases} f''(x) - a^2 f(x) = 0 & \text{for all } x \\ f(0) = \alpha \quad \text{and} & f'(0) = \beta \end{cases}$$

that is of the form

$$f(x) = c_1 \cosh ax + c_2 \sinh ax \qquad \text{for all } x.$$

11. Show that

$$\lim_{n\to\infty} \left[\frac{\ln(1+1/n) - \ln 1}{1/n}\right] = \lim_{n\to\infty} \left[n \ln\left(1 + \frac{1}{n}\right)\right] = 1.$$

(*Hint:* Use the definition of the derivative of the logarithm at $x = 1$.)

12. Use Exercise 11 and the continuity of the exponential functions to show that

$$\lim_{n\to\infty} (1 + 1/n)^n = e.$$

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5.3 THE TRIGONOMETRIC FUNCTIONS

A function $f: \mathbb{R} \to \mathbb{R}$ is called *periodic*, with *period* $T > 0$, if

$$f(x+T) = f(x) \qquad \text{for all } x.$$

So far, with the exception of constant functions, we have not encountered any periodic functions. Since periodic phenomena occur in nature (planets, pendulums, and so on) and since the basic functions of trigonometry are periodic, we need to analyze such functions.

In the same way in which the properties of the logarithm and the exponential functions were deduced from a single differential equation, we will now define and analyze the sine and cosine functions with a single differential equation as our starting point. In this section we will study functions $f: \mathbb{R} \to \mathbb{R}$ that have the property that

$$f''(x) + f(x) = 0 \qquad \text{for all } x. \tag{5.17}$$

Lemma 5.5 Suppose that the function $f: \mathbb{R} \to \mathbb{R}$ is a solution of the differential equation

$$\begin{cases} f''(x) + f(x) = 0 & \text{for all } x \\ f(0) = 0 \quad \text{and} \quad f'(0) = 0. \end{cases} \tag{5.18}$$

Then $f(x) = 0$ for all x.

Proof

Define

$$g(x) \equiv [f(x)]^2 + [f'(x)]^2 \qquad \text{for all } x.$$

Using the Chain Rule, we have for all x,

$$\begin{aligned} g'(x) &= 2f(x)f'(x) + 2f'(x)f''(x) \\ &= 2f'(x)[f(x) + f''(x)] \\ &= 0. \end{aligned}$$

Moreover,

$$g(0) = [f(0)]^2 + [f'(0)]^2 = 0.$$

Thus, by the Identity Criterion, the function $g: \mathbb{R} \to \mathbb{R}$ is identically equal to 0. However,

$$0 \le [f(x)]^2 \le g(x) \qquad \text{for all } x,$$

so that $f(x) = 0$ for all x. ∎

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The Cosine Differential Equation

For fixed numbers α and β, consider the differential equation

$$\begin{cases} f''(x) + f(x) = 0 & \text{for all } x \\ f(0) = \alpha \quad \text{and} & f'(0) = \beta. \end{cases} \tag{5.19}$$

This equation can have at most one solution since if there were two distinct functions that were solutions of (5.19), we see that their difference would be a solution of (5.18) that is not identically zero, which would contradict Lemma 5.5.

We provisionally *assume* that in the case $\alpha = 1$ and $\beta = 0$, there is a solution of the differential equation (5.19). In Chapter 9, we will prove that there is a solution. We denote this unique solution by $C: \mathbb{R} \to \mathbb{R}$. Thus, by definition,

$$\begin{cases} C''(x) + C(x) = 0 & \text{for all } x \\ C(0) = 1 \quad \text{and} & C'(0) = 0. \end{cases} \tag{5.20}$$

The Sine Differential Equation

We define a companion function $S: \mathbb{R} \to \mathbb{R}$ by

$$S(x) = -C'(x) \qquad \text{for all } x.$$

Differentiate the first line of (5.20) to obtain

$$C'''(x) + C'(x) = 0,$$

and hence, since

$$C'''(x) = -S''(x) \qquad \text{and} \qquad C'(x) = -S(x),$$

we have

$$S''(x) + S(x) = 0 \qquad \text{for all } x.$$

On the other hand,

$$S(0) = 0 \text{ since } C'(0) = 0.$$

Moreover,

$$S'(0) = 1 \text{ since } S'(0) = -C''(0) = C(0) = 1.$$

Therefore, the function $S: \mathbb{R} \to \mathbb{R}$ is the unique solution of the differential equation

$$\begin{cases} S''(x) + S(x) = 0 & \text{for all } x \\ S(0) = 0 \quad \text{and} & S'(0) = 1. \end{cases} \tag{5.21}$$

For future reference, it is useful to record the formulas

$$S'(x) = C(x) \qquad \text{and} \qquad C'(x) = -S(x) \qquad \text{for all } x. \tag{5.22}$$

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Three Trigonometric Identities

Theorem 5.6 For all a and b,

$$[S(a)]^2 + [C(a)]^2 = 1 \qquad \text{(Pythagorean Identity)}$$
$$S(a+b) = S(a)C(b) + C(a)S(b) \qquad \text{(Sine addition formula)}$$
$$C(a+b) = C(a)C(b) - S(a)S(b) \qquad \text{(Cosine addition formula)}$$

Proof

In order to prove the Pythagorean Identity, define

$$g(x) \equiv [S(x)]^2 + [C(x)]^2 - 1 \qquad \text{for all } x.$$

Observe that

$$\begin{aligned} g'(x) &= 2S(x)S'(x) + 2C(x)C'(x) \\ &= 2C(x)[S(x) + S''(x)] = 0 \qquad \text{for all } x. \end{aligned}$$

Moreover, $g(0) = 0$ since $C(0) = 1$ and $S(0) = 0$. Therefore, by the Identity Criterion, the function g is identically equal to 0. Thus, the Pythagorean Identity holds.

In order to prove the sine addition formula, fix a real number b and define

$$f(x) \equiv S(x+b) - [S(x)C(b) + C(x)S(b)] \qquad \text{for all } x.$$

Then $f(0) = 0$ and $f'(0) = 0$. Moreover, for all x, $f''(x) + f(x) = 0$ since

$$S''(x+b) + S(x+b) = 0, \qquad S''(x) + S(x) = 0, \qquad \text{and } C''(x) + C(x) = 0.$$

Thus, the function $f\colon \mathbb{R} \to \mathbb{R}$ is a solution of (5.18), and so according to Lemma 5.5, $f(x) = 0$ for all x. This proves the sine addition formula.

Finally, differentiating the above function $f\colon \mathbb{R} \to \mathbb{R}$, we have

$$0 = f'(x) = C(x+b) - [C(x)C(b) - S(x)S(b)] \qquad \text{for all } x,$$

so the cosine addition formula is proved. ∎

Observe that as a consequence of the Pythagorean Identity,

$$|S(x)| \leq 1 \qquad \text{and} \qquad |C(x)| \leq 1 \qquad \text{for all } x. \tag{5.23}$$

The Periodicity of the Sine and Cosine

We will now show that the functions $S\colon \mathbb{R} \to \mathbb{R}$ and $C\colon \mathbb{R} \to \mathbb{R}$ are periodic. The strategy is to show that there is a smallest positive number p at which $C(p) = 0$ and then to use the sine and cosine addition formulas to prove that these functions have period $T = 4p$.

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Theorem 5.7 There is a smallest positive number x at which $C(x) = 0$.

Proof
For any continuous function $f: \mathbb{R} \to \mathbb{R}$ for which $f(0) > 0$ and there is any positive number x at which $f(x) = 0$, there is a smallest positive number x at which $f(x) = 0$ (Exercise 16). Thus, it suffices to show that there is some positive number x at which $C(x) = 0$.

From the Mean Value Theorem, it follows that we can select a number z strictly between 0 and 2 such that $S(2) - S(0) = 2C(z)$. Since $S(0) = 0$ and $|S(2)| \leq 1$, we see that $|2C(z)| \leq 1$. Thus, by the cosine addition formula and the Pythagorean Identity, we have

$$C(2z) = [C(z)]^2 - [S(z)]^2 = 2[C(z)]^2 - 1 \leq 0.$$

Hence $C(0) > 0$ and $C(2z) \leq 0$, and so, by the Intermediate Value Theorem, there is a number x between 0 and $2z$ at which $C(x) = 0$. ∎

Theorem 5.8 Let p be the smallest positive number at which $C(x) = 0$. Then the functions $C: \mathbb{R} \to \mathbb{R}$ and $S: \mathbb{R} \to \mathbb{R}$ both have period $4p$.

Proof
Since $S'(x) = C(x) > 0$ if $0 < x < p$, the function $S: [0, p] \to \mathbb{R}$ is strictly increasing. Hence, since $S(0) = 0$, $S(p) > 0$. But $C(p) = 0$, and therefore, by the Pythagorean Identity, $S(p) = 1$. Thus,

$$C(p) = 0 \qquad \text{and} \qquad S(p) = 1,$$

and so, using the sine and cosine addition formulas,

$$S(x + p) = C(x) \qquad \text{and} \qquad C(x + p) = -S(x) \qquad \text{for all } x. \tag{5.24}$$

Substituting $x + p$ for x in (5.24), we obtain

$$S(x + 2p) = -S(x) \qquad \text{and} \qquad C(x + 2p) = -C(x) \qquad \text{for all } x, \tag{5.25}$$

and now, substituting $x + 2p$ for x in (5.25), we have

$$S(x + 4p) = S(x) \qquad \text{and} \qquad C(x + 4p) = C(x) \qquad \text{for all } x;$$

that is, the functions $S: \mathbb{R} \to \mathbb{R}$ and $C: \mathbb{R} \to \mathbb{R}$ have period $4p$. ∎

As we have already mentioned, none of the functions that we have seen until now have been periodic, except of course constant functions.

We define the number π to be $2p$, where p is the smallest positive number at which $C(p) = 0$. Hence $S: \mathbb{R} \to \mathbb{R}$ and $C: \mathbb{R} \to \mathbb{R}$ have period 2π. Of course, we need to show that this definition of π is in accordance with the usual definition of π as the area of a circle of unit radius. Specifically, we must show that the first positive zero of the solution of the differential equation (5.20) occurs at p, where p is half the area of a circle of unit radius. To do this, we first need to discuss integration, and so we postpone

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a justification of our use of the symbol π until Chapter 7. However, from now on we will denote $S(x)$ by $\sin x$ and denote $C(x)$ by $\cos x$ for all x. The function $S: \mathbb{R} \to \mathbb{R}$ is the familiar *sine* function; the function $C: \mathbb{R} \to \mathbb{R}$ is the familiar *cosine* function.

Solutions of Second-Order Differential Equations

The cosine function was defined to be the unique solution of the differential equation (5.20). In fact, the cosine and sine functions play a central role in the theory of general differential equations. One indication of this is the following.

Theorem 5.9 Let α and β be any numbers. Then there is exactly one solution of the differential equation

$$\begin{cases} f''(x) + f(x) = 0 & \text{for all } x \\ f(0) = \alpha \quad \text{and} & f'(0) = \beta. \end{cases} \tag{5.26}$$

This solution is defined by

$$f(x) = \alpha \cos x + \beta \sin x \qquad \text{for all } x. \tag{5.27}$$

Proof

The differential equation (5.26) can have at most one solution since if there were two distinct solutions, their difference would be a solution of the differential equation (5.18) that is not identically zero, which would contradict Lemma 5.5. From (5.20) and (5.21), it follows that formula (5.27) defines a solution of the differential equation (5.26). ■

The Tangent Function

We conclude this section with a discussion of the tangent function. We begin with two observations about the cosine. First, in view of the definition of $\pi/2$, note that the second identity in (5.25) can be rewritten as

$$\cos(x + \pi) = -\cos x \qquad \text{for all } x.$$

Second, the cosine function is even; that is,

$$\cos(-x) = \cos x \qquad \text{for all } x.$$

This follows from the observation that the cosine function and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(-x)$, are solutions of the differential equation (5.20), a differential equation with a unique solution. Now by definition, $\cos \pi/2 = 0$ and $\cos x > 0$ if $0 \leq x < \pi/2$. It follows that $\cos x > 0$ if $-\pi/2 < x < \pi/2$ and that

$$\cos x = 0 \qquad \text{if and only if } x = \pi/2 + n\pi \text{ for some integer } n.$$

Define $D = \{x \mid x \neq \pi/2 + n\pi, n \text{ an integer}\}$. The *tangent* function, with domain D, is defined by

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{for all } x \text{ in } D.$$

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Thus the tangent, being the quotient of differentiable functions, is differentiable, and from the quotient formula for derivatives, together with (5.22), it follows that

$$\frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2 x} \qquad \text{if } x \neq \frac{\pi}{2} + n\pi,\ n \text{ an integer.} \tag{5.28}$$

Theorem 5.10 The function $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is strictly increasing, is odd, and has as its image all of $\mathbb{R}$.

Proof
Since

$$\frac{d}{dx}[\tan x] = \cos^{-2} x > 0 \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

the function $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is strictly increasing. We leave it as an exercise (Exercise 2), to show that the tangent is an odd function; that is,

$$\tan(-x) = -\tan(x).$$

It remains to be proven that the image of the restriction of the tangent function to the interval $(-\pi/2, \pi/2)$ is all of $\mathbb{R}$. Since the tangent is odd and $\tan 0 = 0$, it will suffice to show that given $c > 0$ there is a solution of the equation

$$\tan x = c, \qquad 0 < x < \frac{\pi}{2}. \tag{5.29}$$

According to the Intermediate Value Theorem, since $\tan 0 = 0$, in order to prove that there is a solution of (5.29) it suffices to find a point x in the interval $(0, \pi/2)$ at which $\tan x > c$. However, since the sine is increasing on the interval $[0, \pi/2]$,

$$\tan x = \frac{\sin x}{\cos x} \geq \frac{\sin \pi/4}{\cos x} \qquad \text{if } \frac{\pi}{4} \leq x < \frac{\pi}{2}.$$

Moreover, since the cosine is continuous and positive on the interval $(0, \pi/2)$ and $\cos \pi/2 = 0$, we can choose a point x in the interval $(\pi/4, \pi/2)$ at which $\cos x < (\sin \pi/4)/c$. At this point, $\tan x > c$. ■

EXERCISES FOR SECTION 5.3

1. Find a formula for $\sin 3a$ in terms of $\sin a$ and $\cos a$. Use it to calculate $\sin \pi/3$ and $\cos \pi/3$. Also calculate $\sin \pi/6$ and $\cos \pi/4$.
2. **a.** Use the uniqueness of the solution of the differential equation (5.20) to show that the cosine is an even function, that is, $\cos(-x) = \cos(x)$ for all x.
   b. Use the uniqueness of the solution of the differential equation (5.21) to show that the sine is an odd function, that is, $\sin(-x) = -\sin(x)$ for all x.
   c. From the oddness of the sine and the evenness of the cosine, conclude that the tangent is an odd function.
3. Derive formulas for $\cos(a - b)$ and $\sin(a - b)$ in terms of $\sin a$, $\sin b$, $\cos a$, and $\cos b$.

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4. For numbers a and b such that $|a| < 1$, prove that the following equation, which is called *Kepler's equation*, has exactly one solution:

$$x = a \sin x + b, \qquad x \text{ in } \mathbb{R}.$$

5. Prove that the following equation has exactly one solution:

$$e^{2x} + \cos x + x = 0, \qquad x \text{ in } \mathbb{R}.$$

6. For numbers a and b, define

$$f(x) = \sin x + ax + b \qquad \text{for all } x.$$

For what values of a is the function $f: \mathbb{R} \to \mathbb{R}$ increasing?

7. Find the maximum and minimum points of the set $\{\sin x + \cos x \mid x \text{ in } \mathbb{R}\}$.

8. Using the definition of derivative, prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{and} \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

9. Let k be a fixed number. Suppose that the function $f: \mathbb{R} \to \mathbb{R}$ is a solution of the differential equation

$$\begin{cases} f''(x) + k^2 f(x) = 0 & \text{for all } x \\ f(0) = 0 \quad \text{and} & f'(0) = 0. \end{cases}$$

Prove that $f(x) = 0$ for all x.

10. Let a, b, and k be fixed numbers. Use Exercise 9 to show that the following differential equation has at most one solution:

$$\begin{cases} f''(x) + k^2 f(x) = 0 & \text{for all } x \\ f(0) = a \quad \text{and} & f'(0) = b. \end{cases}$$

Then verify that if $k \neq 0$, the solution is defined by

$$f(x) = a \cos kx + b \sin kx \qquad \text{for all } x.$$

11. Let a and b be numbers such that $a^2 + b^2 = 1$. Prove that there exists exactly one number θ in the interval $[0, 2\pi)$ such that

$$\begin{cases} \cos \theta = a \\ \sin \theta = b. \end{cases}$$

12. For positive numbers M and T and a number θ_0 in the interval $[0, 2\pi)$, define

$$g(x) = M \sin(Tx + \theta_0) \qquad \text{for all } x.$$

Graph the function $g: \mathbb{R} \to \mathbb{R}$.

13. Let c_1 and c_2 be numbers such that $c_1^2 + c_2^2 = 1$. Define

$$h(x) = c_1 \cos x + c_2 \sin x \qquad \text{for all } x.$$

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Use Exercise 11 and the addition formula for the cosine to show that there is a number θ_0 such that

$$h(x) = \cos(x + \theta_0) \qquad \text{for all } x.$$

14. Define

$$f(x) = \begin{cases} x^2 \sin(1/x) + x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is differentiable and that $f'(0) = 1$. Also prove that there is no neighborhood I of 0 such that the function $f: I \to \mathbb{R}$ is increasing.

15. Does the function $f: \mathbb{R} \to \mathbb{R}$ defined in Exercise 14 have a continuous derivative? Justify your answer.

16. Suppose that the function $g: \mathbb{R} \to \mathbb{R}$ is continuous, $g(0) > 0$, and at some positive number x_0, $g(x_0) = 0$. Prove that there is a smallest positive number p at which $g(x) = 0$. [*Hint:* Define $p = \inf\{x \mid x > 0, g(x) = 0\}$ and prove that $p > 0$ and $g(p) = 0$.]

5.4 THE INVERSE TRIGONOMETRIC FUNCTIONS

The sine, cosine, and tangent functions are all periodic, so none of them has an inverse. However, if we restrict these functions to appropriate intervals, the restrictions will have inverses. To study the inverses of these restricted functions, it is useful to recall the formula for the inverse of a differentiable function that was established in Corollary 4.12.

Proposition 5.11 Let I be an open interval and suppose that the strictly monotone function $f : I \to \mathbb{R}$ is differentiable and such that $f'(x) \neq 0$ for all x in I. Then f has a differentiable inverse function defined on an open interval J which, if we denote the inverse function by $g : J \to \mathbb{R}$, has its derivative given, for x in J, by the formula

$$g'(x) = \frac{1}{f'(g(x))}. \tag{5.30}$$

The Arcsine

Since $\sin: [-\pi/2, \pi/2] \to \mathbb{R}$ is a strictly increasing continuous function with $\sin -\pi/2 = -1$ and $\sin \pi/2 = 1$, it follows from the Intermediate Value Theorem that for each number x in $[-1, 1]$, there is a unique solution to the equation

$$\sin z = x, \qquad z \text{ in } [-\pi/2, \pi/2].$$

We denote this solution by $\arcsin x$, and so we have defined the *arcsine* function, denoted by arcsin: $[-1, 1] \to \mathbb{R}$, as the inverse of sin: $[-\pi/2, \pi/2] \to \mathbb{R}$.

Since

$$\frac{d}{dx}[\sin x] = \cos x \neq 0 \qquad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

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it follows from the inverse function formula (5.30) that the arcsine function is differentiable and that

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\cos(\arcsin x)} \qquad \text{for } -1 < x < 1.$$

However, $\arcsin x$ is in $(-\pi/2, \pi/2)$, so $\cos(\arcsin x) > 0$. Consequently, using the Pythagorean Identity, we have

$$\cos(\arcsin x) = [1 - \sin^2(\arcsin x)]^{1/2} = \sqrt{1 - x^2}.$$

Thus,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}} \qquad \text{if } -1 < x < 1. \tag{5.31}$$

The Arccosine

We now turn to the *arccosine* function. Indeed, since $\cos\colon [0, \pi] \to \mathbb{R}$ is a strictly decreasing continuous function with $\cos 0 = 1$ and $\cos \pi = -1$, it follows from the Intermediate Value Theorem that for each x in $[-1, 1]$ there is a unique solution to the equation

$$\cos z = x, \qquad z \text{ in } [0, \pi]. \tag{5.32}$$

We denote this solution by $\arccos x$, and therefore we have defined the function $\arccos\colon [-1, 1] \to \mathbb{R}$, which is the inverse of $\cos\colon [0, \pi] \to \mathbb{R}$. Since

$$\frac{d}{dx}[\cos x] = -\sin x \neq 0 \qquad \text{if } 0 < x < \pi,$$

it follows from the inverse function formula (5.30) that

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sin(\arccos x)} \qquad \text{if } -1 < x < 1.$$

However, $\arccos x$ belongs to the interval $(0, \pi)$ when x is in $(-1, 1)$, so $\sin(\arccos x) > 0$. Consequently, using the Pythagorean Identity, we see that

$$\sin(\arccos x) = [1 - \cos^2(\arccos x)]^{1/2} = \sqrt{1 - x^2}.$$

Thus,

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}} \qquad \text{if } -1 < x < 1. \tag{5.33}$$

The Arctangent

Finally, we consider the *arctangent* function. According to Theorem 5.10, the function $\tan\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is a strictly increasing function whose image is all of $\mathbb{R}$. It follows that for each number x the equation

$$\tan z = x, \qquad z \text{ in } (-\pi/2, \pi/2),$$

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has a unique solution. We denote this solution by $\arctan x$, and so we have defined the function arctan: $\mathbb{R} \to \mathbb{R}$, the inverse of tan: $(-\pi/2, \pi/2) \to \mathbb{R}$.

From formula (5.28) and the inverse function formula (5.30), we conclude that

$$\frac{d}{dx}[\arctan x] = \cos^2(\arctan x) \qquad \text{for all } x.$$

However, by the Pythagorean Identity,

$$\tan^2 z + 1 = \frac{\sin^2 z}{\cos^2 z} + 1 = \frac{1}{\cos^2 z},$$

so that, setting $z = \arctan x$

$$x^2 + 1 = \tan^2(\arctan x) + 1 = \frac{1}{\cos^2(\arctan x)}.$$

Thus,

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \qquad \text{for all } x. \tag{5.34}$$

EXERCISES FOR SECTION 5.4

1. Prove that $\arcsin x + \arccos x = \pi/2$ if $-1 \leq x \leq 1$.
2. Find the unique solution of the differential equation

$$\begin{cases} F'(x) = x/\sqrt{1-x^4}, & -1 < x < 1 \\ F(0) = 1. \end{cases}$$

3. Suppose that $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are periodic functions of period T. Under what conditions is the sum $f + g: \mathbb{R} \to \mathbb{R}$ also periodic? Under what conditions is the composition $f \circ g: \mathbb{R} \to \mathbb{R}$ periodic?
4. Define $h(x) = 4\sin(x/2)$ for all x. By restricting the function $h: \mathbb{R} \to \mathbb{R}$ to a suitable interval $[a, b]$ such that $h: [a, b] \to \mathbb{R}$ is strictly increasing and $h([a, b]) = [-4, 4]$, find the inverse of $h: [a, b] \to \mathbb{R}$ and calculate its derivative on the interval $(-4, 4)$.
5. Suppose that

$$p(x) = ax^2 + bx + c > 0 \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Find a solution of the differential equation

$$F'(x) = \frac{1}{p(x)} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

(*Hint:* Complete the square.)

6. Prove that

$$\arctan v - \arctan u < v - u \qquad \text{if } u < v.$$

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CHAPTER 6

INTEGRATION: TWO FUNDAMENTAL THEOREMS

6.1 DARBOUX SUMS; UPPER AND LOWER INTEGRALS

For certain functions $f : [a, b] \to \mathbb{R}$ that we call *integrable,* we will define a number called the *integral* of f on $[a, b]$ and denoted by $\int_a^b f$. Four principal goals of this chapter are to

i. Define the concepts of integrable function and integral and then establish a criterion for integrability we call the Archimedes–Riemann Theorem.

ii. Prove that a continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

iii. Prove the First Fundamental Theorem (Integrating Derivatives), which states that for a continuous function $F : [a, b] \to \mathbb{R}$ that has a continuous bounded derivative on the open interval (a, b), the following integration formula holds:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

iv. Prove the Second Fundamental Theorem (Differentiating Integrals) which states that for a continuous function $f : [a, b] \to \mathbb{R}$,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \text{for all } x \text{ in the open interval } (a, b).$$

The significance of the integral depends on the context in which it is being considered. For example, the following interpretation of the integral is appropriate in a geometric context:

- For an integrable function $f : [a, b] \to \mathbb{R}$ having the property that $f(x) \geq 0$ for all x in $[a, b]$, the integral $\int_a^b f$ is the area under the graph of $f : [a, b] \to \mathbb{R}$ and above the interval $[a, b]$.

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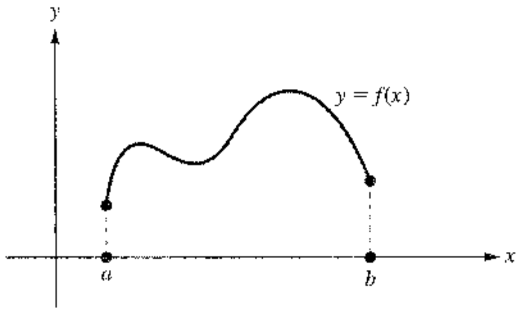

FIGURE 6.1 Area of the shaded region equals $\int_a^b f$.

The integral of $f : [a, b] \to \mathbb{R}$ has many other physical interpretations,[1] but the geometric interpretation of it as an area is sufficient to motivate our definition.

Upper and Lower Darboux Sums

Let a and b be real numbers with $a < b$. If n is a natural number and

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

then $P = \{x_0, \ldots, x_n\}$ is called a *partition* of the interval $[a, b]$. For each index $i \geq 0$, we call x_i a *partition point* of P, and if $i \geq 1$, we call the interval $[x_{i-1}, x_i]$ a *partition interval* of P. The crudest partition of $[a, b]$ occurs when $n = 1$, so that $x_0 = a$ and $x_1 = b$, in which case there are just two partition points and one partition interval of the partition P.

Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that $P = \{x_0, \ldots, x_n\}$ is a partition of its domain $[a, b]$. For each index $i \geq 1$, we define

$$\begin{cases} m_i \equiv \inf\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} \\ \qquad\qquad \text{and} \\ M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}. \end{cases} \tag{6.1}$$

[1] The book *Introduction to Calculus and Analysis*, by R. Courant and Fritz John (New York: Springer-Verlag, 1989), presents many interesting applications of the integral that arise in problems in physics and engineering. The geometric motivation for the definition of the integral of a function as the assignment of an area under a graph is useful. But it is an intuitive motivation since we presently do not have a precise definition of the area. This is similar to the way that the slope of a tangent line was useful in motivating the definition of the derivative of a function, despite the fact that we did not have a precise definition of the tangent line beforehand.

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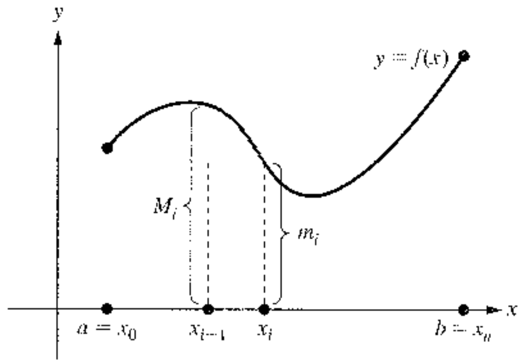

FIGURE 6.2 Approximation of the area over the partition interval $[x_{i-1}, x_i]$.

We then define

$$\begin{cases} L(f, P) \equiv \displaystyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ \qquad\qquad \text{and} \\ U(f, P) \equiv \displaystyle\sum_{i=1}^{n} M_i(x_i - x_{i-1}). \end{cases} \tag{6.2}$$

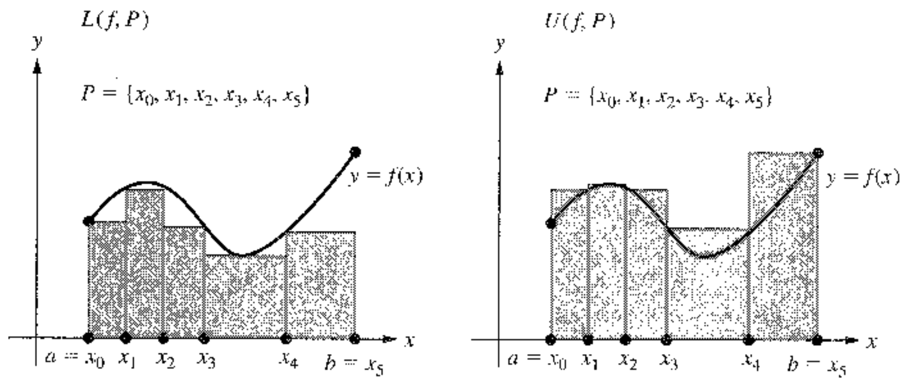

FIGURE 6.3 Upper and lower Darboux sums.

We call $U(f, P)$ the *upper Darboux sum for the function* $f : [a, b] \to \mathbb{R}$ *based on the partition* P, and we call $L(f, P)$ the *lower Darboux sum for the function* $f : [a, b] \to \mathbb{R}$ *based on the partition P.*

It follows directly from the definition of m_i and M_i that

$$m_i \leq M_i \qquad \text{for each index } i \geq 1,$$

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and therefore, for any partition P of $[a, b]$,

$$L(f, P) \leq U(f, P). \tag{6.3}$$

Our intuitive concept of the integral as representing an area suggests that the following property should hold for an integrable function $f : [a, b] \to \mathbb{R}$:

$$L(f, P) \leq \int_a^b f \leq U(f, P) \qquad \text{for every partition } P \text{ of } [a, b]. \tag{6.4}$$

We will define the integral so that this property holds.

In order to define the integral we now show that for a bounded function on a closed bounded interval, any lower Darboux sum is less than or equal to any upper Darboux sum, even when the sums are based on different partitions. It is convenient to break the proof of this into several pieces.

We will frequently use the fact that if $P = \{x_0, \ldots, x_n\}$ is a partition of the interval $[a, b]$, then the length of the interval $[a, b]$ is the sum of the lengths of the partition intervals of P; that is,

$$b - a = \sum_{i=1}^{n} (x_i - x_{i-1}). \tag{6.5}$$

Lemma 6.1 Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m and M have the property that

$$m \leq f(x) \leq M \qquad \text{for all } x \text{ in } [a, b].$$

Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \qquad \text{and} \qquad U(f, P) \leq M(b - a).$$

Proof

Let $P = \{x_0, \ldots, x_n\}$. For each index $i \geq 1$, the number m is a lower bound for the set $\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}$, so that, by the definition of infimum,

$$m \leq m_i.$$

Thus, by the sum of lengths formula (6.5),

$$\begin{aligned} m(b - a) &= m \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} m(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= L(f, P). \end{aligned}$$

Therefore, $m(b - a) \leq L(f, P)$. By a similar argument, $U(f, P) \leq M(b - a)$. ■

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Given a partition P of the interval $[a, b]$, another partition P^* of $[a, b]$ is called a *refinement* of P if each partition point of P is also a partition point of P^*. If $P = \{x_0, \ldots, x_n\}$ and P^* is a refinement of P, then for each index $i \geq 1$, the partition points of P^* that belong to the partition interval $[x_{i-1}, x_i]$ define a partition P_i of the interval $[x_{i-1}, x_i]$. Observe that

$$\sum_{i=1}^{n} L(f, P_i) = L(f, P^*) \qquad \text{and} \qquad \sum_{i=1}^{n} U(f, P_i) = U(f, P^*). \tag{6.6}$$

Lemma 6.2 The Refinement Lemma Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \qquad \text{and} \qquad U(f, P^*) \leq U(f, P).$$

Proof

Let $P = \{x_0, \ldots, x_n\}$. For an index $i \geq 1$, let m_i be as defined by (6.1) and let P_i be the partition of $[x_{i-1}, x_i]$ that is induced by P^*. Then Lemma 6.1, applied to the function $f : [x_{i-1}, x_i] \to \mathbb{R}$, yields

$$m_i(x_i - x_{i-1}) \leq L(f, P_i) \qquad \text{for } 1 \leq i \leq n.$$

Summing these n inequalities and using formula (6.6), we obtain the inequality

$$L(f, P) \leq \sum_{i=1}^{n} L(f, P_i) = L(f, P^*).$$

A similar argument shows that $U(f, P^*) \leq U(f, P)$. ∎

Given two partitions P_1 and P_2 of the interval $[a, b]$, the partition P^* formed by taking the union of the partition points of P_1 and of P_2 is a *common refinement* of P_1 and P_2 since P^* is a refinement of both P_1 and P_2.

Lemma 6.3 Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1 and P_2 are partitions of its domain $[a, b]$. Then

$$L(f, P_1) \leq U(f, P_2). \tag{6.7}$$

Proof

Let P^* be a common refinement of P_1 and P_2. From the Refinement Lemma, it follows that

$$L(f, P_1) \leq L(f, P^*) \qquad \text{and} \qquad U(f, P^*) \leq U(f, P_2).$$

On the other hand, by inequality (6.3),

$$L(f, P^*) \leq U(f, P^*).$$

Thus,

$$L(f, P_1) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P_2). \qquad ∎$$

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Upper and Lower Integrals

Definition Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded. Then we define the lower integral of f on $[a, b]$, which we denote by $\underline{\int_a^b} f$, by

$$\underline{\int_a^b} f \equiv \sup\{L(f, P) \mid P \text{ a partition of the interval } [a, b]\}. \tag{6.8}$$

We define the upper integral of f on $[a, b]$, which we denote by $\overline{\int_a^b} f$, by

$$\overline{\int_a^b} f \equiv \inf\{U(f, P) \mid P \text{ a partition of the interval } [a, b]\}. \tag{6.9}$$

Lemma 6.4 For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f. \tag{6.10}$$

Proof

Let P be a partition of $[a, b]$. Lemma 6.3 asserts that $U(f, P)$ is an upper bound for the collection of all lower Darboux sums for f. Therefore, by the definition of supremum,

$$\underline{\int_a^b} f \leq U(f, P).$$

But this inequality asserts that $\underline{\int_a^b} f$ is a lower bound for the collection of upper Darboux sums for f. Thus, by the definition of infimum,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$ ■

Example 6.5 Let the function $f : [a, b] \to \mathbb{R}$ have a constant value c. We claim that the upper integral and the lower integral both equal $c(b - a)$. Indeed, this follows immediately from the definition. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. For each index $i \geq 1$, if m_i and M_i are as defined by (6.1), then $m_i = c$ and $M_i = c$. From the sum of lengths formula (6.5),

$$c(b - a) = c\left[\sum_{i=1}^{n}(x_i - x_{i-1})\right] = \sum_{i=1}^{n} c(x_i - x_{i-1}) = L(f, P) = U(f, P).$$

Thus, the collection of lower Darboux sums consists of the single number $c(b - a)$, as does the collection of upper Darboux sums. By the definition of the lower and upper integral,

$$\underline{\int_a^b} f = c(b - a) \qquad \text{and} \qquad \overline{\int_a^b} f = c(b - a).$$ ■

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Example 6.6 Consider Dirichlet's function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) \equiv \begin{cases} 0 & \text{if the point } x \text{ in } [0, 1] \text{ is rational} \\ 1 & \text{if the point } x \text{ in } [0, 1] \text{ is irrational.} \end{cases}$$

Let $P = \{x_0, \dots, x_n\}$ be a partition of its domain $[0, 1]$. Since the rationals and the irrationals are dense in $\mathbb{R}$, it follows that for each index $i \geq 1$, if m_i and M_i are as defined by (6.1), then $m_i = 0$ and $M_i = 1$. Thus, the collection of lower Darboux sums consists of the single number 0. Therefore, by the definition of supremum,

$$\underline{\int_a^b} f = 0.$$

On the other hand, the collection of upper Darboux sums consists of the single number 1, and hence, by the definition of infimum,

$$\overline{\int_a^b} f = 1.$$

■

EXERCISES FOR SECTION 6.1

1. Consider the partition $P = \{0, 1/4, 1/2, 1\}$ of the interval $[0, 1]$. Compute $L(f, P)$ and $U(f, P)$ for the following three choices of function $f : [0, 1] \to \mathbb{R}$:
   a. $f(x) = x$ for all x in $[0, 1]$.
   b. $f(x) = 10$ for all x in $[0, 1]$.
   c. $f(x) = -x^2$ for all x in $[0, 1]$.
2. For an interval $[a, b]$ and a positive number δ, use the Archimedean Property of $\mathbb{R}$ to show that there is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that each partition interval $[x_{i-1}, x_i]$ of P has length less than δ.
3. Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

4. Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that

$$f(x) \geq 0 \qquad \text{for all } x \text{ in } [a, b].$$

   Prove that $\underline{\int_a^b} f \geq 0$.
5. Suppose that the two bounded functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ have the property that

$$g(x) \leq f(x) \qquad \text{for all } x \text{ in } [a, b].$$

   a. For P a partition of $[a, b]$, show that $L(g, P) \leq L(f, P)$.
   b. Use part (a) to show that $\underline{\int_a^b} g \leq \underline{\int_a^b} f$.

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6. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

7. Define

$$f(x) \equiv \begin{cases} x & \text{if the point } x \text{ in } [0, 1] \text{ is rational} \\ 0 & \text{if the point } x \text{ in } [0, 1] \text{ is irrational.} \end{cases}$$

Prove that $\underline{\int}_a^b f = 0$ and $\overline{\int}_a^b f \geq 1/2$.

6.2 THE ARCHIMEDES–RIEMANN THEOREM

Definition Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded. Then we say that $f : [a, b] \to \mathbb{R}$ is *integrable*, or that f is integrable on $[a, b]$, provided that

$$\underline{\int}_a^b f = \overline{\int}_a^b f.$$

When this is so, the integral of the function $f : [a, b] \to \mathbb{R}$, denoted by $\int_a^b f$, is defined by

$$\int_a^b f \equiv \underline{\int}_a^b f = \overline{\int}_a^b f.$$

We have shown in Example 6.5 that a function $f : [a, b] \to \mathbb{R}$ that has a constant value c is integrable and that its integral equals $c(b - a)$. This is as it should be if the integral of a positive function has its original intuitive interpretation as representing an area. We also have seen in Example 6.6 that Dirichlet's function is not integrable. The fact that Dirichlet's function is not integrable is not so surprising since there is no obvious way to assign an area to the region under the graph of a wild function.[2]

In preparation for the proof of a general criterion for showing that a function is integrable, we establish the following useful inequalities relating upper and lower Darboux sums and upper and lower integrals.

Lemma 6.7 For a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \underline{\int}_a^b f \leq \overline{\int}_a^b f \leq U(f, P). \tag{6.11}$$

[2] The integral that we have defined is often called the *Riemann integral* in order to distinguish it from other types of integrals. There is an integral called the *Lebesgue integral* that is defined in a similar manner to the Riemann integral; the difference is that instead of decomposing the interval $[a, b]$ into the union of intervals, the interval $[a, b]$ is decomposed into unions of so-called *measurable* sets. An interval is an example of a measurable set. Thus, for the concept of integral in the sense of Lebesgue, there are more upper sums and lower sums, and hence better approximations of the integral are possible. Dirichlet's function is integrable in the sense of Lebesgue. The book *Real Analysis*, 3rd. ed., by H. L. Royden (New York: Macmillan, 1988), a clear exposition of the Lebesgue integral.

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As a consequence, we also have the following three inequalities:

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(f, P) - L(f, P), \tag{6.12}$$

$$0 \leq U(f, P) - \overline{\int_a^b} f \leq U(f, P) - L(f, P) \tag{6.13}$$

and

$$0 \leq \underline{\int_a^b} f - L(f, P) \leq U(f, P) - L(f, P). \tag{6.14}$$

Proof

Since the lower integral is an upper bound for the collection of lower Riemann sums and the upper integral is a lower bound for the collection of upper Darboux sums, we have

$$L(f, P) \leq \underline{\int_a^b} f \qquad \text{and} \qquad \overline{\int_a^b} f \leq U(f, P).$$

But, according to Lemma 6.4,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Therefore, the inequality (6.11) holds, and the last three inequalities immediately follow from this one. ■

Theorem 6.8 The Archimedes–Riemann Theorem[3] Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0. \tag{6.15}$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n\to\infty} U(f, P_n) = \int_a^b f. \tag{6.16}$$

[3] This theorem is attributed to the Greek mathematician Archimedes because he first devised and implemented the strategy to compute the area of a nonpolygonal geometric object by constructing outer and inner polygonal approximations of the object. It is attributed to the German mathematician Bernard Riemann because he, in 1845, placed the approximation strategy of Archimedes in a general, rigorous mathematical context applicable to problems much more general than the computation of area. Riemann's contribution was made more than 2000 years after Archimedes computed the area of parabolic and circular regions by the construction of ingenious elementary geometric devices. Archimedes calculated the area of a circle of radius 1 and provided accurate error bounds for his approximation; he calculated π with an error bound of $1/500$.

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Proof

First, suppose that there is a sequence of partitions such that (6.15) holds. We will apply Lemma 6.7. Indeed, for an index n, substitute P_n for P in the inequality (6.12). Then, using (6.15), we obtain the inequality

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0. \tag{6.17}$$

Thus, the lower integral equals the upper integral, so that, by definition, the function f is integrable on $[a, b]$.

We now prove the converse assertion. Assume that f is integrable on $[a, b]$. By definition, this means that

$$\underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f. \tag{6.18}$$

Fix a natural number n. By the definition of lower integral, $\underline{\int_a^b} f$ is the least upper bound of the collection of lower Darboux sums for f. Thus, the number $\left[\underline{\int_a^b} f\right] - 1/n$, which is smaller than $\underline{\int_a^b} f$, is not an upper bound for this collection, and so there is a partition P' of $[a, b]$ such that

$$\left[\underline{\int_a^b} f\right] - \frac{1}{n} < L(f, P'),$$

and therefore, by (6.18),

$$\left[\int_a^b f\right] - \frac{1}{n} < L(f, P'). \tag{6.19}$$

Reasoning similarly for the upper integral, there is a partition P'' of $[a, b]$ such that

$$U(f, P'') < \left[\int_a^b\right] f + \frac{1}{n}. \tag{6.20}$$

By the Refinement Lemma, the above two estimates continue to hold for the common refinement of P' and P'', which we denote by P_n. Substituting P_n for P' in the first estimate (6.19) and P_n for P'' in the second estimate (6.19) yields

$$0 \le U(f, P_n) - L(f, P_n) < \left[\int_a^b f + \frac{1}{n}\right] - \left[\int_a^b f - \frac{1}{n}\right] = \frac{2}{n}.$$

Thus,

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

It remains to show that if the function f is integrable on $[a, b]$ and $\{P_n\}$ is a sequence of partitions such that (6.15) holds, then both sequences of Darboux sums converge to the integral. However, if (6.15) holds, then the convergence of these sequences of Darboux sums follows immediately from the inequalities (6.13) and (6.14), with P_n substituted for P and the integral substituted for the upper and lower integrals. ■

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In view of the Archimedes–Riemann Theorem it is useful to attach a name to a sequences of partitions for which (6.15) holds.

Definition Let the function $[a, b]: \mathbb{R} \to \mathbb{R}$ be bounded and for each natural number n let P_n be a partition of its domain $[a, b]$. Then $\{P_n\}$ is said to be an *Archimedean sequence of partitions* for f on $[a, b]$ provided that

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Thus, the Archimedean–Riemann Theorem can be reworded as follows: A bounded function f on $[a, b]$ is integrable if and only if there is an Archimedean sequence of partitions for f on $[a, b]$, and, moreover, for any such Archimedean sequence of partitions, the corresponding sequences upper and lower of Darboux sums converge to the integral of f on $[a, b]$.

Regular Partitions

Definition For a natural number n, the partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$ defined by

$$x_i = a + i\frac{(b-a)}{n} \qquad \text{for } 0 \le i \le n$$

is called the *regular partition* of $[a, b]$ into n partition intervals.

A regular partition of $[a, b]$ into n partition intervals is characterized by the fact that all of its partition intervals have the same length, namely, $(b - a)/n$.

The Gap of a Partition

Definition For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, we define the *gap of* P, denoted by gap P, to be the length of the largest partition interval of P; that is,

$$\text{gap } P \equiv \max_{1 \le i \le n} [x_i - x_{i-1}].$$

Observe that for a partition P and a positive number ϵ, gap $P < \epsilon$ if and only if each partition interval of P has length less than ϵ.

Integrability of Monotone Functions

Example 6.9 A monotonically increasing function $f: [a, b] \to \mathbb{R}$ is integrable. We will use the Archimedes–Riemann Theorem to verify this. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. First observe that since the function is monotonically increasing, for any index $i \ge 1$ and partition interval $[x_{i-1}, x_i]$,

$$m_i \equiv \inf\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} = f(x_{i-1})$$

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and

$$M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} = f(x_i).$$

For a natural number n, let P_n be the regular partition of $[a, b]$ into n partition intervals of equal length $(b-a)/n$. Then

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i)\left(\frac{b-a}{n}\right) \\ &= \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \left(\frac{b-a}{n}\right)(f(b) - f(a)). \end{aligned}$$

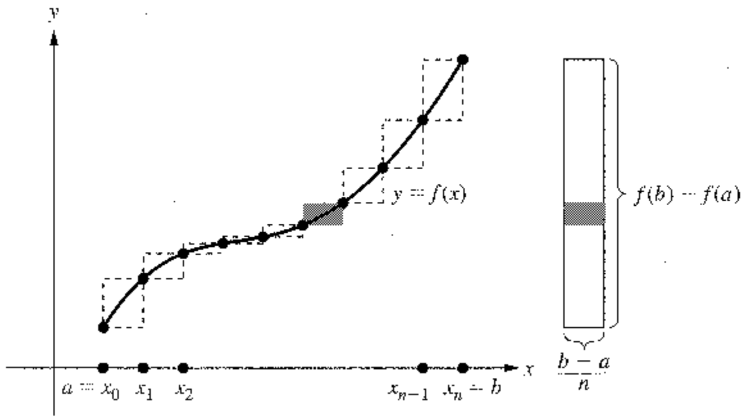

FIGURE 6.4 $U(f, P_n) - L(f, P_n) = \left(\frac{b-a}{n}\right)(f(b) - f(a)).$

Therefore,

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = \lim_{n\to\infty} \frac{[f(b) - f(a)][b-a]}{n} = 0.$$

Thus, this sequence of regular partitions is an Archimedean sequence of partitions for f on $[a, b]$. It follows from the Archimedes–Riemann Theorem that f is integrable on $[a, b]$. ∎

Integrability of Step Functions

Definition A function $f : [a, b] \to \mathbb{R}$ is called a *step function* provided that there is a partition $P^* = \{z_0, \ldots, z_k\}$ of its domain $[a, b]$ and numbers $c_1, \ldots, c_k$ such that for $1 \le i \le k$,

$$f(x) = c_i \qquad \text{for all } x \text{ in the open partition interval } (z_{i-1}, z_i).$$

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Example 6.10 A step function $f:[a, b] \to \mathbb{R}$ is integrable. We will use the Archimedes–Riemann Theorem to verify this. Select a partition $P^* = \{z_0, \ldots, z_k\}$ of the interval $[a, b]$ such that for an index $i \geq 1$, f is constant on the interval (z_{i-1}, z_i). Since a step function has only a finite number of functional values, it is bounded. Therefore, we can choose $M \geq 0$ such that

$$-M \leq f(x) \leq M \qquad \text{for all } x \text{ in } [a, b].$$

We will show that for any partition P of $[a, b]$,

$$U(f, P) - L(f, P) \leq 4(k+1)M \cdot \text{gap } P. \tag{6.21}$$

From this estimate it follows that if $\{P_n\}$ is any sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \text{gap } P_n = 0,$$

then

$$0 \leq \lim_{n\to\infty} U(f, P_n) - L(f, P_n) \leq \lim_{n\to\infty} 4(k+1)M \cdot \text{gap } P_n = 0, \tag{6.22}$$

so that $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. Thus, by the Archimedes–Riemann Theorem, f is integrable on $[a, b]$. To establish the inequality (6.21), let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$. We call an index i with $1 \leq i \leq n$ a *crossing index* if the interval $[x_{i-1}, x_i]$ contains a partition point of the partition P^*. Denote the set of crossing indices by $\mathcal{C}$. Since there are $k+1$ partition points of P^* and each partition point of P^* can belong to at most two partition intervals of P, we see that there are at most $2(k+1)$ crossing indices. Also note that for any index $i \geq 1$,

$$[M_i - m_i][x_i - x_{i-1}] \leq 2M[x_i - x_{i-1}] \leq 2M \cdot \text{gap } P.$$

Therefore, we have the following estimate of the contribution to $U(f, P) - L(f, P)$ made by the crossing indices:

$$\sum_{i \text{ in } \mathcal{C}} [M_i - m_i][x_i - x_{i-1}] \leq 4(k+1)M \cdot \text{gap } P. \tag{6.23}$$

However, observe that if i is not a crossing index, then since f is a step function, f is constant on the interval $[x_{i-1}, x_i]$ and therefore this index makes no contribution to the difference of the Darboux sums; that is,

$$U(f, P) - L(f, P) = \sum_{i \text{ in } \mathcal{C}} [M_i - m_i][x_i - x_{i-1}]. \tag{6.24}$$

From the estimate (6.23) and the equality (6.24) we obtain the required estimate (6.21). ■

Leibnitz Notation

For an integrable function $f:[a, b] \to \mathbb{R}$, we have denoted the value of the integral by the symbol $\int_a^b f$. The value of the integral is also often denoted by symbols such as

$$\int_a^b f(x)\,dx \qquad \text{or} \qquad \int_a^b f(t)\,dt.$$

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We will see that this alternate notation, involving Leibnitz symbols, is often more economical and suggestive.

The preceding two examples indicate the way the Archimedes–Riemann Theorem can be used to establish integrability; integrability can be established if there are appropriate estimates for the difference between the upper and lower Darboux sums. To use the Archimedes–Riemann Theorem to evaluate the integral is more difficult: This requires information about the actual values of the Darboux sums rather than about the difference of such sums. Such information is difficult to obtain. However, we now present an example where it can be obtained and so the integral can be determined.

In the next example, we need the following formula for the sum of squares of consecutive integers to evaluate the Darboux sums. We leave the inductive proof of the following summation formula as an exercise. For each natural number n,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{6.25}$$

Example 6.11 Define $f(x) = x^2$ for all x in $[0, 1]$. Since the function $f : [0, 1] \to \mathbb{R}$ is monotonically increasing, by Example 6.9, the function f is integrable on $[0, 1]$. We will show that

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

For each natural number n, let P_n be the regular partition of $[0, 1]$ into n partition intervals of equal length. In Example 6.9 we showed that $\{P_n\}$ is an Archimedean sequence of partitions for f on $[0, 1]$. Thus, by the Archimedes–Riemann Theorem,

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} U(f, P_n). \tag{6.26}$$

For each index $i \geq 1$,

$$M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\} = f(x_i) = \frac{i^2}{n^2} \qquad \text{and} \qquad x_i - x_{i-1} = \frac{1}{n},$$

so that,

$$M_i(x_i - x_{i-1}) = \frac{i^2}{n^3}.$$

Using the above sum of squares formula (6.25), we have

$$U(f, P_n) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Therefore,

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty}\left[\frac{2n^3 + 3n^2 + n}{6n^3}\right] = \frac{1}{3}. \qquad \blacksquare$$

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EXERCISES FOR SECTION 6.2

1. For any function $f : [a, b] \to \mathbb{R}$ and partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$, prove that

$$\sum_{i=1}^{n}(f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

2. Let P_1 and P_2 be partitions of $[a, b]$. Show that if P_1 is a refinement of P_2, then gap $P_1 \leq$ gap P_2. Is the converse true?

3. Let the function $f : [a, b] \to \mathbb{R}$ be monotonically decreasing and let P_n be the regular partition of $[a, b]$ into n intervals of equal length $(b - a)/n$.

a. Show that

$$U(f, P_n) - L(f, P_n) = \frac{[f(a) - f(b)][b - a]}{n}.$$

b. Use part (a) and the Archimedes–Riemann Theorem to show that f is integrable on $[a, b]$.

4. a. Prove that for a natural number n,

$$\sum_{1}^{n} i = \frac{n(n+1)}{2}.$$

b. Use part (a) and the Archimedes–Riemann Theorem to show that $\int_a^b x\,dx = (b - a)/2$.

5. Use part (a) of Exercise 4 and the Archimedes–Riemann Theorem to find the values of the following two integrals.

a. $\displaystyle\int_0^1 [x + 1]\,dx$

b. $\displaystyle\int_0^1 [4x + 1]\,dx$

6. Use the Archimedes–Riemann Theorem to show that for $0 \leq a < b$

a. $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$

b. $\displaystyle\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$

7. Suppose that the function $f : [a, b] \to \mathbb{R}$ has the value c at all points in the interval $[a, b]$ except at the point $x = a$. Use the Archimedes–Riemann Theorem to show that f is integrable on $[a, b]$ and that its integral equals $c(b - a)$.

8. Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for f on $[a, b]$ and has the additional property that for each index n, P_{n+1} is a refinement of P_n. For such a sequence show that the sequence of upper Darboux sums is monotonically decreasing and the sequence of lower Darboux sums is monotonically increasing. (*Hint:* Use the Refinement Lemma.)

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9. Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$. (*Hint:* Use the Refinement Lemma.)

10. Define

$$f(x) = \begin{cases} x & \text{if } 2 \le x \le 3 \\ 2 & \text{if } 3 < x \le 4. \end{cases}$$

Prove that the function $f : [2, 4] \to \mathbb{R}$ is integrable.

11. For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, show that

$$\sum_{i=1}^{n} [x_i - x_{i-1}]^2 \le [b - a] \cdot \text{gap } P.$$

12. Suppose that the function $f : [a, b] \to \mathbb{R}$ is Lipschitz; that is, that there is a constant $c \ge 0$ such that

$$|f(u) - f(v)| \le c|u - v| \qquad \text{for all points } u, v \text{ in } [a, b].$$

For a partition P of $[a, b]$, prove that

$$0 \le U(f, P) - L(f, P) \le c[b - a] \cdot \text{gap } P.$$

(*Hint:* Use the Extreme Value Theorem on each partition interval and the summation estimate in Exercise 11.)

13. Use the Darboux sum difference estimate in Exercise 12 and the Archimedes–Riemann Theorem to show that a Lipschitz function is integrable.

6.3 ADDITIVITY, MONOTONICITY, AND LINEARITY

Theorem 6.12 Additivity over Intervals Let the function $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let c be a point in the open interval (a, b). The f is integrable both on $[a, c]$ and on $[c, b]$, and furthermore,

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{6.27}$$

Proof

Since the function f is integrable on $[a, b]$, by the Archimedes–Riemann Theorem, there is an Archimedean sequence of partitions for f on $[a, b]$; that is, a sequence $\{P_n\}$ of partitions of its domain $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0, \tag{6.28}$$

and therefore,

$$\lim_{n\to\infty} U(f, P_n) = \int_a^b f. \tag{6.29}$$

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Using the Refinement Lemma, we can suppose that the point c belongs to each partition P_n. For an index n, let P_n' be the partition that P_n induces on $[a, c]$ and let P_n'' be the partition that P_n induces on $[c, b]$. It follows from the definition of the Darboux sum that

$$U(f, P_n) = U(f, P_n') + U(f, P_n'')$$

and

$$L(f, P_n) = L(f, P_n') + L(f, P_n''),$$

and therefore,

$$U(f, P_n) - L(f, P_n) = [U(f, P_n') - L(f, P_n')] + [U(f, P_n'') - L(f, P_n'')].$$

Since each of the terms in brackets on the right is nonnegative, from the limit (6.28) it follows (Exercise 3) that $\{P_n'\}$ is an Archimedean sequence of partitions for f on $[a, c]$ and $\{P_n''\}$ is an Archimedean sequence of partitions for f on $[c, b]$. According to the Archimedes–Riemann Theorem, f is integrable on $[a, c]$ and on $[c, b]$, and, moreover,

$$\lim_{n\to\infty} U(f, P_n') = \int_a^c f \qquad \text{and} \qquad \lim_{n\to\infty} U(f, P_n'') = \int_c^b f.$$

Thus, from these two limits, the limit (6.29), and the sum property of convergent sequences, we have

$$\begin{aligned}
\int_a^b f &= \lim_{n\to\infty} U(f, P_n) \\
&= \lim_{n\to\infty} \left[U(f, P_n') + U(f, P_n'')\right] \\
&= \lim_{n\to\infty} U(f, P_n') + \lim_{n\to\infty} U(f, P_n'') \\
&= \int_a^c f + \int_c^b f.
\end{aligned}$$

■

Theorem 6.13 Monotonicity of the Integral Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable and that

$$f(x) \le g(x) \qquad \text{for all } x \text{ in } [a, b].$$

Then

$$\int_a^b f \le \int_a^b g.$$

Proof

By the Archimedes–Riemann Theorem and the Refinement Lemma, there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ that is both an Archimedean sequence of partitions for f on $[a, b]$ and for g on $[a, b]$. Therefore,

$$\lim_{n\to\infty} U(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n\to\infty} U(g, P_n) = \int_a^b g.$$

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Since

$$f(x) \leq g(x) \qquad \text{for all } x \text{ in } [a,\ b],$$

it follows directly from the definition of the upper Darboux sum that for each index n,

$$U(f,\ P_n) \leq L(f,\ P_n).$$

By the order preservation property of convergent sequences,

$$\int_a^b f = \lim_{n\to\infty} U(f,\ P_n) \leq \lim_{n\to\infty} U(g,\ P_n) = \int_a^b g. \qquad \blacksquare$$

To facilitate the proof of the linearity property of the integral, it is useful to first compare Darboux sums for $f+g$ and αf with Darboux sums for f and g.

Lemma 6.14 Let $f:[a,\ b]\to\mathbb{R}$ and $g:[a,\ b]\to\mathbb{R}$ be bounded functions and let P be a partition of their domain $[a,\ b]$. Then

$$L(f,\ P)+L(g,\ P) \leq L(f+g,\ P) \quad \text{and} \quad U(f+g,\ P) \leq L(f,\ P)+U(g,\ P). \tag{6.30}$$

Moreover, for any number α,

$$\begin{aligned} U(\alpha f,\ P) &= \alpha U(f,\ P) \quad \text{and} \quad L(\alpha f,\ P) = \alpha L(f,\ P) \quad \text{if } \alpha \geq 0 \\ U(\alpha f,\ P) &= \alpha L(f,\ P) \quad \text{and} \quad L(\alpha f,\ P) = \alpha U(f,\ P) \quad \text{if } \alpha < 0. \end{aligned} \tag{6.31}$$

Proof

Both (6.30) and (6.31) will follow from an examination of the contributions to the Darboux sums for f and αf made by each portion interval of the partition P.

Choose an interval I_i of the partition P and for a bounded function $h:[a,\ b]\to\mathbb{R}$ define

$$M_i(h) \equiv \sup\{h(x) \mid x \text{ in } I_i\} \qquad \text{and} \qquad m_i(h) \equiv \inf\{h(x) \mid x \text{ in } I_i\}.$$

Then, for any x in I_i,

$$f(x)+g(x) \leq M_i(f)+M_i(g),$$

so that by the definition of supremum

$$M_i(f+g) \leq M_i(f)+M_i(g).$$

Multiplying this inequality by the length of the interval I_i and summing the resulting inequalities over the intervals of the partition P yields the second inequality in (6.30). The first inequality follows from a similar argument.

The comparison of Darboux sums in (6.31) follows from the following relation between the contributions to the Darboux sums for f and αf made by each

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interval I_i of the partition P (Exercise 4):

$$\begin{array}{llll} M_i(\alpha f) = \alpha M_i(f) & \text{and} & m_i(\alpha f) = \alpha m_i(f) & \text{if } \alpha \geq 0 \\ M_i(\alpha f) = \alpha m_i(f) & \text{and} & m_i(\alpha f) = \alpha M_i(f) & \text{if } \alpha < 0. \end{array} \tag{6.32}$$

■

Theorem 6.15 Linearity of the Integral Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be integrable. Then for any two numbers α and β, the function $\alpha f + \beta g : [a, b] \to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g. \tag{6.33}$$

Proof

By the Archimedes–Riemann Theorem and the Refinement Lemma, there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ that is both an Archimedean sequence of partitions for f on $[a, b]$ and an Archimedean sequence of partitions for g on $[a, b]$.

Case 1: $\beta = 0$. For an index n, substituting P_n for P in the relations (6.31), we have

$$U(\alpha f, P_n) - L(\alpha f, P_n) = |\alpha| \left[U(f, P_n) - L(f, P_n)\right].$$

Therefore, since $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$, it is also an Archimedean sequence of partitions for αf on $[a, b]$. According to the Archimedes–Riemann Theorem, αf is integrable on $[a, b]$. Now, by (6.31), for each index n,

$$U(\alpha f, P_n) = \begin{cases} \alpha U(f, P_n) & \text{if } \alpha > 0 \\ \alpha L(f, P_n) & \text{if } \alpha \leq 0. \end{cases}$$

However, by the Archimedes–Riemann Theorem, the sequences of upper and lower Darboux sums associated with Archimedean sequences of partitions each converge to the value of the integral, and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$ and also for αf on $[a, b]$. Consequently, if $\alpha > 0$,

$$\int_a^b \alpha f = \lim_{n\to\infty} U(\alpha f, P_n) = \alpha \lim_{n\to\infty} U(f, P_n) = \alpha \int_a^b f,$$

while if $\alpha \leq 0$, we also have

$$\int_a^b \alpha f = \lim_{n\to\infty} U(\alpha f, P_n) = \alpha \lim_{n\to\infty} L(f, P_n) = \alpha \int_a^b f.$$

This concludes the proof of the case where $\beta = 0$.

Case 2: $\alpha = \beta = 1$. For an index n, substituting P_n for P in (6.30), we obtain

$$L(f, P_n) + L(g, P_n) \leq L(f + g, P_n) \leq U(f + g, P_n) \leq L(f, P_n) + U(g, P_n). \tag{6.34}$$

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However, once more using the Archimedes–Riemann Theorem, since the sequences of upper and lower Darboux sums associated with Archimedean sequences of partitions each converge to the value of the integral and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$ and also for g on $[a, b]$, from (6.34) it follows that

$$\lim_{n\to\infty} L(f+g, P_n) = \int_a^b f + \int_a^b g \quad \text{and} \quad \lim_{n\to\infty} U(f+g, P_n) = \int_a^b f + \int_a^b g.$$

Invoking the Archimedes–Riemann Theorem for the final time in the proof, we conclude that $\{P_n\}$ is an Archimedean sequence of partitions for $f+g$ on $[a, b]$ and

$$\int_a^b [f+g] = \int_a^b f + \int_a^b g.$$

This concludes the proof of the case where $\alpha = \beta = 1$.

The general case follows from the two special cases. Indeed, by case 1, both αf and βg are integrable on $[a, b]$, and hence, by case 2, so is $\alpha f + \beta g$. By using the integral formula in case 2 and then twice using that for case 1, we conclude that

$$\int_a^b [\alpha f + \beta g] = \int_a^b \alpha f + \int_a^b \beta g = \alpha \int_a^b f + \beta \int_a^b g. \qquad \blacksquare$$

Corollary 6.16 Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $|f| : [a, b] \to \mathbb{R}$ are integrable. Then

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx. \tag{6.35}$$

Proof
For all x in $[a, b]$,

$$-|f(x)| \le f(x) \le |f(x)|.$$

Thus, using the monotonicity and linearity of integration, it follows that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

which is equivalent to (6.35). $\blacksquare$

EXERCISES FOR SECTION 6.3

1. Suppose that the functions f, g, f^2, g^2, and fg are integrable on the closed bounded interval $[a, b]$. Prove that $[f-g]^2$ also is integrable on $[a, b]$ and that $\int_a^b [f-g]^2 \ge 0$. Use this to prove that

$$\int_a^b fg \le \frac{1}{2}\left[\int_a^b f^2 + \int_a^b g^2\right].$$

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2. (The Cauchy–Schwarz Inequality for Integrals) Suppose that the functions f, g, f^2, g^2, and fg are integrable on the closed, bounded interval $[a, b]$. Prove that

$$\int_a^b fg \leq \sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}.$$

(*Hint:* For each number λ, define $p(\lambda) = \int_a^b [f - \lambda g]^2$. Show that $p(\lambda)$ is a quadratic polynomial for which $p(\lambda) \geq 0$ for all λ and therefore that its discriminant is not positive.)

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers. Show that if

$$\lim_{n\to\infty} [a_n + b_n] = 0, \text{ then } \lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty} b_n = 0.$$

4. Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x \mid x \text{ in } S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \qquad \text{and} \qquad \inf \alpha S = \alpha \inf S \qquad \text{if } \alpha \geq 0,$$

while

$$\sup \alpha S = \alpha \inf S \qquad \text{and} \qquad \inf \alpha S = \alpha \sup S \qquad \text{if } \alpha < 0.$$

Use this to prove (6.31).

5. Under the assumptions of Theorem 6.13 show that if P_1 and P_2 are partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$. Use this to provide a direct proof of the theorem.

6. Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on $[a, c]$ and on $[c, b]$, then it is integrable on $[a, b]$.

6.4 CONTINUITY AND INTEGRABILITY

Our primary goal in this section is to show that a continuous function on a closed bounded interval is integrable. To do so, it is useful to first isolate the following estimate of the difference of upper and lower Darboux sums for a continuous function.

Lemma 6.17 Let the function $f : [a, b] \to \mathbb{R}$ be continuous and let P be a partition of its domain $[a, b]$. Then there is a partition interval of P that contains two points u and v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a]. \tag{6.36}$$

Proof

Let $P = \{x_0, \ldots, x_n\}$. For an index $i \geq 1$, since f is continuous on the closed bounded partition interval $[x_{i-1}, x_i]$, the Extreme Value Theorem asserts that it assumes a maximum value and a minimum value on this interval. That is, there are points u_i and v_i in $[x_{i-1}, x_i]$ such that

$$f(u_i) = m_i \equiv \inf\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}$$

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and

$$f(v_i) = M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}.$$

Choose an index i_0 such that

$$M_{i_0} - m_{i_0} = \max_{1 \le i \le n} [M_i - m_i],$$

and define

$$u \equiv u_{i_0} \qquad \text{and} \qquad v \equiv v_{i_0}.$$

Then

$$M_i - m_i \le M_{i_0} - m_{i_0} = f(v) - f(u) \qquad \text{for } i \le i \le n.$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} [M_i - m_i][x_{i-1} - x_i] \\ &\le \sum_{i=1}^{n} [f(v) - f(u)][x_{i-1} - x_i] \\ &= [f(v) - f(u)] \sum_{i=1}^{n} [x_{i-1} - x_i] \\ &= [f(v) - f(u)][b - a]. \end{aligned}$$

■

Recall that Theorem 3.17 asserts that a continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is uniformly continuous; that is, for any two sequences $\{u_n\}$ and $\{v_n\}$ in its domain $[a, b]$,

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0 \qquad \text{if } \lim_{n\to\infty} [u_n - v_n] = 0.$$

This is the property of a continuous function on a closed bounded interval that implies that it is integrable.

Theorem 6.18 A continuous function on a closed bounded interval, $f : [a, b] \to \mathbb{R}$, is integrable.

Proof

To prove the theorem, we will use the Archimedes–Riemann Theorem. Let $\{P_n\}$ be any sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \text{gap } P_n = 0. \tag{6.37}$$

We will show that the sequence $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. By the preceding lemma, for each index n, we can choose a partition interval in the partition P_n that contains two points u_n and v_n for which the following

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estimate holds:

$$0 \leq U(f, P_n) - L(f, P_n) \leq [f(v_n) - f(u_n)][b - a]. \tag{6.38}$$

Observe that since u_n and v_n belong to a common partition interval of P_n,

$$|v_n - u_n| \leq \text{gap } P_n. \tag{6.39}$$

From this estimate and (6.37), we conclude that $\{u_n\}$ and $\{v_n\}$ are sequences in the closed bounded interval $[a, b]$ having the property that

$$\lim_{n\to\infty} [u_n - v_n] = 0.$$

But a continuous function on a closed bounded interval is uniformly continuous. Thus,

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0.$$

This limit, together with the inequality (6.38), implies that

$$0 \leq \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] \leq \lim_{n\to\infty} [f(v_n) - f(u_n)][b - a] = 0.$$

Thus, the sequence $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. According to the Archimedes–Riemann Theorem, f is integrable on $[a, b]$. ■

There is a slight generalization of the preceding theorem that will be needed for the formulation, in the following section, of the First Fundamental Theorem (Integrating Derivatives).

Theorem 6.19 Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded on the closed interval $[a, b]$ and is continuous on the open interval (a, b). Then f is integrable on $[a, b]$ and the value of the integral, $\int_a^b f$, does not depend on the values of f at the endpoints of the interval.

Proof

Since $f : [a, b] \to \mathbb{R}$ is bounded, we can choose $M \geq 0$ such that

$$-M \leq f(x) \leq M \qquad \text{for all } x \text{ in } [a, b]. \tag{6.40}$$

Choose sequences $\{a_n\}$ and $\{b_n\}$ such that for each index n,

$$a < a_n < b_n < b$$

and

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b. \tag{6.41}$$

Fix an index n. Then the function $f : [a_n, b_n] \to \mathbb{R}$ is continuous. By the preceding theorem, f is integrable on $[a_n, b_n]$. It is therefore a consequence of the Archimedes–Riemann Theorem (Exercise 4) that there is a partition P_n^* of $[a_n, b_n]$

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such that

$$0 \leq U(f, P_n^*) - L(f, P_n^*) < 1/n. \tag{6.42}$$

Define P_n to be the partition of the whole interval $[a, b]$ obtained by appending the endpoints a and b to the set of partition points of P_n^*.

Observe that

$$U(f, P_n) - L(f, P_n) = U(f, P_n^*) - L(f, P_n^*) + A_n + B_n, \tag{6.43}$$

where A_n denotes the contribution to the difference of Darboux sums from the initial partition interval $[a, a_n]$ and B_n denotes the contribution to the difference of Darboux sums from the final partition interval $[b_n, b]$. It follows from (6.40) that

$$0 \leq A_n \leq 2M[a_n - a] \qquad \text{and} \qquad 0 \leq B_n \leq 2M[b_n - b].$$

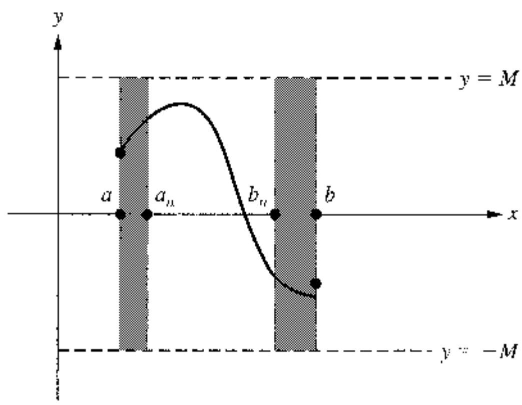

FIGURE 6.5 Area of left (right) shaded region is greater than $A_n(B_n)$.

Thus,

$$0 \leq U(f, P_n) - L(f, P_n) \leq [U(f, P_n^*) - L(f, P_n^*)] + 2M[a_n - a] + 2M[b_n - b].$$

However, by the choices of the sequences $\{a_n\}$, $\{b_n\}$, and $\{P_n^*\}$,

$$\lim_{n\to\infty} [U(f, P_n^*) - L(f, P_n^*)] = 0, \lim_{n\to\infty} [a_n - a] = 0, \quad \text{and} \quad \lim_{n\to\infty} [b_n - b] = 0.$$

Therefore,

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Thus, $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$. According to the Archimedes–Riemann Theorem, the function f is integrable on $[a, b]$ and, moreover,

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n).$$

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But the difference between $U(f, P_n)$ and $U(f, P_n^*)$ is exactly the contributions to $U(f, P_n)$ from the first and last intervals of the partition P_n. Arguing as above, we have

$$\lim_{n\to\infty} [U(f, P_n) - U(f, P_n^*)] = 0.$$

Thus,

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n^*).$$

However, each upper Darboux sum $U(f, P_n^*)$ depends only on the value of f on $[a_n, b_n]$ and so is independent of the values of f at $x = a$ and $x = b$. Therefore, so is the integral. ■

Example 6.20 Define

$$f(x) = \begin{cases} \sin(1/x) & \text{if } 0 < x \leq 1 \\ 4 & \text{if } x = 0. \end{cases}$$

Then the function $f : [0, 1] \to \mathbb{R}$ is bounded and $f : (0, 1) \to \mathbb{R}$ is continuous. The preceding theorem implies that the function $f : [0, 1] \to \mathbb{R}$ is integrable. ■

EXERCISES FOR SECTION 6.4

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all x in $[a, b]$.
   b. If $f : [a, b] \to \mathbb{R}$ is integrable, then $f : [a, b] \to \mathbb{R}$ is continuous.
   c. If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f \geq 0$.
   d. A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.
   e. A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.
2. Define

$$f(x) = \begin{cases} x & \text{if the point } x \text{ in } [0, 1] \text{ is rational} \\ -x & \text{if the point } x \text{ in } [0, 1] \text{ is irrational.} \end{cases}$$

   Prove that the function $f : [0, 1] \to \mathbb{R}$ is not integrable.
3. Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has $\int_a^b f = 0$. Prove that there is some point x_0 in the interval $[a, b]$ at which $f(x_0) = 0$. (*Hint:* Use the Extreme Value Theorem and the Intermediate Value Theorem.)
4. Let the function $f : [a, b] \to \mathbb{R}$ be integrable and let $\epsilon > 0$. Use the Archimedes–Riemann Theorem to show that there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

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5. Suppose the continuous function $f : [a, b] \to \mathbb{R}$ has the property that

$$\int_c^d f \leq 0 \qquad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all x in $[a, b]$. Is this true if we require only integrability of the function?

6. Suppose that the function $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all x in $[0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point x_0 in $[0, 1]$ at which $f(x_0) > 0$.

7. For a point x in the interval $[1, 2]$, define $f(x) = 0$ if x is irrational and define $f(x) = 1/n$ if x is rational and is expressed as $x = m/n$ for natural numbers m and n having no common positive integer divisor other than 1. Prove that the function $f : [1, 2] \to \mathbb{R}$ is integrable. (*Hint:* First prove that given $\epsilon > 0$, there are only a finite number of points x in the interval $[1, 2]$ at which $f(x) \geq \epsilon$.)

8. Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that it is continuous except at one point x_0 in the open interval (a, b). Prove that $f : [a, b] \to \mathbb{R}$ is integrable. (*Hint:* Modify the proof of Theorem 6.19.)

9. (The Triangle Inequality for Integrals) Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous. Prove that

$$\int_a^b |f + g| \leq \int_a^b |f| + \int_a^b |g|.$$

6.5 THE FIRST FUNDAMENTAL THEOREM: INTEGRATING DERIVATIVES

In the preceding section, we used the Archimedes–Riemann Theorem to prove that a continuous function on a closed bounded interval is integrable. In fact, we proved a little more, namely, that a function f on a closed interval $[a, b]$ that is continuous and bounded on the open interval (a, b) is integrable on $[a, b]$. However, except in very special cases, so far we have no general method to actually determine the value of the integral. The reason for this is that, while it has been possible (for monotone functions, for step functions, and for continuous functions) to estimate the *difference* of upper and lower Darboux sums, it is not possible to evaluate upper and lower Darboux sums themselves except in quite special cases.[4] Therefore, the Archimedes–Riemann Theorem is of limited value as a tool to actually determine the value of an integral. In this section, we will prove the First Fundamental Theorem (Integrating Derivatives), which states that for a continuous function $F : [a, b] \to \mathbb{R}$ that has a continuous bounded derivative

[4] There is an ingenious trick, due to Pierre de Fermat, for evaluating the integral of any power function $f(x) = x^r$ on an interval $[a, b]$ by the actual computation of Darboux sums; see Exercise 13 of Section 7.4. However, the theory of integration could not proceed by requiring ingenious tricks for each example.

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on the open interval (a, b), the following integration formula holds:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

This theorem presents a method for evaluating integrals that does not require the explicit evaluation of upper and lower Darboux sums. It is a fundamental result of mathematical analysis and, indeed, of science. In order to prove this theorem, it is useful to first isolate the following simple result.

Lemma 6.21 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every partition P of $[a, b]$,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof

By assumption, A is an upper bound for the collection of lower Darboux sums. Thus, by the definition of supremum,

$$\underline{\int_a^b} f \leq A.$$

By assumption, A is a lower bound for the collection of upper Darboux sums. Thus, by the definition of infimum,

$$A \leq \overline{\int_a^b} f.$$

But f is assumed to be integrable on $[a, b]$; that is,

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Therefore,

$$A \leq \int_a^b f \leq A. \qquad \blacksquare$$

Theorem 6.22 The First Fundamental Theorem: Integrating Derivatives Let the function $F : [a, b] \to \mathbb{R}$ be continuous on the closed interval $[a, b]$ and be differentiable on the open interval (a, b). Moreover, suppose that its derivative

$$F' : (a, b) \to \mathbb{R} \text{ is both continuous and bounded.}$$

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Then

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{6.44}$$

Proof

The function $F' : (a, b) \to \mathbb{R}$ is continuous and bounded. Thus, by Theorem 6.19, it is integrable on $[a, b]$ in the sense that any extension of F' to the closed interval $[a, b]$ is integrable and the value of the resulting integral does not depend on the values of the extension at the endpoints of the interval. By the preceding lemma, in order to verify the integration formula (6.44) it suffices to verify that for each partition P of the interval $[a, b]$,

$$L(F', P) \le F(b) - F(a) \le U(F', P). \tag{6.45}$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Fix an index $i \ge 1$. By assumption, the function $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continuous on the closed interval $[x_{i-1}, x_i]$ and differentiable on the open interval (x_{i-1}, x_i). By the Mean Value Theorem, there is a point c_i in the open interval (x_{i-1}, x_i) at which

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}). \tag{6.46}$$

Since the point c_i belongs to the interval $[x_{i-1}, x_i]$,

$$m_i \equiv \inf\{F'(x) \mid x \text{ in } [x_{i-1}, x_i]\} \le F'(c_i) \le \sup\{F'(x) \mid x \text{ in } [x_{i-1}, x_i]\} \equiv M_i.$$

Multiplying this last inequality by $x_i - x_{i-1}$ and substituting the Mean Value Formula (6.46), we obtain

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

Summing these n inequalities, we obtain the following inequality:

$$\sum_{i=1}^n m_i(x_i - x_{i-1}) \le \sum_{i=1}^n [F(x_i) - F(x_{i-1})] \le \sum_{i=1}^n M_i(x_i - x_{i-1}).$$

The left-hand sum is $L(f, P)$, the right-hand sum is $R(f, P)$, and moreover,

$$\sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(b) - F(a).$$

Thus, we have the required inequality (6.45). ∎

For a function $f : [a, b] \to \mathbb{R}$ that is continuous and bounded on the open interval (a, b), the First Fundamental Theorem (Integrating Derivatives) asserts that if it is possible to find an *antiderivative* $F : [a, b] \to \mathbb{R}$ for the function f, then the integral is given by the formula

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{6.47}$$

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By an antiderivative of f we mean a continuous function $F:[a, b] \to \mathbb{R}$ having a derivative on the open interval (a, b) such that

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } (a, b). \tag{6.48}$$

Example 6.23 For $r > 0$,

$$\int_0^1 x^r \, dx = \frac{1}{r+1}. \tag{6.49}$$

Define $f(x) = x^r$ for all x in $[0, 1]$. The function $f:[0, 1] \to \mathbb{R}$ is continuous, so it is integrable. From the formula for differentiating power functions, we immediately recognize that the function $F:[0, 1] \to \mathbb{R}$ defined by

$$F(x) = \frac{x^{r+1}}{r+1} \qquad \text{for all } x \text{ in } [0, 1]$$

is an antiderivative of f. Formula (6.49) follows from the First Fundamental Theorem (Integrating Derivatives). ■

Example 6.24 We wish to evaluate

$$\int_0^1 \left[\frac{1}{1+x^4}\right] dx.$$

Define $f(x) = 1/(1+x^4)$ for all x in $[0, 1]$. Since the function $f:[0, 1] \to \mathbb{R}$ is continuous, it is also integrable. In order to apply the First Fundamental Theorem (Integrating Derivatives), we need to find a continuous function $F:[0, 1] \to \mathbb{R}$ such that $F:(0, 1) \to \mathbb{R}$ is differentiable and

$$F'(x) = \frac{1}{1+x^4} \qquad \text{for all } x \text{ in } (0, 1). \tag{6.50}$$

Even if we carefully sift through all of our differentiation results, no solution of the differential equation (6.50) comes to mind. We are unable to recognize an antiderivative.[5] Hence we cannot directly apply the First Fundamental Theorem (Integrating Derivatives) to evaluate the above integral. ■

Example 6.25 Define

$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

[5] In fact, it will follow from the Second Fundamental Theorem (Differentiating Integrals), which we will prove in the next section, that this function has an antiderivative. But the antiderivative is not recognizable as an "elementary function." To properly define what is meant by an "elementary function" requires a background in modern algebra that is outside the intended scope of this book; see the article "Integration in Finite Terms" by Maxwell Rosenlicht in *American Mathematical Monthly*, Nov. 1972.

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There is now no possibility of applying the First Fundamental Theorem (Integrating Derivatives) in the evaluation of $\int_2^6 f$ since the function $f : [2, 6] \to \mathbb{R}$ does not have an antiderivative (Exercise 4). Since this function is a step function, it is integrable. It is not difficult to see that $\int_2^6 f = 4$ (Exercise 3). ■

The above examples illustrate both the power and the limitations of the First Fundamental Theorem (Integrating Derivatives). It replaces the problem of calculating $\int_a^b f$ with the problem of finding an antiderivative for f. Frequently one can recognize an antiderivative, but there are cases when an antiderivative is not readily recognizable and there are cases where there is no antiderivative.

EXERCISES FOR SECTION 6.5

1. Let m and b be positive numbers. Find the value of $\int_0^1 [mx + b]\,dx$ in the following three ways:

a. Using elementary geometry, interpreting $\int_0^1 [mx + b]\,dx$ as an area.

b. Calculating upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes–Riemann Theorem.

c. Using the First Fundamental Theorem (Integrating Derivatives).

2. Use the First Fundamental Theorem (Integrating Derivatives) to evaluate each of the following integrals:

a. $\displaystyle\int_1^2 \left[\frac{1}{x^2} + x + \cos x\right] dx$

b. $\displaystyle\int_0^1 x\sqrt{4 - x^2}\,dx$

c. $\displaystyle\int_1^3 x\sqrt{10 - x}\,dx$

d. $\displaystyle\int_0^\pi \cos^2 x\,dx$

3. Define

$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

Show that $\int_a^b f = 4$.

4. Define

$$f(x) = \begin{cases} 4 & \text{if } 2 \le x < 3 \\ 0 & \text{if } 3 \le x \le 6. \end{cases}$$

a. Use the Identity Criterion to show that if there is an antiderivative $F : [2, 6] \to \mathbb{R}$ for f, then there are numbers c_1 and c_2 such that

$$F(x) = \begin{cases} 4x + c_1 & \text{if } 2 \le x < 3 \\ c_2 & \text{if } 3 \le x \le 6. \end{cases}$$

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b. From (a) show that by the continuity of F at $x = 3$, $12 + c_1 = c_2$.

c. Show that the function F cannot be differentiable at $x = 3$ and therefore that the function f does not have an antiderivative.

5. The monotonicity property of the integral implies that if the functions $g : [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to \mathbb{R}$ are continuous and $g(x) \leq h(x)$ for all $x \geq 0$, then

$$\int_0^x g \leq \int_0^x h \qquad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem (Integrating Derivatives) to show that each of the following inequalities implies its successor:

$$\begin{aligned} \cos x &\leq 1 && \text{if } x \geq 0 \\ \sin x &\leq x && \text{if } x \geq 0 \\ 1 - \cos x &\leq \frac{x^2}{2} && \text{if } x \geq 0 \\ x - \sin x &\leq \frac{x^3}{6} && \text{if } x \geq 0. \end{aligned}$$

Thus,

$$x - \frac{x^3}{6} \leq \sin x \leq x \qquad \text{if } x \geq 0.$$

6. Show that in the First Fundamental Theorem (Integrating Derivatives) it is necessary to assume that the function $F : [a, b] \to \mathbb{R}$ is continuous at the endpoints of the interval. At what step in the proof of this theorem is continuity at the endpoints used?

6.6 THE SECOND FUNDAMENTAL THEOREM: DIFFERENTIATING INTEGRALS

Given a function $f : (a, b) \to \mathbb{R}$, is there a differentiable function $F : (a, b) \to \mathbb{R}$ such that

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } (a, b)?$$

Such a function F is called an *antiderivative* of f on (a, b).

i. For certain functions $f : (a, b) \to \mathbb{R}$, such as polynomials, we can explicitly find an antiderivative.

ii. A step function that is not actually constant on (a, b) does not have an antiderivative.

iii. For many functions $f : (a, b) \to \mathbb{R}$, we have been unable to decide whether there is an antiderivative. In particular, in Chapter 5, we supposed that there was a solution of the equation

$$F'(x) = 1/x \qquad \text{for all } x > 0,$$

but at that stage, we had not developed the tools to prove that the function $f(x) = 1/x$, $x > 0$, has an antiderivative.

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The importance of the concept of antiderivative became evident in the statement of the First Fundamental Theorem (Integrating Derivatives). However, quite independently of the problem of explicitly evaluating integrals, studying the question of the existence of an antiderivative is the first step toward the study of general differential equations. In this section, we will prove that if the function $f : (a, b) \to \mathbb{R}$ is continuous, then it has an antiderivative for which there is an explicit integral formula. Before proving this result, we derive two preliminary results that are of independent interest.

Theorem 6.26 The Mean Value Theorem for Integrals Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a}\int_a^b f = f(x_0).$$

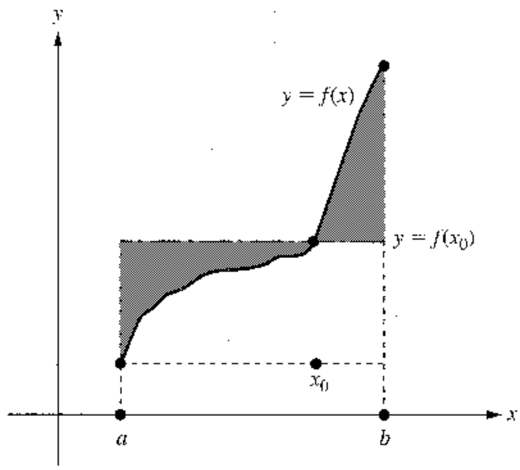

FIGURE 6.6 $f(x_0) = \frac{1}{b-a}\int_a^b f.$

Proof

Since the function $f : [a, b] \to \mathbb{R}$ is continuous, we can use the Extreme Value Theorem to choose points x_m and x_M in the interval $[a, b]$ at which $f : [a, b] \to \mathbb{R}$ attains minimum and maximum values, respectively. Thus,

$$f(x_m) \le f(x) \le f(x_M) \qquad \text{for all } x \text{ in } [a, b].$$

The monotonicity property of integration implies that

$$f(x_m)(b-a) \le \int_a^b f(x)\,dx \le f(x_M)(b-a),$$

and so

$$f(x_m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(x_M).$$

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Thus, by the Intermediate Value Theorem, there is a point x_0 between x_m and x_M such that

$$f(x_0) = \frac{1}{b-a}\int_a^b f. \qquad \blacksquare$$

Proposition 6.27 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \qquad \text{for all } x \text{ in } [a, b].$$

Then the function $F : [a, b] \to \mathbb{R}$ is continuous.

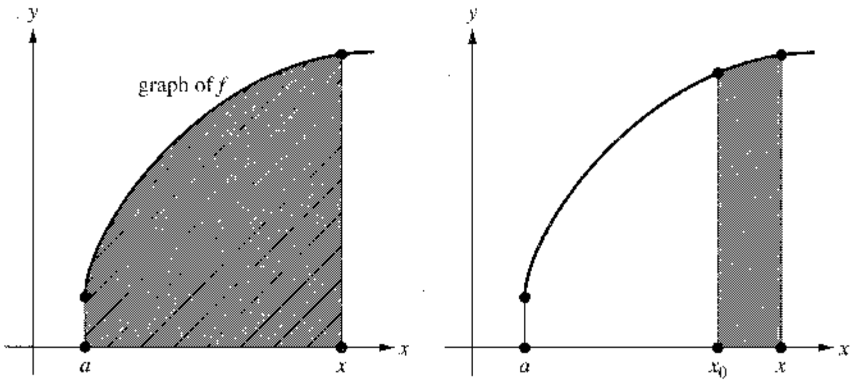

FIGURE 6.7 Shaded area equals $F(x) - F(x_0)$.

Proof

The proof will rely on the additivity over intervals property of the integral we proved as Theorem 6.12. By that theorem, for each point x in $[a, b]$, the function f is integrable on the closed bounded interval $[a, x]$. Hence, the function $F : [a, b] \to \mathbb{R}$ is properly defined.

Now since the function $f : [a, b] \to \mathbb{R}$ is integrable, it is, by assumption, bounded. Choose $M > 0$ such that

$$-M \le f(x) \le M \qquad \text{for all } x \text{ in } [a, b]. \tag{6.51}$$

We claim that

$$|F(u) - F(v)| \le M|u - v| \qquad \text{for all points } u \text{ and } v \text{ in } [a, b]. \tag{6.52}$$

Indeed, let u and v be points in $[a, b]$ with $u < v$. Once more using Theorem 6.12, we have

$$F(v) = \int_a^v f = \int_a^u f + \int_u^v f = F(u) + \int_u^v f,$$

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so that

$$F(v) - F(u) = \int_u^v f. \tag{6.53}$$

But, by the choice of M,

$$-M \leq f(x) \leq M \qquad \text{if } u \leq x \leq v,$$

so that, by the monotonicity of the integral,

$$-M(v-u) \leq \int_u^v f \leq M(v-u),$$

and therefore, by (6.53),

$$-M(v-u) \leq F(v) - F(u) \leq M(v-u).$$

Thus, the inequality (6.52) holds when $u < v$, and since this inequality remains unchanged when u and v are interchanged, it also holds when $v < u$.

Finally, inequality (6.52) immediately implies the continuity of the function $F : [a, b] \to \mathbb{R}$. ■

Example 6.28 Define

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2. \end{cases}$$

Now define

$$F(x) = \int_0^x f \qquad \text{if } 0 \leq x \leq 2.$$

By the First Fundamental Theorem (Integrating Derivatives) and the additivity over intervals property of the integral,

$$F(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ F(1) + \int_1^x f = 3/2 + x^2/2 & \text{if } 1 < x \leq 2. \end{cases}$$

As the preceding theorem predicted, the function $F : [0, 2] \to \mathbb{R}$ is continuous. ■

When we strengthen the assumption in Proposition 6.27 by replacing the integrability of $f : [a, b] \to \mathbb{R}$ with the continuity of $f : [a, b] \to \mathbb{R}$, we obtain another cornerstone of mathematical analysis.

Theorem 6.29 The Second Fundamental Theorem: Differentiating Integrals Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x f\right] = f(x) \qquad \text{for all } x \text{ in } (a, b).$$

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Proof
Define

$$F(x) \equiv \int_a^x f \qquad \text{for all } x \text{ in } [a, b].$$

We have already verified that the function $F : [a, b] \to \mathbb{R}$ is properly defined and is, in fact, continuous. Let x_0 be a point in (a, b). We must show that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Let x be a point in (a, b) with $x \neq x_0$. By the additivity over intervals property of the integral, Theorem 6.12,

$$F(x) - F(x_0) = \int_{x_0}^{x} f \qquad \text{if } x > x_0,$$

while

$$F(x) - F(x_0) = -\int_{x}^{x_0} f \qquad \text{if } x < x_0.$$

Consequently, applying the Mean Value Theorem for Integrals, we see that we can select a point $c(x)$ between x_0 and x such that

$$\frac{F(x) - F(x_0)}{x - x_0} = f(c(x)). \tag{6.54}$$

But the function $f : [a, b] \to \mathbb{R}$ is continuous at x_0, so that

$$\lim_{x \to x_0} f(c(x)) = f(x_0).$$

Thus,

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} f(c(x)) = f(x_0). \qquad \blacksquare$$

As we have previously mentioned, *the principal importance of the Second Fundamental Theorem is that it provides the crucial first step on the road to studying quite general differential equations.* We shall turn to this aspect in Section 7.2. Also, as we shall see in Section 7.3, this theorem can be used to verify various classical techniques for replacing complicated integrals by ones for which we can directly apply the First Fundamental Theorem (Integrating Derivatives) by inspection. Finally, in Section 7.4, we will see that the Second Fundamental Theorem is important in analyzing the errors that arise when we use approximation techniques for estimating integrals.

In the remainder of this section, we will consider some variations of the Second Fundamental Theorem.

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Corollary 6.30 Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_x^b f\right] = -f(x) \qquad \text{for all } x \text{ in } (a, b).$$

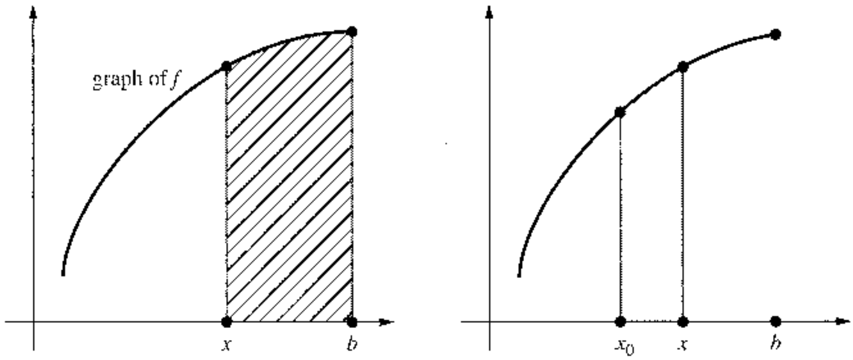

FIGURE 6.8 Shaded area equals $-[F(x) - F(x_0)]$ where $F(x) = \int_x^b f$.

Proof

By the additivity over intervals property of the integral, Theorem 6.12, for each x in $[a, b]$,

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

Observe that $\int_a^b f$ is independent of x in $[a, b]$. Thus, by the Second Fundamental Theorem (Differentiating Integrals),

$$\frac{d}{dx}\left[\int_x^b f\right] = \frac{d}{dx}\left[\int_a^b f - \int_a^x f\right] = -\frac{d}{dx}\left[\int_a^x f\right] = -f(x). \qquad \blacksquare$$

Motivated by the previous result, we extend the meaning of the symbol $\int_a^b f$ as follows.

Definition Let the function $f : [a, b] \to \mathbb{R}$ be integrable. Let c and d be numbers in $[a, b]$ such that $c < d$. We define

$$\int_d^c f \equiv -\int_c^d f \qquad \text{and} \qquad \int_c^c f = 0.$$

The above definition was made in order to have the following convenient extension of the additivity-over-intervals property of the integral: Let I be a closed bounded interval

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and suppose that the function $f : I \to \mathbb{R}$ is integrable. Then for any three points x_1, x_2, and x_3 in I,

$$\int_{x_1}^{x_3} f = \int_{x_1}^{x_2} f + \int_{x_2}^{x_3} f.$$

We leave the proof of this as an exercise.

Corollary 6.31 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Fix a point x_0 in I. Then

$$\frac{d}{dx}\left[\int_{x_0}^{x} f\right] = f(x) \qquad \text{for all } x \text{ in } I.$$

Proof

By the above extension of the additivity-over-intervals property of the integral, for each x in $[a, b]$,

$$\int_{x_0}^{x} f = \int_{x_0}^{a} f + \int_{a}^{x} f.$$

Observe that $\int_{x_0}^{a} f$ is independent of x in $[a, b]$. Thus, by the Second Fundamental Theorem (Differentiating Integrals),

$$\frac{d}{dx}\left[\int_{x_0}^{x} f\right] = \frac{d}{dx}\left[\int_{x_0}^{a} f + \int_{a}^{x} f\right] = \frac{d}{dx}\left[\int_{a}^{x} f\right] = f(x). \qquad \blacksquare$$

Corollary 6.32 Let I be an open interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Let J be an open interval and suppose that the function $\varphi : J \to \mathbb{R}$ is differentiable and that $\varphi(J) \subseteq I$. Fix a point x_0 in I. Then

$$\frac{d}{dx}\left[\int_{x_0}^{\varphi(x)} f\right] = f(\varphi(x))\varphi'(x) \qquad \text{for all } x \text{ in } J.$$

Proof

Define

$$G(x) \equiv \int_{x_0}^{\varphi(x)} f \qquad \text{for all } x \text{ in } J$$

and

$$F(x) \equiv \int_{x_0}^{x} f \qquad \text{for } x \text{ in } I.$$

By the Second Fundamental Theorem (Differentiating Integrals),

$$G = F \circ \varphi : J \to \mathbb{R}$$

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is the composition of differentiable functions, so the result follows from the Chain Rule: For all x in J,

$$\frac{d}{dx}\left[\int_{x_0}^{\varphi(x)} f\right] = \frac{d}{dx}[(F \circ \varphi)(x)] = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x). \qquad \blacksquare$$

The Second Fundamental Theorem was proved under the assumption that the function $f : [a, b] \to \mathbb{R}$ is continuous. In fact, if $f : [a, b] \to \mathbb{R}$ is merely integrable and we define $F(x) \equiv \int_a^x f$ for all x in $[a, b]$, then $F'(x) = f(x)$ at each point x in (a, b) at which the function $f : [a, b] \to \mathbb{R}$ is continuous (Exercise 10).

EXERCISES FOR SECTION 6.6

1. Calculate the following derivatives:

a. $\dfrac{d}{dx}\left(\displaystyle\int_0^x x^2t^2\,dt\right)$

b. $\dfrac{d}{dx}\left(\displaystyle\int_1^{e^x} \ln t\,dt\right)$

c. $\dfrac{d}{dx}\left(\displaystyle\int_{-x}^x e^{t^2}\,dt\right)$

d. $\dfrac{d}{dx}\left(\displaystyle\int_1^x \cos(x+t)\,dt\right)$

2. For each of the following integrable functions $f : [a, b] \to \mathbb{R}$ define

$$F(x) = \int_a^x f(t)\,dt \qquad \text{for all } x \text{ in } [a, b]$$

and find a formula for $F(x)$, $a \le x \le b$, that does not involve integrals.

a. $f : [1, 4] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } 1 \le x \le 3 \\ 6 & \text{if } 3 < x \le 4. \end{cases}$$

b. $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2. \end{cases}$$

c. $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } -1 \le x < 0 \\ x+1 & \text{if } 0 \le x \le 1. \end{cases}$$

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3. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Define the function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \int_{-x}^{x} [f(t) + f(-t)]\, dt \qquad \text{for all } x.$$

Find $H''(x)$.

4. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Prove that

$$f(x) = f(0) + f'(0)x + \int_0^x (x - t) f''(t)\, dt \qquad \text{for all } x.$$

(*Hint:* Use the Identity Criterion.)

5. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x - t) f(t)\, dt \qquad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x.

6. Define

$$F(x) = \int_1^x \frac{1}{2\sqrt{t} - 1}\, dt \qquad \text{for all } x \geq 1.$$

Prove that if $c > 0$, then there is a unique solution to the equation

$$F(x) = c, \qquad x > 1.$$

7. Show that the Mean Value Theorem for Integrals does not hold if we replace the assumption that $f : [a, b] \to \mathbb{R}$ is continuous with the assumption that $f : [a, b] \to \mathbb{R}$ is integrable.

8. For numbers $a_1, \ldots, a_n$, define $p(x) = a_1 x + a_2 x^2 + \cdots + a_n x^n$ for all x. Suppose that

$$\frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that there is some point x in the interval $(0, 1)$ such that $p(x) = 0$.

9. Show that the conclusion of the Mean Value Theorem for Integrals can be strengthened so that we can choose the point x_0 to be in (a, b), not just in $[a, b]$.

10. The Second Fundamental Theorem has a somewhat more general form than we have stated: *For an integrable function $f : [a, b] \to \mathbb{R}$, we define $F(x) \equiv \int_a^x f$ for all x in $[a, b]$. Then at each point x_0 in (a, b) at which the function $f : [a, b] \to \mathbb{R}$ is continuous, the function $F : [a, b] \to \mathbb{R}$ is differentiable and $F'(x_0) = f(x_0)$.* Use the monotonicity property of integration to prove this.

11. Let the function $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that the function $F : [a, b] \to \mathbb{R}$ is continuous, that $F : (a, b) \to \mathbb{R}$ is differentiable, and that $F'(x) = f(x)$ for

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all x in (a, b). Use the Second Fundamental Theorem to prove that

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = 0 \qquad \text{for all } x \text{ in } (a, b)$$

and from this derive a new proof of the First Fundamental Theorem.

12. Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that α and β are any real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \qquad \text{for all } x \text{ in } [a, b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all x in (a, b). Then use the Identity Criterion to provide another proof of the linearity of the integral in the case where the functions are assumed to be continuous, not just integrable.

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CHAPTER 7*

INTEGRATION: FURTHER TOPICS

7.1 SOLUTIONS OF DIFFERENTIAL EQUATIONS

Proposition 7.1 Let I be an open interval containing the point x_0 and suppose that the function $f : I \to \mathbb{R}$ is continuous. For any number y_0, the differential equation

$$\begin{cases} F'(x) = f(x) & \text{for all } x \text{ in } I \\ F(x_0) = y_0 \end{cases}$$

has a unique solution $F : I \to \mathbb{R}$ given by the formula

$$F(x) = y_0 + \int_{x_0}^{x} f \qquad \text{for all } x \text{ in } I.$$

Proof

By definition, $F(x_0) = y_0$. By the Second Fundamental Theorem (Differentiating Derivatives), $F'(x) = f(x)$ for all x in I. Thus, $F : I \to \mathbb{R}$ is a solution of the differential equation. The Identity Criterion, Proposition 4.20, implies that there is only one solution. ■

Recall that in Section 5.1 we provisionally *assumed* that there was a differentiable function $F : (0, \infty) \to \mathbb{R}$ that was a solution of the differential equation

$$\begin{cases} F'(x) = 1/x & \text{for all } x > 0 \\ F(1) = 0. \end{cases} \tag{7.1}$$

We can now *prove* that there is a solution.

Proposition 7.2 Define

$$F(x) \equiv \int_{1}^{x} \frac{1}{t}\, dt \qquad \text{for all } x > 0.$$

Then the function $F : (0, \infty) \to \mathbb{R}$ is the solution of the differential equation (7.1).

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Proof

This proposition follows from Proposition 7.1 in the case where $f(x) = 1/x$ for all x in $I \equiv (0, \infty)$. ■

We defined the natural logarithm $\ln : (0, \infty) \to \mathbb{R}$ to be the unique solution of the differential equation (7.1) provided that there is a solution. We now have the explicit integral formula

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \text{for all } x > 0.$$

We now consider the following general linear differential equation depending only on the function and its first derivatives.

Given a continuous function $h : \mathbb{R} \to \mathbb{R}$ and numbers a, x_0, and y_0, find a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} F'(x) + aF(x) = h(x) & \text{for all } x \\ F(x_0) = y_0. \end{cases} \tag{7.2}$$

Theorem 7.3 Suppose that the function $h : \mathbb{R} \to \mathbb{R}$ is continuous. Then the differential equation (7.2) has precisely one solution, given by the formula

$$F(x) = y_0 e^{-a(x-x_0)} + e^{-ax} \int_{x_0}^{x} e^{at} h(t)\,dt \qquad \text{for all } x. \tag{7.3}$$

Proof

Since for any number x, $e^{ax} \neq 0$, we see that the function $F : \mathbb{R} \to \mathbb{R}$ is a solution of (7.2) if and only if

$$\begin{cases} e^{ax}[F'(x) + aF(x)] = e^{ax} h(x) & \text{for all } x \\ F(x_0) = y_0. \end{cases}$$

But

$$e^{ax}[F'(x) + aF(x)] = \frac{d}{dx}(e^{ax} F(x)) \qquad \text{for all } x,$$

and $F(x_0) = y_0$ if and only if $e^{ax_0} F(x_0) = e^{ax_0} y_0$. Thus, the function $F : \mathbb{R} \to \mathbb{R}$ is a solution of (7.2) if and only if

$$\begin{cases} (d/dx)(e^{ax} F(x)) = e^{ax} h(x) & \text{for all } x \\ e^{ax_0} F(x_0) = e^{ax_0} y_0. \end{cases} \tag{7.4}$$

Now, by Proposition 7.1, there is a unique solution to the differential equation

$$\begin{cases} g'(x) = e^{ax} h(x) & \text{for all } x \text{ in } I \\ g(x_0) = e^{ax_0} y_0 \end{cases} \tag{7.5}$$

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that is defined by

$$g(x) = e^{ax_0}y_0 + \int_{x_0}^{x} e^{at}h(t)\,dt \qquad \text{for all } x.$$

But (7.4) means exactly that the function $g(x) = e^{ax}F(x)$ is a solution of the differential equation (7.5). Therefore,

$$e^{ax}F(x) = e^{ax_0}y_0 + \int_{x_0}^{x} e^{at}f(t)\,dt \qquad \text{for all } x.$$

Thus, there is a unique solution to the differential equation (7.2) that is defined by the formula (7.3). ∎

Example 7.4 Find the unique solution of the equation

$$\begin{cases} F'(x) - F(x) = x & \text{for all } x \\ F(0) = 2. \end{cases}$$

According to Theorem 7.3, the unique solution is given by the formula

$$F(x) = 2e^x + e^x \int_0^x e^{-t}t\,dt \qquad \text{for all } x.$$

But the integral in this formula can be evaluated by using the First Fundamental Theorem (Integrating Derivatives). After doing so, we obtain

$$F(x) = 3e^x - x - 1 \qquad \text{for all } x.$$ ∎

EXERCISES FOR SECTION 7.1

1. Find the unique solution of each of the following differential equations:

   a. $\begin{cases} F'(x) + F(x) = x & \text{for all } x \\ F(0) = 1. \end{cases}$

   b. $\begin{cases} F'(x) + 4F(x) = e^x & \text{for all } x \\ F(2) = 31. \end{cases}$

   c. $\begin{cases} F'(x) + F(x) = x^2 & \text{for all } x \\ F(0) = -1. \end{cases}$

2. For numbers c and a, consider the differential equation

$$\begin{cases} F'(x) = c(a - F(x)) & \text{for all } x. \\ F(0) = 0. \end{cases}$$

Prove that the unique solution is given by the formula

$$F(x) = a(1 - e^{-cx}) \qquad \text{for all } x.$$

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3. For an open interval I containing 0 and a function $f : I \to \mathbb{R}$ that has a continuous derivative, prove that

$$f(x) = f(0) + \int_0^x f'(t)\,dt \qquad \text{for all } x \text{ in } I.$$

Use this formula to obtain explicit representations of the arcsine, the arccosine, and the arctangent functions as integrals.

4. Prove that

$$\int_1^{12} \frac{1}{x}\,dx = 2\int_1^2 \frac{1}{x}\,dx + \int_1^3 \frac{1}{x}\,dx.$$

5. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$f(x) = \int_0^x f(t)\,dt \qquad \text{for all } x.$$

Prove that $f(x) = 0$ for all x.

6. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and that $g(x) > 0$ for all x. Define

$$h(x) = \int_0^x \frac{1}{g(t)}\,dt \qquad \text{for all } x$$

and let $J = h(\mathbb{R})$. Prove that if $f : J \to \mathbb{R}$ is the inverse of $h : \mathbb{R} \to \mathbb{R}$, then $f : J \to \mathbb{R}$ is a solution of the nonlinear differential equation

$$\begin{cases} f'(x) = g(f(x)) & \text{for all } x \text{ in } J \\ f(0) = 0. \end{cases}$$

7.2 INTEGRATION BY PARTS AND BY SUBSTITUTION

For a continuous function $f : [a, b] \to \mathbb{R}$ such that its restriction to the open interval (a, b) has a continuous bounded derivative, the First Fundamental Theorem (Integrating Derivatives) asserts that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This formula, together with the product formula for differentiation, provides a useful method for calculating integrals which we now describe.

Integration by Parts

Theorem 7.5 Suppose that the functions $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that both h and g have continuous bounded derivatives on the open

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interval (a, b). Then

$$\int_a^b h(x)g'(x)\,dx = h(b)g(b) - h(a)g(a) - \int_a^b g(x)h'(x)\,dx. \tag{7.6}$$

Proof

The product function $hg : [a, b] \to \mathbb{R}$ is continuous, $hg : (a, b) \to \mathbb{R}$ is differentiable, and according to the product rule for derivatives,

$$(hg)'(x) = h(x)g'(x) + g(x)h'(x) \qquad \text{for all } x \text{ in } (a, b).$$

Thus, by the First Fundamental Theorem (Integrating Derivatives), we have

$$\int_a^b [hg' + gh'] = \int_a^b (hg)' = h(b)g(b) - h(a)g(a), \tag{7.7}$$

and, on the other hand, the linearity property of the integral implies that

$$\int_a^b (hg' + gh') = \int_a^b hg' + \int_a^b gh'. \tag{7.8}$$

Formula (7.6) follows from (7.7) and (7.8). ■

Example 7.6 By reformulating

$$\int_0^1 xe^x\,dx \qquad \text{as} \qquad \int_0^1 x\frac{d}{dx}(e^x)\,dx,$$

$$\int_0^\pi x\cos x\,dx \qquad \text{as} \qquad \int_0^\pi x\frac{d}{dx}(\sin x)\,dx,$$

and

$$\int_1^2 \ln x\,dx \qquad \text{as} \qquad \int_1^2 \ln x\frac{d}{dx}(x)\,dx,$$

each of the integrals on the left-hand side can be evaluated using integration by parts and the First Fundamental Theorem (Integrating Derivatives). ■

Integration by Substitution

Theorem 7.7 Let the function $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that the function $g : [c, d] \to \mathbb{R}$ is also continuous, that $g : (c, d) \to \mathbb{R}$ has a bounded continuous derivative, and, moreover, that the image of $g : (c, d) \to \mathbb{R}$ is contained in the interval (a, b). Then

$$\int_c^d f(g(x))g'(x)\,dx = \int_{g(c)}^{g(d)} f(x)\,dx. \tag{7.9}$$

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Proof
Define the function $H : [c, d] \to \mathbb{R}$ by

$$H(x) \equiv \int_c^x (f \circ g)g' - \int_{g(c)}^{g(x)} f \qquad \text{for all } x \text{ in } [c,\ d].$$

Since the composition of continuous functions is continuous, it follows from Proposition 6.27 that the function $H : [c,\ d] \to \mathbb{R}$ is continuous. Moreover, from the Second Fundamental Theorem (Differentiating Integrals) and Corollary 6.32, we see that

$$H'(x) = f(g(x))g'(x) - f(g(x))g'(x) = 0 \qquad \text{for all } x \text{ in } (c, d).$$

The Identity Criterion implies that the function $H : [c, d] \to \mathbb{R}$ is constant. In particular, since $H(c) = 0$, $H(d) = 0$ also; that is, formula (7.9) holds. ■

The Geometric Significance of the Number π

For an integrable function $f : [a,\ b] \to \mathbb{R}$ having the property that $f(x) \geq 0$ for all x in $[a,\ b]$, we *define* the area bounded by the graph of $f : [a,\ b] \to \mathbb{R}$ and the x axis to be the integral $\int_a^b f(x)\,dx$. Of course, the integral itself was defined in order to make this definition reasonable. In particular, since the graph of the function

$$f(x) = \sqrt{1 - x^2}, \qquad 0 \leq x \leq 1,$$

represents the arc of the circle of unit radius centered at the origin that lies in the first quadrant, the value of the integral

$$\int_0^1 \sqrt{1 - x^2}\,dx$$

is one-quarter of the area bounded by a circle of unit radius.

Recall that in Section 5.2 we provided an analytic definition of the number π as follows: We assumed that the following trigonometric differential equation had a solution which we denoted by $\cos x$:

$$\begin{cases} f''(x) + f(x) = 0 & \text{for all } x \\ f(0) = 1 \quad \text{and} & f'(0) = 0. \end{cases}$$

Then the number $\pi/2$ was defined to be the smallest positive number at which $\cos x = 0$.

Of course, the number π has the geometric significance of being the area of a circle of unit radius. The next formula reconciles the analytic definition of π with its usual geometric significance.

Proposition 7.8

$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\,dx.$$

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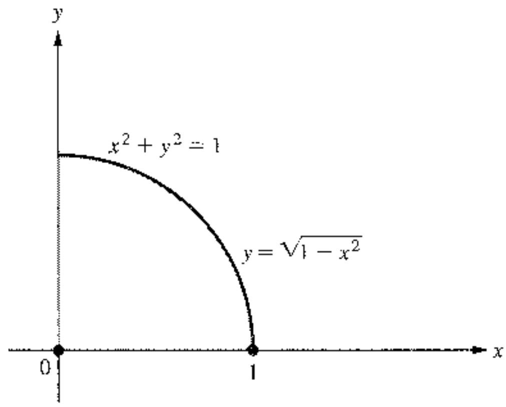

FIGURE 7.1 Area $= \int_0^1 \sqrt{1-x^2}\,dx = \pi/4$.

Proof

Since $\sin 0 = 0$, $\sin \pi/2 = 1$, and the function $\sin x$ is increasing and differentiable on the interval $[0, \pi/2]$, the substitution property of integration, Theorem 7.7, yields

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \sqrt{1-\sin^2 x}\cos x\,dx.$$

On the other hand, since $\cos x \geq 0$ on the interval $[0, \pi/2]$, by the Pythagorean Identity and the cosine addition formula,[1]

$$\sqrt{1-\sin^2 x}\cos x = \cos^2 x = (1+\cos 2x)/2 \qquad \text{for all } x \text{ in } [0, \pi/2].$$

Thus,

$$\int_0^{\pi/2} \sqrt{1-\sin^2 x}\cos x\,dx = \int_0^{\pi/2} (1+\cos 2x)/2\,dx.$$

Since

$$\frac{d}{dx}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right) = \frac{1+\cos 2x}{2} \qquad \text{for } 0 \leq x \leq \frac{\pi}{2},$$

we can apply the First Fundamental Theorem (Integrating Derivatives) to conclude that since $\sin 0 = \sin \pi = 0$,

$$\int_0^{\pi/2} \left[\frac{1+\cos 2x}{2}\right] dx = \left(\frac{x}{2} + \frac{\sin 2x}{4}\right)\Bigg|_{x=0}^{x=\pi/2} = \frac{\pi}{4}. \qquad \blacksquare$$

[1] Recall that in Section 5.2 the Pythagorean Identity and the cosine addition formula were derived solely on the basis that the cosine is the unique solution of the above trigonometric differential equation.

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Example 7.9 Consider the integral

$$\int_0^1 e^{x^2}\,dx.$$

Define $f(x) \equiv e^{x^2}$ for all x in $[0, 1]$. If we attempt to write $f:[0,1] \to \mathbb{R}$ as $f = hg':[0,1] \to \mathbb{R}$ so that $\int_0^1 gh'$ can be calculated, we will find that we cannot do so. Similarly, if we try substitution, we quickly encounter difficulties in choosing a function $g:[c, d] \to \mathbb{R}$ that will make $\int_c^d (f \circ g)g'$ easy to integrate. ■

The examples we have considered illustrate both the power and the limitations of the techniques of integration by parts and by substitution. The object, of course, is to replace one integration problem with another in which we can directly, by inspection, apply the First Fundamental Theorem. When we cannot make such a simplification, we need to develop methods for *estimating* integrals. We will turn to this in Section 7.4.

EXERCISES FOR SECTION 7.2

1. Evaluate the following integrals:

a. $\displaystyle\int_1^2 xe^{x^2}\,dx$

b. $\displaystyle\int_0^1 (1-x)^2\sqrt{2+x}\,dx$

c. $\displaystyle\int_2^3 x^3e^{x^2}\,dx$

d. $\displaystyle\int_2^\pi x^2\cos x\,dx$

2. Evaluate the following integrals:

a. $\displaystyle\int_1^e (\ln x)^2\,dx$

b. $\displaystyle\int_4^5 \frac{1+x}{1-x}\,dx$

c. $\displaystyle\int_4^9 \frac{1}{1-x^2}\,dx$

d. $\displaystyle\int_3^4 \left(\frac{1}{x^2-2x}+\frac{1}{1+\sqrt{x}}\right)dx$

3. Prove that for any two natural numbers n and m,

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 (1-x)^m x^n\,dx.$$

4. Suppose that the function $f:\mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Fix a number a. Prove that

$$\int_a^x f''(t)(x-t)\,dt = -(x-a)f'(a) + f(x) - f(a) \qquad \text{for all } x.$$

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5. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Prove that for any two numbers a and b,

$$\int_a^b xf''(x)\,dx = bf'(b) + f(a) - af'(a) - f(b).$$

6. Prove that the area of a circle of radius r is πr^2.

7. Calculate the three integrals in Example 7.6.

8. Suppose that the function $f : [0, \infty) \to \mathbb{R}$ is continuous and strictly increasing and that $f : (0, \infty) \to \mathbb{R}$ is differentiable. Moreover, assume $f(0) = 0$. Consider the formula

$$\int_0^x f + \int_0^{f(x)} f^{-1} = xf(x) \qquad \text{for all } x \geq 0.$$

Provide a geometric interpretation of this formula in terms of areas. Then prove this formula. (*Hint:* Differentiate the formula and apply the Identity Criterion.)

9. Suppose that the function $f : [0, \infty) \to \mathbb{R}$ is continuous and strictly increasing, with $f(0) = 0$ and $f([0, \infty)) = [0, \infty)$. Then define

$$F(x) = \int_0^x f \qquad \text{and} \qquad G(x) = \int_0^x f^{-1} \qquad \text{for all } x \geq 0.$$

a. Prove Young's Inequality:

$$ab \leq F(a) + G(b) \qquad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

(*Hint:* A sketch will help, as will the formula in Exercise 8.)

b. Now use Young's Inequality, with $f(x) = x^{p-1}$ for all $x \geq 0$ and $p > 1$ fixed, to prove that if the number q is chosen to have the property that $1/p + 1/q = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

7.3 THE CONVERGENCE OF DARBOUX AND RIEMANN SUMS

Recall that for a bounded function $f : [a, b] \to \mathbb{R}$ we have called a sequence $\{P_n\}$ of partitions of the domain $[a, b]$ an Archimedean sequence of partitions for f on $[a, b]$ provided that

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0. \tag{7.10}$$

The Archimedes–Riemann Theorem states that the function $f : [a, b] \to \mathbb{R}$ is integrable if there exists an Archimedean sequence of partitions for f on $[a, b]$, and moreover, for any such Archimedean sequence,

$$\lim_{n \to \infty} L(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n \to \infty} U(f, P_n) = \int_a^b f. \tag{7.11}$$

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We will now prove a theorem, called the Darboux Sum Convergence Theorem, that asserts that for a function f that is integrable on $[a, b]$, if a sequence of partitions $\{P_n\}$ has the property that

$$\lim_{n\to\infty} \text{gap } P_n = 0,$$

then this sequence is an Archimedean sequence of partitions for f on $[a, b]$ and therefore

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

In order to prove the Darboux Sum Convergence Theorem, it is useful to first establish a preliminary result comparing Darboux sums based on two different partitions. For an integrable function $f : [a, b] \to \mathbb{R}$ and a partition P of the interval $[a, b]$, the Refinement Lemma asserts that if the partition P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \leq \int_a^b f \leq U(f, P^*) \leq U(f, P). \tag{7.12}$$

The estimate (7.12) certainly requires that P^* be a refinement of P. However, if P^* is not a refinement of P, we have the following precise estimate comparing the Darboux sums based on P^* with those based on P: If the gap of P^* is small, then Darboux sums based on P^* compare almost as well with those of P as in the case where P^* is a refinement of P.

Lemma 7.10 Darboux Sum Comparison Lemma Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and let $M \geq 0$ be such that

$$-M \leq f(x) \leq M \qquad \text{for all } x \text{ in } [a, b].$$

Let P be a partition of $[a, b]$ that has k partition points. Let P^* be any other partition of $[a, b]$. Then

$$L(f, P) - E \leq L(F, P^*) \qquad \text{and} \qquad U(f, P^*) \leq U(f, P) + E, \tag{7.13}$$

where

$$E \equiv k \cdot M \cdot \text{gap } P^*.$$

Proof
Let

$$P^* = \{x_0, \ldots, x_n\} \qquad \text{and} \qquad P = \{z_0, \ldots, z_{k-1}\}.$$

As usual, for $1 \leq i \leq n$, we set $M_i \equiv \sup\{f(x) \mid x \text{ in } [x_{i-1}, x_i]\}$ and observe that by the choice of M, $M_i \leq M$, so that by the definition of gap,

$$M_i(x_i - x_{i-1}) \leq M \cdot \text{ gap } P^*. \tag{7.14}$$

For $1 \leq i \leq n$, we call the index i a *crossing index* provided that the open ith partition interval of P^*, (x_{i-1}, x_i), contains a partition point z_j of the partition P. Denote by C the set of crossing indices among the indices $\{1, \ldots, n\}$. Since P has $k - 2$ partition points other than endpoints and each partition point z_j of P can

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belong to at most one open interval (x_{i-1}, x_i), there are fewer than k crossing indices. Therefore, by the estimate (7.14), we have the following estimate of the contribution to $U(f, P^*)$ from the crossing indices:

$$\sum_{i \text{ in } C} M_i(x_i - x_{i-1}) \le k \cdot M \cdot \text{gap } P^*. \tag{7.15}$$

We now estimate the contribution to $U(f, P^*)$ from the noncrossing indices. If the index i, $1 \le i \le n$, is not a crossing index, then the open interval (x_{i-1}, x_i) does not contain any partition points of P, and therefore the interval $[x_{i-1}, x_i]$ of the partition P^* is contained in a partition interval of P. Thus, $[x_{i-1}, x_i]$ is a partition interval of the common refinement P' of P and P^*. Therefore,

$$\sum_{i \text{ not in } C} M_i(x_i - x_{i-1}) \le U(f, P').$$

However, by the Refinement Lemma,

$$U(f, P') \le U(f, P).$$

Thus,

$$\sum_{i \text{ not in } C} M_i(x_i - x_{i-1}) \le U(f, P). \tag{7.16}$$

We add the estimates of the contributions to $U(f, P^*)$ from the noncrossing indices and the crossing indices in $\{1, \ldots, n\}$ to obtain the following upper estimate for $U(f, P^*)$:

$$\begin{aligned} U(f, P^*) &\equiv \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= \sum_{i \text{ not in } C} M_i(x_i - x_{i-1}) + \sum_{i \text{ in } C} M_i(x_i - x_{i-1}) \\ &\le U(f, P) + k \cdot M \cdot \text{gap } P^*. \end{aligned}$$

By an entirely similar argument, we obtain the corresponding lower estimate for $L(f, P^*)$:

$$L(f, P) - k \cdot M \cdot \text{gap } P^* \le L(f, P^*).$$

The two estimates stated in (7.13) have been established. ■

The following simple lemma follows from the Archimedes–Riemann Theorem (Exercise 7).

Lemma 7.11 Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Then for each positive number ϵ, there is a partition P of the interval $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

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Theorem 7.12 The Darboux Sum Convergence Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded. Then the following two assertions are equivalent:

i. The function $f : [a, b] \to \mathbb{R}$ is integrable.

ii. For any sequence $\{P_n\}$ of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \text{gap } P_n = 0,$$

we have

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0, \tag{7.17}$$

and therefore,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \qquad \text{and} \qquad \lim_{n\to\infty} U(f, P_n) = \int_a^b f. \tag{7.18}$$

Proof

Assume that (ii) holds. Choose $\{P_n\}$ to be a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \text{gap } P_n = 0.$$

Then, since (ii) holds,

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0. \tag{7.19}$$

According to the Archimedes–Riemann Theorem, the function f is integrable on $[a, b]$.

To prove the converse, assume that f is integrable on $[a, b]$. Let $\{P_n\}$ be any sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \text{gap } P_n = 0.$$

We wish to prove (7.17). To do so, by the definition of the convergence of a sequence, we choose $\epsilon > 0$ and must show that there is an index N such that

$$0 \le U(f, P_n) - L(f, P_n) < \epsilon \qquad \text{if } n \ge N. \tag{7.20}$$

Since $\epsilon/2 > 0$, by Lemma 7.11, we can choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon/2.$$

Let k be the number of points in the partition P and let $M \ge 0$ be such that

$$-M \le f(x) \le M \qquad \text{for all } x \text{ in } [a, b].$$

Then, by the preceding Darboux Sum Comparison Lemma, substituting P_n for P^* for each index n, we have

$$L(f, P) - E_n \le L(f, P_n) \qquad \text{and} \qquad U(f, P_n) \le U(f, P) + E_n, \tag{7.21}$$

where

$$E_n \equiv k \cdot M \cdot \text{gap } P_n.$$

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It follows that for each index n,

$$\begin{aligned}0 \le U(f, P_n) - L(f, P_n) &\le [U(f, P) + E_n] - [L(f, P) - E_n]\\ &= [U(f, P) - L(f, P)] + 2E_n\\ &< \epsilon/2 + [2k \cdot M \cdot \text{gap } P_n].\end{aligned}$$

That is, for each index n,

$$0 \le U(f, P_n) - L(f, P_n) < \epsilon/2 + [2k \cdot M \cdot \text{gap } P_n]. \qquad (7.22)$$

By assumption,

$$\lim_{n\to\infty} \text{gap } P_n = 0,$$

so that we also have

$$\lim_{n\to\infty} [2k \cdot M \cdot \text{gap } P_n] = 0.$$

Thus, we can choose an index N such that

$$0 \le [2k \cdot M \cdot \text{gap } P_n] < \epsilon/2 \qquad \text{if } n \ge N.$$

From the estimate (7.22) and this choice of N, we see that the required estimate (7.20) also holds for this choice of index N. Thus, $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$, and therefore, by the Archimedes–Riemann Theorem, (7.18) holds. ■

Definition Consider a function $f : [a, b] \to \mathbb{R}$, and let $P = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$. For each index $i \ge 1$, let c_i be a point in the interval $[x_{i-1}, x_i]$. Then the sum

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \qquad (7.23)$$

is called a *Riemann sum* for the function $f : [a, b] \to \mathbb{R}$ based on the partition P.

It is convenient to denote the Riemann sum (7.23) by the symbol $R(f, P, C)$, where C denotes the choice of points $\{c_1, \ldots, c_n\}$ from the partition intervals in P.[2] It is clear that if the function $f : [a, b] \to \mathbb{R}$ is bounded, then for each partition P of $[a, b]$ and choice C,

$$L(f, P) \le R(f, P, C) \le U(f, P).$$

Considering this inequality and the Darboux Sum Convergence Theorem, the following convergence result is not surprising.

[2] When we use the symbol $R(f, P, C)$, we implicitly suppose that C is chosen as above.

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Theorem 7.13 Riemann Sum Convergence Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. For each natural number n, let P_n be a partition of $[a, b]$ and let $R(f, P_n, C_n)$ be a Riemann sum. If

$$\lim_{n\to\infty} \text{gap } P_n = 0,$$

then

$$\lim_{n\to\infty} R(f, P_n, C_n) = \int_a^b f. \tag{7.24}$$

Proof

For each index n,

$$L(f, P_n) \le R(f, P_n, C_n) \le U(f, P_n).$$

According to the Darboux Sum Convergence Theorem,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f \quad \text{and} \quad \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Thus,

$$\int_a^b f = \lim_{n\to\infty} L(f, P_n) \le \lim_{n\to\infty} R(f, P_n, C_n) \le \lim_{n\to\infty} U(f, P_n) = \int_a^b f,$$

so that (7.24) holds. ■

Example 7.14 For each natural number n, define P_n to be the regular partition of $[0, 1]$ into n partition intervals of equal length. Consider the Riemann sum for the integral $\int_0^1 \sqrt{x}\, dx$ based on the partition P_n that we obtain by letting $c_i = i/n$ for $1 \le i \le n$. From the preceding Riemann Sum Convergence Theorem and the First Fundamental Theorem (Integrating Derivatives), it follows that

$$\lim_{n\to\infty} \left[\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}\right] = \lim_{n\to\infty} \frac{1}{n} \left[\sqrt{\frac{1}{n}} + \cdots + \sqrt{\frac{n}{n}}\right] = \int_0^1 \sqrt{x}\, dx = \frac{2}{3}.$$

■

EXERCISES FOR SECTION 7.3

1. For a fixed positive number β, find

$$\lim_{n\to\infty} \left[\frac{1^\beta + 2^\beta + \cdots + n^\beta}{n^{\beta+1}}\right].$$

2. Find

$$\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right].$$

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3. Find

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{k}{n^2+k^2}\right].$$

4. Find

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n\cdot n}}+\frac{1}{\sqrt{n(n+1)}}+\cdots+\frac{1}{\sqrt{n(n+n)}}\right].$$

5. Let $b>1$. Find the value of the Riemann sum for $\int_1^b[1/\sqrt{x}]\,dx$ that one obtains for the partition $P=\{x_0,\ldots,x_n\}$ of $[1,b]$ by choosing $c_i=[(\sqrt{x_i}+\sqrt{x_{i-1}})/2]^2$ for $1\le k\le n$.

6. Suppose that the function $f:[0,1]\to\mathbb{R}$ is integrable. Prove that

$$\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)+f(1)\right]=\int_0^1 f.$$

7. Prove Lemma 7.11.

8. Supose that the function $f:[a,\,b]\to\mathbb{R}$ is integrable. In order for a sequence $\{P_n\}$ of partitions of the domain $[a,\,b]$ to be an Archimedean sequence of partitions for f on $[a,\,b]$, is it necessary that $\lim_{n\to\infty}\text{gap } P_n=0$?

9. Suppose that the function $f:[a,\,b]\to\mathbb{R}$ is continuous and let P be any partition of its domain $[a,\,b]$. Show that there is a Riemann sum $R(f,P,C)$ that equals $\int_a^b f$. (*Hint:* Use the Mean Value Theorem for Integrals.)

10. Suppose that the function $f:[a,\,b]\to\mathbb{R}$ is Lipschitz; that is, there is a number c such that

$$|f(u)-f(v)|\le c|u-v|\qquad\text{for all } u \text{ and } v \text{ in } [a,\,b].$$

Let P be a partition of $[a,\,b]$ and let $R(f,P,C)$ be a Riemann sum based on P. Prove that

$$\left|R(f,P,C)-\int_a^b f\right|\le c\cdot(b-a)\cdot[\text{gap } P].$$

11. Let p and n be natural numbers with $n\ge 2$. Prove that

$$\sum_{k=1}^{n-1}k^p\le\frac{n^{p+1}}{p+1}\le\sum_{k=1}^{n}k^p.$$

(*Hint:* Use an induction argument on n.)

12. For a natural number p, use Exercise 11 to prove that

$$\int_0^1 x^p\,dx=\frac{1}{p+1}.$$

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13. (Fermat's Method for Computing $\int_1^b x^\beta\,dx$.) Let $b > 1$ and $\beta \neq -1$. Define $f(x) = x^\beta$ for all x in $[1, b]$. For each natural number n, let $P_n = \{x_0, \ldots, x_n\}$ be the partition of $[1, b]$ defined by $x_i = b^{i/n}$ for $0 \le i \le n$.

a. Show that

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \frac{b^{1/n} - 1}{b^{1/n}}\left[\frac{1 - (b^{\beta+1})^{(n+1)/n}}{1 - b^{(\beta-1)/n}} - 1\right].$$

b. Show that

$$\lim_{n\to\infty} \frac{1 - (b^{1/n})^{\beta+1}}{1 - b^{1/n}} = \beta + 1.$$

c. Use (a) and (b) to show that

$$\int_1^b x^\beta\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \frac{b^{\beta+1} - 1}{\beta + 1}.$$

7.4 THE APPROXIMATION OF INTEGRALS

There are many functions $f : [a, b] \to \mathbb{R}$ that are integrable, but it is not possible by substitution, by integration by parts, or by any other device to reduce the calculation of $\int_a^b f(x)\,dx$ to an application by inspection of the First Fundamental Theorem (Integrating Derivatives). While in the preceding section we determined that sequences of Darboux and Riemann sums converge to the integral provided that the sequences of gaps of the associated sequence of partitions converge to 0, we did not provide any estimate of the difference between the integral and a particular Darboux or Riemann sum. The object of this section is to specify approximations of the value of the integral and to specify a bound for the error that has arisen in the approximation procedures.

Local and Global Errors

The idea is to take a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$. Then for each index $i \ge 1$, we approximate

$$\int_{x_{i-1}}^{x_i} f(x)\,dx$$

by some A_i and define E_i by

$$E_i \equiv \int_{x_{i-1}}^{x_i} f(x)\,dx - A_i.$$

Define $A \equiv \sum_{i=1}^n A_i$ and $E \equiv \sum_{i=1}^n E_i$, so that

$$\int_a^b f(x)\,dx - A = E.$$

The E_i's are referred to as the *local errors;* E is called the *global error*. Once we have estimated the local errors, an estimate for the global error usually follows easily. We will look at two approximation methods, the Trapezoid Rule and Simpson's Rule.

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The Trapezoid Rule

For the Trapezoid Rule, consider a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$. For each index $i \geq 1$, we approximate the integral $\int_{x_{i-1}}^{x_i} f(x)\,dx$ by

$$A_i \equiv \frac{1}{2}(x_i - x_{i-1})(f(x_{i-1}) + f(x_i)).$$

If $f(x_i) \geq 0$ and $f(x_{i-1}) \geq 0$, the above A_i is simply the area of the trapezoid having vertices $(x_{i-1}, 0)$, $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_i, 0)$. We first estimate the local error.

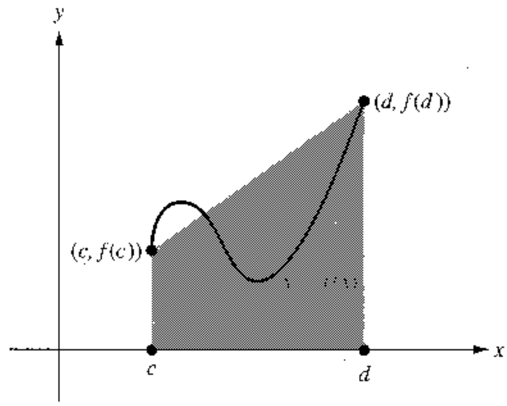

FIGURE 7.2 The trapozoid approximation of $\int_c^d f$.

Theorem 7.15 The Local Error for the Trapezoid Rule Suppose that the function $f : [c, d] \to \mathbb{R}$ is continuous and that its restriction to the open interval (c, d) has two derivatives. Then there is a point ζ in the open interval (c, d) at which

$$\int_c^d f(x)\,dx - \frac{(d-c)(f(c)+f(d))}{2} = \frac{-(d-c)^3}{12} f''(\zeta). \tag{7.25}$$

Proof

Let $m \equiv (d+c)/2$ be the midpoint of the interval $[c, d]$. Then if $t_0 \equiv (d-c)/2$, we have $m - t_0 = c$ and $m + t_0 = d$. For $0 \leq t \leq t_0$, let $E(t)$ be the error for the direct trapezoidal approximation of the integral of f on the interval $[m-t, m+t]$. This auxiliary function $E : [0, t_0] \to \mathbb{R}$ is thus defined by

$$E(t) = \left[\int_{m-t}^{m+t} f(x)\,dx\right] - t[f(m+t) + f(m-t)] \qquad \text{for } 0 \leq t \leq t_0.$$

Observe that the left-hand side of formula (7.25) is precisely $E(t_0)$. Now define another function $H : [0, t_0] \to \mathbb{R}$ by

$$H(t) = E(t) - \left(\frac{t}{t_0}\right)^3 E(t_0) \qquad \text{for } 0 \leq t \leq t_0.$$

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It follows from Proposition 6.27 that the function $E : [0, t_0] \to \mathbb{R}$ is continuous, so the function $H : [0, t_0] \to \mathbb{R}$ also is continuous. Moreover, the Second Fundamental Theorem and the Chain Rule yield

$$\begin{aligned} E'(t) &= f(m+t)+f(m-t)-[f(m+t)+f(m-t)]-t[f'(m+t)-f'(m-t)] \\ &= -t[f'(m+t)-f'(m-t)] \qquad \text{for } 0 < t < t_0. \end{aligned}$$

Hence,

$$H'(t) = -t[f'(m+t)-f'(m-t)] - \frac{3t^2}{t_0^3}E(t_0) \qquad \text{for } 0 < t < t_0.$$

It is clear that $H(0) = H(t_0) = 0$. We can apply Rolle's Theorem to the function $H : [0, t_0] \to \mathbb{R}$ to choose a point t_* in $(0, t_0)$ at which $H'(t_*) = 0$; that is,

$$-t_*[f'(m+t_*) - f'(m-t_*)] - \frac{3t_*^2}{t_0^3}E(t_0) = 0.$$

Now apply the Mean Value Theorem to the function $f' : [m - t_*, m + t_*] \to \mathbb{R}$ and choose a point ζ in $(m - t_*, m + t_*)$ at which

$$f'(m+t_*) - f'(m-t_*) = 2t_* f''(\zeta).$$

Substituting the last equality in the preceding equation, we arrive at

$$2t_*^2\left[f''(\zeta) + \frac{3}{2t_0^3}E(t_0)\right] = 0.$$

But $t_* \neq 0$, so that since $t_0 = [d-c]/2$, we have

$$E(t_0) = \frac{-(d-c)^3}{12}f''(\zeta). \tag{7.26}$$

As we already noted, $E(t_0)$ is the left-hand side of formula (7.25). Thus, formula (7.25) follows from the preceding formula (7.26). ∎

Theorem 7.16 The Global Error for the Trapezoid Rule Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that its restriction to the open interval (a, b) has a bounded second derivative. For a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$,

$$\int_a^b f = \sum_{i=1}^n \frac{(x_i - x_{i-1})(f(x_{i-1}) + f(x_i))}{2} + E \tag{7.27}$$

with

$$|E| \leq \frac{M[\text{gap } P]^2(b-a)}{12}, \tag{7.28}$$

where

$$M = \sup\{|f''(x)| \mid x \text{ in } (a, b)\}.$$

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Proof

For each index $i \geq 1$, we can apply the local error estimate for the Trapezoid Rule on the partition interval $[x_{i-1}, x_i]$ to choose a point ζ_i in the open interval (x_{i-1}, x_i) such that formula (7.25) holds with $x_{i-1} = c, x_i = d$, and $\zeta_i = \zeta$. Summing these n formulas, we see that (7.27) holds where

$$E = -\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^3 f''(\zeta_i)}{12}.$$

However, by the Triangle Inequality and the definitions of gap P and M,

$$\begin{aligned}
|E| &\leq \frac{[\text{gap } P]^2}{12} \sum_{i=1}^{n} (x_i - x_{i-1})|f''(\zeta_i)| \\
&\leq \frac{[\text{gap } P]^2 M}{12} \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= \frac{M[\text{gap } P]^2 (b-a)}{12}.
\end{aligned}$$

■

Example 7.17 We apply the Trapezoid Rule to estimate $\ln 2 = \int_1^2 1/t\, dt$. Define $f(t) = 1/t$ for $1 \leq t \leq 2$ and note that

$$0 \leq f''(t) \leq 2 \qquad \text{for } 1 \leq t \leq 2.$$

For a natural number n, let P_n be the partition obtained by dividing the interval $[1, 2]$ into n subintervals of equal length. Then (Exercise 3)

$$\ln 2 = \frac{1}{n}\left[\frac{1}{2} + \frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n-1} + \frac{1}{4}\right] + E_n,$$

where

$$0 \leq |E_n| \leq \frac{1}{6n^2}.$$

Taking $n = 10$, a brief calculation yields 0.6937714 as a lower approximation of $\ln 2$, and the error is at most $1/600$. ■

From the Local Error Estimate for the Trapezoid Rule, we note that the Trapezoid Rule gives the *exact value* of the integral when $f: \mathbb{R} \to \mathbb{R}$ has a line as its graph. Moreover, when

$$f''(x) \geq 0 \qquad \text{for all } x \text{ in } (a, b),$$

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which means that the function $f : [a, b] \to \mathbb{R}$ is convex, the Trapezoid Rule gives an *upper* approximation to the integral. Each of these observations is clear geometrically.

Simpson's Rule

The second approximation method that we will consider is called Simpson's Rule. Though it is not as easy to motivate geometrically, this rule is, in general, more accurate than the Trapezoid Rule. Given an integrable function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{x_0, \ldots, x_n\}$ of the interval $[a, b]$, for each index $i \geq 1$, Simpson's Rule approximates the integral $\int_{x_{i-1}}^{x_i} f$ by

$$A_i \equiv \frac{x_i - x_{i-1}}{6}\left[f(x_{i-1}) + 4f\left(\frac{x_{i-1} + x_i}{2}\right) + f(x_i)\right]. \tag{7.29}$$

One way to compare approximation methods is to see for what functions the approximation of the integral agrees precisely with the value of the integral. For the Trapezoid Rule, there is exact agreement provided that the function $f : [a, b] \to \mathbb{R}$ is a polynomial of degree less than 2; that is, the graph of $f : [a, b] \to \mathbb{R}$ is a line. We will show that for Simpson's Rule there is exact agreement provided that the function $f : [a, b] \to \mathbb{R}$ is a polynomial of degree[3] less than 4.

In order to derive the local error estimate for Simpson's Rule, it is first convenient to note the following slight extension of Rolle's Theorem.

Theorem 7.18 A Generalized Rolle's Theorem Suppose that the function $g : [c, d] \to \mathbb{R}$ is continuous. For a natural number n, suppose that its restriction to the open interval (c, d) has $n + 1$ derivatives. Let x_0 be a point in (c, d) at which

$$g(x_0) = g'(x_0) = \cdots = g^{(n)}(x_0) = 0.$$

Then for each point $x \neq x_0$ in $[c, d]$ at which $g(x) = 0$, there is a point ζ strictly between x_0 and x at which

$$g^{(n+1)}(\zeta) = 0.$$

Proof

We will assume that $x > x_0$. If we apply Rolle's Theorem to the function $g : [x_0, x] \to \mathbb{R}$, we can choose a point z_1 in (x_0, x) at which $g'(z_1) = 0$. Now apply Rolle's Theorem to the function $g' : [x_0, z_1] \to \mathbb{R}$ to choose a point z_2 in (x_0, z_1) at which $g''(z_2) = 0$. By continuing this procedure n times, we find a point $z_{n+1} = \zeta$ in (x_0, x) at which $g^{(n+1)}(\zeta) = 0$. ■

[3] The formula in Simpson's Rule is motivated as follows: For a function $f : [c, d] \to \mathbb{R}$, there is a unique quadratic polynomial $p : \mathbb{R} \to \mathbb{R}$ that agrees with the function $f : [c, d] \to \mathbb{R}$ at $x = c$, $x = (c + d)/2$, and $x = d$, and it can be shown that for this polynomial,

$$\int_c^d p = \frac{(d - c)}{6}\left[f(c) + 4f\left(\frac{c + d}{2}\right) + f(d)\right].$$

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Theorem 7.19 The Local Error for Simpson's Rule Suppose that the function $f : [c, d] \to \mathbb{R}$ is continuous and that its restriction to the open interval (c, d) has four derivatives. Then there is a point ζ in the open interval (c, d) at which

$$\int_c^d f(x)\,dx - \frac{(d-c)}{6}\left[f(c) + 4f\left(\frac{c+d}{2}\right) + f(d)\right] = -\frac{1}{2880}(d-c)^5 f^{(4)}(\zeta). \tag{7.30}$$

Proof

Let $m \equiv (d+c)/2$ be the midpoint of the interval $[c, d]$. Then, if $t_0 \equiv (d-c)/2$, we have $m - t_0 = c$ and $m + t_0 = d$. For $0 \le t \le t_0$, let $E(t)$ be the error for the direct Simpson approximation of the integral of f on the interval $[m-t, m+t]$. This auxiliary function $E : [0, t_0] \to \mathbb{R}$ is thus defined by

$$E(t) = \int_{m-t}^{m+t} f - \frac{t}{3}[f(m+t) + 4f(m) + f(m-t)] \qquad \text{for } -t_0 \le t \le t_0.$$

Observe that the left-hand side of formula (7.30) is $E(t_0)$. Now define the function $H : [-t_0, t_0] \to \mathbb{R}$ by

$$H(t) = E(t) - \left(\frac{t}{t_0}\right)^5 E(t_0) \qquad \text{for } -t_0 \le t \le t_0. \tag{7.31}$$

Proposition 6.27 implies that the function $E : [-t_0, t_0] \to \mathbb{R}$ is continuous, so the function $H : [-t_0, t_0] \to \mathbb{R}$ also is continuous. It is clear that $H(0) = H(t_0) = 0$, and we will now show that $H'(0) = H''(0) = 0$ so that we can apply the Generalized Rolle's Theorem. Using the Second Fundamental Theorem (Differentiating Integrals) and the Chain Rule, we have the following straightforward calculation of derivatives at each point t in $(-t_0, t_0)$:

$$\begin{aligned}
E'(t) &= f(m+t) + f(m-t) - \frac{1}{3}[f(m+t) + 4f(m) + f(m-t)] \\
&\quad - \frac{t}{3}[f'(m+t) - f'(m-t)] \\
&= \frac{2}{3}[f(m+t) - 2f(m) + f(m-t)] - \frac{t}{3}[f'(m+t) - f'(m-t)], \\
E''(t) &= \frac{2}{3}[f'(m+t) - f'(m-t)] - \frac{1}{3}[f'(m+t) - f'(m-t)] \\
&\quad - \frac{t}{3}[f''(m+t) + f''(m-t)] \\
&= \frac{1}{3}[f'(m+t) - f'(m-t)] - \frac{t}{3}[f''(m+t) + f''(m-t)],
\end{aligned}$$

and

$$E'''(t) = -\frac{t}{3}[f'''(m+t) - f'''(m-t)].$$

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These calculations, together with (7.31), show that $H(0) = H'(0) = H''(0) = 0$ and that

$$H'''(t) = -\frac{t}{3}[f'''(m+t) - f'''(m-t)] - \frac{60t^2}{t_0^5}E(t_0) \qquad \text{for } -t_0 < t < t_0.$$

Consequently, since $H(t_0) = 0$, we can apply the Generalized Rolle's Theorem with $n = 2$ to choose a point t_* in $(0, t_0)$ at which $H'''(t_*) = 0$; that is,

$$-\frac{t_*}{3}[f'''(m+t_*) - f'''(m-t_*)] - \frac{60t_*^2}{t_0^5}E(t_0) = 0. \tag{7.32}$$

Finally, we can apply the Mean Value Theorem to the function $f''' : [m - t_*, m + t_*] \to \mathbb{R}$ to choose a point ζ in $(m - t_*, m + t_*)$ at which

$$f'''(m+t_*) - f'''(m-t_*) = 2t_* f^{(4)}(\zeta).$$

Substituting the last equality in (7.32) leads to

$$t_*^2\left[-\frac{2}{3}f^{(4)}(\zeta) - \frac{60}{t_0^5}E(t_0)\right] = 0, \tag{7.33}$$

so since $t_* \neq 0$ and $t_0 = (d-c)/2$, we have

$$E(t_0) = -\frac{1}{2880}(d-c)^5 f^{(4)}(\zeta). \tag{7.34}$$

As we already observed, $E(t_0)$ is the left-hand side of formula (7.30). Thus, formula (7.30) follows from formula (7.34). ■

Corollary 7.20 For a polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree at most 3 and numbers $c < d$,

$$\int_c^d p(x)\,dx = \frac{1}{6}(d-c)\left[p(c) + 4p\left(\frac{c+d}{2}\right) + p(d)\right].$$

Proof
Since $p^{(4)}(\zeta) = 0$ for every number ζ, the result immediately follows from the local error bound for Simpson's Rule. ■

Theorem 7.21 The Global Error for Simpson's Rule Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that its restriction to the open interval (a, b) has a bounded fourth derivative. Let $P = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$. Then

$$\int_a^b f = \frac{1}{6}\sum_{i=1}^{n}(x_i - x_{i-1})\left[f(x_i) + 4f\left(\frac{x_i + x_{i-1}}{2}\right) + f(x_{i-1})\right] + E, \tag{7.35}$$

with

$$|E| \leq \frac{M[\text{gap } P]^4(b-a)}{2880},$$

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where

$$M \equiv \sup\left\{|f^{(4)}(x)| \mid a < x < b\right\}.$$

Proof

For each index $i \geq 1$, we can apply the local error estimate for Simpson's Rule on the partition interval $[x_{i-1}, x_i]$ to choose a point ζ_i in the open interval (x_{i-1}, x_i) such that formula (7.30) holds with $x_{i-1} = c, x_i = d$, and $\zeta_i = \zeta$. Summing these n formulas, we see that (7.35) holds where

$$E = -\frac{1}{2880}\sum_{i=1}^{n}(x_i - x_{i-1})^5 f^{(4)}(\zeta_i).$$

However, the Triangle Inequality and the definitions of gap P and M yield

$$\begin{aligned}|E| &\leq \frac{[\text{gap } P]^4}{2880}\sum_{i=1}^{n}(x_i - x_{i-1})|f^{(4)}(\zeta_i)| \\ &\leq \frac{M[\text{gap } P]^4}{2880}\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= \frac{M[\text{gap } P]^4(b-a)}{2880}.\end{aligned}$$

■

EXERCISES FOR SECTION 7.4

1. Use the First Fundamental Theorem (Integrating Derivatives) to compute each of the following integrals. Then compute the approximations using the Trapezoid Rule and Simpson's Rule with the partition $P = \{c, d\}$. Then compare the actual errors generated against the error estimates provided by Theorems 7.15 and 7.19.

a. $\int_1^2 (2x + 3)\, dx$

b. $\int_0^1 x^2\, dx$

c. $\int_0^1 x^4\, dx$

2. For each of the following integrals, verify Corollary 7.20 by direct computation.

a. $\int_0^1 (x^2 + x^3)\, dx$

b. $\int_2^3 (x + 1)^2\, dx$

3. Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that its restriction to the open interval (a, b) has a bounded second derivative. Let n be a natural number. Show that

$$\int_a^b f(x)\, dx = \left(\frac{b-a}{n}\right)\left[\frac{f(a)}{2} + \sum_{k=1}^{n-1} f\left(a + \frac{k}{n}(b-a)\right) + \frac{f(b)}{2}\right] + E,$$

where

$$|E| \leq \frac{(b-a)^3}{12n^2}\sup\{|f''(x)| x \text{ in } (a, b)\}.$$

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4. Use Exercise 3 with $n = 3$ to estimate $\ln 4$ and give an upper bound for the error.

5. Use Exercise 3 with $n = 4$ to estimate $\int_0^1 \sqrt{1+x^2}\,dx$ and give an upper bound for the error.

6. Use Simpson's Rule with $P = \{1,\ 3/2,\ 2\}$ to estimate $\ln 2$ and give an upper bound for the error.

7. An approximation rule similar to the Trapezoid Rule is the Midpoint Rule. This rule approximates the integral $\int_c^d f(x)\,dx$ by $(d-c)f([d+c]/2)$. Prove that if the function $f:[c,\ d] \to \mathbb{R}$ is continuous and its restriction $f:(c,d) \to \mathbb{R}$ has a second derivative, then for some point ζ in (c,d),

$$\int_c^d f(x)\,dx = (d-c)f\left(\frac{c+d}{2}\right) + \frac{1}{24}(d-c)^3 f''(\zeta).$$

[*Hint:* Let $m = (c+d)/2$, let $t_0 = (d-c)/2$ and define

$$H(t) = \left[\int_{m-t}^{m+t} f\right] - 2tf(m) - \left(\frac{t}{t_0}\right)^3 \int_{m-t_0}^{m+t_0} f - 2\frac{t^3}{t_0^2}(f)m \qquad \text{for } -t_0 \le t \le t_0.$$

Apply the Generalized Rolle's Theorem with $n=1$ to the function $H:[-t_0,\ t_0] \to \mathbb{R}$.]

8. For a continuous function $f:[a,\ b] \to \mathbb{R}$ and a partition P of its domain $[a,b]$, show that both the Trapezoidal Rule and Simpson's Rule approximations of the integral of f on $[a,b]$ are Riemann sums.

9. Find a global error estimate for the Midpoint Rule.

10. Use the local error estimates for the Midpoint Rule (Exercise 7) and the Trapezoid Rule for the evaluation of

$$\int_{\ln a}^{\ln b} e^x\,dx$$

to prove that if $0 < a < b$, then

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

(The geometric mean is less than the logarithmic mean, which is less than the arithmetic mean.)

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CHAPTER 8

APPROXIMATION BY TAYLOR POLYNOMIALS

8.1 TAYLOR POLYNOMIALS

Polynomials are the simplest kinds of functions. In this chapter, we will study the manner in which general functions can be approximated by polynomials.

Order of Contact of Two Functions

Definition Let I be a neighborhood of the point x_0. Two functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to *have contact of order* 0 at x_0 provided that $f(x_0) = g(x_0)$. For a natural number n, the functions f and g are said to *have contact of order* n at x_0 provided that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ have n derivatives and

$$f^{(k)}(x_0) = g^{(k)}(x_0) \qquad \text{for } 0 \le k \le n.$$

Example 8.1 Define $f(x) = \sqrt{2 - x^2}$ and $g(x) = e^{1-x}$ for $0 < x < \sqrt{2}$. Then

$$f(1) = g(1) \text{ and } f'(1) = g'(1), \text{ but } f''(1) \neq g''(1).$$

Hence the functions $f : (0, \sqrt{2}) \to \mathbb{R}$ and $g : (0, \sqrt{2}) \to \mathbb{R}$ have contact of order 1 at $x_0 = 1$ but do not have contact of order 2 there. At the point $(1, 1)$, which lies on both graphs, the tangent lines to the functions are the same. ■

For any natural number j, and any number x_0,

$$\frac{d}{dx}(x - x_0)^j = j(x - x_0)^{j-1} \text{ for all } x.$$

Thus, successively taking derivatives, we see that for each pair of nonnegative integers k and ℓ,

$$\left.\frac{d^k}{dx^k}[(x - x_0)^\ell]\right|_{x=x_0} = \begin{cases} k! & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases} \tag{8.1}$$

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Proposition 8.2 Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to \mathbb{R}$ has n derivatives. Then there is a unique polynomial of degree at most n that has contact of order n with the function $f : I \to \mathbb{R}$ at x_0. This polynomial is defined by the formula

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{8.2}$$

Proof

If $n = 0$, the result is clear; there is only one constant function whose value at x_0 is $f(x_0)$. So suppose that $n \geq 1$. From the differentiation of powers formula (8.1), it follows that

$$\frac{d^k}{dx^k}[p_n(x)]\bigg|_{x=x_0} = f^{(k)}(x_0) \qquad \text{for } 0 \leq k \leq n,$$

so the function f and the polynomial p_n have contact of order n at x_0.

It remains to prove uniqueness. However, if we take a general polynomial of degree at most n, written in powers of $x - x_0$ as

$$p(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n,$$

then, again from the differentiation of powers formula (8.1), it is clear that

$$\frac{d^k}{dx^k}[p(x)]\bigg|_{x=x_0} = k!c_k \qquad \text{for } 0 \leq k \leq n,$$

so that if the polynomial p has contact of order n with f, we must have $k!c_k = f^{(k)}(x_0)$ for $0 \leq k \leq n$; that is, $p = p_n$. ■

The polynomial p_n defined by (8.2) is called the nth *Taylor polynomial* for the function $f : I \to \mathbb{R}$ at the point x_0.

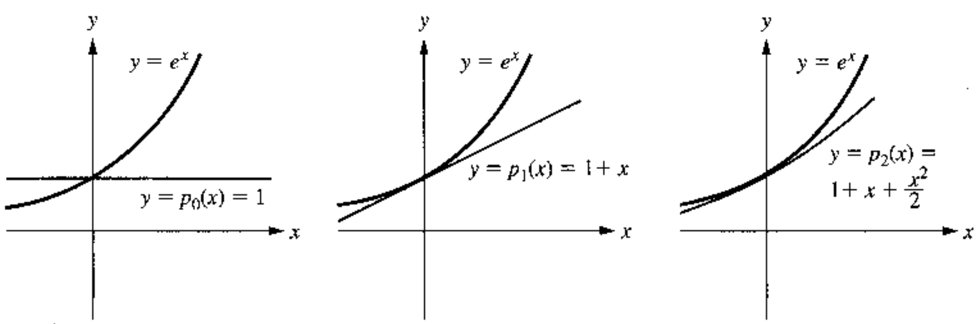

FIGURE 8.1 $p_0(x)$, $p_1(x)$ and $p_2(x)$ for $f(x) = e^x$.

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Examples of Taylor Polynomials

Example 8.3 Define $f(x) = e^x$ for all x. For each natural number k,

$$f^{(k)}(x) = e^x \qquad \text{for all } x.$$

Thus, the nth Taylor polynomial for $f : \mathbb{R} \to \mathbb{R}$ at $x = 0$ is defined by

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$ ∎

Example 8.4 Define $f(x) = \ln(1 + x)$ for $x > -1$. For each natural number k,

$$f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(1+x)^k} \qquad \text{for all } x > -1.$$

Thus, the nth Taylor polynomial for $f : (-1, \infty) \to \mathbb{R}$ at $x = 0$ is defined by

$$p_n(x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n+1}}{n} x^n.$$ ∎

Example 8.5 Define $f(x) = \cos x$ for all x. For each natural number k,

$$f^{(2k)}(x) = (-1)^k \cos x \qquad \text{and} \qquad f^{(2k+1)}(x) = (-1)^{k+1} \sin x \qquad \text{for all } x.$$

Thus, for each nonnegative integer n, the Taylor polynomials for the cosine at $x = 0$ are given by

$$p_{2n}(x) = p_{2n+1}(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n}.$$ ∎

Example 8.6 Define $f(x) = \sqrt{x}$ for $x > 0$. For each natural number k,

$$f^{(k)}(x) = \frac{1}{2}\left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - k + 1\right) x^{1/2-k} \qquad \text{for all } x > 0.$$

Thus, the third Taylor polynomial for the function $f : (0, \infty) \to \mathbb{R}$ at $x = 1$ is

$$p_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$ ∎

For two functions that have a high order of contact at a point, it is reasonable to expect that near this point the difference between the functional values will be small. In particular, if I is a neighborhood of the point x_0 and p_n is the nth Taylor polynomial for the function $f : I \to \mathbb{R}$ at x_0, one expects that for another point x in I, the difference $f(x) - p_n(x)$ can be estimated and shown to be small if x is close to x_0 and n is large.

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What is really surprising is that frequently it happens that

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0,$$

even when the point x is far away from x_0. As we will show in Section 8.6, it can also happen that the Taylor polynomials for certain functions do not provide good approximations at any point x other than x_0, no matter how large the index[1] n. We define $R_n(x) \equiv f(x) - p_n(x)$ for all x in I, so that

$$f(x) = p_n(x) + R_n(x) \qquad \text{for all } x \text{ in } I,$$

and call $R_n : I \to \mathbb{R}$ the nth *remainder*. In Section 8.2, we will begin a rigorous analysis of this remainder.

EXERCISES FOR SECTION 8.1

1. For each of the following pairs of functions, determine its highest order of contact at the indicated point:
a. $f(x) = x^2$ and $g(x) = \sin x$ for all x; $x_0 = 0$.
b. $f(x) = e^{x^2}$ and $g(x) = 1 + 2x^2$ for all x; $x_0 = 0$.
c. $f(x) = \ln x$ and $g(x) = (x-1)^3 + \ln x$ for all $x > 0$; $x_0 = 1$.
d. $f(x) = \ln x$ and $g(x) = (x-1)^{200} + \ln x$ for all $x > 0$; $x_0 = 1$.

2. Compute the third Taylor polynomial for each of the following functions at the indicated point:
a. $f(x) = \int_0^x 1/(1+t^2)\,dt$ for all x; $x_0 = 0$.
b. $f(x) = \sin x$ for all x; $x_0 = 0$.
c. $f(x) = \sin x + x^{200}$ for all x; $x_0 = 0$.
d. $f(x) = \sqrt{2-x}$ for all $x < 2$; $x_0 = 1$.

3. Define $f(x) = x^6 e^x$ for all x. Find the sixth Taylor polynomial for the function f at $x = 0$.

4. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has three derivatives and that the third Taylor polynomial at $x = 0$ is $p_3(x) = 1 + 4x - x^2 + x^3/6$. Show that there is a neighborhood of the point 0 such that $f : I \to \mathbb{R}$ is positive, strictly increasing, and has a strictly increasing derivative.

5. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a second derivative and that

$$\begin{cases} f''(x) + f(x) = e^{-x} & \text{for all } x \\ f(0) = 0 \quad \text{and} & f'(0) = 2. \end{cases}$$

Find the fourth Taylor polynomial for $f : \mathbb{R} \to \mathbb{R}$ at $x = 0$.

6. By replacing x by $x_0 + (x - x_0)$ and using the Binomial Formula, show that any polynomial p can be expressed in powers of $x - x_0$ in the form

$$p(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n.$$

[1] In Section 8.6, we present a function $f : \mathbb{R} \to \mathbb{R}$ that has derivatives of all orders and has $f(x) > 0$ if $x \neq 0$, and yet all Taylor polynomials at $x = 0$ are the same constant function whose value is 0.

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8.2 THE LAGRANGE REMAINDER THEOREM

We devote this section to establishing an estimate of the difference between a function and its nth Taylor polynomial. For immediate reference, we restate here a consequence of the Cauchy Mean Value Theorem (Theorem 4.24).

Lemma 8.7 Let I be an open interval and let n be a nonnegative integer and suppose that the function $f: I \to \mathbb{R}$ has $n+1$ derivatives. Suppose also that at the point x_0 in I,

$$f^{(k)}(x_0) = 0 \qquad \text{for } 0 \leq k \leq n.$$

Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 at which

$$f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^n.$$

The general remainder theorem is a simple extension of the above lemma.

Theorem 8.8 The Lagrange Remainder Theorem Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f: I \to \mathbb{R}$ has $n+1$ derivatives. Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{8.3}$$

Proof

Consider the nth Taylor polynomial for the function f at x_0,

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Since the functions f and p_n have contact of order n at x_0, it follows that if we define the function $R: I \to \mathbb{R}$ by

$$R(x) \equiv f(x) - p_n(x) \qquad \text{for all } x \text{ in } I,$$

then

$$R(x_0) = R'(x_0) = \cdots = R^{(n)}(x_0) = 0.$$

According to the preceding lemma, if $x \neq x_0$ is in I, then there is a point c strictly between x and x_0 such that

$$R(x) = \frac{R^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n-1}.$$

But since p_n is a polynomial of degree at most n, its $n+1$ derivative is identically 0, and therefore

$$R^{(n-1)}(c) = f^{(n+1)}(c) - p_n^{(n+1)}(c) = f^{(n+1)}(c).$$

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From the previous equation it follows that

$$f(x) - p_n(x) = R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

and therefore we have established the formula (8.3). ∎

Corollary 8.9 Suppose that p is a polynomial of degree at most n and let x_0 be any point. Then the nth Taylor polynomial for p at x_0 is p itself.

Proof

Fix $x \neq x_0$. According to the Lagrange Remainder Theorem, we can choose a point c strictly between x and x_0 such that

$$p(x) - p_n(x) = \frac{p^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

But p is a polynomial of degree at most n, so $p^{n+1}(c) = 0$. Thus, $p_n(x) = p(x)$. ∎

The Number e Is Irrational

We will use the Lagrange Remainder Theorem to get a precise estimate of the number e. First, we establish the following very crude estimate of e:

$$e < 4. \tag{8.4}$$

To verify this, observe that the function $\ln : (0, \infty) \to \mathbb{R}$ is strictly increasing, so that

$$e < 4 \qquad \text{if and only if } 1 = \ln e < \ln 4.$$

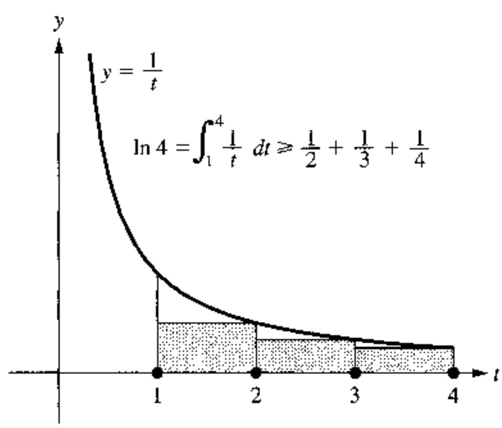

FIGURE 8.2 Lower Darboux sum for $\int_1^4 1/t\, dt$ and partition $P = \{1, 2, 3, 4\}$.

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However, using the integral representation of the natural logarithm and the fact that the integral is greater or equal to any lower Darboux sum, if we take the partition $P = \{1, 2, 3, 4\}$ of the interval $[1, 4]$ and set $f(t) = 1/t$ for $1 \leq t \leq 4$, we have

$$\begin{aligned}
\ln 4 &= \int_1^4 \frac{1}{t}\, dt \\
&\geq L(f, P) \\
&= 1/2(1-0) + 1/3(1-0) + 1/4(1-0) \\
&> 1.
\end{aligned}$$

Theorem 8.10 For each natural number n and each nonzero number x, there is a point c strictly between 0 and x such that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}. \tag{8.5}$$

In particular,

$$0 < e^x - \left[1 + x + \cdots + \frac{x^n}{n!}\right] < \frac{4}{(n+1)!} \qquad \text{if } 0 \leq x \leq 1. \tag{8.6}$$

Proof

Formula (8.5) follows directly from the Lagrange Remainder Theorem and Example 8.3. We just showed that $e < 4$, and since the function $\exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing, it follows that

$$1 = e^0 \leq e^c < e^1 < 4 \qquad \text{if } 0 < c < x \leq 1.$$

Thus, estimate (8.6) follows from (8.5). ∎

Proposition 8.11 The number e is irrational.

Proof

We will argue by contradiction. Suppose that e is rational. Then there are natural numbers n_0 and m_0 such that $e = n_0/m_0$. Then (8.6), with $x = 1$, becomes

$$0 < \frac{n_0}{m_0} - \left[2 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right] \leq \frac{4}{(n+1)!} \qquad \text{for every natural number } n.$$

Now multiply this inequality by $n!$ to get

$$0 < \frac{n!n_0}{m_0} - n!\left[2 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right] \leq \frac{4}{n+1} \qquad \text{for every natural number } n.$$

However, if $n > 4$ and $n \geq m_0$, the above inequality implies the existence of an integer in the interval $(0, 4/5)$. In Section 1.2 we showed that there is no integer in the interval $(0, 1)$. This contradiction proves that e is irrational. ∎

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Euler's Constant: The Growth of the Natural Logarithm

Recall that in Section 2.2 we showed that the sequence

$$\left\{1+\frac{1}{2}+\cdots+\frac{1}{n}\right\}$$

is strictly increasing but unbounded above; that is, the harmonic series diverges. On the other hand, in Section 5.1 we showed that the sequence

$$\{\ln n\}$$

also is strictly increasing but unbounded above. In fact, these two results are related in an interesting manner described in the following proposition.

Proposition 8.12

$$\lim_{n\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1)\right]=\gamma \qquad \text{where } 0<\gamma\le 1.$$

Proof

For each natural number n, define

$$c_n \equiv 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n+1).$$

We will show that the sequence $\{c_n\}$ is strictly increasing and bounded above by 1. The conclusion of the proposition will then be a consequence of the Monotone Convergence Theorem.

The crux of the proof is the following inequality, whose proof we temporarily postpone. For each natural number k,

$$0<\frac{1}{k}-[\ln(k+1)-\ln k]<\frac{1}{2k^2}. \tag{8.7}$$

From the left-hand inequality (8.7), with $k=n+1$, we see that for each index n,

$$c_{n+1}-c_n=\frac{1}{n+1}-[\ln(n+2)-\ln(n+1)]>0,$$

so that the sequence $\{c_n\}$ is monotonically increasing.

We sum these first n of the right-hand inequalities (8.7) to obtain

$$\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n}[\ln(k+1)-\ln k]<\sum_{k=1}^{n}\left[\frac{1}{2k^2}\right].$$

Now observe that

$$\sum_{k=1}^{n}[\ln(k+1)-\ln k]=\ln(n+1)=\ln 1=\ln(n+1),$$

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so the preceding inequality can be rewritten as

$$c_n \leq \sum_{k=1}^{n} \left[\frac{1}{2k^2}\right] \qquad \text{for each index } n. \tag{8.8}$$

We have the following estimate of the right-hand side of this inequality:

$$\begin{aligned}
\sum_{k=1}^{n} \left[\frac{1}{2k^2}\right] &= \frac{1}{2} + \sum_{k=2}^{n} \left[\frac{1}{2k^2}\right] \\
&< \frac{1}{2} + \sum_{k=2}^{n} \left[\frac{1}{2k(k-1)}\right] \\
&= \frac{1}{2} + \frac{1}{2} \sum_{k=2}^{n} \left[\frac{1}{k-1} - \frac{1}{k}\right] \\
&= \frac{1}{2} + \frac{1}{2}\left[1 - \frac{1}{n}\right] \\
&< 1.
\end{aligned}$$

Thus, by (8.8), $\{c_n\}$ is bounded above by 1. The Monotone Convergence Theorem implies that $\{c_n\}$ converges to a number γ, and $\gamma \leq 1$.

It remains to verify the inequality (8.7). Indeed, since $\ln(k+1) - \ln k = \ln(1 + 1/k)$, (8.7) is a consequence of the following inequality whose proof we leave as an exercise in the use of the Lagrange Remainder Theorem (Exercise 4):

$$0 < x - \ln(1+x) < \frac{x^2}{2} \qquad \text{if } x > 0. \tag{8.9}$$

■

The number γ defined in the statement of Proposition 8.12 is called *Euler's constant:* it is not known whether γ is rational or irrational. Since

$$\lim_{n\to\infty} [\ln(n+1) - \ln n] = \lim_{n\to\infty} \ln(1 + 1/n) = \ln 1 = 0,$$

Euler's constant γ can also be expressed as

$$\lim_{n\to\infty} \left[1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right] = \gamma.$$

EXERCISES FOR SECTION 8.2

1. Prove that

$$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2} \qquad \text{if } x > 0.$$

In particular, show that $1.375 < \sqrt{2} < 1.5$.

2. Prove that

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1+x)^{1/3} < 1 + \frac{x}{3} \qquad \text{if } x > 0.$$

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3. Expand the polynomial $p(x) = x^5 - x^3 + x$ in powers of $x - 1$.

4. Use the Lagrange Remainder Formula applied to the function $f(x) = \ln(1 + x)$ at $x = 0$ to verify the inequality (8.9).

5. Prove that for every pair of numbers x and h,

$$|\sin(x + h) - (\sin x + h \cos x)| \leq \frac{h^2}{2}.$$

6. Let I be a neighborhood of the point x_0 and let n be a natural number. Suppose that the function $f : I \to \mathbb{R}$ has $n + 1$ derivatives. Show that the Lagrange Remainder Theorem is equivalent to the following: For each number h such that $x_0 + h$ is in I, there is a number θ, strictly between 0 and 1, such that

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \frac{1}{(n + 1)!} f^{(n+1)}(x_0 + \theta h) h^{n+1}.$$

7. Using Corollary 8.9, show that if p is a polynomial and the number x_0 is a root of p (that is, $p(x_0) = 0$) then there is a polynomial q such that $p(x) = (x - x_0)q(x)$ for all x.

8. A number x_0 is said to be a *root of order k of the polynomial p* provided that k is a natural number such that $p(x) = (x - x_0)^k r(x)$, where r is a polynomial and $r(x_0) \neq 0$. Prove that x_0 is a root of order k of the polynomial p if and only if

$$p(x_0) = p'(x_0) = \cdots = p^{(k-1)}(x_0) = 0 \qquad \text{and} \qquad p^{(k)}(x_0) \neq 0.$$

9. **a.** Show that for a natural number n,

$$(1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + x^n.$$

b. Use part (a) to provide another proof of the Binomial Formula.

10. Suppose that each of the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ has $n + 1$ continuous derivatives. Prove that f and g have contact of order n at 0 if and only if

$$\lim_{x \to 0} \frac{f(x) - g(x)}{x^n} = 0.$$

11. Use the Lagrange Remainder Theorem to verify the following criterion for identifying local extreme points: Let I be a neighborhood of the point x_0 and let n be a natural number. Suppose that the function $f : I \to \mathbb{R}$ has $n + 1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Assume that $f^{(k)}(x_0) = 0$ if $1 \leq k \leq n$ and that $f^{(n+1)}(x_0) \neq 0$.

a. If $n + 1$ is even and $f^{(n+1)}(x_0) > 0$, then x_0 is a local minimizer.

b. If $n + 1$ is even and $f^{(n+1)}(x_0) < 0$, then x_0 is a local maximizer.

c. If $n + 1$ is odd, then x_0 is neither a local maximizer nor a local minimizer.

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12. Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has a continuous third derivative with $f'''(x) > 0$ for all x in I.

a. Prove that if $x_0+h \neq x_0$ is in I, there is a unique number $\theta = \theta(h)$ in the interval $(0, 1)$ such that

$$f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0+\theta h)\frac{h^2}{2}.$$

b. Prove that

$$\lim_{h\to 0} \theta(h) = \frac{1}{3}.$$

8.3 THE CONVERGENCE OF TAYLOR POLYNOMIALS

For a sequence of numbers $\{a_k\}$ that is indexed by the nonnegative integers, we define

$$s_n = \sum_{k=0}^{n} a_k \qquad \text{for every nonnegative integer } n$$

and obtain a new sequence $\{s_n\}$. The sequence $\{s_n\}$ is called the *sequence of partial sums* for the series $\sum_{k=0}^{\infty} a_k$, and a_k is called the kth *term* of the series $\sum_{k=0}^{\infty} a_k$. We write

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k\right]$$

if the sequence $\{s_n\}$ converges. If the sequence $\{s_n\}$ does not converge, then we say that the series $\sum_{k=0}^{\infty} a_k$ *diverges*.

Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has derivatives of all orders. The nth Taylor polynomial for f at x_0 is defined by

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

In conformity with the above series notation, if x is a point in I at which

$$\lim_{n\to\infty} p_n(x) = f(x), \tag{8.10}$$

we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{8.11}$$

This formula is called a *Taylor series expansion* of the function $f : I \to \mathbb{R}$ about the point x_0. By its very definition, (8.11) holds at x if and only if

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0. \tag{8.12}$$

In this section, we will use the Lagrange Remainder Theorem to provide a general criterion for determining the validity of the Taylor series expansion in a neighborhood of the expansion point. To begin, we prove a useful preliminary result.

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Lemma 8.13 For any number c,

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0.$$

Proof

Choose k to be a natural number such that $k \geq 2|c|$. Then if $n \geq k$,

$$\begin{aligned} 0 &\leq \left|\frac{c^n}{n!}\right| \\ &= \left[\frac{|c|}{1}\cdots\frac{|c|}{k}\right]\left[\frac{|c|}{k+1}\cdots\frac{|c|}{n}\right] \\ &\leq |c|^k\left(\frac{1}{2}\right)^{n-k} \\ &= |c|^k 2^k\left(\frac{1}{2}\right)^n. \end{aligned}$$

But $\lim_{n\to\infty}(1/2)^n = 0$, and so $\lim_{n\to\infty} c^n/n! = 0$ also. ■

Theorem 8.14 Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has derivatives of all orders. Suppose also that there are positive numbers r and M such that the interval $[x_0 - r,\ x_0 + r]$ is contained in I and that for every natural number n and every point x in $[x_0 - r,\ x_0 + r]$,

$$|f^{(n)}(x)| \leq M^n. \tag{8.13}$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \qquad \text{if } |x - x_0| \leq r. \tag{8.14}$$

Proof

The nth Taylor polynomial p_n for f at x_0 is defined by

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

and, according to the Lagrange Remainder Theorem, for each point x in I, there is a point c strictly between x and x_0 such that

$$|f(x) - p_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x - x_0|^{n+1}.$$

In view of inequality (8.13), it follows that for every natural number n and every point x in $[x_0 - r,\ x_0 + r]$,

$$|f(x) - p_n(x)| \leq \frac{M^{n+1}}{(n+1)!}|x - x_0|^{n+1} \leq \frac{c^{n+1}}{(n+1)!}, \tag{8.15}$$

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where $c = Mr$. According to Lemma 8.13, $\lim_{n\to\infty} c^n/n! = 0$. Thus, from (8.15), we see that

$$\lim_{n\to\infty} [f(x) - p_n(x)] = 0 \qquad \text{if } |x - x_0| \le r.$$

This is precisely assertion (8.14). ∎

Corollary 8.15

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for all } x. \tag{8.16}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad \text{for all } x. \tag{8.17}$$

Proof

First, we will prove (8.16). Define $f(x) = e^x$ for all x and let $x_0 = 0$. Fix $r > 0$. If we define $M = e^r$, it follows that for every natural number n and every point x in the interval $[-r, r]$,

$$|f^{(n)}(x)| \le M \le M^n.$$

According to Theorem 8.14, if $|x| < r$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

But the choice of $r > 0$ was arbitrary, and so the Taylor expansion (8.16) is verified.

The proof of (8.17) is similar. We define $f(x) = \cos x$ for all x and observe that $|f^{(n)}(x)| \le 1$ for every natural number n and every number x. Then the proof proceeds as above. ∎

EXERCISES FOR SECTION 8.3

1. Show that the Taylor expansion of the following functions at the given points converges for all points x:

a. $f(x) = \sin x$ at the point $x_0 = 0$.

b. $f(x) = \cos x$ at the point $x_0 = \pi$.

2. Define $f(x) = 1/x$ if $0 < x < 2$.

a. Find p_n, the nth Taylor polynomial at $x_0 = 1$.

b. Use the Geometric Sum Formula to show that for every natural number n,

$$f(x) - p_n(x) = \frac{(1-x)^{n+1}}{x} \qquad \text{if } 0 < x < 2.$$

c. Use part (b) to prove that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k \qquad \text{if } |x - 1| < 1.$$

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3. Suppose that the function $F:\mathbb{R}\to\mathbb{R}$ has derivatives of all orders and that

$$\begin{cases} F'(x)-F(x)=0 & \text{for all } x \\ F(0)=2. \end{cases}$$

Find a formula for the coefficients of the nth Taylor polynomial for F at $x=0$. Show that the Taylor expansion converges at every point.

4. Suppose that the function $F:\mathbb{R}\to\mathbb{R}$ has derivatives of all orders and that

$$\begin{cases} F''(x)-F'(x)-F(x)=0 & \text{for all } x \\ F(0)=1 \quad \text{and} \quad F'(0)=1. \end{cases}$$

Find a recursive formula for the coefficients of the nth Taylor polynomial for F at $x=0$. Show that the Taylor expansion converges at every point.

5. For a pair of numbers α and β, suppose that the function $f:\mathbb{R}\to\mathbb{R}$ has derivatives of all orders and that

$$f''(x)+\alpha f'(x)+\beta f(x)=0 \qquad \text{for all } x.$$

a. Show that for every natural number n,

$$f^{(n+2)}(x)+\alpha f^{(n+1)}(x)+\beta f^{(n)}(x)=0 \qquad \text{for all } x.$$

b. Use part (a) to show that

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \qquad \text{for all } x.$$

8.4 A POWER SERIES FOR THE LOGARITHM

In this section, we will analyze the validity of the Taylor expansion of the natural logarithm. In order to do so, it is convenient first to translate the natural logarithm and consider the function $f:(-1,\infty)\to\mathbb{R}$ defined by

$$f(x)=\ln(1+x) \qquad \text{if } x>-1.$$

A direct calculation of derivatives shows that for each natural number k,

$$f^{(k)}(x)=\frac{(-1)^{k+1}(k-1)!}{(1+x)^k} \qquad \text{for all } x>-1.$$

In particular, the nth Taylor polynomial for f at $x=0$ is defined by

$$p_n(x)=x-\frac{x^2}{2}+\cdots+\frac{(-1)^{n+1}}{n}x^n. \tag{8.18}$$

Rather than trying to use the Lagrange Remainder Theorem to study the difference $f(x)-p_n(x)$, it is better to jointly exploit the integral formula for the natural logarithm and the Geometric Sum Formula in order to derive a more explicit formula for the difference. Indeed, observe that for each natural number n, the Geometric Sum Formula

$$\frac{1}{1-r}=1+r+\cdots+r^{n-1}+\frac{r^n}{1-r} \qquad \text{if } r\neq 1$$

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becomes, if one substitutes $1 - t$ for r,

$$\frac{1}{t} = 1 + (1-t) + \cdots + (1-t)^{n-1} + \frac{(1-t)^n}{t} \qquad \text{if } t \neq 0.$$

Thus, using the integral representation for $\ln x$, the preceding formula, the linearity of integration and formula (8.18), we have

$$\begin{aligned}
\ln(1+x) &= \int_1^{1+x} \frac{1}{t}\,dt \\
&= \int_1^{1+x} [1 + (1-t) + \cdots + (1-t)^{n-1}]\,dt + \int_1^{1+x} \frac{(1-t)^n}{t}\,dt \\
&= x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n+1}}{n}x^n + \int_1^{1+x} \frac{(1-t)^n}{t}\,dt \\
&= p_n(x) + \int_1^{1+x} \frac{(1-t)^n}{t}\,dt \qquad \text{if } x > -1.
\end{aligned} \tag{8.19}$$

Theorem 8.16

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \qquad \text{if } -1 < x \leq 1. \tag{8.20}$$

Proof

Formula (8.19) implies that for each natural number n,

$$\ln(1+x) - \sum_{k=1}^{n} (-1)^{k+1}\frac{x^k}{k} = \int_1^{1+x} \frac{(1-t)^n}{t}\,dt \qquad \text{if } x > -1.$$

Thus, to verify (8.20), we must show that

$$\lim_{n\to\infty}\left[\int_1^{1+x} \frac{(1-t)^n}{t}\,dt\right] = 0 \qquad \text{if } -1 < x \leq 1. \tag{8.21}$$

First, suppose $0 \leq x \leq 1$. Then for each natural number n,

$$\begin{aligned}
\left|\int_1^{1+x} \frac{(1-t)^n}{t}\,dt\right| &= \int_1^{1+x} \frac{(t-1)^n}{t}\,dt \\
&\leq \int_1^{1+x} (t-1)^n\,dt \\
&= \frac{x^{n+1}}{n+1} \\
&\leq \frac{1}{n+1}.
\end{aligned}$$

Since $\lim_{n\to\infty} 1/n = 0$, we see that (8.21) holds if $0 \leq x \leq 1$.

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Now suppose that $-1 < x < 0$. Then for each natural number n,

$$\begin{aligned}\left|\int_1^{1+x} \frac{(1-t)^n}{t}\,dt\right| &= \int_{1+x}^{1} \frac{(1-t)^n}{t}\,dt \\ &\leq \frac{1}{1+x}\int_{1+x}^{1} (1-t)^n\,dt \\ &= \left(\frac{1}{1+x}\right)\frac{|x|^{n+1}}{n+1} \\ &\leq \left(\frac{1}{1+x}\right)\frac{1}{n+1}.\end{aligned}$$

Since $\lim_{n\to\infty} 1/n = 0$, we also see that (8.21) holds if $-1 < x < 0$. ∎

In spite of the fact that the function $f(x) = \ln(1+x)$ has derivatives of all orders for all $x > -1$, the Taylor expansion (8.20) is not valid if $x > 1$. To see this, first observe that as a consequence of the Binomial Formula, for each natural number n,

$$x^{n+1} = [1+(x-1)]^{n+1} \geq 1+(n+1)(x-1) \qquad \text{for } x > 1.$$

Using this, the monotonicity property of integration and the integral representation of the remainder (8.19), it follows that if $x > 1$ and n is any natural number, then, by formula (8.19),

$$\begin{aligned}|\ln(1+x) - p_n(x)| &= \int_1^{1+x} \frac{(1-t)^n}{t}\,dt \\ &\geq \frac{1}{1+x}\int_1^{1+x} (1-t)^n\,dt \\ &= \frac{1}{1+x}\cdot\frac{x^{n+1}}{n+1} \\ &= \frac{1}{1+x}\cdot\frac{[1+(x-1)]^{n+1}}{n+1} \\ &\geq \frac{1}{1+x}\cdot\frac{1+(n+1)(x-1)}{n+1} \\ &\geq \frac{x-1}{1+x}.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\left[\ln(1+x) - \sum_{k=1}^{n}\frac{(-1)^{k+1}x^k}{k}\right] \neq 0 \qquad \text{if } x > 1.$$

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EXERCISES FOR SECTION 8.4

1. Prove that for each natural number n,

$$|\ln(1+x) - p_n(x)| \le \frac{1}{1+x} \cdot \frac{|x|^{n+1}}{n+1} \qquad \text{if } -1 < x \le 0,$$

and

$$|\ln(1+x) - p_n(x)| \le \frac{x^{n+1}}{n+1} \qquad \text{if } 0 \le x \le 1.$$

Estimate $\ln(1.1)$ with an error of at most 10^{-4}.

2. Show that with the error estimates in the previous exercise, we need $n = 10{,}000$ to estimate $\ln 2$ with an error bound of 10^{-4}. How does the identity $\ln 2 = \ln 4/3 - \ln 2/3$ allow us to estimate $\ln 2$ more efficiently?

3. Explain how the identity

$$s = \left(1 + \frac{s-1}{s+1}\right) \Big/ \left(1 - \frac{s-1}{s+1}\right) \qquad \text{if } s \neq 0$$

allows us to efficiently compute the value of $\ln(1+x)$ if $0 < x < 1$ and x is close to 1.

4. Verify the integral inequalities in the proof of Theorem 8.16.

5. At what points x in the interval $(-1, 1]$ can one use the Lagrange Remainder Theorem to verify the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k!}?$$

8.5 THE CAUCHY INTEGRAL REMAINDER THEOREM

If I is a neighborhood of the point x_0 and the function $f : I \to \mathbb{R}$ is differentiable, then, by the Mean Value Theorem, for each point x in I, there is a point c strictly between x and x_0 such that

$$f(x) = f(x_0) + f'(c)(x - x_0). \tag{8.22}$$

If we further assume that the derivative $f' : I \to \mathbb{R}$ is continuous, then, by the First Fundamental Theorem (Integrating Derivatives),

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt. \tag{8.23}$$

The proof of the Lagrange Remainder Theorem was rooted in the Mean Value Theorem as expressed in (8.22). The proof of the following Cauchy Integral Remainder Theorem will exploit the First Fundamental Theorem (Integrating Derivatives) as expressed in (8.23).

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Theorem 8.17 The Cauchy Integral Remainder Formula Let I be a neighborhood of the point x_0 and n be a natural number. Suppose that the function $f : I \to \mathbb{R}$ has $n+1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each point x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n \, dt. \tag{8.24}$$

Proof

By the First Fundamental Theorem (Integrating Derivatives),

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt. \tag{8.25}$$

Integrating by parts, we see that

$$\begin{aligned}
\int_{x_0}^{x} f'(t)\, dt &= -\int_{x_0}^{x} f'(t)\frac{d}{dt}(x-t)\, dt \\
&= -f'(t)(x-t)\Big|_{t=x_0}^{t=x} + \int_{x_0}^{x} f''(t)(x-t)\, dt \\
&= f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t)\, dt. \qquad (8.26)
\end{aligned}$$

From (8.25) and (8.26) we obtain (8.24) when $n = 1$. The general formula follows by induction. The inductive step depends on observing that if $1 \le k \le n-1$, then

$$\begin{aligned}
\frac{1}{k!}\int_{x_0}^{x} f^{(k+1)}(t)(x-t)^k\, dt &= \frac{-1}{(k+1)!}\int_{x_0}^{x} f^{(k+1)}(t)\frac{d}{dt}[(x-t)^{k+1}]\, dt \\
&= \frac{1}{(k+1)!} f^{(k+1)}(x_0)(x-x_0)^{k+1} \\
&\quad + \frac{1}{(k+1)!}\int_{x_0}^{x} f^{(k+2)}(t)(x-t)^{k+1}\, dt.
\end{aligned}$$ ■

The Binomial Formula

Recall from Section 1.3 that for each natural number n and pair of numbers a and b, we have

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{8.27}$$

This formula can be proved directly using elementary algebra, as we did in Section 1.3. Another proof can be given by first applying Corollary 8.9, where a polynomial of degree n equals any nth Taylor polynomial, to show that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \text{for all } x. \tag{8.28}$$

Then, if $a \neq 0$, substitute $x = b/a$ in (8.28) and multiply by a^n to obtain (8.27).

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Newton's Binomial Expansion

We will now extend formula (8.28) to the case of exponents that are not necessarily natural numbers. Of course, if β is not a nonnegative integer, then the function $(1+x)^\beta$ is not a polynomial, so the right-hand side of (8.28), rather than being a polynomial, is an infinite series. In order to find an infinite series expansion of $(1+x)^\beta$, it is useful to extend the definition of the binomial coefficients. For each natural number n and each number β, we define

$$\binom{\beta}{k} \equiv \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$$

and define

$$\binom{\beta}{0} \equiv 1.$$

Theorem 8.18 Newton's Binomial Expansion Let β be any real number. Then

$$(1+x)^\beta = \sum_{k=0}^{\infty} \binom{\beta}{k} x^k \qquad \text{if } -1 < x < 1. \tag{8.29}$$

To verify the Binomial Expansion is to prove that

$$\lim_{n\to\infty} \left[(1+x)^\beta - \sum_{k=0}^{n} \binom{\beta}{k} x^k \right] = 0 \qquad \text{if } -1 < x < 1.$$

The proof of Newton's Binomial Expansion requires some preparation. So we will first establish three preparatory lemmas. The first crucial step in the proof is to show that the Binomial Expansion is a Taylor expansion about $x = 0$ so that the Cauchy Integral Remainder Formula provides an integral representation for the remainders. We summarize this step in the first preparatory lemma.

Lemma 8.19 For any number β and any natural number n, if $x > -1$, then

$$(1+x)^\beta - \sum_{k=0}^{n} \binom{\beta}{k} x^k = (n+1)\binom{\beta}{n+1} \int_0^x (1+t)^{\beta-n-1}(x-t)^n\, dt. \tag{8.30}$$

Proof

Define the function $f : (-1, \infty) \to \mathbb{R}$ by

$$f(x) = (1+x)^\beta \qquad \text{if } x > -1.$$

Observe that for each natural number k,

$$f^{(k)}(x) = \beta(\beta-1)\cdots(\beta-k+1)(1+x)^{\beta-k} \qquad \text{if } x > -1,$$

so that

$$\frac{f^{(k)}(x)}{k!} = \binom{\beta}{k}(1+x)^{\beta-k}$$

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and, in particular,

$$\frac{f^{(k)}(0)}{k!} = \binom{\beta}{k}.$$

Thus, the nth Taylor polynomial for $f : (-1, \infty) \to \mathbb{R}$ at $x = 0$ is

$$p_n(x) = \sum_{k=0}^{n} \binom{\beta}{k} x^k.$$

According to the Cauchy Integral Remainder Theorem, for each natural number n and each number $x > -1$,

$$\begin{aligned} f(x) - p_n(x) &= \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n \, dt \\ &= \frac{1}{n!} \int_0^x \binom{\beta}{n+1} (n+1)!(1+t)^{\beta-n-1}(x-t)^n \, dt \\ &= (n+1)\binom{\beta}{n+1} \int_0^x (1+t)^{\beta-n-1}(x-t)^n \, dt. \end{aligned}$$

■

In order to analyze the size of the right-hand side of (8.30) when n is large, it is convenient first to prove the following lemma, which will also be useful in Chapter 9.

Lemma 8.20 The Ratio Lemma for Sequences Suppose that $\{c_n\}$ is a sequence of nonzero numbers with the property that

$$\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \ell.$$

i. If $\ell < 1$, then

$$\lim_{n\to\infty} c_n = 0.$$

ii. If $\ell > 1$, then the sequence $\{c_n\}$ is unbounded.

Proof

First, suppose that $0 \le \ell < 1$. Define $\alpha = (\ell + 1)/2$. Since $\ell < \alpha$, we can choose a natural number N such that

$$\frac{|c_{n+1}|}{|c_n|} \le \alpha \qquad \text{for all indices } n \ge N.$$

For each index k, if we successively apply the preceding inequality k times, we see that

$$|c_{N+k}| \le |c_N|\alpha^k.$$

Hence, if we define $M = |c_N|\alpha^{-N}$, it follows that

$$|c_n| \le M\alpha^n \qquad \text{for all indices } n \ge N.$$

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However, $0 \leq \alpha < 1$, so $\lim_{n\to\infty} \alpha^n = 0$. The preceding inequality implies that $\lim_{n\to\infty} c_n = 0$.

Now suppose that $\ell > 1$. Define $\beta = (\ell + 1)/2$. Since $\beta < \ell$, we can choose a natural number N such that

$$\frac{|c_{n+1}|}{|c_n|} \geq \beta \qquad \text{for all indices } n \geq N.$$

Hence, for each natural number k,

$$|c_{N+k}| \geq |c_N|\beta^k, \tag{8.31}$$

and since, by the Binomial Formula,

$$\beta^k = (1 + (\beta - 1))^k \geq 1 + k(\beta - 1),$$

(8.31) implies that the sequence $\{c_n\}$ is unbounded. ■

The following is the last lemma we need to prove Newton's Binomial Expansion.

Lemma 8.21 Let β be any number. Then

$$\lim_{n\to\infty} n \binom{\beta}{n} x^n = 0 \qquad \text{if } |x| < 1.$$

Proof
Observe that for each natural number n,

$$(n+1)\binom{\beta}{n+1} \Big/ n\binom{\beta}{n} = \frac{n+1}{n} \cdot \frac{\beta - n}{n+1}.$$

Thus,

$$\lim_{n\to\infty} \left| (n+1)\binom{\beta}{n+1} |x|^{n+1} \Big/ n\binom{\beta}{n} |x|^n \right| = |x|.$$

The conclusions follow immediately from the Ratio Lemma for Sequences. ■

Proof of Newton's Binomial Expansion
First, we consider the case where $-1 < x < 0$. When we write $(x - t) = -(t - x)$ and interchange the limits of integration, formula (8.30) becomes

$$f(x) - p_n(x) = (-1)^{n+1}(n+1)\binom{\beta}{n+1} \int_x^0 \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1}\, dt. \tag{8.32}$$

But observe that

$$0 \leq \left(\frac{t-x}{1+t}\right) \leq -x = |x| \qquad \text{if } -1 < x \leq t \leq 0;$$

so for each natural number n,

$$0 \leq \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1} \leq |x|^n \qquad \text{if } -1 < x \leq t \leq 0. \tag{8.33}$$

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From (8.32) and (8.33), it follows that for each natural number n,

$$
\begin{aligned}
|f(x) - p_n(x)| &= (n+1)\binom{\beta}{n+1}\int_x^0 \left(\frac{t-x}{1+t}\right)^n (1+t)^{\beta-1}\,dt \\
&\le (n+1)\binom{\beta}{n+1}\int_x^0 |x|^n\,dt \\
&\le (n+1)\binom{\beta}{n+1}|x|^n\int_x^0 dt \\
&= (n+1)\binom{\beta}{n+1}|x|^{n+1} \qquad \text{if } -1 < x < 0. \qquad (8.34)
\end{aligned}
$$

According to Lemma 8.21, if $|x| < 1$,

$$\lim_{n\to\infty}(n+1)\binom{\beta}{n+1}|x|^{n+1} = 0,$$

so from (8.34) we conclude that the Binomial Expansion is valid if $-1 < x < 0$.

It remains to consider the case when $0 < x < 1$. In this case, from (8.30) we obtain

$$
\begin{aligned}
|f(x) - p_n(x)| &= (n+1)\binom{\beta}{n+1}\int_0^x (1+t)^{\beta-n-1}(x-t)^n\,dt \\
&= (n+1)\binom{\beta}{n+1}\int_0^x \left[\frac{x-t}{1+t}\right]^n (1+t)^{\beta-1}\,dt \\
&\le (n+1)\binom{\beta}{n+1}x^n\int_0^x (1+t)^{\beta-1}\,dt \\
&= (n+1)\binom{\beta}{n+1}x^{n+1}\left[\frac{(1+x)^\beta - 1}{\beta x}\right]. \qquad (8.35)
\end{aligned}
$$

Again using Lemma 8.21, from (8.35) we conclude that the Binomial Expansion is valid if $0 < x < 1$. ■

EXERCISES FOR SECTION 8.5

1. Verify all the details in the derivations of the inequalities (8.33), (8.34), and (8.35).
2. Show that for $\beta = -1$, the Binomial Expansion reduces to the Geometric Series.
3. Show that for β a natural number, the Binomial Expansion reduces to the Binomial Formula.
4. Prove that if the functions $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ are continuous, with $h(x) \ge 0$ for all x in $[a, b]$, then there is a point c in (a, b) such that

$$\int_a^b h(x)g(x)\,dx = g(c)\int_a^b h(x)\,dx.$$

5. Use Exercise 4 to show that the Cauchy Integral Remainder Theorem implies the Lagrange Remainder Theorem if $f^{(n+1)} : I \to \mathbb{R}$ is assumed to be continuous.

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6. Apply the Cauchy Integral Remainder Theorem in the analysis of the expansion

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} \qquad \text{if } -1 < x \leq 1.$$

7. Show that for $0 \leq x < 1$, the Lagrange Remainder Theorem can be used to verify the Binomial Expansion.
8. Prove that the Binomial Expansion does not converge if $|x| > 1$.
9. For what values of r does the sequence $\{n^3 r^n\}$ converge?

8.6 A NONANALYTIC, INFINITELY DIFFERENTIABLE FUNCTION

We will now present an explicit example of a function $f : \mathbb{R} \to \mathbb{R}$ that has derivatives of all orders and yet the only point at which its Taylor expansion about $x = 0$ agrees with its functional value is at $x = 0$.

Theorem 8.22 Define

$$f(x) = \begin{cases} e^{-(1/x^2)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then the function $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders. However, the only point at which

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{8.36}$$

is at $x = 0$.

Proof

To prove the theorem it will suffice to prove that $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and that for each natural number n, $f^{(n)}(0) = 0$. Once this is proved simply observe that the right-hand side of (8.36) is identically zero and $f(x) = 0$ if and only if $x = 0$.

Step 1: We claim that for any polynomial $q : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to 0} q\left(\frac{1}{x}\right) e^{-(1/x^2)} = 0. \tag{8.37}$$

In order to verify this, it suffices to show that for each natural number n,

$$\lim_{x \to 0} \frac{e^{-(1/x^2)}}{x^n} = 0. \tag{8.38}$$

Indeed, let n be a natural number. From formula (8.5) we conclude that

$$e^b > \frac{b^n}{n!} \qquad \text{if } b > 0,$$

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so that

$$e^{(1/x^2)} \geq \frac{1}{n!x^{2n}} \qquad \text{if } x \neq 0,$$

and hence

$$0 \leq \left| \frac{e^{-(1/x^2)}}{x^n} \right| \leq n!|x|^n \qquad \text{if } x \neq 0.$$

This inequality implies (8.38), which in turn implies (8.37).

Step 2: We will argue by induction to show that for each natural number n, there is a polynomial $q_n : \mathbb{R} \to \mathbb{R}$ such that

$$f^{(n)}(x) = q_n\left(\frac{1}{x}\right) e^{-(1/x^2)} \qquad \text{if } x \neq 0. \tag{8.39}$$

Indeed, $f'(x) = (2/x^3)e^{-(1/x^2)}$ if $x \neq 0$, so (8.39) holds when $n = 1$, where $q_1(t) = 2t^3$. Suppose (8.39) holds with $n = k$. Then

$$f^{(k+1)}(x) = \left[q_k'\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + q_k\left(\frac{1}{x}\right)\left(\frac{2}{x^3}\right)\right] e^{-(1/x^2)} \quad \text{if } x \neq 0,$$

so (8.39) holds if $n = k+1$, where $q_{k+1}(t) = q_k'(t)(-t^2) + q_k(t)(2t^3)$. The Principle of Mathematical Induction implies that (8.39) holds for all natural numbers.

From step 2 it follows that the function $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders at every point $x \neq 0$. To complete the proof, we will show, again by induction, that for each natural number n,

$$f^{(n)}(0) = 0. \tag{8.40}$$

Indeed, if $n = 1$, then, using (8.37) with $q(t) = t$ for all t, it follows that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{1}{x} e^{-(1/x^2)} = 0.$$

Now suppose that k is a natural number such that $f^{(k)}(0) = 0$. Then, using (8.39) together with (8.37), with $q(t) = tq_k(t)$ for all t, it follows that

$$\lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x\to 0} \frac{1}{x} q_k\left(\frac{1}{x}\right) e^{-(1/x^2)} = 0,$$

so $f^{(k+1)}(0) = 0$. The Principle of Mathematical Induction implies that for each natural number n, $f^{(n)}(0) = 0$. ∎

A function $g : \mathbb{R} \to \mathbb{R}$ that has derivatives of all orders is said to be *infinitely differentiable.* A function $g : \mathbb{R} \to \mathbb{R}$ that has derivatives of all orders such that

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)x^k}{k!} \qquad \text{for all } x$$

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is said to be *analytic*. We have exhibited a function $f : \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable and not analytic.

EXERCISES FOR SECTION 8.6

1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be as in Theorem 8.22. Explicitly compute $f''(x)$.
2. Let the function $f : \mathbb{R} \to \mathbb{R}$ be as in Theorem 8.22. Show that there is no positive number M such that for each natural number n,

$$|f^{(n)}(x)| \leq M^n \qquad \text{for all } x.$$

3. For n a natural number, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *n times continuously differentiable* provided that $f : \mathbb{R} \to \mathbb{R}$ has an nth derivative and $f^{(n)} : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$h(x) = \int_0^x |t|\, dt \qquad \text{for all } x.$$

Show that the function $h : \mathbb{R} \to \mathbb{R}$ is once continuously differentiable but is not twice continuously differentiable. For each natural number n, find a function that is n times continuously differentiable but is not $n+1$ times continuously differentiable.
4. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and that for each natural number n there are positive numbers c_n and δ_n such that

$$|g(x)| \leq c_n |x|^n \qquad \text{if } |x| < \delta_n.$$

Prove that for each natural number n, $g^{(n)}(0) = 0$.

8.7 THE WEIERSTRASS APPROXIMATION THEOREM

As we saw in the preceding section, even if a function has derivatives of all orders, the Taylor polynomial for the function computed at a point x_0 in its domain may not provide a good approximation for the function at any point other than x_0. Nevertheless, there is the following remarkable theorem.

Theorem 8.23 The Weierstrass Approximation Theorem Let I be a closed bounded interval and suppose that the function $f : I \to \mathbb{R}$ is continuous. Then for each positive number ϵ, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - p(x)| < \epsilon \qquad \text{for all points } x \text{ in } I. \tag{8.41}$$

What is remarkable about this theorem is that there is no assumption about differentiability. For instance, we allow the possibility that there is no point at which the function $f : I \to \mathbb{R}$ is differentiable. Of course, the polynomial that satisfies (8.41) is not, in general, a Taylor polynomial.

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The proof of the Approximation Theorem that we present is due to Serguei Bernstein, and it is quite ingenious. At its roots lie the following three identities:

For each natural number n and any number x,

$$\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} = 1, \tag{8.42}$$

$$\sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k(1-x)^{n-k} = x, \tag{8.43}$$

and if $n \geq 2$,

$$\sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} x^k(1-x)^{n-k} = x^2. \tag{8.44}$$

The first identity (8.42) follows from the Binomial Formula (8.27) by setting $a = x$ and $b = 1 - x$. Identities (8.43) and (8.44) are consequences of (8.42). Indeed, if in the identity (8.42) we replace n by $n - 1$ and multiply both sides by x, then since

$$\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k} \qquad \text{if } 1 \leq k \leq n \text{, while } \frac{k}{n} = 0 \text{ if } k = 0,$$

we obtain (8.43). Similarly, if in the identity (8.42) we replace n by $n - 2$ and multiply both sides by x^2, since

$$\binom{n-2}{k-2} = \frac{k(k-1)}{n(n-1)}\binom{n}{k} \qquad \text{if } 2 \leq k \leq n \text{, while } \frac{k}{n} = \frac{k-1}{n} = 0 \text{ if } k = 0, 1$$

we obtain (8.44).

For a natural number n and an integer k such that $0 \leq k \leq n$, it is notationally convenient to define

$$g_k(x) = x^k(1-x)^{n-k} \qquad \text{for all } x.$$

Lemma 8.24 For each number x and each natural number $n \geq 2$,

$$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k(1-x)^{n-k} = \frac{x(1-x)}{n}. \tag{8.45}$$

Proof

For each integer $n \geq 2$, it follows from (8.42) and (8.43) that

$$\begin{aligned}
&\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} g_k(x) \\
&= \sum_{k=0}^{n} \left[x^2 - \frac{2xk}{n} + \frac{k^2}{n^2}\right] \binom{n}{k} g_k(x) \\
&= x^2 \sum_{k=0}^{n} \binom{n}{k} g_k(x) - 2x \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} g_k(x) + \sum_{k=0}^{n} \frac{k^2}{n^2}\binom{n}{k} g_k(x) \\
&= x^2 - 2x^2 + \sum_{k=0}^{n} \frac{k^2}{n^2}\binom{n}{k} g_k(x).
\end{aligned}$$

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On the other hand, from (8.43) and (8.44) we have

$$\begin{aligned}\sum_{k=0}^{n} \frac{k^2}{n^2}\binom{n}{k} g_k(x) &= \frac{n-1}{n}\sum_{k=0}^{n} \frac{k^2}{n(n-1)}\binom{n}{k} g_k(x)\\ &= \frac{n-1}{n}\sum_{k=0}^{n}\left[\frac{k(k-1)}{n(n-1)} + \frac{k}{n(n-1)}\right]\binom{n}{k} g_k(x)\\ &= \left[\frac{n-1}{n}\right]\left[x^2 + \frac{x}{n-1}\right].\end{aligned}$$

Hence,

$$\sum_{k=0}^{n}\left(x - \frac{k}{n}\right)^2\binom{n}{k} g_k(x) = x^2 - 2x^2 + \left[\frac{n-1}{n}\right]\left[x^2 + \frac{x}{n-1}\right] = \frac{x(1-x)}{n}.$$

■

Proof of the Weierstrass Approximation Theorem

We will first consider the case where $I = [0, 1]$; the general case follows easily from this case. Let $\epsilon > 0$. We will find a polynomial $p(x)$ for which the estimate (8.41) holds. Recall that we have proven that a continuous function on a closed bounded interval is uniformly continuous. Using the ϵ-δ characterization of uniform continuity, we can choose $\delta > 0$ such that

$$|f(u) - f(v)| < \frac{\epsilon}{2} \qquad \text{for all points } u \text{ and } v \text{ in } I \text{ such that } |u - v| \le \delta. \tag{8.46}$$

Also, it follows from the Extreme Value Theorem that the function $f : I \to \mathbb{R}$ is bounded. Thus, we can choose a number $M > 0$ such that

$$|f(x)| \le M \qquad \text{for all } x \text{ in } I. \tag{8.47}$$

Using the Archimedean Property of $\mathbb{R}$, we can select a natural number n such that

$$n > \frac{4M}{\epsilon\delta^2}. \tag{8.48}$$

Define the polynomial $p : \mathbb{R} \to \mathbb{R}$ by

$$p(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \qquad \text{for all } x.$$

We will show that for this choice of polynomial, the required approximation property (8.41) holds. Indeed, let x be a point in I and k be an integer, $1 \le k \le n$. Either $|x - k/n| < \delta$ or $|x - k/n| \ge \delta$. If $|x - k/n| < \delta$, it follows from (8.46) that $|f(x) - f(k/n)| < \epsilon/2$. If $|x - k/n| \ge \delta$, then, in view of (8.47),

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| \le 2M \le \frac{2M}{\delta^2}\left(x - \frac{k}{n}\right)^2.$$

Thus,

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| \le \frac{\epsilon}{2} + \frac{2M}{\delta^2}\left(x - \frac{k}{n}\right)^2 \qquad \text{for } 0 \le k \le n. \tag{8.49}$$

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From (8.42) it follows that

$$f(x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k},$$

so

$$f(x) - p(x) = \sum_{k=0}^{n} \left[f(x) - f\left(\frac{k}{n}\right) \right] \binom{n}{k} x^k (1-x)^{n-k}.$$

Using the Triangle Inequality, (8.49), (8.42), and (8.45), it follows that

$$\begin{aligned}
|f(x) - p(x)| &\le \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \sum_{k=0}^{n} \left[\frac{\epsilon}{2} + \frac{2M}{\delta^2}\left(x - \frac{k}{n}\right)^2 \right] \binom{n}{k} x^k (1-x)^{n-k} \\
&= \frac{\epsilon}{2} + \frac{2M}{\delta^2} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\
&= \frac{\epsilon}{2} + \frac{2M}{n\delta^2} x(1-x) \\
&< \frac{\epsilon}{2} + \frac{2M}{n\delta^2}.
\end{aligned}$$

Consequently, since n satisfies (8.48), $|f(x) - p(x)| < \epsilon$.

This proves the theorem if $I = [0, 1]$. Now let $I = [a, b]$. Define $g(t) = a + t(b - a)$ if $0 \le t \le 1$. Then the composite function $f \circ g : [0, 1] \to \mathbb{R}$ is continuous. By what we have just proven, there is a polynomial $q : \mathbb{R} \to \mathbb{R}$ such that $|f(g(t)) - q(t)| < \epsilon$ for all points t in $[0, 1]$. Now define $p(x) = q[(x-a)/(b-a)]$ for all x. Then $p : \mathbb{R} \to \mathbb{R}$ is a polynomial for which the approximation property (8.41) holds. ∎

EXERCISES FOR SECTION 8.7

1. Show that if $n \ge k \ge 1$, then

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

Use this, together with (8.42) with n replaced by $n - 1$, to verify (8.43).

2. Show that if $n \ge k \ge 2$, then

$$\frac{k(k-1)}{n(n-1)}\binom{n}{k} = \binom{n-2}{k-2}.$$

Use this, together with (8.42) with n replaced by $n - 2$, to verify (8.44).

3. In the proof of the Approximation Theorem, where did we use the fact that $g_k(x) \ge 0$ for all x in $[0, 1]$ and $0 \le k \le n$?

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4. Show that the Approximation Theorem does not hold if we replace I by a bounded open interval (a, b) by showing that if $f(x) = 1/(b-x)$ for all x, then $f : (a, b) \to \mathbb{R}$ cannot be uniformly approximated by polynomials.

5. Show that the Approximation Theorem does not hold if we replace I by $\mathbb{R}$ by showing that if $f(x) = e^x$ for all x, then $f : \mathbb{R} \to \mathbb{R}$ cannot be uniformly approximated by polynomials.

6. Define $f(x) = |x - 1/2|$ for $0 \le x \le 1$. Use the proof of the Approximation Theorem to find an explicit polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - p(x)| < 1/4$ for all x in $[0, 1]$.

7. Verify the assertion about the composition that was made on the last line of the proof of the Approximation Theorem.

8. Suppose that the function $h : [-1, 1] \to \mathbb{R}$ is continuous. Prove that there is a sequence $\{p_k : \mathbb{R} \to \mathbb{R}\}$ of polynomials having the property that

$$h(x) = \sum_{k=1}^{\infty} p_k(x) \qquad \text{for all points } x \text{ in } [-1, 1].$$

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CHAPTER 9

SEQUENCES AND SERIES OF FUNCTIONS

9.1 SEQUENCES AND SERIES OF NUMBERS

In Chapter 2, we studied the convergence of sequences of numbers. In particular, we proved the following important result.

Theorem 9.1 The Monotone Convergence Theorem A monotone sequence of numbers converges if and only if it is bounded.

This is a criterion for convergence that is intrinsic to the sequence itself; it does not require any information about the proposed limit. But, as the name suggests, the Monotone Convergence Theorem does require that the sequence be monotone. The sequence $\{(-1)^n\}$ shows that, in general, it is not true that any bounded sequence converges. We will now establish a criterion for convergence that applies to *all* sequences of numbers irrespective of monotonicity properties.

The Cauchy Convergence Criterion for Sequences

Definition A sequence of numbers $\{a_n\}$ is said to be a *Cauchy sequence* provided that for each positive number ϵ there is an index N such that

$$|a_n - a_m| < \epsilon \qquad \text{if } n \geq N \text{ and } m \geq N.$$

We will prove that a sequence of numbers converges if and only if it is a Cauchy sequence. This too is a criterion for convergence that is intrinsic to the sequence itself; it does not require any information about the proposed limit. Moreover, monotonicity is not required. We will prove this result in several steps.

Proposition 9.2 Every convergent sequence is Cauchy.

Proof

Suppose that $\{a_n\}$ is a sequence that converges to the number a. Let $\epsilon > 0$. We need to find an index N such that

$$|a_n - a_m| < \epsilon \qquad \text{if } n \geq N \text{ and } m \geq N.$$

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But since $\{a_n\}$ converges to a, we can choose an index N such that

$$|a_k - a| < \frac{\epsilon}{2} \qquad \text{for every index } k \geq N.$$

Thus, if $n \geq N$ and $m \geq N$, setting

$$a_n - a_m = (a_n - a) + (a - a_m),$$

by the Triangle Inequality,

$$\begin{aligned} |a_n - a_m| &= |(a_n - a) + (a - a_m)| \\ &\leq |a_n - a| + |a_m - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

■

Lemma 9.3 Every Cauchy sequence is bounded.

Proof

Suppose that $\{a_n\}$ is a Cauchy sequence. For $\epsilon = 1$, we can choose an index N such that

$$|a_n - a_m| < 1 \qquad \text{if } n \geq N \text{ and } m \geq N.$$

In particular, we have

$$|a_n - a_N| < 1 \qquad \text{if } n \geq N.$$

But, setting

$$a_n = a_N + (a_n - a_N),$$

by the Triangle Inequality,

$$|a_n| = |a_N + (a_n - a_N)| \leq |a_N| + |a_n - a_N|.$$

Consequently, we see that

$$|a_n| \leq |a_N| + 1 \qquad \text{if } n \geq N.$$

Define $M = \max\{|a_N| + 1, |a_1|, |a_2|, \ldots, |a_{N-1}|\}$. Then

$$|a_n| \leq M \qquad \text{for every index } n.$$

■

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Theorem 9.4 The Cauchy Convergence Criterion for Sequences A sequence of numbers converges if and only if it is a Cauchy sequence.

Proof

According to Proposition 9.2, every convergent sequence is a Cauchy sequence. The converse remains to be proven. Suppose that $\{a_n\}$ is a Cauchy sequence. The preceding lemma asserts that $\{a_n\}$ is bounded. Thus, by the Sequential Compactness Theorem, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a number a.

We claim that the whole sequence $\{a_n\}$ converges to a. Indeed, let $\epsilon > 0$. We need to find an index N such that

$$|a_n - a| < \epsilon \qquad \text{if } n \geq N.$$

Since $\{a_n\}$ is a Cauchy sequence, we can choose an index N such that

$$|a_n - a_m| < \frac{\epsilon}{2} \qquad \text{if } n \geq N \text{ and } m \geq N. \tag{9.1}$$

On the other hand, since the subsequence $\{a_{n_k}\}$ converges to a, there is an index K such that

$$|a_{n_k} - a| < \frac{\epsilon}{2} \qquad \text{if } k \geq K. \tag{9.2}$$

Now choose any index k such that $k \geq K$ and $n_k \geq N$. Using the inequalities (9.1) and (9.2) together with the Triangle Inequality, it follows that if $n \geq N$, then

$$\begin{aligned} |a_n - a| &= |(a_n - a_{n_k}) + (a_{n_k} - a)| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

■

Convergence Tests for Series

Recall that for a sequence of numbers $\{a_k\}$ that is indexed by the natural numbers, we define

$$s_n = \sum_{k=1}^{n} a_k \qquad \text{for every index } n$$

and obtain a new sequence $\{s_n\}$. The sequence $\{s_n\}$ is called the *sequence of partial sums* for the series $\sum_{k=1}^{\infty} a_k$, and a_k is called the kth *term* of the series $\sum_{k=1}^{\infty} a_k$. We write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \left[\sum_{k=1}^{n} a_k\right]$$

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if the sequence $\{s_n\}$ converges. If the sequence $\{s_n\}$ does not converge, we say that the series $\sum_{k=1}^{\infty} a_k$ *diverges.*[1]

Proposition 9.5 Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof

Define $\{s_n\}$ to be the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_n$ and define s to be the limit of the sequence $\{s_n\}$. Since $\lim_{n\to\infty} s_n = s$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Thus, by the difference property of convergent sequences,

$$\lim_{n\to\infty} [s_n - s_{n-1}] = 0.$$

However, for each index $n \geq 2$, $a_n = s_n - s_{n-1}$ and hence $\lim_{n\to\infty} a_n = 0$. ■

As we have already seen in Chapter 2, the Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge despite the fact that $\lim_{n\to\infty} 1/n = 0$. Thus, the convergence of the sequence of terms $\{a_n\}$ to 0 is a necessary, but not sufficient, condition for the series $\sum_{n=1}^{\infty} a_n$ to converge. The remainder of this section will be devoted to presenting conditions on the terms of a series that are sufficient to ensure that the series converges.

Recall that one of the first results we proved about convergent sequences was that

$$\lim_{n\to\infty} r^n = 0 \qquad \text{if } |r| < 1. \tag{9.3}$$

This limit, together with the Geometric Sum Formula, is exactly what is needed to establish the convergence of the Geometric Series $\sum_{k=0}^{\infty} r^k$ provided that $|r| < 1$.

Proposition 9.6 For a number r such that $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

---

[1] For a sequence of numbers $\{a_k\}$ indexed by the nonnegative integers, we define $s_n = \sum_{k=0}^{n} a_k$ for every nonnegative integer n and then define

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \left[\sum_{k=0}^{n} a_k\right]$$

if the sequence $\{s_n\}$ converges. This change in the initial index for the terms of a series has no material effect on the theory.

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Proof

The Geometric Sum Formula asserts that for each index n,

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

But $|r| < 1$, so $\lim_{n\to\infty} r^{n+1} = 0$, and hence, by the linearity property of convergent sequences of numbers,

$$\lim_{n\to\infty}\left[\sum_{k=0}^{n} r^k\right] = \lim_{n\to\infty}\left[\frac{1 - r^{n+1}}{1 - r}\right] = \frac{1}{1 - r}.$$ ■

Given two sequences $\{a_k\}$ and $\{b_k\}$ and two numbers α and β, observe that for each index n,

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha\sum_{k=1}^{n} a_k + \beta\sum_{k=1}^{n} b_k.$$

Thus, from the linearity property of convergent sequences, it follows that if the two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, then so also is the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$, and moreover,

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha\sum_{k=1}^{\infty} a_k + \beta\sum_{k=1}^{\infty} b_k.$$

We have two principal general criteria for a sequence of numbers to converge, namely, the Monotone Convergence Theorem and the Cauchy Convergence Criterion. Applying these criteria to series—that is, to sequences of partial sums—we obtain criteria for the convergence of series. First we examine consequences of the Monotone Convergence Theorem.

Theorem 9.7 Suppose that $\{a_k\}$ is a sequence of nonnegative numbers. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums is bounded; that is, there is a positive number M such that

$$a_1 + \cdots + a_n \leq M \qquad \text{for every index } n.$$

Proof

Since the terms of the series $\sum_{k=1}^{\infty} a_k$ are nonnegative, the sequence of partial sums is monotonically increasing. The Monotone Convergence Theorem asserts that the sequence of partial sums converges if and only if the sequence of partial sums is bounded. ■

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Corollary 9.8 The Comparison Test Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of numbers such that for index k,

$$0 \leq a_k \leq b_k.$$

i. The series $\sum_{k=1}^{\infty} a_k$ converges if the series $\sum_{k=1}^{\infty} b_k$ converges.
ii. The series $\sum_{k=1}^{\infty} b_k$ diverges if the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof
Observe that for each index n,

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k.$$

These inequalities show that if the partial sums of the series $\sum_{k=1}^{\infty} b_k$ are bounded, then so are the partial sums of $\sum_{k=1}^{\infty} a_k$, and, on the other hand, if the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are not bounded, then those of $\sum_{k=1}^{\infty} b_k$ are also not bounded. The result follows from these two observations and Theorem 9.7. ■

Example 9.9 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\, 2^k}. \tag{9.4}$$

For each index k, $1/[\sqrt{k}\, 2^k] \leq 1/2^k$. Since the Geometric Series converges for $r = 1/2$, it follows from the Comparison Test that the series (9.4) also converges. ■

Example 9.10 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}. \tag{9.5}$$

For each index k, $1/\sqrt{k} \geq 1/k$. Since the Harmonic Series $\sum_{k=1}^{\infty} 1/k$ diverges, it follows from the Comparison Test that the series (9.5) also diverges. ■

Corollary 9.11 The Integral Test Let $\{a_k\}$ be a sequence of nonnegative numbers and suppose that the function $f : [1, \infty) \to \mathbb{R}$ is continuous and monotonically decreasing and has the property that

$$f(k) = a_k \qquad \text{for every index } k.$$

Then the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence of integrals $\{\int_1^n f(x)\, dx\}$ is bounded.

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Proof

Since the function f is continuous, its restriction to each bounded interval is integrable. Moreover, since f is monotonically decreasing, for each index k and each point x in the interval $[k, k+1]$,

$$a_k = f(k) \geq f(x) \geq f(k+1) = a_{k+1},$$

so the monotonicity property of integration implies that

$$a_k \geq \int_k^{k+1} f(x)\,dx \geq a_{k+1}.$$

By the additivity-over-intervals property of integration, we conclude that for each index n,

$$\sum_{k=1}^{n} a_k \geq \int_1^{n+1} f(x)\,dx \geq \sum_{k=2}^{n+1} a_k.$$

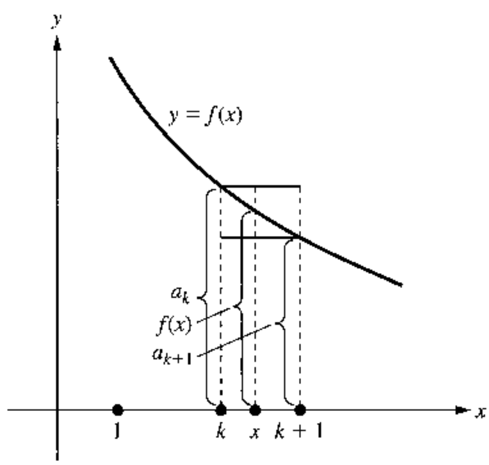

FIGURE 9.1 $a_k \geq \int_k^{k+1} f(x)\,dx \geq a_{k+1}$.

These inequalities imply that the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k$ is bounded if and only if the sequence $\{\int_1^n f(x)\,dx\}$ is bounded. We assumed that the terms a_k are nonnegative. Therefore, in view of Theorem 9.7, it follows that the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence $\{\int_1^n f(x)\,dx\}$ is bounded. ■

Example 9.12 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)\ln(k+1)}. \tag{9.6}$$

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Using the First Fundamental Theorem (Integrating Derivatives), we see that for every index n,

$$\int_1^n \frac{1}{(x+1)\ln(x+1)}\,dx = \ln[\ln(n+1)] - \ln(\ln 2).$$

Since the sequence $\{\ln[\ln(n+1)] - \ln(\ln 2)\}$ is not bounded, it follows from the Integral Test that the series (9.6) diverges. ■

Corollary 9.13 The p-Test For a positive number p, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Proof
Define $f(x) = x^{-p}$ for $x \geq 1$. The function $f : [1, \infty) \to \mathbb{R}$ is continuous and monotonically decreasing. For each index n, the First Fundamental Theorem (Integrating Derivatives) implies that

$$\int_1^n f(x)\,dx = \begin{cases} (n^{1-p} - 1)/(1-p) & \text{if } p \neq 1 \\ \ln n & \text{if } p = 1. \end{cases}$$

Hence the sequence $\{\int_1^n f(x)\,dx\}$ is bounded if and only if $p > 1$. According to the Integral Test, the series $\sum_{k=1}^{\infty} 1/k^p$ converges if and only if $p > 1$. ■

Example 9.14 Consider the series

$$\sum_{k=1}^{\infty} \frac{k}{e^k}.$$

From Theorem 8.10 it follows that for each $b > 0$ there is a point c in the open interval $(0, c)$ at which

$$e^b = 1 + b + \frac{b^2}{2} + \frac{b^3}{6} + \frac{e^c}{24},$$

and therefore,

$$e^b > \frac{b^3}{6} \qquad \text{if } b > 0.$$

Thus, for each index k, $k/e^k < 6/k^2$. The p-Test implies that the series $\sum_{k=1}^{\infty} 1/k^2$ converges and hence so does the series $\sum_{k=1}^{\infty} 6/k^2$. The Comparison Test now implies that the series $\sum_{k=1}^{\infty} k/e^k$ also converges. ■

When the terms of a series fail to be of one sign, it is not possible to directly invoke the Monotone Convergence Theorem in order to check convergence since the associated

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sequence of partial sums is not a monotone sequence. However, for series whose terms *alternate in sign* we can indirectly use the Monotone Convergence Criterion to obtain the following convergence test.

Theorem 9.15 The Alternating Series Test Suppose that $\{a_k\}$ is a monotonically decreasing sequence of nonnegative numbers that converges to 0. Then the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

converges.

Proof
Define

$$s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k \qquad \text{for every index } n.$$

In order to prove that the sequence of partial sums $\{s_n\}$ converges, we will first show that the subsequence $\{s_{2n}\}$ converges. Indeed, for each index n, observe that since the sequence $\{a_k\}$ is monotonically decreasing,

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0,$$

and since the sequence $\{a_k\}$ also consists of nonnegative numbers,

$$s_{2n} = \sum_{k=1}^{n}(a_{2k-1} - a_{2k}) = a_1 - \sum_{k=1}^{n-1}(a_{2k} - a_{2k+1}) - a_{2n} \leq a_1.$$

We conclude that $\{s_{2n}\}$ is monotonically increasing and bounded above by a_1. By the Monotone Convergence Theorem, the sequence $\{s_{2n}\}$ converges. Define $s = \lim_{n\to\infty} s_{2n}$. But

$$s_{2n+1} = s_{2n} + a_{2n+1} \qquad \text{for every index } n.$$

Since $\lim_{n\to\infty} a_n = 0$, it follows that the sequence $\{s_{2n+1}\}$ converges to the same limit s.

We claim that the whole sequence $\{s_n\}$ converges to s. Indeed, let $\epsilon > 0$. We can choose indices N_1 and N_2 such that

$$|s_{2n} - s| < \epsilon \qquad \text{if } n \geq N_1$$

and

$$|s_{2n+1} - s| < \epsilon \qquad \text{if } n \geq N_2.$$

Define $N = \max\{2N_1,\ 2N_2 + 1\}$. Then

$$|s_n - s| < \epsilon \qquad \text{if } n \geq N. \qquad ■$$

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Example 9.16 From the Alternating Series Test, it follows that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges. In fact, in Section 8.4 we proved that it converges to $\ln 2$. ■

For series whose terms are neither of one sign nor alternating in sign, it is natural to apply the Cauchy Convergence Criterion for Sequences to the sequence of partial sums in order to determine if the series converges.

It is sometimes useful, particularly when considering series, to restate the definition of a Cauchy sequence as follows: A sequence $\{s_n\}$ is a Cauchy sequence provided that for each positive number ϵ there is an index N such that for each index $n \geq N$ and any natural number k,

$$|s_{n+k} - s_n| < \epsilon.$$

Theorem 9.17 The Cauchy Convergence Criterion for Series The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for each positive number ϵ there is an index N such that

$$|a_{n+1} + \cdots + a_{n+k}| < \epsilon \qquad \text{for all indices } n \geq N \text{ and all natural numbers } k.$$

Proof

Apply the Cauchy Convergence Criterion for Sequences to the sequence of partial sums. ■

Definition The series $\sum_{k=1}^{\infty} a_k$ is said to *converge absolutely* provided that the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Corollary 9.18 The Absolute Convergence Test An absolutely convergent series converges; that is, the series $\sum_{k=1}^{\infty} a_k$ converges if the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Proof

From the Triangle Inequality we conclude that for each pair of natural numbers n and k,

$$\left| \sum_{j=n+1}^{n+k} a_j \right| \leq \sum_{j=n+1}^{n+k} |a_j|.$$

Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, it follows from the Cauchy Convergence Criterion for Series that the sequence of partial sums for this series is a Cauchy sequence. The preceding inequality implies that the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k$ also is a Cauchy sequence. Once more using the Cauchy Convergence Criterion for Series, it follows that the series $\sum_{k=1}^{\infty} a_k$ converges. ■

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Example 9.19 The series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$$

converges. To verify this, first observe that by the p-test, with $p = 2$, the series $\sum_{k=1}^{\infty} 1/k^2$ converges. Since for every index k, $|\sin k| \leq 1$, it follows from the Comparison Test that the series $\sum_{k=1}^{\infty} |\sin k|/k^2$ also converges. The Absolute Convergence Test now implies that the series $\sum_{k=1}^{\infty} \sin k/k^2$ converges. ∎

A series that converges but does not converge absolutely is said to *converge conditionally.* The series $\sum_{k=1}^{\infty} (-1)^{k+1}/k$ converges conditionally since by the Alternating Series Test it converges, but the Harmonic Series does not converge.

Theorem 9.20 For the series $\sum_{k=1}^{\infty} a_k$, suppose that there is a number r with $0 \leq r < 1$ and an index N such that

$$|a_{n+1}| \leq r|a_n| \qquad \text{for all indices } n \geq N. \tag{9.7}$$

Then the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof

First, observe that for each natural number k, if we apply the inequality (9.7) successively k times, we obtain

$$|a_{N+k}| \leq r^k |a_N|.$$

From this inequality and the Geometric Sum Formula, we conclude that for each natural number k,

$$\begin{aligned}
|a_1| + \cdots + |a_{N+k}| &= |a_1| + \cdots + |a_{N-1}| + |a_N| + \cdots + |a_{N+k}| \\
&\leq |a_1| + \cdots + |a_{N-1}| + |a_N|[1 + r + \cdots + r^k] \\
&= |a_1| + \cdots + |a_{N-1}| + |a_N| \left[\frac{1 - r^{k+1}}{1 - r}\right] \\
&\leq |a_1| + \cdots + |a_{N-1}| + |a_N| \left[\frac{1}{1 - r}\right]. \qquad (9.8)
\end{aligned}$$

Define

$$M \equiv |a_1| + \cdots + |a_{N-1}| + |a_N| \left[\frac{1}{1 - r}\right].$$

Then, from the preceding inequality (9.8), it follows that

$$|a_1| + \cdots + |a_n| \leq M \qquad \text{for every index } n.$$

This means that the sequence of partial sums of the series $\sum_{k=1}^{\infty} |a_k|$ is bounded. According to Theorem 9.7, the series $\sum_{k=1}^{\infty} |a_k|$ converges. ∎

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Recall that in Section 8.5 we established Lemma 8.20, the Ratio Test for Sequences. We have the following companion result for series.

Corollary 9.21 The Ratio Test for Series For the series $\sum_{k=1}^{\infty} a_k$, suppose that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell.$$

i. If $\ell < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
ii. If $\ell > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof
First, suppose that $\ell < 1$. Define $r = (1 + \ell)/2$. Then $\ell < r$ since $\ell < 1$, and so we can choose an index N such that

$$\frac{|a_{n+1}|}{|a_n|} < r \qquad \text{for all indices } n \geq N.$$

Also, $r < 1$ since $\ell < 1$. The conclusion now follows from Theorem 9.20.

Now suppose that $\ell > 1$. Then it follows from the Ratio Lemma for Sequences that the sequence $\{a_n\}$ does not converge to 0. Thus, the series $\sum_{n=1}^{\infty} a_n$ diverges. ■

The theory of convergent and divergent series is an important area of mathematics. The present section gives but a very brief glimpse of some ways in which the Monotone Convergence Theorem and the Cauchy Convergence Criterion for sequences of numbers can be applied to obtain criteria that are sufficient for a series to converge.

EXERCISES FOR SECTION 9.1

1. Examine the following series for convergence:

a. $\displaystyle\sum_{k=1}^{\infty} \frac{a^k}{k^p}$, where $a > 0$ and $p > 0$

b. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+3}$

c. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

d. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

e. $\displaystyle\sum_{k=1}^{\infty} ke^{-k^2}$

f. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+1}{k^2+1}\right)^3$

g. $\displaystyle\sum_{k=1}^{\infty} k \sin\left(\frac{1}{k}\right)$

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2. For any positive number α, prove that the series

$$\sum_{k=1}^{\infty} \frac{k^\alpha}{e^k}$$

converges.

3. Fix a positive number α and consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)[\ln(k+1)]^\alpha}.$$

For what values of α does this series converge?

4. Under the assumptions of the Alternating Series Test, define

$$s = \sum_{k=1}^{\infty} (-1)^{k+1} a_k.$$

Prove that for every index n,

$$\left| s - \sum_{k=1}^{n} (-1)^{k+1} a_k \right| \leq a_{n+1}.$$

5. Use the Cauchy Convergence Criterion for Series to provide another proof of the Alternating Series Test.

6. If a sequence converges, then each of its subsequences converges to the same limit. However, the convergence of a subsequence does not imply the convergence of the whole sequence. Based on these two observations, prove that

$$\text{if } \sum_{k=1}^{\infty} a_k = \ell, \text{ then } \sum_{k=1}^{\infty} (a_{2k} + a_{2k-1}) = \ell,$$

but that the converse does not necessarily hold. (*Hint:* Consider the series $\sum_{k=1}^{\infty} (-1)^k$.)

7. (The Cauchy Root Test) For the series $\sum_{k=1}^{\infty} a_k$, suppose that there is a number r with $0 \leq r < 1$ and a natural number N such that

$$|a_k|^{1/k} < r \qquad \text{for all indices } k \geq N.$$

Prove that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

8. Suppose that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive numbers such that

$$\lim_{n \to \infty} \left(\frac{a_k}{b_k} \right) = \ell \qquad \text{and} \qquad \ell > 0.$$

Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty} b_k$ converges.

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9.2 POINTWISE CONVERGENCE OF SEQUENCES OF FUNCTIONS

Chapter 3 was devoted to the study of sequences of numbers. In Section 9.1, we studied sequences of numbers and then series of numbers. We now turn to the study of sequences of *functions*.

Definition Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, we say that the sequence $\{f_n : D \to \mathbb{R}\}$ *converges pointwise* to $f : D \to \mathbb{R}$, or that $\{f_n\}$ *converges pointwise* on D to f, provided that for each point x in D,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Example 9.22 For each natural number n, define

$$f_n(x) = x^n \qquad \text{for } 0 \le x \le 1.$$

Since $\{f_n(1)\}$ is a constant sequence whose constant value is 1.

$$\lim_{n\to\infty} f_n(1) = 1.$$

On the other hand,

$$\lim_{n\to\infty} x^n = 0 \qquad \text{if } 0 \le x < 1.$$

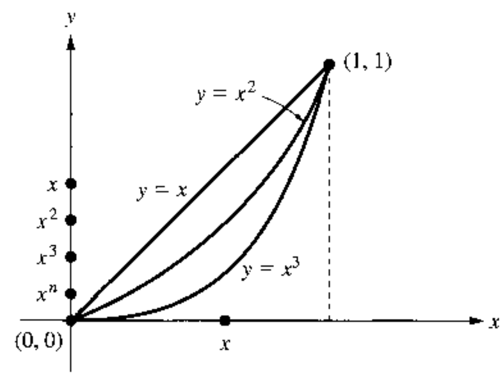

FIGURE 9.2 $\lim_{n\to\infty} x^n = 0$ if $0 \le x < 1$; $\lim_{n\to\infty} 1^n = 0$.

Thus, the sequence of functions $\{f_n\}$ converges pointwise on $[0, 1]$ to the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \le x < 1. \end{cases}$$

Observe that this is an example of a sequence of continuous functions that converges pointwise to a discontinuous function. ∎

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Example 9.23 For each natural number n, define

$$f_n(x) = e^{-nx^2} \qquad \text{for all } x.$$

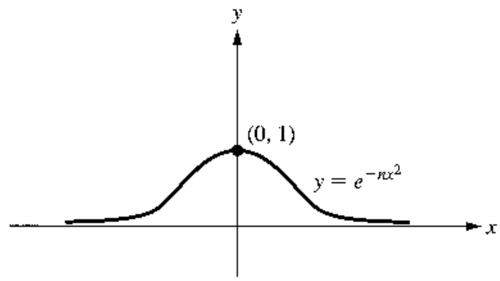

FIGURE 9.3 $\lim_{n\to\infty} e^{-nx^2} = 0$ if $x \neq 0$; $\lim_{n\to\infty} e^{-n[0]^2} = 1$.

Then $\{f_n(0)\}$ is a constant sequence whose constant value is 1, so

$$\lim_{n\to\infty} f_n(0) = 1.$$

On the other hand, since $e^b > 1 + b$ if $b > 0$, it follows that

$$\frac{1}{e^b} < \frac{1}{1+b} \qquad \text{for all } b > 0.$$

Thus, for each index n and each $x \neq 0$,

$$0 < f_n(x) < \frac{1}{1+nx^2},$$

so

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{if } x \neq 0.$$

Thus, the sequence of functions $\{f_n\}$ converges pointwise on $\mathbb{R}$ to the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Observe that this is an example of a sequence of functions, each of which is differentiable on $\mathbb{R}$, that converges pointwise on $\mathbb{R}$ to a function that is not differentiable at $x = 0$. ■

Example 9.24 A number x of the form $x = k/2^n$, for an integer k and a natural number n, is called a *dyadic rational*. For each natural number n and each number x in the interval $[0, 1]$, define

$$f_n(x) = \begin{cases} 1 & \text{if } x = k/2^n \text{ for some natural number } k \\ 0 & \text{otherwise.} \end{cases}$$

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It is clear that for a dyadic rational number $x = k/2^N$, $f_n(x) = 1$ for each index $n \geq N$. Thus, the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a dyadic rational} \\ 0 & \text{otherwise.} \end{cases}$$

This is an example of a sequence of integrable functions that converges pointwise on a closed bounded interval to a function that is not integrable (Exercise 5). ■

Example 9.25 For a natural number n, define $f_n : [0, 1] \to \mathbb{R}$ to be the function such that $f_n(0) = f(2/n) = f_n(1) = 0$, $f_n(1/n) = n$, and f_n is linear on the intervals $[0, 1/n]$, $[1/n, 2/n]$, and $[2/n, 0]$.

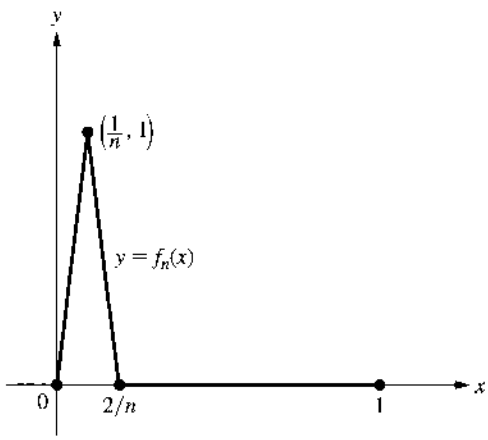

FIGURE 9.4 $f_n(x) = 0$ if $2/n \leq x \leq 1$; $\int_0^1 f_n = 1$.

Since $\{f_n(0)\}$ is a constant sequence whose constant value is 0,

$$\lim_{n\to\infty} f_n(0) = 0.$$

On the other hand, for any positive number x in the interval $[0, 1]$, by the Archimedean Property of $\mathbb{R}$, there is a natural number N such that $1/2N < x$. Therefore, for each index $n \geq N$, $f_n(x) = 0$. Thus,

$$\lim_{n\to\infty} f_n(x) = 0.$$

Thus, the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to 0 (by this we mean to the function that is identically equal to 0 on $[0, 1]$). Observe that $\int_0^1 f = 0$, while for each index n, $\int_0^1 f_n = 1$. ■

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Example 9.26 For each natural number n, define

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \qquad \text{for } 0 \le x \le 1.$$

According to the Taylor series expansion formula (8.16), the sequence of functions $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the function e^x. In infinite series notation, this is expressed as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for } 0 \le x \le 1.$$

There is no pathology in this example. ■

EXERCISES FOR SECTION 9.2

1. For each natural number n and each number x, define

$$f_n(x) = \frac{1 - |x|^n}{1 + |x|^n}.$$

Find the function $f : \mathbb{R} \to \mathbb{R}$ to which the sequence $\{f_n : \mathbb{R} \to \mathbb{R}\}$ converges pointwise.

2. For each natural number n and each number $x \ge 2$, define

$$f_n(x) = \frac{1}{1 + x^n}.$$

Find the function $f : [2, \infty) \to \mathbb{R}$ to which the sequence $\{f_n : [2, \infty) \to \mathbb{R}\}$ converges pointwise.

3. For each natural number n and each number x in $(0, 1)$, define

$$f_n(x) = \frac{1}{nx + 1}.$$

Find the function $f : (0, 1) \to \mathbb{R}$ to which the sequence $\{f_n : (0, 1) \to \mathbb{R}\}$ converges pointwise.

4. For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = \frac{x}{nx + 1}.$$

Find the function $f : [0, 1] \to \mathbb{R}$ to which the sequence $\{f_n : [0, 1] \to \mathbb{R}\}$ converges pointwise.

5. Prove that the dyadic rationals are dense in $\mathbb{R}$. From this, conclude that the limit function in Example 9.24 is not integrable.

6. For each natural number n and each number x in $(-1, 1)$, define

$$p_n(x) = x + x(1 - x^2) + \cdots + x(1 - x^2)^n.$$

Prove that the sequence $\{p_n : (-1, 1) \to \mathbb{R}\}$ converges pointwise.

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9.3 UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

For a sequence of functions $\{f_n : D \to \mathbb{R}\}$ that converges pointwise to the function $f : D \to \mathbb{R}$, we wish to determine the properties possessed by the individual f_n's that are inherited by the limit function $f : D \to \mathbb{R}$. Three natural questions come to mind.

Question A *Suppose that each function $f_n : D \to \mathbb{R}$ is continuous. Is the limit function $f : D \to \mathbb{R}$ also continuous?*

Answer: No. Example 9.22 describes a sequence of polynomials that converges pointwise on the interval $[0, 1]$ to a discontinuous function.

Question B *If $D = I$ is an open interval and each function $f_n : I \to \mathbb{R}$ is differentiable, is the limit function $f : I \to \mathbb{R}$ also differentiable? If it is, is*

$$\lim_{n\to\infty}\left[\frac{df_n}{dx}(x)\right] = \frac{df}{dx}(x)?$$

Answer: No. Example 9.23 describes a sequence of exponential functions that converges pointwise on $\mathbb{R}$ to a nondifferentiable function.

Question C *If $D = [a, b]$ and each function $f_n : [a, b] \to \mathbb{R}$ is integrable, is the limit function $f : [a, b] \to \mathbb{R}$ also integrable? If it is, is*

$$\lim_{n\to\infty}\left[\int_a^b f_n\right] = \int_a^b f?$$

Answer: No. Example 9.24 describes a sequence of step functions that converges pointwise on the interval $[0, 1]$ to a nonintegrable function. Moreover, as Example 9.25 shows, even if the limit function is integrable, it is not necessarily the case that the limit of the integrals equals the integral of the limit.

Despite the discouraging responses to the above three questions, all is not lost. If we strengthen the assumption of pointwise convergence to what we will call *uniform convergence*, then the first and third questions have affirmative answers, and Question *B* has a quite satisfactory answer. What is of equal importance is that in many interesting situations we can verify uniform convergence.

Definition Given a function $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, the sequence $\{f_n : D \to \mathbb{R}\}$ is said to *converge uniformly* to $f : D \to \mathbb{R}$, or $\{f_n\}$ is said to *converge uniformly* on D to f, provided that for each positive number ϵ there is an index N such that

$$|f(x) - f_n(x)| < \epsilon \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } D. \tag{9.9}$$

It is clear from the above definition that uniform convergence implies pointwise convergence; however, the converse is not true. To understand the distinction between uniform and pointwise convergence, observe that the sequence $\{f_n : D \to \mathbb{R}\}$ converges pointwise to $f : D \to \mathbb{R}$ provided that for each fixed point x in D the sequence of numbers $\{f_n(x)\}$ converges to the number $f(x)$; thus, for a given point x in D and a

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positive number ϵ, there is an index N such that $|f_n(x) - f(x)| < \epsilon$ if $n \geq N$. *The index N that responds to the ϵ challenge may depend on the point x.* In the case of uniform convergence on a set D, for a given positive number ϵ, there is an index N that simultaneously responds to the ϵ challenge for all the points in D.

In terms of graphs, the sequence $\{f_n : D \to \mathbb{R}\}$ converges uniformly to $f : D \to \mathbb{R}$ if for each positive number ϵ there is a natural number N such that if $n \geq N$, the graph of the function $f_n : D \to \mathbb{R}$ lies between the graphs of the functions $f + \epsilon : D \to \mathbb{R}$ and $f - \epsilon : D \to \mathbb{R}$.

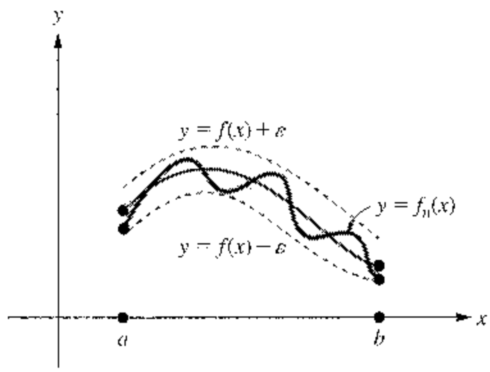

FIGURE 9.5 $f(x) - \epsilon < f_n(x) < f(x) + \epsilon$ if $a \leq x \leq b$ and $n \geq N$.

We now revisit two of the examples of pointwise convergence considered in the preceding section and analyze them for uniform convergence.

Example 9.27 Let the sequence $\{f_n : [0, 1] \to \mathbb{R}\}$ and the function $f : [0, 1] \to \mathbb{R}$ be as in Example 9.22. The convergence is not uniform. Indeed, for $\epsilon = 1/2$, there is no index N possessing the property that

$$|f_n(x) - f(x)| < \frac{1}{2} \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } [0, 1]$$

because no matter what index number N is chosen, by taking $x = (3/4)^{1/(N+1)}$, we have

$$f_{N+1}(x) - f(x) = \frac{3}{4} > \frac{1}{2}.$$

■

Example 9.28 Let the sequence of functions $\{f_n : [0, 1] \to \mathbb{R}\}$ and the function $f : [0, 1] \to \mathbb{R}$ be as in Example 9.26. We claim that $\{f_n\}$ converges uniformly to f on $[0, 1]$. To verify this claim, we need the estimate (8.6) obtained from the Lagrange Remainder Theorem applied to the exponential function. According to that estimate,

$$|f(x) - f_n(x)| \leq \frac{4}{(n+1)!} \qquad \text{for all indices } n \text{ and all points } x \text{ in } [0, 1]. \quad (9.10)$$

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Now let $\epsilon > 0$. By the Archimedean Property of $\mathbb{R}$, we can choose a natural number N such that $N > 4/\epsilon$. Thus,

$$|f(x) - f_n(x)| < \epsilon \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } [0, 1]. \qquad \blacksquare$$

In Section 9.1, we proved the Cauchy Convergence Criterion for the convergence of a sequence of numbers. There is a similar criterion for the uniform convergence of a sequence of functions.

Definition The sequence of functions $\{f_n : D \to \mathbb{R}\}$ is said to be *uniformly Cauchy*, or $\{f_n\}$ is said to be *uniformly Cauchy* on D, provided that for each positive number ϵ there is an index N such that

$$|f_{n+k}(x) - f_n(x)| < \epsilon \tag{9.11}$$

for every index $n \geq N$, every natural number k, and every point x in D.

Theorem 9.29 The Weierstrass Uniform Convergence Criterion The sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f: D \to \mathbb{R}$ if and only if the sequence $\{f_n: D \to \mathbb{R}\}$ is uniformly Cauchy.

Proof

Suppose that $\{f_n\}$ converges uniformly to f on D. We will show that $\{f_n\}$ is uniformly Cauchy on D. Indeed, let $\epsilon > 0$. We can select an index N such that

$$|f_n(x) - f(x)| < \epsilon/2 \qquad \text{for all indices } n \geq N \text{ and every point } x \text{ in } D.$$

Using the Triangle Inequality, it follows that

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &= |f_{n+k}(x) - f(x) + f(x) - f_n(x)| \\ &\leq |f_{n+k}(x) - f(x)| + |f_n(x) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for every index $n \geq N$, every natural number k, and every point x in D. Thus, the sequence $\{f_n : D \to \mathbb{R}\}$ is a uniformly Cauchy sequence.

To prove the converse, we suppose that the sequence of functions $\{f_n\}$ is uniformly Cauchy on D. Let x be a point in D. Then clearly the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence, and so, by the Cauchy Convergence Criterion for sequences of numbers, $\{f_n(x)\}$ converges. Denote the limit by $f(x)$. This defines a function $f : D \to \mathbb{R}$ that is the only candidate for a function to which $\{f_n\}$ can converge uniformly.

We now prove that $\{f_n\}$ converges uniformly to f on D. Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly Cauchy, we can select a natural number N such that

$$|f_{n+k}(x) - f_n(x)| < \epsilon/2 \tag{9.12}$$

for every index $n \geq N$, every natural number k, and every point x in D.

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Let x be a point in D. Choose $n \geq N$. Now observe that from inequality (9.12) we have

$$f_n(x) - \epsilon/2 < f_{n+k}(x) < f_n(x) + \epsilon/2 \qquad \text{for every natural number } k. \quad (9.13)$$

But

$$\lim_{k\to\infty} f_{n+k}(x) = f(x),$$

so that from (9.13) we obtain

$$f_n(x) - \epsilon/2 \leq f(x) \leq f_n(x) + \epsilon/2.$$

Hence,

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } D.$$

It follows that $\{f_n\}$ converges uniformly to f on D. ■

Example 9.30 For each natural number n and each number x with $|x| \leq 1$, define

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k2^k}.$$

Observe, using the Triangle Inequality and the Geometric Sum Formula, that for each pair of natural numbers n and k and each number x with $|x| \leq 1$,

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &\leq \frac{|x|^{n+1}}{(n+1)2^{n+1}} + \cdots + \frac{|x|^{n+k}}{(n+k)2^{n+k}} \\ &\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}} \\ &\leq \frac{1}{2^n}. \end{aligned} \qquad (9.14)$$

But $\lim_{n\to\infty}(1/2)^n = 0$, and this, together with the inequality (9.14), implies that the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ is uniformly Cauchy. According to the Weierstrass Uniform Convergence Criterion, there is a function $f : [-1, 1] \to \mathbb{R}$ to which the sequence $\{f_n : [-1, 1] \to \mathbb{R}\}$ converges uniformly. ■

EXERCISES FOR SECTION 9.3

1. For each natural number n and each number x, define

$$f_n(x) = \frac{1 - |x|^n}{1 + |x|^n}.$$

Find the function $f : \mathbb{R} \to \mathbb{R}$ to which the sequence $\{f_n : \mathbb{R} \to \mathbb{R}\}$ converges pointwise. Prove that the convergence is not uniform.

2. For each natural number n and each number $x \geq 2$, define

$$f_n(x) = \frac{1}{1 + x^n}.$$

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Find the function $f : [2, \infty) \to \mathbb{R}$ to which the sequence $\{f_n : [2, \infty) \to \mathbb{R}\}$ converges pointwise. Prove that the convergence is uniform.

3. For each natural number n and each number x in $(0, 1)$, define

$$f_n(x) = \frac{1}{nx + 1}.$$

Find the function $f : (0, 1) \to \mathbb{R}$ to which the sequence $\{f_n : (0, 1) \to \mathbb{R}\}$ converges pointwise. Prove that the convergence is not uniform.

4. For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = \frac{x}{nx + 1}.$$

Find the function $f : [0, 1] \to \mathbb{R}$ to which the sequence $\{f_n : [0, 1] \to \mathbb{R}\}$ converges pointwise. Prove that the convergence is uniform.

5. Determine whether the sequences in Examples 9.23, 9.24, and 9.25 converge uniformly.

6. Suppose that the sequences $\{f_n : D \to \mathbb{R}\}$ and $\{g_n : D \to \mathbb{R}\}$ converge uniformly to the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$, respectively. For any two numbers α and β, prove that the sequence $\{\alpha f_n + \beta g_n : D \to \mathbb{R}\}$ converges uniformly to the function $\alpha f + \beta g : D \to \mathbb{R}$.

7. For each natural number n, let the function $f_n : \mathbb{R} \to \mathbb{R}$ be bounded. Suppose that the sequence $\{f_n\}$ converges uniformly to f on $\mathbb{R}$. Prove that the limit function $f : \mathbb{R} \to \mathbb{R}$ also is bounded.

8. Let $\{a_n\}$ be a bounded sequence of numbers. For each natural number n and each number x, define

$$f_n(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!}.$$

Prove that for each $r > 0$, the sequence of functions $\{f_n : [-r, r] \to \mathbb{R}\}$ is uniformly convergent.

9.4 THE UNIFORM LIMIT OF FUNCTIONS

We will now provide some affirmative answers to the three questions raised at the beginning of Section 9.3 by strengthening the assumption of pointwise convergence to that of uniform convergence.

Uniformly Convergent Sequences of Continuous Functions

Theorem 9.31 Suppose that $\{f_n : D \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to the function $f : D \to \mathbb{R}$. Then the limit function $f : D \to \mathbb{R}$ also is continuous.

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Proof

Let x_0 be a point in D. We will use the ϵ-δ criterion for continuity in order to prove that the function $f : D \to \mathbb{R}$ is continuous at x_0. Indeed, let $\epsilon > 0$. It is necessary to find a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \qquad \text{for all points } x \text{ in } D \text{ such that } |x - x_0| < \delta. \tag{9.15}$$

Since the sequence $\{f_n : D \to \mathbb{R}\}$ converges uniformly to the function $f : D \to \mathbb{R}$, we can choose an index N such that

$$|f_n(x) - f(x)| < \epsilon/3 \qquad \text{for all indices } n \geq N \text{ and all points } x \text{ in } D.$$

Using this inequality, with $n = N$, and the Triangle Inequality, we see that

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + |f_N(x) - f_N(x_0)| + \frac{\epsilon}{3} \qquad \text{for all points } x \text{ in } D. \end{aligned} \tag{9.16}$$

By assumption, the function $f_N : D \to \mathbb{R}$ is continuous at x_0. Hence, we can choose $\delta > 0$ such that

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3} \qquad \text{for all points } x \text{ in } D \text{ such that } |x - x_0| < \delta. \tag{9.17}$$

The inequalities (9.16) and (9.17) show that for the above choice of δ, the required criterion (9.15) holds. Thus, the function $f : D \to \mathbb{R}$ is continuous at the point x_0. ■

Uniformly Convergent Sequences of Integrable Functions

Theorem 9.32 Suppose that $\{f_n : [a, b] \to \mathbb{R}\}$ is a sequence of integrable functions that converges uniformly to the function $f : [a, b] \to \mathbb{R}$. Then the limit function $f : [a, b] \to \mathbb{R}$ also is integrable. Moreover,

$$\lim_{n\to\infty} \left[\int_a^b f_n\right] = \int_a^b f.$$

Proof

We begin with a preliminary observation: It follows from the definition of the lower and upper integrals (Exercise 6) that they possess the following monotonicity properties: If $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are bounded functions and

$$h(x) \leq g(x) \qquad \text{for all } x \text{ in } [a, b],$$

then

$$\underline{\int_a^b} h \leq \underline{\int_a^b} g \qquad \text{and} \qquad \overline{\int_a^b} h \leq \overline{\int_a^b} g.$$

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In order to show that $f : [a, b] \to \mathbb{R}$ is integrable, that is, that its upper integral equals its lower integral, it suffices to show that

$$\overline{\int_a^b} f - \underline{\int_a^b} f < \epsilon \qquad \text{for every } \epsilon > 0. \tag{9.18}$$

Let $\epsilon > 0$. Then $\epsilon' \equiv \epsilon/4[b-a]$ also is positive. Since $\{f_n\}$ converges uniformly to f on $[a, b]$, there is a function f_n such that

$$f_n(x) - \epsilon' \leq f(x) \leq f_n(x) + \epsilon' \qquad \text{for all points } x \text{ in } [a, b]. \tag{9.19}$$

From the monotonicity property of the upper integral and integrability of f_n on $[a, b]$, we conclude that

$$\overline{\int_a^b} f \leq \overline{\int_a^b} [f_n + \epsilon'] = \overline{\int_a^b} f_n + \frac{\epsilon}{4} = \int_a^b f_n + \frac{\epsilon}{4},$$

so that

$$\overline{\int_a^b} f \leq \int_a^b f_n + \frac{\epsilon}{4}.$$

Similarly, using the monotonicity property of the lower integral, we obtain the estimate

$$\int_a^b f_n - \frac{\epsilon}{4} \leq \underline{\int_a^b} f.$$

Thus,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq \left[\int_a^b f_n + \frac{\epsilon}{4}\right] - \left[\int_a^b f_n - \frac{\epsilon}{4}\right] < \epsilon.$$

The criterion (9.18) has been verified, so that f is integrable on $[a, b]$.

It remains to be verified that

$$\lim_{n\to\infty}\left[\int_a^b f_n\right] = \int_a^b f.$$

Let $\epsilon > 0$. We need to find a natural number N such that

$$\left|\int_a^b f_n - \int_a^b f\right| < \epsilon \qquad \text{for all indices } n \geq N. \tag{9.20}$$

For $\epsilon' = \epsilon/6[b-a)]$, by the uniform convergence of $\{f_n\}$ to f on $[a, b]$, there is an index N such that (9.19) holds for every index $n \geq N$. The linearity and monotonicity properties of integration show that the inequality (9.19) implies the integral inequality (9.20). ■

Uniformly Convergent Sequences of Differentiable Functions

The answer to Question B in Section 9.3 regarding the differentiability of the limit of differentiable functions requires more care than the answers to the other two questions.

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The uniform limit of differentiable functions need not be differentiable (Exercise 1). However, there are quite reasonable circumstances under which it is differentiable, and the derivative of the limit is the limit of the derivatives.

A function $f : I \to \mathbb{R}$, defined on an open interval, is called *continuously differentiable* provided that it is differentiable and its derivative is continuous.

Theorem 9.33 Let I be an open interval. Suppose that $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions that has the following two properties:

i. The sequence $\{f_n\}$ converges pointwise on I to the function f, and
ii. The derived sequence $\{f_n'\}$ converges uniformly on I to the function g.

Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and

$$f'(x) = g(x) \qquad \text{for all } x \text{ in } I.$$

Proof

Fix a point x_0 in I. According to the First Fundamental Theorem (Integrating Derivatives), for each index n and each point x in I,

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f_n'. \tag{9.21}$$

Now, Theorem 9.32 implies that for each point x in I,

$$\lim_{n\to\infty}\left[\int_{x_0}^{x} f_n'\right] = \int_{x_0}^{x} g. \tag{9.22}$$

Also, since by assumption the sequence $\{f_n\}$ converges pointwise to the function f on I, for each point x in I,

$$\lim_{n\to\infty}[f_n(x) - f_n(x_0)] = f(x) - f(x_0). \tag{9.23}$$

From (9.21), (9.22), and (9.23), it follows that

$$f(x) - f(x_0) = \int_{x_0}^{x} g \qquad \text{for all } x \text{ in } I. \tag{9.24}$$

By assumption, for each natural number n, the function $f_n' : I \to \mathbb{R}$ is continuous, so by Theorem 9.31, the uniform limit $g : I \to \mathbb{R}$ also is continuous. From (9.24) and the Second Fundamental Theorem (Differentiating Integrals), we see that

$$f'(x) = g(x) \qquad \text{for all } x \text{ in } I. \qquad \blacksquare$$

Theorem 9.34 Let I be an open interval. Suppose that $\{f_n : I \to \mathbb{R}\}$ is a sequence of continuously differentiable functions that has the following two properties:

i. The sequence $\{f_n\}$ converges pointwise on I to the function f, and
ii. The derived sequence $\{f_n'\}$ is uniformly Cauchy on I.

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Then the function $f : I \to \mathbb{R}$ is continuously differentiable, and for each x in I,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

Proof

The Weierstrass Uniform Convergence Criterion implies that there is a function $g : I \to \mathbb{R}$ to which the sequence $\{f_n' : I \to \mathbb{R}\}$ converges uniformly. The conclusion now follows from Theorem 9.33. ∎

The property of uniform convergence can frequently be verified. However, there are many interesting cases in which a sequence of functions fails to converge uniformly but nevertheless the limit function inherits properties possessed by the individual functions in the approximation sequence.[2] We will describe one instance of this: the following verification of a classical formula for π.

Proposition 9.35 The Newton–Gregory Formula

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \tag{9.25}$$

Proof

Since for each number x,

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2},$$

it follows from the First Fundamental Theorem (Integrating Derivatives) that

$$\frac{\pi}{4} = \arctan 1 - \arctan 0 = \int_0^1 \frac{1}{1+x^2}\,dx. \tag{9.26}$$

Let n be a natural number. Substituting $-x^2 = r$ in the Geometric Sum Formula, we see that for each number x,

$$\frac{1}{1+x^2} = 1 - x^2 + \cdots + (-1)^n x^{2n} + \frac{(-1)^{n+1}x^{2n+2}}{1+x^2},$$

so that

$$\int_0^1 \frac{1}{1+x^2}\,dx = 1 - \frac{1}{3} + \cdots + \frac{(-1)^n}{2n+1} + \int_0^1 \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}\,dx. \tag{9.27}$$

The monotonicity property of the integral gives the estimate

$$\left|\int_0^1 \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}\,dx\right| \le \int_0^1 x^{2n+2}\,dx = \frac{1}{2n+3},$$

[2] One of the motivations for the development of a more general theory of integration based on the Lebesgue integral was the study of the question of when, in the absence of uniform convergence, the integral of the limit is the limit of the integrals: see the book *Real Analysis*, by H. L. Royden.

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from which it follows that

$$\lim_{n\to\infty}\left[\int_0^1 \frac{(-1)^{n+1}x^{2n+2}}{1+x^2}\,dx\right] = 0.$$

Thus, (9.25) follows from (9.26) and (9.27). ■

For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = \sum_{k=0}^{n}(-1)^k x^{2k}$$

and define $f(x) = 1/(1+x^2)$. The Newton–Gregory Formula can be restated as

$$\lim_{n\to\infty}\left[\int_0^1 f_n\right] = \int_0^1 \lim_{n\to\infty} f_n.$$

We proved this without proving that the sequence of functions $\{f_n : [0, 1] \to \mathbb{R}\}$ converges uniformly to the function $f : [0, 1] \to \mathbb{R}$. In fact, we do not even have pointwise convergence on the whole interval $[0, 1]$ since the sequence $\{f_n(1)\}$ does not converge to $f(1)$.

EXERCISES FOR SECTION 9.4

1. For each natural number n and each number x in $(-1, 1)$, define

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

and define $f(x) = |x|$. Prove that the sequence $\{f_n\}$ converges uniformly on the open interval $(-1, 1)$ to the function f. Check that each function f_n is continuously differentiable, whereas the limit function f is not differentiable at $x = 0$. Does this contradict Theorem 9.33?

2. For each natural number n and each number x in $[0, 1]$, define

$$f_n(x) = nxe^{-nx^2}.$$

Prove that the sequence $\{f_n\}$ converges pointwise on the interval $[0, 1]$ to the constant function 0, but that the sequence of integrals $\{\int_0^1 f_n\}$ does not converge to 0. Does this contradict Theorem 9.32?

3. Prove that if $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of continuously differentiable functions such that the sequence of derivatives $\{f_n' : \mathbb{R} \to \mathbb{R}\}$ is uniformly convergent and the sequence $\{f_n(0)\}$ is also convergent, then $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is pointwise convergent. Is the assumption that the sequence $\{f_n(0)\}$ converges necessary?

4. Give an example of a sequence of differentiable functions $\{f_n : (-1, 1) \to \mathbb{R}\}$ that converges uniformly but for which $\{f_n'(0)\}$ is unbounded.

5. Under the assumptions of Theorem 9.33, show that for each interval $[\alpha, \beta]$ contained in I, the sequence $\{f_n : [\alpha, \beta] \to \mathbb{R}\}$ converges uniformly to $f : [\alpha, \beta] \to \mathbb{R}$.

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6. Let $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be bounded functions such that

$$h(x) \leq g(x) \qquad \text{for all } x \text{ in } [a, b].$$

Show that if P is a partition of the domain $[a, b]$, then

$$L(h, P) \leq L(g, P) \qquad \text{and} \qquad U(h, P) \leq U(g, P).$$

Use these inequalities to establish the monotonicity properties of the upper and lower integrals stated in the proof of Theorem 9.32.

9.5 POWER SERIES

In the study of Taylor series, we began with an infinitely differentiable function; we then constructed a Taylor series, which we analyzed for convergence to the given function. We will now change our point of view. In this section we will *define* a function by a power series expansion and study the properties of such a function.

Definition Given a sequence of real numbers $\{c_k\}$ indexed by the nonnegative integers, we define the *domain of convergence* of the series $\sum_{k=0}^{\infty} c_k x^k$ to be the set of all numbers x such that the series $\sum_{k=0}^{\infty} c_k x^k$ converges. Denote the domain of convergence by D. We then define a function $f : D \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} c_k x^k \right] = \sum_{k=0}^{\infty} c_k x^k \qquad \text{for all } x \text{ in } D. \tag{9.28}$$

We refer to (9.28) as a *power series expansion* and call the set D the *domain of convergence of the expansion*.

Example 9.36 Consider the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k+2}. \tag{9.29}$$

Fix a number x. Since

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{k+1}}{k+3} \Big/ \frac{(-1)^k x^k}{k+2} \right| = |x|,$$

it follows from the Ratio Test for Series that the power series (9.29) converges if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$, from the Alternating Series Test we conclude that (9.29) converges. For $x = -1$, the Integral Test shows that the series diverges. Thus, the domain of convergence of the power series (9.29) is the interval $(-1, 1]$. ∎

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Example 9.37 Consider the power series

$$\sum_{k=0}^{\infty} k!x^k.$$

For any nonzero number x, the terms of the series $\sum_{k=0}^{\infty} k!x^k$ do not converge to 0, and hence the series does not converge. Thus, the domain of convergence of the power series $\sum_{k=0}^{\infty} k!x^k$ consists of the single point $x = 0$. ■

Example 9.38 Consider the power series

$$\sum_{k=0}^{\infty} \frac{1}{(1+k!)} x^k.$$

Fix a number x. Since

$$\lim_{k\to\infty} \left[\frac{1}{(1+(k+1)!)} x^{k+1} \Big/ \frac{1}{(1+k!)} x^k \right] = 0,$$

the Ratio Test for Series shows that the domain of convergence of this series is the set of all real numbers. ■

The principal objective of this section is to show that if the function $f : (-r, r) \to \mathbb{R}$ is defined by the power series expansion

$$f(x) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} c_k x^k \right] = \sum_{k=0}^{\infty} c_k x^k \qquad \text{for } |x| < r,$$

then $f : (-r, r) \to \mathbb{R}$ is differentiable, and moreover,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\lim_{n\to\infty} \sum_{k=0}^{n} c_k x^k \right] \\ &= \lim_{n\to\infty} \left[\frac{d}{dx} \left[\sum_{k=0}^{n} c_k x^k \right] \right] \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} k c_k x^{k-1} \right] \\ &= \sum_{k=1}^{\infty} k c_k x^{k-1} \qquad \text{if } |x| < r. \end{aligned}$$

The above computation is known as *term-by-term differentiation* of a series expansion. It is not at all obvious that the passage from the first line to the second is justified, that is, that the derivative of the sum is the sum of the derivatives when the sum is infinite. Once this computation is justified, it follows easily that the function

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$f:(-r,r)\to\mathbb{R}$ has derivatives of all orders and that term-by-term differentiation of all orders is valid.

Uniform Convergence of Power Series

For sequences of functions, we have distinguished between pointwise convergence and uniform convergence. It is necessary to make a similar distinction for the convergence of power series.

Definition Let A be a subset of the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$. Define the function $f:A\to\mathbb{R}$ by

$$f(x)=\sum_{k=0}^{\infty} c_k x^k \qquad \text{for all } x \text{ in } A,$$

and for each natural number n, define the function $s_n:A\to\mathbb{R}$ by

$$s_n(x)=\sum_{k=0}^{n} c_k x^k \qquad \text{for all } x \text{ in } A.$$

The series $\sum_{k=0}^{\infty} c_k x^k$ is said to be *convergent uniformly* on the set A provided that the sequence of partial sums $\{s_n\}$ converges uniformly on A to the function f.

The following lemma will be the most important ingredient in the proof that term-by-term differentiation of a power series is justified.

Lemma 9.39 Let A be a subset of the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$. Assume the following: There is a positive number M and a number α with $0\le\alpha<1$ such that

$$|c_k x^k|\le M\alpha^k \qquad \text{for all indices } k \text{ and all } x \text{ in } A. \tag{9.30}$$

Then the power series $\sum_{k=0}^{\infty} c_k x^k$ is uniformly convergent on A.

Proof

Define $\{s_n:A\to\mathbb{R}\}$ to be the sequence of partial sums for the series $\sum_{k=0}^{\infty} c_k x^k$ on the set A. By the definition of uniform convergence on a set, we must show that the sequence of functions $\{s_n:A\to\mathbb{R}\}$ is uniformly convergent. However, the Weierstrass Uniform Convergence Criterion asserts that a sequence of functions converges uniformly if and only if the sequence is uniformly Cauchy. Thus, it suffices to show that the sequence of partial sums is uniformly Cauchy on A.

Let $\epsilon>0$. We need to find an index N such that for each index $n\ge N$ and every natural number k,

$$|s_{n+k}(x)-s_n(x)|<\epsilon \qquad \text{for all } x \text{ in } A. \tag{9.31}$$

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However, successively using the Triangle Inequality, assumption (9.30), and the Geometric Sum Formula, we see that for any pair of natural numbers k and n,

$$\begin{aligned}
|s_{n+k}(x) - s_n(x)| &= |c_{n+k}x^{n+k} + \cdots + c_{n+1}x^{n+1}| \\
&\leq |c_{n+k}x^{n+k}| + \cdots + |c_{n+1}x^{n+1}| \\
&\leq M[\alpha^{n+k} + \cdots + \alpha^{n+1}] \\
&= M\alpha^{n+1}[1 + \cdots + \alpha^{k-1}] \\
&= M\alpha^{n+1}\left[\frac{1-\alpha^k}{1-\alpha}\right] \\
&\leq M\alpha^{n+1}\left[\frac{1}{1-\alpha}\right] \qquad \text{for all } x \text{ in } A. \qquad (9.32)
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha^n = 0$, we can choose a natural number N such that

$$\alpha^n < \frac{\epsilon}{M}(1-\alpha) \qquad \text{for all indices } n \geq N.$$

With this choice of index N, it follows from the inequality (9.32) that the required inequality (9.31) holds. ■

In order to use the above lemma to justify term-by-term differentiation of a power series, it is useful to observe (Exercise 9) that if the numbers α and β have the property $0 \leq \alpha < \beta$, then there is a number c such that

$$k\alpha^k \leq c\beta^k \qquad \text{for every nonnegative integer } k. \qquad (9.33)$$

Proposition 9.40 Suppose that the nonzero number x_0 is in the domain of convergence of the power series $\sum_{k=0}^\infty c_k x^k$. Let r be any positive number less than $|x_0|$. Then the interval $[-r, r]$ is in the domain of convergence of the power series $\sum_{k=0}^\infty c_k x^k$ and also in the domain of convergence of the derived power series $\sum_{k=1}^\infty kc_k x^{k-1}$. Moreover, each of the power series

$$\sum_{k=0}^\infty c_k x^k \qquad \text{and} \qquad \sum_{k=1}^\infty kc_k x^{k-1}$$

converges uniformly on the interval $[-r, r]$.

Proof

First, we show that the power series $\sum_{k=0}^\infty c_k x^k$ converges uniformly on the interval $[-r, r]$. Since the series $\sum_{k=0}^\infty c_k x_0^k$ converges, the terms of this series converge to 0, so, in particular, the terms are bounded. Thus, we can choose a number M such that

$$|c_k x_0^k| \leq M \qquad \text{for every index } k.$$

Define $\alpha \equiv r/|x_0|$ and observe that $0 \leq \alpha < 1$. Moreover, writing x as $x = (x/x_0)x_0$, we see that

$$|c_k x^k| \leq M\alpha^k \qquad \text{for every index } k \text{ and } x \text{ in } [-r, r]. \qquad (9.34)$$

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The uniform convergence of the series $\sum_{k=0}^{\infty} c_k x^k$ on the interval $[-r, r]$ now follows from Lemma 9.39.

Now we consider the derived series $\sum_{k=1}^{\infty} kc_k x^{k-1}$. Observe that the coefficient of x^k in the series $\sum_{k=1}^{\infty} kc_k x^{k-1}$ is $(k+1)c_{k+1}$. Thus, by Lemma 9.39, in order to establish the uniform convergence of the derived series on the interval $[-r, r]$, it suffices to find a number α' with $0 \leq \alpha' < 1$ and a positive number M' such that

$$|(k+1)c_{k+1}x^k| \leq M'[\alpha']^k \qquad \text{for every index } k \text{ and } x \text{ in } [-r, r]. \tag{9.35}$$

In order to do so, observe that since $|x| \leq r$ if we use the estimate (9.34) at $x = r$, we have the following estimate for every index k and x in $[-r, r]$:

$$\begin{aligned} |(k+1)c_{k+1}x^k| &\leq (k+1)|c_{k+1}|r^k \\ &= \frac{(k+1)}{r}|c_{k+1}|r^{k+1} \\ &\leq \frac{(k+1)}{r}M\alpha^{k+1} \\ &\leq \left[\frac{(k+1)}{\alpha k r}M\right]k\alpha^k \\ &\leq \left[\frac{2M}{\alpha r}\right]k\alpha^k. \end{aligned} \tag{9.36}$$

However, since $0 \leq \alpha < 1$, if we set $\alpha' = [\alpha+1]/2$; then $0 \leq \alpha < \alpha' < 1$. By (9.4), we can choose $c > 0$ such that

$$k\alpha^k \leq c[\alpha']^k \qquad \text{for every index } k. \tag{9.37}$$

From this inequality and inequality (9.36) we see that inequality (9.35) holds for this choice of α' if we set

$$M' \equiv c \cdot \left[\frac{2M}{\alpha r}\right].$$

■

Let D be the domain of convergence of the power series expansion $\sum_{k=0}^{\infty} c_k x^k$. From Proposition 9.40, it follows that $D = \mathbb{R}$ if D is unbounded. If D is bounded, we define

$$r = \sup D,$$

and it follows that

$$(-r, r) \subseteq D \subseteq [r, r].$$

Because of this, we call the number r the *radius of convergence* of the series $\sum_{k=0}^{\infty} c_k x^k$.

We leave it as an exercise for the reader to verify that if the sequence $\{|a_n|^{1/n}\}$ converges to α, then $D = \mathbb{R}$ if $\alpha = 0$, and $r = \alpha^{-1}$ if $\alpha > 0$ (Exercise 14).

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Term-by-Term Differentiation of Power Series

Theorem 9.41 Let r be a positive number such that the interval $(-r, r)$ lies in the domain of convergence of the series $\sum_{k=0}^{\infty} c_k x^k$. Define

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{if } |x| < r.$$

Then the function $f: (-r, r) \to \mathbb{R}$ has derivatives of all orders. For each natural number n,

$$\frac{d^n}{dx^n}[f(x)] = \sum_{k=0}^{\infty} \frac{d^n}{dx^n}[c_k x^k] \qquad \text{if } |x| < r,$$

so that, in particular,

$$\frac{f^{(n)}(0)}{n!} = c_n.$$

Proof
It is clear that it will be sufficient to prove that

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}(c_k x^k) \qquad \text{if } |x| < r,$$

and hence

$$f'(0) = c_1.$$

The general result follows by induction since, according to Theorem 9.40, the derived series of any power series that converges on $(-r, r)$ is another power series that also converges on $(-r, r)$.

Choose R to be any positive number less than r. Since the series $\sum_{k=0}^{\infty} c_k x^k$ converges at each point between R and r, according to Theorem 9.40, each of the series

$$\sum_{k=0}^{\infty} c_k x^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} k c_k x^{k-1}$$

converges uniformly on the interval $[-R, R]$.

For each natural number n, define

$$s_n(x) = \sum_{k=0}^{n} c_k x^k \qquad \text{if } |x| < R.$$

Then each of the sequences of functions

$$\{s_n : (-R, R) \to \mathbb{R}\} \qquad \text{and} \qquad \{s_n' : (-R, R) \to \mathbb{R}\}$$

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is uniformly convergent. Theorem 9.34 implies that

$$\lim_{n\to\infty} s_n'(x) = f'(x) \qquad \text{if } |x| < R;$$

that is,

$$\sum_{k=0}^{\infty} kc_k x^{k-1} = f'(x) \qquad \text{if } |x| < R.$$

Since for each point x in the interval $(-r, r)$ we can choose a positive number R less than r, with $|x| < R$, it follows that

$$f'(x) = \sum_{k=0}^{\infty} kc_k x^{k-1} \qquad \text{for all points } x \text{ in the interval } (-r, r). \qquad \blacksquare$$

The above theorem implies that a function defined by a power series expansion on the interval $(-r, r)$ coincides with its Taylor series expansion about 0; this is a uniqueness result for the coefficients of a power series expansion.

The Trigonometric Differential Equation Revisited

$$\begin{cases} F''(x) + F(x) = 0 & \text{for all } x \\ F(0) = 1, \qquad F'(0) = 0. \end{cases} \tag{9.38}$$

Recall that in Section 5.2 we proved that the above Trigonometric Differential Equation had at most one solution and, we *provisionally assumed* that it had a solution. We will now remove this provisional assumption and present a solution that is expressed as a power series.

Theorem 9.42 For each number x, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

converges. Define

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad \text{for all } x. \tag{9.39}$$

Then the function $F : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and satisfies the differential equation (9.38).

Proof

From the Ratio Test for Series, it follows that the domain of convergence of the series $\sum_{k=0}^{\infty}[(-1)^k/(2k)!]x^{2k}$ is the set of all real numbers.Thus, the above function

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$F : \mathbb{R} \to \mathbb{R}$ is properly defined. Moreover, by Theorem 9.41, it follows that for all x,

$$F'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1}$$

and

$$F''(x) = \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k-2)!} x^{2k-2} = -F(x).$$

Thus, the power series expansion (9.39) defines a function $F : \mathbb{R} \to \mathbb{R}$ that satisfies the differential equation (9.38). ■

EXERCISES FOR SECTION 9.5

1. Determine the domains of convergence of each of the following power series:

a. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k5^k}$ **b.** $\displaystyle\sum_{k=1}^{\infty} k!x^k$ **c.** $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k+1)!}$

2. Prove that

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \qquad \text{if } |x| < 1.$$

3. Prove that

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} k x^{k-1} \qquad \text{if } |x| < 1.$$

4. Prove that

$$\frac{1}{(1+x^2)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} k x^{2k-2} \qquad \text{if } |x| < 1.$$

5. Prove that

$$x = \sum_{k=0}^{\infty} \left(1 - \frac{1}{x}\right)^k \qquad \text{if } |1 - x| < |x|.$$

6. Define $f(x) = 1/(1-x)^3$ if $|x| < 1$. Find a power series expansion for the function $f : (-1, 1) \to \mathbb{R}$.

7. Suppose that the domain of convergence of the power series $\sum_{k=0}^{\infty} c_k x^k$ contains the interval $(-r, r)$. Define

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{if } |x| < r.$$

Let the interval $[a, b]$ be contained in the interval $(-r, r)$. Prove that

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} \frac{c_k}{k+1} \left[b^{k+1} - a^{k+1}\right].$$

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8. Obtain a series expansion for the integral

$$\int_0^{1/2} 1/(1+x^4)\,dx$$

and justify your calculation.

9. Prove the inequality (9.33). (*Hint:* For numbers α and β, with $0 \le \alpha \le \beta$, define $r \equiv \alpha/\beta$ and observe that since $0 \le r < 1$ it follows from the Ratio Lemma for Sequences that $\lim_{n\to\infty} kr^k = 0$. In particular, we can choose $c > 0$ such that $kr^k \le c$ for every index k.)

10. For each number x, define

$$h(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

Prove that for any pair of numbers α and β, the function

$$f = \alpha h + \beta g : \mathbb{R} \to \mathbb{R}$$

is a solution of the differential equation

$$\begin{cases} f''(x) - f(x) = 0, & x \text{ in } \mathbb{R} \\ f(0) = \alpha \quad \text{and} & f'(0) = \beta. \end{cases}$$

11. Prove that if $0 \le \alpha < 1$, then $\lim_{n\to\infty} n\alpha^n = 0$.

12. Rewrite the Geometric Sum Formula as follows: For each natural number n,

$$\frac{1}{1-x} - (1 + x + \cdots + x^n) = \frac{x^{n+1}}{1-x} \qquad \text{if } x \ne 1.$$

Differentiate this identity to obtain

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) - [1 + 2x + \cdots + nx^{n-1}] = \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}.$$

Now use this identity and Exercise 11 to directly justify term-by-term differentiation of the Geometric Series.

13. Prove that the series

$$\sum_{k=0}^{\infty} \frac{1}{1+|x|^k}$$

converges if and only if $|x| > 1$. In particular, show that the series converges at $x = 2$ but not at $x = 1$. Does this contradict Theorem 9.40? (*Hint:* This is not a power series.)

14. Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = \alpha$.

a. If $\alpha > 0$, show that $\sum_{n=0}^{\infty} a_n x^n$ converges if $|x| < 1/\alpha$ and diverges if $|x| > 1/\alpha$.

b. If $\alpha = 0$, show that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \ne 0$.

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9.6 A CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTION

Weierstrass presented the first example of a *continuous* function $f:\mathbb{R}\to\mathbb{R}$ that has the remarkable property that there is no point at which it is differentiable: Such a function is said to be *nowhere differentiable*. We will analyze such an example, where f is defined by an expansion

$$f(x)=\sum_{k=0}^{\infty}h_k(x) \qquad \text{for all } x$$

and the function f inherits all the nondifferentiability possessed by the individual h_k's.

We first prove a preliminary proposition regarding the construction of continuous functions as series of continuous functions.

Proposition 9.43 Suppose that $\sum_{k=0}^{\infty}c_k$ is a convergent sequence of nonnegative numbers. For each nonnegative integer k, let $h_k:\mathbb{R}\to\mathbb{R}$ be a continuous function such that

$$|h_k(x)|\le c_k \qquad \text{for all } x. \tag{9.40}$$

Define

$$f(x)=\lim_{n\to\infty}\left[\sum_{k=0}^{n}h_k(x)\right]=\sum_{k=0}^{\infty}h_k(x) \qquad \text{for all } x. \tag{9.41}$$

Then the function $f:\mathbb{R}\to\mathbb{R}$ is continuous.

Proof

The proof rests on the Cauchy Convergence Criterion for the convergence of sequences of numbers and the Weierstrass Uniform Convergence Criterion for the uniform convergence of sequences of functions.

For each number x and each natural number n, define

$$f_n(x)=\sum_{k=0}^{n}h_k(x).$$

Since each function h_k is continuous, each function f_n is also continuous. We will prove that the sequence of functions $\{f_n\}$ is uniformly Cauchy on $\mathbb{R}$. Once this is proven, it follows from the Weierstrass Uniform Convergence Criterion that $\{f_n\}$ converges uniformly on $\mathbb{R}$. Then, by Theorem 9.31, we can conclude that the limit function f, being the uniform limit of a sequence of continuous functions, is continuous.

By assumption (9.40) and the Triangle Inequality, for each index n, natural number k, and any number x,

$$\begin{aligned}|f_{n+k}(x)-f_n(x)|&=|h_{n+k}(x)+\cdots+h_{n+1}(x)|\\&\le|h_{n+k}(x)|+\cdots+|h_{n+1}(x)|\\&\le c_{n+k}+\cdots+c_{n+1}.\end{aligned} \tag{9.42}$$

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The Cauchy Convergence Criterion for sequences of numbers, applied to the sequence of partial sums of the series $\sum_{k=0}^{\infty} c_k$, asserts that the sequence of partial sums is a Cauchy sequence. Thus, from the estimate (9.42) we conclude that the sequence of functions $\{f_n\}$ is uniformly Cauchy on $\mathbb{R}$. ■

We will now make a particular choice of the h_k's so that the function f defined by (9.41) fails, at each point, to be differentiable.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic, with period p, provided that

$$f(x + p) = f(x) \qquad \text{for all } x$$

Observe that if a function has period p and k is any integer, then it also has period kp.

It is convenient to introduce the following descriptive terminology: For a positive number ℓ, we define the *tent function of base length* 2ℓ to be the periodic function of period 2ℓ, $h : \mathbb{R} \to \mathbb{R}$ whose values on the interval $[-\ell, \ell]$ are defined by

$$h(x) = |x| \qquad \text{for } -\ell \leq x \leq \ell.$$

For an integer m, we call the interval $[m\ell, (m+1)\ell]$ an *interval of monotonicity* for this tent function.

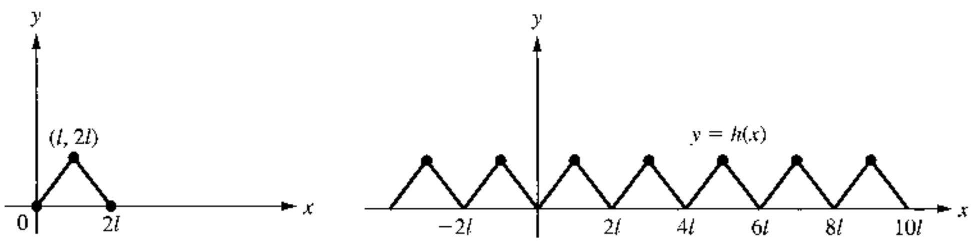

FIGURE 9.6 The tent function of base length 2ℓ.

Lemma 9.44 For $\ell > 0$, let $h : \mathbb{R} \to \mathbb{R}$ be the tent function of base length 2ℓ. Let x_0 be any number. Then either the interval $[x_0, x_0 + \ell/2]$ or the interval $[x_0 - \ell/2, x_0]$ is contained in an interval of monotonicity for the function h.

Proof

Recall Theorem 1.8, which asserts that for any number c there is a unique integer belonging to the interval $[c, c+1)$. We apply this theorem, with $c = [x_0/\ell] - 1$, to choose an integer m such that

$$[x_0/\ell] - 1 \leq m < x_0/\ell.$$

The left-hand side of this inequality yields $x_0 \leq (m+1)\ell$. This, with the right-hand side of the inequality, yields

$$m\ell < x_0 \leq (m+1)\ell.$$

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Consider the midpoint z of the interval $[m\ell,\ (m+1)\ell]$. Since x_0 belongs to the interval $(m\ell,\ (m+1)\ell]$, either x_0 belongs to the left-hand interval $(m\ell,\ z]$ or it belongs to the right-hand interval $(z,\ (m+1)\ell]$. In the first case, the interval $[x_0,\ x_0+\ell/2]$ is contained in the interval $[m\ell,\ (m+1)\ell]$, while in the second case, $[x_0-\ell/2,\ x_0]$ is contained in the interval $[m\ell,\ (m+1)\ell]$. Of course, $[m\ell,\ (m+1)\ell]$ is an interval of monotonicity for the tent function h. ■

We will need the following two observations regarding $h:\mathbb{R}\to\mathbb{R}$, the tent function of base length 2ℓ: If u and v belong to an interval of monotonicity for the function h, then

$$\frac{h(u)-h(v)}{u-v}=\pm 1. \tag{9.43}$$

Since h has period 2ℓ, any integer multiple of 2ℓ is also a period for h; that is, for any integer j and any number u,

$$h(u+j[2\ell])=h(u). \tag{9.44}$$

Theorem 9.45 For each nonnegative integer k, let $h_k:\mathbb{R}\to\mathbb{R}$ be the tent function of base length $2\ell_k$, where $\ell_k=(1/4)^k$. Define the function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\sum_{k=1}^{\infty}h_k(x) \qquad \text{for all } x.$$

Then

i. the function f is continuous, but
ii. there is no point at which the function f is differentiable.

Proof
By the definition of tent function, for each nonnegative integer k and any number x,

$$|h_k(x)|\le \ell_k=(1/4)^k.$$

Therefore, since the Geometric Series $\sum_{k=0}^{\infty}(1/4)^k$ converges, it follows from Proposition 9.43 that $f:\mathbb{R}\to\mathbb{R}$ is continuous.

Let x_0 be any number. We will show that f is not differentiable at x_0 by choosing a sequence of numbers $\{x_n\}$, with each $x_n\neq x_0$, that converges to x_0 but for which the limit

$$\lim_{n\to\infty}\frac{f(x_n)-f(x_0)}{x_n-x_0}$$

does not exist.

Let n be a natural number. We apply Lemma 9.44, with $\ell=\ell_n$. Thus, either the interval $[x_0,\ x_0+\ell_n/2]$ or the interval $[x_0-\ell_n/2,\ x_0]$ is contained in an interval

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of monotonicity for the function h_n. In the first case define $x_n = x_0 - \ell_n/2$, and in the second case define $x_n = x_0 + \ell_n/2$. Hence, since the points x_0 and x_n belong to an interval of monotonicity for the function h_n, by (9.43),

$$\frac{h_n(x_n) - h_n(x_0)}{x_n - x_0} = \pm 1.$$

For $k > n$, the function $h_k : \mathbb{R} \to \mathbb{R}$ has period $2\ell_k$. Therefore, since

$$\frac{\ell_n}{2} = j[2\ell_k], \qquad \text{where } j \equiv \frac{\ell_n}{4\ell_k} = 4^{k-n-1} \text{ is a natural number,}$$

it follows from (9.44) that

$$h_k(x_0) - h_k(x_n) = h_k(x_0) - h_k(x_0 \pm j[2\ell_k]) = 0.$$

Consequently,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \sum_{k=0}^{\infty}\left[\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}\right] = \sum_{k=0}^{n}\left[\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0}\right]. \tag{9.45}$$

On the other hand, for an integer k, $0 \le k < n$, since the ratio of base lengths, ℓ_k/ℓ_n, is a natural number, any monotonicity interval for h_n is contained in a monotonicity interval for h_k. Thus, again using (9.43),

$$\frac{h_k(x_n) - h_k(x_0)}{x_n - x_0} = \pm 1 \qquad \text{for } 0 \le k \le n.$$

We conclude that the right-hand side of (9.45) is the sum of $n + 1$ numbers each of which equals $+1$ or -1. Thus,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \begin{cases} \text{an odd integer} & \text{if } n \text{ is even} \\ \text{an even integer} & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence, the limit

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

does not exist. Thus, since the sequence $\{x_n\}$ converges to x_0, with each $x_n \neq x_0$, the function f is not differentiable at the point x_0. ■

EXERCISES FOR SECTION 9.6

1. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ has period p. Show that for each integer k, the function g also has period kp.

2. Suppose that $\{t_n\}$ is a sequence such that t_k is an odd integer if the index k is even, and an even integer if the index k is odd. Show that $\{t_n\}$ does not converge.

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3. Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be tent functions of base lengths 2ℓ and $2\ell'$, respectively, where ℓ' is an integer multiple of ℓ. Show that each monotonicity interval for g is contained in a monotonicity interval for h.
4. Find a function $g : \mathbb{R} \to \mathbb{R}$ that is continuously differentiable but for which there is no point at which it has a second derivative.

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CHAPTER

10

THE EUCLIDEAN SPACE $\mathbb{R}^n$

10.1 THE LINEAR STRUCTURE OF $\mathbb{R}^n$ AND THE SCALAR PRODUCT

For each natural number n, we denote by $\mathbb{R}^n$ the set of n-tuples of real numbers

$$\mathbf{u} = (u_1, \ldots, u_n),$$

where u_i is a real number for each index i with $1 \leq i \leq n$. We call u_i the ith *component of* $\mathbf{u}$ and refer to the members of $\mathbb{R}^n$ as *points in* $\mathbb{R}^n$. Moreover, we shall consider two points $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ to be *equal* provided that they have the same components; that is,

$$\mathbf{u} = \mathbf{v} \qquad \text{if and only if } u_i = v_i \text{ for each index } i \text{ with } 1 \leq i \leq n.$$

For any two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we define the *sum* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, by the formula

$$\mathbf{u} + \mathbf{v} \equiv (u_1 + v_1, \ldots, u_n + v_n).$$

Also, for each real number α, we define the point $\alpha\mathbf{u}$, called the *scalar multiple* of the point $\mathbf{u}$ by α, by the formula

$$\alpha\mathbf{u} \equiv (\alpha u_1, \ldots, \alpha u_n).$$

The point in $\mathbb{R}^n$ all of whose components are zero is denoted by $\mathbf{0}$. In algebraic contexts, it is called *zero;* in geometric contexts, it is called the *origin*. Finally, for each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we define their *difference*, denoted by $\mathbf{u} - \mathbf{v}$, by the formula

$$\mathbf{u} - \mathbf{v} \equiv \mathbf{u} + (-\mathbf{v}).$$

Of course, $\mathbb{R}^1$ is simply the familiar set of real numbers $\mathbb{R}$, and the definitions of addition of points and of multiplication of points in $\mathbb{R}^1$ by real numbers are exactly the definitions of addition and multiplication of real numbers that we have been using in $\mathbb{R}$. In the Preliminaries, we codified the properties of addition and multiplication in $\mathbb{R}$ as the Field Axioms. Using these Field Axioms and the definition of equality of two points in $\mathbb{R}^n$ as meaning the equality of corresponding components, we obtain the following.

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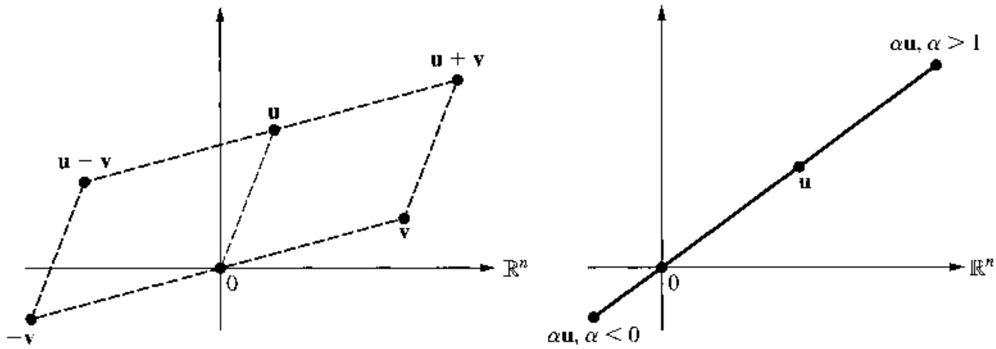

FIGURE 10.1 Addition, subtraction, and scalar multiplication.

Proposition 10.1 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be points in $\mathbb{R}^n$. Then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \mathbf{u}+(\mathbf{v}+\mathbf{w}),\\ \mathbf{u}+\mathbf{0} &= \mathbf{u},\\ \mathbf{u}-\mathbf{u} &= \mathbf{0},\\ \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u},\end{aligned}$$

and if α and β are real numbers, then

$$\begin{aligned}\alpha(\mathbf{u}+\mathbf{v}) &= \alpha\mathbf{u}+\alpha\mathbf{v},\\ (\alpha+\beta)(\mathbf{u}) &= \alpha\mathbf{u}+\beta\mathbf{u},\\ (\alpha\beta)\mathbf{u} &= \alpha(\beta\mathbf{u}).\end{aligned}$$

Proof

Each of these equalities follows from the observation that the Field Axioms for $\mathbb{R}$ imply that we have componentwise equality. ■

The Scalar Product

Definition Let $\mathbf{u}$ and $\mathbf{v}$ be points in $\mathbb{R}^n$. The *scalar product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\langle\mathbf{u},\ \mathbf{v}\rangle$, is defined by the formula

$$\langle\mathbf{u},\ \mathbf{v}\rangle \equiv u_1v_1+\cdots+u_nv_n.$$

The scalar product is also denoted by $\mathbf{u}\cdot\mathbf{v}$ and is often called the *dot product* or the *inner product*. The following algebraic properties of the scalar product are extensions of the commutative and distributive properties of the addition and multiplication of real numbers.

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Proposition 10.2 Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be points in $\mathbb{R}^n$. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle, \qquad (\textit{Symmetry})$$

and if α and β are any real numbers, then

$$\langle \alpha\mathbf{u} + \beta\mathbf{w}, \mathbf{v} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle. \qquad (\textit{Linearity})$$

Proof

The commutative property of multiplication of real numbers implies that

$$\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} v_i u_i,$$

which is the first identity. The distributive property of addition and multiplication of the real numbers implies that

$$\sum_{i=1}^{n}(\alpha u_i + \beta w_i)v_i = \alpha \sum_{i=1}^{n} u_i v_i + \beta \sum_{i=1}^{n} w_i v_i,$$

which is the second identity. ∎

The Norm, and the Distance between Two Points

Definition

i. Let $\mathbf{w}$ be a point in $\mathbb{R}^n$. Then the *norm* of $\mathbf{w}$, denoted by $\|\mathbf{w}\|$, is defined by the formula

$$\|\mathbf{w}\| \equiv \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} = \sqrt{\sum_{i=1}^{n} w_i^2}.$$

ii. Let $\mathbf{u}$ and $\mathbf{v}$ be points in $\mathbb{R}^n$. Then the *distance* between the points $\mathbf{u}$ and $\mathbf{v}$, denoted by $\operatorname{dist}(\mathbf{u}, \mathbf{v})$, is defined by the formula

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) \equiv \|\mathbf{u} - \mathbf{v}\|.$$

It follows that the norm of the point $\mathbf{w}$ is the distance from the origin to $\mathbf{w}$. Moreover, the distance between the points $\mathbf{u}$ and $\mathbf{v}$ can be expressed in terms of the scalar product by the formula

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = (\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle)^{1/2}.$$

When we consider the set $\mathbb{R}^n$ together with the concepts of addition, scalar multiplication, and distance between points that we have introduced so far, it is customary to refer to $\mathbb{R}^n$ as *Euclidean n-space*. Moreover, for a point $\mathbf{u}$ in $\mathbb{R}^n$, it is often convenient to identify with $\mathbf{u}$ all the *segments* from a point $\mathbf{p}$ in $\mathbb{R}^n$ to a point $\mathbf{p} + \mathbf{u}$ in $\mathbb{R}^n$ and to refer to this collection of segments as the *vector* $\mathbf{u}$. By *the vector* $\mathbf{u}$ *based at the point* $\mathbf{p}$ we mean the segment from the point $\mathbf{p}$ to the point $\mathbf{p} + \mathbf{u}$. Addition, scalar multiplication, and the scalar product extend to vectors. The norm of the point $\mathbf{u}$ is called the *length* of

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the vector **u**; it is equal to the distance between the endpoints of any segment associated with the vector **u**.

Of course, in the dimensions $n = 1$, $n = 2$, and $n = 3$, the reader is already quite familiar with the geometric meaning of addition and scalar multiplication. The norm and the scalar product also have a geometric significance. In the case where $n = 1$, the scalar product is just the usual multiplication of real numbers, and the norm of a number is simply its absolute value. In the case where $n = 2$, if $\mathbf{u} = (u_1, u_2)$ is a point in the plane $\mathbb{R}^2$, then the Pythagorean Theorem asserts that the norm of **u**, given by

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2},$$

is the distance from the origin to the point **u**. It also is the length of any of the segments associated with the vector **u**.

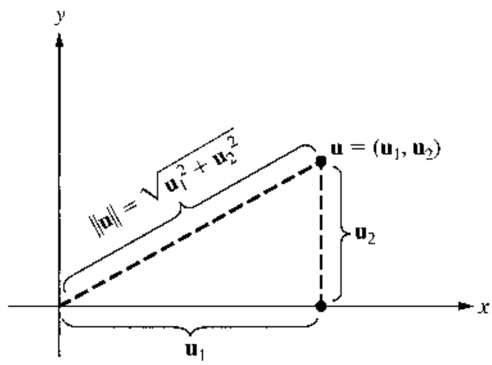

FIGURE 10.2 The Pythagorean Theorem.

The geometric significance of the scalar product of two points (or two vectors) in the plane $\mathbb{R}^2$ is described by the following proposition.

Proposition 10.3 Let **u** and **v** be nonzero vectors in the plane $\mathbb{R}^2$. Then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta, \tag{10.1}$$

where θ is the radian measure of the angle between the vector **u** based at the origin and the vector **v** based at the origin.

Proof

First, recall that if **u** is any nonzero point in the plane and θ is the radian measure of the angle with vertex at **0** formed by the points **u**, **0**, and $(1, 0)$, then the components of **u** can be expressed, in terms of the norm of **u** and the angle θ, by the formula

$$\mathbf{u} = (\|\mathbf{u}\| \cos\theta, \|\mathbf{u}\| \sin\theta).$$

Let θ_1 be the radian measure of the angle with vertex at **0** formed by the points **u**, **0**, and $(1, 0)$; let θ_2 be the radian measure of the angle with vertex at **0** formed by the

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points $\mathbf{v}$, $\mathbf{0}$, and $(1, 0)$. We can suppose that $\theta_2 > \theta_1$. It follows from the definition of the scalar product and the cosine addition formula that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= u_1v_1 + u_2v_2\\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)\\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta_2 - \theta_1)\\ &= \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta,\end{aligned}$$

where θ is the angle with vertex at $\mathbf{0}$ determined by $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$. ∎

From formula (10.1), it follows that for two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane $\mathbb{R}^2$, the scalar product of $\mathbf{u}$ and $\mathbf{v}$ is zero if and only if vector $\mathbf{u}$ based at the origin is orthogonal (perpendicular) to the vector $\mathbf{v}$ based at the origin. This leads us to make the following definition of orthogonality of two vectors in $\mathbb{R}^n$.

Orthogonality and the Orthogonal Projection

Definition Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be *orthogonal* provided that $\langle \mathbf{u}, \mathbf{v}\rangle = 0$.

Lemma 10.4 For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the following assertions are equivalent:

i. The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
ii. The Pythagorean Identity holds; that is,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof
By the definition of the norm,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle,$$

and we can use the linearity and the symmetry of the scalar product to simplify the right-hand side to obtain the identity

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle.$$

The equivalence of (i) and (ii) follows from this identity. ∎

In the case where the scalar product $\langle \mathbf{u}, \mathbf{v}\rangle \neq 0$, it is useful to obtain an estimate of the size of $\langle \mathbf{u}, \mathbf{v}\rangle$ in terms of the norms of $\mathbf{u}$ and $\mathbf{v}$. Recall that for any real number θ, $|\cos\theta| \leq 1$. Therefore, it follows from formula (10.1) that if $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in the plane $\mathbb{R}^2$, then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|. \tag{10.2}$$

Since we have established formula (10.1) only for vectors in the plane $\mathbb{R}^2$, this argument is insufficient to verify inequality (10.2) for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ when $n > 2$. Nevertheless,

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the inequality (10.2) holds for vectors in $\mathbb{R}^n$ even if $n > 2$. Before proving this, it is convenient to establish the following lemma which, for a vector $\mathbf{v} \neq \mathbf{0}$ and any vector $\mathbf{u}$ expresses $\mathbf{u}$ as $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$, where $\mathbf{w}$ is orthogonal to $\mathbf{v}$. The vector $\lambda\mathbf{v}$ is called the *orthogonal projection* of $\mathbf{u}$ along $\mathbf{v}$.

Lemma 10.5 For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, with $\mathbf{v} \neq \mathbf{0}$, define $\lambda = \langle \mathbf{u}, \mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle$. Then the vector $\mathbf{w} \equiv \mathbf{u} - \lambda\mathbf{v}$ is orthogonal to the vector $\mathbf{v}$ and

$$\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}.$$

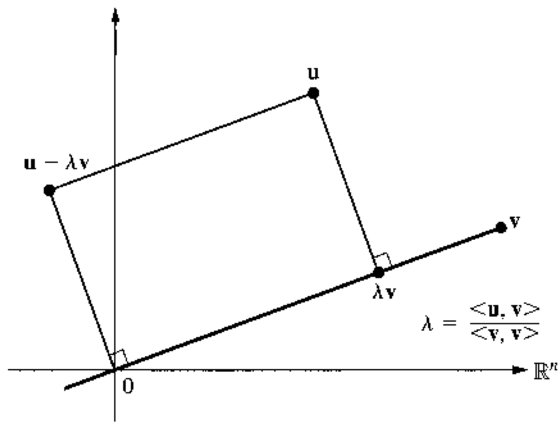

FIGURE 10.3 $\lambda\mathbf{v}$ is the orthogonal projection of $\mathbf{u}$ along $\mathbf{v}$.

Proof

By the linearity of the scalar product and the definition of λ,

$$\langle \mathbf{u} - \lambda\mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \lambda\langle \mathbf{v}, \mathbf{v}\rangle = 0.$$

■

The Cauchy–Schwarz Inequality

Theorem 10.6 The Cauchy–Schwarz Inequality For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|. \tag{10.3}$$

Proof

If the vector $\mathbf{v} = \mathbf{0}$, then certainly the Cauchy–Schwarz Inequality holds since each side of the inequality is 0, so suppose that $\mathbf{v} \neq \mathbf{0}$. Define $\lambda = \langle \mathbf{u}, \mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle$. By Lemma 10.5, the vector $\mathbf{u} - \lambda\mathbf{v}$ is orthogonal to the vector $\mathbf{v}$ and hence is also

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orthogonal to $\lambda\mathbf{v}$. Thus, since the scalar product of a vector and itself is always nonnegative, the linearity of the scalar product yields

$$0 \le \langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{u} - \lambda\mathbf{v}\rangle = \langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{u}\rangle = \|\mathbf{u}\|^2 - \frac{\langle \mathbf{u},\ \mathbf{v}\rangle^2}{\|\mathbf{v}\|^2}.$$

Multiply this inequality by $\|\mathbf{v}\|^2$ to obtain $\langle \mathbf{u},\ \mathbf{v}\rangle^2 \le \|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2$, which is equivalent to the Cauchy–Schwarz Inequality. ∎

The Triangle Inequality

In the study of functions of several variables, it is often necessary to estimate the norms of sums and differences of vectors and also to estimate the distance between points. In order to do so, we now extend the most useful of the inequalities that we have already established for real numbers, namely, the Triangle Inequality. Recall that in Section 1.3 we proved that for any real numbers a and b, we have the following upper bound for $|a+b|$:

$$|a + b| \le |a| + |b|.$$

The length of a vector $\mathbf{u}$ in $\mathbb{R}^n$ extends the concept of the absolute value of a real number, so the following inequality is a direct extension of the Triangle Inequality for pairs of numbers.

Theorem 10.7 The Triangle Inequality For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{10.4}$$

Proof
If we square both sides of inequality (10.4), it is clear that this inequality holds if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 \le (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \tag{10.5}$$

But

$$\|\mathbf{u} + \mathbf{v}\|^2 = \langle \mathbf{u} + \mathbf{v},\ \mathbf{u} + \mathbf{v}\rangle = \langle \mathbf{u},\ \mathbf{u}\rangle + 2\langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{v},\ \mathbf{v}\rangle,$$

so (10.5) can be rewritten as

$$\langle \mathbf{u},\ \mathbf{u}\rangle + 2\langle \mathbf{u},\ \mathbf{v}\rangle + \langle \mathbf{v},\ \mathbf{v}\rangle \le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2. \tag{10.6}$$

However, the Cauchy–Schwarz Inequality asserts that $|\langle \mathbf{u},\ \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$, so certainly $\langle \mathbf{u},\ \mathbf{v}\rangle \le \|\mathbf{u}\|\,\|\mathbf{v}\|$. This is what is needed to verify inequality (10.6). ∎

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EXERCISES FOR SECTION 10.1

1. Consider the two points $\mathbf{u} = (1, 3, -2)$ and $\mathbf{v} = (2, 2, 4)$ in $\mathbb{R}^3$. Find the norm of $\mathbf{u}$ and the norm of $\mathbf{v}$ and show that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. Show that

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

2. Find the maximum value of

$$\frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}}$$

as (x, y, z) varies among nonzero points in $\mathbb{R}^3$.

3. For a point $\mathbf{u}$ in $\mathbb{R}^n$, show that
a. $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = 0$ and that
b. for any number α, $\|\alpha\mathbf{u}\| = |\alpha|\,\|\mathbf{u}\|$.

4. For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, verify the identity

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle.$$

Show that when $n = 2$, this identity is equivalent to the Law of Cosines from trigonometry.

5. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove that

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2}{4}.$$

This identity is called the Polarization Identity.

6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Prove that if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \alpha\mathbf{u}$ for some number α, then the Cauchy–Schwarz Inequality becomes an equality. Then prove the converse: If $|\langle \mathbf{u}, \mathbf{v}\rangle| = \|\mathbf{u}\|\,\|\mathbf{v}\|$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \alpha\mathbf{u}$ for some number α.

7. For a natural number n and real numbers $a_1, \ldots, a_n$, verify that

$$|a_1 + \cdots + a_n| \le \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2}.$$

8. Let $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ be nonzero points in the plane $\mathbb{R}^2$ and let θ be the radian measure of the angle with vertex at $\mathbf{0}$ formed by $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$.
a. Prove that

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\langle \mathbf{u}, \mathbf{v}\rangle)^2 = (\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta)^2.$$

b. Express the left-hand side of the above equation in components to obtain

$$|ad - bc| = |\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta|.$$

c. Use (b) to verify that $|ad - bc|/2$ is the area of the triangle with vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ and that, as a consequence, $|ad - bc|$ is the area of the parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{u}+\mathbf{v}$, and $\mathbf{v}$.

9. Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and suppose that $\|\mathbf{u}\| < 1$. Show that if $\mathbf{v}$ is in $\mathbb{R}^n$ and $\|\mathbf{v}-\mathbf{u}\| < 1 - \|\mathbf{u}\|$, then $\|\mathbf{v}\| < 1$.

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10. Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Suppose that the points $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are at a distance less than r from the point $\mathbf{u}$. Prove that if $0 \leq t \leq 1$, then the point $t\mathbf{v} + (1-t)\mathbf{w}$ is also at a distance less than r from $\mathbf{u}$.

11. For points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, define the function $p\colon \mathbb{R} \to \mathbb{R}$ by $p(t) = \|\mathbf{u} + t\mathbf{v}\|^2$ for t in $\mathbb{R}$. Show that $p(t)$ is a quadratic polynomial that attains only nonnegative values. Use this to show that the discriminant is nonpositive and thus provide another proof of the Cauchy–Schwarz Inequality.

12. The points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ are said to be an *orthonormal set* if $\|\mathbf{u}_i\| = 1$ for $1 \leq i \leq k$ and $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ if $1 \leq i \leq k$, $1 \leq j \leq k$, and $i \neq j$. Suppose that the points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$ are an orthonormal set. For $\mathbf{u} = \alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k$, show that

$$\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{k} \alpha_i^2}.$$

13. Given two continuous functions $f\colon [0, 1] \to \mathbb{R}$ and $g\colon [0, 1] \to \mathbf{R}$, we define the scalar product of f and g, denoted by $\langle f, g \rangle$, by the formula

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

a. Verify that this scalar product has the properties of the scalar product in $\mathbb{R}^n$ listed in Proposition 10.2.

b. Follow the proof of the Cauchy–Schwarz Inequality for points in $\mathbb{R}^n$ to prove that

$$\left| \int_0^1 f(x)g(x)\, dx \right| \leq \sqrt{\int_0^1 [f(x)]^2\, dx}\, \sqrt{\int_0^1 [g(x)]^2\, dx}.$$

10.2 CONVERGENCE OF SEQUENCES IN $\mathbb{R}^n$

Recall that in Chapter 2 we studied the concept of convergence of sequences of real numbers. A sequence of real numbers $\{x_k\}$ is defined as *converging* to the real number x provided that *for each positive number ϵ there is an index K such that*

$$|x - x_k| < \epsilon \qquad \text{for all indices } k \geq K.$$

We denote the convergence of the sequence $\{x_k\}$ to x by writing

$$\lim_{k \to \infty} x_k = x,$$

and we call x the *limit* of the sequence $\{x_k\}$. It is immediately clear that

$$\lim_{k \to \infty} x_k = x \qquad \text{if and only if } \lim_{k \to \infty} |x - x_k| = 0. \tag{10.7}$$

By a *sequence of points in* $\mathbb{R}^n$ we mean a function from the set of natural numbers into $\mathbb{R}^n$. It is customary to denote such a sequence by a symbol such as $\{\mathbf{u}_k\}$, indicating that for each index k, the functional value of k is $\mathbf{u}_k$. If A is a subset of $\mathbb{R}^n$, by a *sequence*

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in A we mean a sequence $\{\mathbf{u}_k\}$ of points in $\mathbb{R}^n$ having the property that $\mathbf{u}_k$ belongs to the set A for each index k. The aim of this section is to extend the concept of convergence of sequences of real numbers to that of convergence of sequences of points in $\mathbb{R}^n$ and to establish various properties of such sequences.

In Section 10.1, we defined the distance $\text{dist}(\mathbf{u}, \mathbf{v})$ between two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ by the formula

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

In the case where $n = 1$ and u and v are real numbers, the distance formula becomes

$$\text{dist}(u, v) = |u - v|,$$

so that in view of criterion (10.7) it is reasonable to extend the concept of convergence as follows.

Definition Let $\{\mathbf{u}_k\}$ be a sequence of points in $\mathbb{R}^n$ and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then the sequence $\{\mathbf{u}_k\}$ is said to *converge* to $\mathbf{u}$ provided that for each positive number ϵ there is an index K such that

$$\text{dist}(\mathbf{u}_k, \mathbf{u}) < \epsilon \qquad \text{for all indices } k \geq K.$$

In conformity with the notation established for sequences of real numbers, if $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$, we denote the convergence of $\{\mathbf{u}_k\}$ to the point $\mathbf{u}$ by writing

$$\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u},$$

and we call $\mathbf{u}$ the *limit* of the sequence $\{\mathbf{u}_k\}$.

The Triangle Inequality, stated as inequality (10.4), asserts that the length of the sum of two vectors is less than or equal to the sum of the lengths of the individual vectors. It is quite useful to rewrite the Triangle Inequality in an equivalent form related to the distance between two points as follows.

Corollary 10.8 Another Version of the Triangle Inequality For any points $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$,

$$\text{dist}(\mathbf{u}, \mathbf{v}) \leq \text{dist}(\mathbf{u}, \mathbf{w}) + \text{dist}(\mathbf{w}, \mathbf{v}). \tag{10.8}$$

Proof

We write $\mathbf{u} - \mathbf{v} = (\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})$, so that by inequality (10.4),

$$\begin{aligned}
\text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| \\
&= \|(\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v})\| \\
&\leq \|\mathbf{u} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{v}\| \\
&= \text{dist}(\mathbf{u}, \mathbf{w}) + \text{dist}(\mathbf{w}, \mathbf{v}).
\end{aligned}$$ ∎

It follows from the definition of convergence that a sequence $\{\mathbf{u}_k\}$ in $\mathbb{R}^n$ converges to the point $\mathbf{u}$ in $\mathbb{R}^n$ if and only if the sequence of real numbers $\{\text{dist}(\mathbf{u}_k, \mathbf{u})\}$ converges

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to 0. Observe that a sequence can have at most one limit since if $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ and also to $\mathbf{u}'$, then according to the Triangle Inequality,

$$0 \leq \text{dist}(\mathbf{u}, \mathbf{u}') \leq \text{dist}(\mathbf{u}, \mathbf{u}_k) + \text{dist}(\mathbf{u}_k, \mathbf{u}') \qquad \text{for every index } k.$$

Consequently,

$$0 \leq \text{dist}(\mathbf{u}, \mathbf{u}') \leq \lim_{k\to\infty} [\text{dist}(\mathbf{u}, \mathbf{u}_k) + \text{dist}(\mathbf{u}_k, \mathbf{u}')] = 0,$$

so that dist$(\mathbf{u}, \mathbf{u}')$ and hence $\mathbf{u} = \mathbf{u}'$.

The Componentwise Convergence Criterion for Convergent Sequences

In order to establish the properties of this extended concept of convergence, rather than directly using the definition, it is convenient to proceed by exploiting what we already know about the convergence of sequences of real numbers. Recall that in Section 2.1 we proved the sum, product, and quotient properties for convergent sequences of real numbers. We also established the Comparison Lemma (Lemma 2.9), which implies that if $\{c_k\}$ is a sequence of nonnegative numbers, $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$, and $\mathbf{u}$ is a point in $\mathbb{R}^n$ such that there is an index K with the property that

$$\text{dist}(\mathbf{u}_k, \mathbf{u}) \leq c_k \qquad \text{for all indices } k \geq K,$$

then

$$\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u} \qquad \text{if } \lim_{k\to\infty} c_k = 0.$$

Definition For each index i with $1 \leq i \leq n$, we define the *ith component projection function* $p_i : \mathbb{R}^n \to \mathbb{R}$ by

$$p_i(\mathbf{u}) \equiv u_i \qquad \text{for } \mathbf{u} = (u_1, \ldots, u_n) \text{ in } \mathbb{R}^n.$$

It follows directly from this definition that

$$\mathbf{u} = (p_1(\mathbf{u}), \ldots, p_n(\mathbf{u})) \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n,$$

so a point in $\mathbb{R}^n$ is completely determined by the values of the component projection functions at that point.

We frequently make use of the following linearity property of the projection function: For each index i with $1 \leq i \leq n$, each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, and each pair of real numbers α and β,

$$p_i(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha p_i(\mathbf{u}) + \beta p_i(\mathbf{v}).$$

This property follows from the very definitions of sum and scalar multiple. We also make use of the inequality

$$|p_i(\mathbf{u})| \leq \|\mathbf{u}\| \qquad \text{for each index } i \text{ with } 1 \leq i \leq n,$$

which follows from the definition of $\|\mathbf{u}\|$ in terms of the components of $\mathbf{u}$.

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Definition A sequence of points $\{\mathbf{u}_k\}$ in $\mathbb{R}^n$ is said to *converge componentwise* to the point $\mathbf{u}$ in $\mathbb{R}^n$ provided that for each index i with $1 \leq i \leq n$,

$$\lim_{k\to\infty} p_i(\mathbf{u}_k) = p_i(\mathbf{u}).$$

Theorem 10.9 The Componentwise Convergence Criterion Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ if and only if $\{\mathbf{u}_k\}$ converges componentwise to $\mathbf{u}$.

Proof
First we suppose that the sequence $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. Fix an index i with $1 \leq i \leq n$. Then

$$0 \leq |p_i(\mathbf{u}_k) - p_i(\mathbf{u})| = |p_i(\mathbf{u}_k - \mathbf{u})| \leq \|\mathbf{u}_k - \mathbf{u}\| \qquad \text{for every index } k.$$

Since, by definition, the sequence of real numbers $\{\|\mathbf{u}_k - \mathbf{u}\|\}$ converges to 0, it follows that

$$0 \leq \lim_{k\to\infty} |p_i(\mathbf{u}_k) - p_i(\mathbf{u})| \leq \lim_{k\to\infty} \|\mathbf{u}_k - \mathbf{u}\| = 0;$$

that is, the sequence $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$. Thus, $\{\mathbf{u}_k\}$ converges componentwise to $\mathbf{u}$.

To prove the converse, suppose that the sequence $\{\mathbf{u}_k\}$ converges componentwise to $\mathbf{u}$. Then, by definition, for each index i with $1 \leq i \leq n$,

$$\lim_{k\to\infty} p_i(\mathbf{u}_k - \mathbf{u}) = 0.$$

But then by the product and addition properties of convergent real sequences, it follows that

$$\lim_{k\to\infty} [(p_1(\mathbf{u}_k - \mathbf{u}))^2 + \cdots + (p_n(\mathbf{u}_k - \mathbf{u}))^2] = 0.$$

This last assertion means precisely that

$$\lim_{k\to\infty} \|\mathbf{u}_k - \mathbf{u}\|^2 = 0,$$

and hence, by the continuity of the square root function,

$$\lim_{k\to\infty} \|\mathbf{u}_k - \mathbf{u}\| = 0;$$

that is, the sequence $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. ■

Theorem 10.10 Let $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ be sequences in $\mathbb{R}^n$ such that $\{\mathbf{u}_k\}$ converges to the point $\mathbf{u}$ and $\{\mathbf{v}_k\}$ converges to the point $\mathbf{v}$. Then for any two real numbers α and β,

$$\lim_{k\to\infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

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Proof

From the Componentwise Convergence Theorem, it follows that for each index i with $1 \leq i \leq n$,

$$\lim_{k\to\infty} p_i(\mathbf{u}_k) = p_i(\mathbf{u}) \qquad \text{and} \qquad \lim_{k\to\infty} p_i(\mathbf{v}_k) = p_i(\mathbf{v}). \tag{10.9}$$

Observe that the linearity property of convergent real sequences implies that

$$\lim_{k\to\infty} [\alpha p_i(\mathbf{u}_k) + \beta p_i(\mathbf{v}_k)] = \alpha p_i(\mathbf{u}) + \beta p_i(\mathbf{v}),$$

which, by the linearity of the projections, means that

$$\lim_{k\to\infty} p_i(\alpha\mathbf{u}_k + \beta\mathbf{v}_k) = p_i(\alpha\mathbf{u} + \beta\mathbf{v}).$$

Thus, the sequence $\{\alpha\mathbf{u}_k + \beta\mathbf{v}_k\}$ converges componentwise to $\alpha\mathbf{u} + \beta\mathbf{v}$, and so, by the Componentwise Convergence Criterion, the sequence $\{\alpha\mathbf{u}_k + \beta\mathbf{v}_k\}$ converges to the point $\alpha\mathbf{u} + \beta\mathbf{v}$. ∎

EXERCISES FOR SECTION 10.2

1. Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Prove that

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$$

for every point $\mathbf{v}$ in $\mathbb{R}^n$.

2. Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Suppose that for every $\mathbf{v}$ in $\mathbb{R}^n$,

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle.$$

Prove that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. [*Hint:* For each index i with $1 \leq i \leq n$ and each point $\mathbf{u}$ in $\mathbb{R}^n$, $p_i(\mathbf{u}) = \langle \mathbf{u}, \mathbf{e}_i\rangle$, where $\mathbf{e}_i$ is the point in $\mathbb{R}^n$ whose ith component is 1 and whose other components are 0.]

3. Suppose that $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Prove that the sequence of real numbers $\{\|\mathbf{u}_k\|\}$ converges to $\|\mathbf{u}\|$.

4. Suppose that $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Assume also that $\mathbf{u}_k \neq \mathbf{0}$ for all k and that $\mathbf{u} \neq \mathbf{0}$. Define $\mathbf{v}_k = (1/\|\mathbf{u}_k\|)\mathbf{u}_k$ and $\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u}$. Prove that the sequence $\{\mathbf{v}_k\}$ converges to $\mathbf{v}$.

5. Suppose that $\{\mathbf{u}_k\}$ is a sequence of points in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$ and that $\|\mathbf{u}\| = r > 0$. Prove that there is an index K such that

$$\|\mathbf{u}_k\| > r/2 \qquad \text{if } k \geq K.$$

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6. Suppose that $\{\mathbf{u}_k\}$ is a sequence in $\mathbb{R}^n$ that converges to the point $\mathbf{u}$. Let $\mathbf{v}$ be a point in $\mathbb{R}^n$ that is orthogonal to each $\mathbf{u}_k$. Prove that $\mathbf{v}$ is also orthogonal to $\mathbf{u}$.
7. Use the Triangle Inequality for the sum of points in $\mathbb{R}^n$ to give a direct proof of Theorem 10.10.
8. A sequence of points $\{\mathbf{u}_k\}$ in $\mathbb{R}^n$ is said to be a *Cauchy sequence* provided that for each positive number ϵ there is an index K such that

$$\text{dist}(\mathbf{u}_k, \mathbf{u}_\ell) < \epsilon \qquad \text{if } k \geq K \text{ and } \ell \geq K.$$

   a. Prove that $\{\mathbf{u}_k\}$ is a Cauchy sequence if and only if each component sequence is a Cauchy sequence.
   b. Prove that a sequence in $\mathbb{R}^n$ converges if and only if it is a Cauchy sequence. (*Hint:* For sequences of real numbers, this was proved in Section 9.1.)

10.3 OPEN SETS AND CLOSED SETS IN $\mathbb{R}^n$

The Interior of a Set

The first nine chapters of this book were devoted to the study of real-valued functions of a single real variable. In this study, a prominent part was played by functions that have as their domains intervals of real numbers. In the study of functions that have as their domains subsets of $\mathbb{R}^n$, it turns out that it is necessary to study functions that have as their domains quite general subsets of $\mathbb{R}^n$; there is no special class of subsets of $\mathbb{R}^n$ that play the same distinguished role as do the intervals as subsets of $\mathbb{R}$. As preparation for the study of such functions, in the present section we consider some special types of subsets of $\mathbb{R}^n$, among which are open subsets and closed subsets.

Definition Given a point $\mathbf{u}$ in $\mathbb{R}^n$ and a positive number r, we call the set

$$\mathcal{B}_r(\mathbf{u}) \equiv \{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{u}, \mathbf{v}) < r\}$$

the *open ball* of radius r about $\mathbf{u}$.

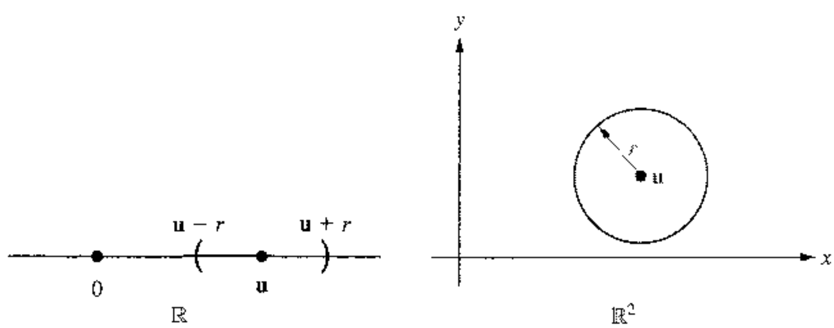

FIGURE 10.4 Open balls in $\mathbb{R}$ and in $\mathbb{R}^2$.

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In the case where $n = 1$, the open ball of radius r about a point u in $\mathbb{R}$ is simply the open interval $\{v \text{ in } \mathbb{R} \mid u - r < v < u + r\}$. In the case where $n = 2$, the open ball of radius r about the point $\mathbf{u} = (x_0, y_0)$ in $\mathbb{R}^2$ consists of all points in the plane $\mathbb{R}^2$ that lie inside the circle of radius r centered at the point (x_0, y_0).

Definition Let A be a subset of $\mathbb{R}^n$. A point $\mathbf{u}$ in $\mathbb{R}^n$ is called an *interior point* of A provided that there is an open ball about $\mathbf{u}$ that is contained in A. The set of all interior points of A is called the *interior* of A and is denoted by int A.

It is clear that the interior of a set is contained in the set. But there may be points in the set that are not interior points, and a set may have no interior points.

Example 10.11 Let a and b be real numbers with $a < b$. Define A to be the interval $(a, b] = \{u \text{ in } \mathbb{R} \mid a < u \leq b\}$. For a point u in (a, b), define r to be the smaller of the positive numbers $u - a$ and $b - u$. Then $\mathcal{B}_r(u) \subseteq (a, b]$. Thus, u is an interior point of A. On the other hand, the point b is in the set A but is not an interior point of A since every open ball about the point b contains points greater than b. Thus, the interior of A is the interval (a, b); that is,

$$\text{int}(a, b] = (a, b). \qquad \blacksquare$$

Example 10.12 Consider the set Q of rational real numbers. Then the density of the irrational numbers is equivalent to the assertion that Q has no interior points; that is,[1]

$$\text{int } Q = \emptyset. \qquad \blacksquare$$

Open Sets in $\mathbb{R}^n$

Definition A subset A of $\mathbb{R}^n$ is said to be *open in* $\mathbb{R}^n$ provided that every point in A is an interior point of A; that is,

$$\text{int } A = A.$$

It follows immediately that $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ and that the empty set $\emptyset$ is also an open subset of $\mathbb{R}^n$. Moreover, if a and b are real numbers with $a < b$, then, by the discussion in Example 10.11, we see that the interval $(a, b) = \{u \text{ in } \mathbb{R} \mid a < u < b\}$ is open in $\mathbb{R}$. Thus, our previous use of the adjective *open* for intervals of real numbers is consistent with the general definition we have just given. The next proposition shows that we have consistently used the adjective *open* in the term *open balls*.

[1] Recall that the symbol $\emptyset$ denotes the set that has no members; it is empty.

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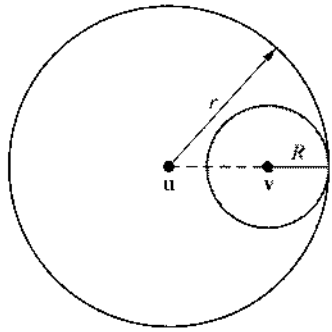
Proposition 10.13 Every open ball in $\mathbb{R}^n$ is open in $\mathbb{R}^n$.



$R = r - \text{dist}(\mathbf{u}, \mathbf{v})$

FIGURE 10.5 $\mathcal{B}_R(\mathbf{v}) \subseteq \mathcal{B}_r(\mathbf{u})$ if $R = r - \text{dist}(\mathbf{u}, \mathbf{v})$.

Proof

Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let r be a positive real number. Consider the open ball $\mathcal{B}_r(\mathbf{u})$. We must show that every point in $\mathcal{B}_r(\mathbf{u})$ is an interior point of $\mathcal{B}_r(\mathbf{u})$. Let $\mathbf{v}$ be a point in $\mathcal{B}_r(\mathbf{u})$. Define $R = r - \text{dist}(\mathbf{u}, \mathbf{v})$ and observe that R is positive. We claim that

$$\mathcal{B}_R(\mathbf{v}) \subseteq \mathcal{B}_r(\mathbf{u}). \tag{10.10}$$

Indeed, if $\mathbf{w}$ is in $\mathcal{B}_R(\mathbf{v})$, then

$$\text{dist}(\mathbf{w}, \mathbf{v}) < R = r - \text{dist}(\mathbf{u}, \mathbf{v}),$$

so using the Triangle Inequality, we have

$$\begin{aligned}\text{dist}(\mathbf{w}, \mathbf{u}) &\leq \text{dist}(\mathbf{w}, \mathbf{v}) + \text{dist}(\mathbf{v}, \mathbf{u}) \\ &< [r - \text{dist}(\mathbf{u}, \mathbf{v})] + \text{dist}(\mathbf{v}, \mathbf{u}) \\ &= r.\end{aligned}$$

Thus, the inclusion (10.10) holds; so $\mathbf{v}$ is an interior point of $\mathcal{B}_r(\mathbf{u})$. ∎

Closed Sets in $\mathbb{R}^n$

In Section 2.2, we defined what it means for a set of real numbers to be *closed*. The definition extends to subsets of $\mathbb{R}^n$ as follows.

Definition A subset A of $\mathbb{R}^n$ is said to be *closed in* $\mathbb{R}^n$ provided that whenever $\{\mathbf{u}_k\}$ is a sequence of points in A that converges to a point $\mathbf{u}$ in $\mathbb{R}^n$, then $\mathbf{u}$ belongs to the set A.

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Example 10.14 Let a and b be real numbers with $a < b$. Then the interval $[a, b] = \{u \text{ in } \mathbb{R} \mid a \leq u \leq b\}$ is closed. This is the content of Theorem 2.22. ■

Example 10.15 Define

$$A = \{(x, y) \text{ in } \mathbb{R}^2 \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

Then the set A is closed in $\mathbb{R}^2$. This follows from Example 10.14 and the Componentwise Convergence Criterion. ■

The Complementing Characterization of Open and Closed Sets

If A is a subset of $\mathbb{R}^n$, the *complement of A in $\mathbb{R}^n$*, denoted by $\mathbb{R}^n \backslash A$, is defined to be the set of all points in $\mathbb{R}^n$ that do not belong to A; that is,

$$\mathbb{R}^n \backslash A \equiv \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} \text{ does not belong to } A\}.$$

Given a collection $\{A_s\}_{s \in S}$ of subsets of $\mathbb{R}^n$ indexed by a set S, from the definition of union, intersection, and complement, it follows that

$$\mathbb{R}^n \backslash \bigcap_{s \in S} A_s = \bigcup_{s \in S} (\mathbb{R}^n \backslash A_s) \qquad \text{and} \qquad \mathbb{R}^n \backslash \bigcup_{s \in S} A_s = \bigcap_{s \in S} (\mathbb{R}^n \backslash A_s).$$

Such formulas are often referred to as *DeMorgan's Laws.*

Theorem 10.16 The Complementing Characterization A subset of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ if and only if its complement in $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.

Proof
First, suppose that A is an open subset of $\mathbb{R}^n$. Thus, every point in A is an interior point of A, so a sequence in $\mathbb{R}^n \backslash A$ cannot converge to a point in A. It follows that a sequence in $\mathbb{R}^n \backslash A$ that converges must converge to a point in $\mathbb{R}^n \backslash A$. Thus, $\mathbb{R}^n \backslash A$ is closed in $\mathbb{R}^n$.

To prove the converse, suppose that A is a subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \backslash A$ is closed in $\mathbb{R}^n$. We must show that every point in A is an interior point of A. Let $\mathbf{u}$ be a point in A. Suppose that $\mathbf{u}$ is not an interior point of A. Let k be a natural number. Then the open ball $B_{1/k}(\mathbf{u})$ is not a subset of A, so we can choose a point, which we label $\mathbf{u}_k$, such that

$$\mathbf{u}_k \text{ belongs to } \mathbb{R}^n \backslash A \qquad \text{and} \qquad \operatorname{dist}(\mathbf{u}_k, \mathbf{u}) < 1/k.$$

Thus, the sequence $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$. But $\mathbb{R}^n \backslash A$ is closed, so $\mathbf{u}$ belongs to $\mathbb{R}^n \backslash$A. This contradiction shows that $\mathbf{u}$ is an interior point of A. ■

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Intersections and Unions of Collections of Open and Closed Sets

Proposition 10.17

i. The union of a collection of open subsets of $\mathbb{R}^n$ is open in $\mathbb{R}^n$.
ii. The intersection of a collection of closed subsets of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.

Proof of (i)
Suppose that $\mathcal{O} = \cup_{s\in S}\mathcal{O}_s$, where each $\mathcal{O}_s$ is an open subset of $\mathbb{R}^n$. We claim that $\mathcal{O}$ is open; that is, that each point of $\mathcal{O}$ is an interior point of $\mathcal{O}$. Indeed, let $\mathbf{u}$ be in $\mathcal{O}$. Then $\mathbf{u}$ is in $\mathcal{O}_s$ for some s in S. Since $\mathcal{O}_s$ is open, there is an open ball about $\mathbf{u}$ that is contained in $\mathcal{O}_s$ and hence is contained in the union $\cup_{s\in S}\mathcal{O}_s = \mathcal{O}$. Thus, $\mathbf{u}$ is an interior point of $\mathcal{O}$. ■

Proof of (ii)
Suppose that $C = \bigcap_{s\in S} C_s$, where each C_s is closed in $\mathbb{R}^n$. Observe, by DeMorgan's Laws, that

$$\mathbb{R}^n\backslash C = \mathbb{R}^n\backslash \bigcap_{s\in S} C_s = \bigcup_{s\in S}(\mathbb{R}^n\backslash C_s).$$

From part (i) and the Complementing Characterization, it follows that C is closed in $\mathbb{R}^n$. ■

It is not always true that the intersection of a collection of open sets is again open. For instance, for each natural number k the interval of real numbers $(-1/k,\ 1/k)$ is an open subset of $\mathbb{R}$, yet the intersection of this collection of open sets is the set consisting of the single point 0. But a set consisting of a single point is clearly not open, so the intersection is not open. However, we will now prove that the intersection of a *finite* collection of open sets is open.

Proposition 10.18

i. The intersection of a finite collection of open subsets of $\mathbb{R}^n$ is open in $\mathbb{R}^n$.
ii. The union of a finite collection of closed subsets of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.

Proof of (i)
Suppose that $\mathcal{O} = \bigcap_{i=1}^k \mathcal{O}_i$ for some natural k, where each $\mathcal{O}_i$ is open in $\mathbb{R}^n$. Let $\mathbf{u}$ be a member of $\mathcal{O}$. If $1 \le i \le k$, $\mathbf{u}$ belongs to $\mathcal{O}_i$ and $\mathcal{O}_i$ is open in $\mathbb{R}^n$, so there is a positive number r_i such that $\mathcal{B}_{r_i}(\mathbf{u}) \subseteq \mathcal{O}_i$. Define $r = \min\{r_1, \ldots, r_k\}$. Then r is positive, and the open ball about the point u, $\mathcal{B}_r(\mathbf{u})$, is contained in each $\mathcal{O}_i$ and therefore is contained in the intersection $\bigcap_{i=1}^k \mathcal{O}_i = \mathcal{O}$. Thus, $\mathbf{u}$ is an interior point of $\mathcal{O}$. Therefore, every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$, so $\mathcal{O}$ is open in $\mathbb{R}^n$. ■

Proof of (ii)
Suppose that $C = \bigcup_{i=1}^k C_i$ for some natural number k, where each C_i is closed in $\mathbb{R}^n$. Observe, by DeMorgan's Laws, that $\mathbb{R}^n\backslash C = \bigcap_{i=1}^k(\mathbb{R}^n\backslash C_i)$. From part (i) and the Complementing Characterization, it follows that $\bigcap_{i=1}^k C_i$ is closed in $\mathbb{R}^n$. ■

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The Boundary and the Exterior of a Set

Definition Let A be a subset of $\mathbb{R}^n$.

i. A point $\mathbf{u}$ in $\mathbb{R}^n$ is called an *exterior point* of A provided that there is an open ball about $\mathbf{u}$ that is contained in $\mathbb{R}^n \setminus A$. The set of all exterior points of A is called the *exterior* of A and is denoted by ext A.

ii. A point $\mathbf{u}$ in $\mathbb{R}^n$ is called a *boundary point* of A provided that each open ball about $\mathbf{u}$ contains a point in A and also contains a point not in A. The set of all boundary points of A is called the *boundary* of A and is denoted by bd$A$.

It is clear that given a subset A of $\mathbb{R}^n$ and any point $\mathbf{u}$ in $\mathbb{R}^n$, there is an open ball about $\mathbf{u}$ that is contained in A, or there is an open ball about $\mathbf{u}$ that is contained in $\mathbb{R}^n \setminus A$, or every open ball about $\mathbf{u}$ contains a point in A and also contains a point in $\mathbb{R}^n \setminus A$; furthermore, these possibilities are mutually exclusive. This means precisely that $\mathbb{R}^n$ is decomposed into the following disjoint union:

$$\mathbb{R}^n = \operatorname{int} A \cup \operatorname{ext} A \cup \operatorname{bd} A. \tag{10.11}$$

Directly from the definitions of interior, exterior, and boundary, we see that if A is a subset of $\mathbb{R}^n$, then

$$\operatorname{int} A = \operatorname{ext}(\mathbb{R}^n \setminus A) \qquad \text{and} \qquad \operatorname{bd} A = \operatorname{bd}(\mathbb{R}^n \setminus A). \tag{10.12}$$

Proposition 10.19 Let A be a subset of $\mathbb{R}^n$. Then

i. A is open in $\mathbb{R}^n$ if and only if $A \cap \operatorname{bd} A = \emptyset$;

ii. A is closed in $\mathbb{R}^n$ if and only if $\operatorname{bd} A \subseteq A$.

Proof of (i)
First, let us suppose that A is open in $\mathbb{R}^n$. Then $A = \operatorname{int} A$. Thus, since $\operatorname{int} A \cap \operatorname{bd} A = \emptyset$, it follows that $A \cap \operatorname{bd} A = \emptyset$. To prove the converse, suppose that $A \cap \operatorname{bd} A = \emptyset$. Then since $A \cap \operatorname{ext} A = \emptyset$ and $\operatorname{int} A \subseteq A$, it follows from the decomposition (10.11) that $A = \operatorname{int} A$; that is, A is an open subset of $\mathbb{R}^n$. ∎

Proof of (ii)
The Complementing Characterization asserts that A is closed in $\mathbb{R}^n$ if and only if $\mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$. However, from part (i) it follows that $\mathbb{R}^n \setminus A$ is open in $\mathbb{R}^n$ if and only if $(\mathbb{R}^n \setminus A) \cap \operatorname{bd}(\mathbb{R}^n \setminus A) = \emptyset$. Since $\operatorname{bd} A = \operatorname{bd}(\mathbb{R}^n \setminus A)$, we conclude that A is closed in $\mathbb{R}^n$ if and only if $(\mathbb{R}^n \setminus A) \cap \operatorname{bd} A = \emptyset$. This proves (ii) since clearly $\operatorname{bd} A \cap (\mathbb{R}^n \setminus A) = \emptyset$ if and only if $\operatorname{bd} A \subseteq A$. ∎

We conclude this chapter by introducing a construction called the *Cartesian product*, which builds subsets of $\mathbb{R}^n$ from subsets of $\mathbb{R}$.

Definition For each index i with $1 \leq i \leq n$, let A_i be a subset of $\mathbb{R}$. The *Cartesian product* of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the subset of $\mathbb{R}^n$ defined by the formula

$$A_1 \times A_2 \times \cdots \times A_n \equiv \{\mathbf{u} = (u_1, \ldots, u_i, \ldots, u_n) \text{ in } \mathbb{R}^n \mid u_i \text{ in } A_i \qquad \text{for } 1 \leq i \leq n\}.$$

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If each A_i is a closed bounded interval $I_i = [a_i, b_i]$, the Cartesian product is called a *generalized rectangle*. We leave it as an exercise to verify that a generalized rectangle is a closed subset of $\mathbb{R}^n$. In fact, the Cartesian product of any n closed subsets of $\mathbb{R}$ is closed in $\mathbb{R}^n$, and the Cartesian product of any n open subsets of $\mathbb{R}$ is open in $\mathbb{R}^n$ (Exercises 10 and 11).

EXERCISES FOR SECTION 10.3

1. Determine which of the following subsets of $\mathbb{R}$ are open in $\mathbb{R}$, closed in $\mathbb{R}$, or neither open nor closed in $\mathbb{R}$. Justify your conclusions.
a. $A = (0, \infty)$
b. $A = \mathbb{Q}$, the set of rational numbers
c. $A = \{u \text{ in } \mathbb{R} \mid u^2 > 4\}$
d. $A = \{u \text{ in } \mathbb{R} \mid u^2 \geq 4\}$
e. $A = [0, \infty)$

2. Determine which of the following subsets A of $\mathbb{R}^2$ are open in $\mathbb{R}^2$, closed in $\mathbb{R}^2$, or neither open nor closed in $\mathbb{R}^2$. Justify your conclusions.
a. $A = \{\mathbf{u} = (x, y) \mid x^2 > y\}$
b. $A = \{\mathbf{u} = (x, y) \mid x^2 + y^2 = 1\}$
c. $A = \{\mathbf{u} = (x, y) \mid x \text{ is rational}\}$
d. $A = \{\mathbf{u} = (x, y) \mid x \geq 0,\ y \geq 0\}$

3. Let r be a positive number and define $\mathcal{O} = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| > r\}$. Prove that $\mathcal{O}$ is open in $\mathbb{R}^n$ by showing that every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$. [*Hint:* For $\mathbf{u}$ in $\mathcal{O}$, define $R = \|\mathbf{u}\| - r$ and show that $\mathcal{B}_R(\mathbf{u}) \subseteq \mathcal{O}$.]

4. Let r be a positive number and define $F = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| \leq r\}$. Use the Componentwise Convergence Criterion to prove that F is closed.

5. Let r be a positive number and define $\mathcal{O} = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| > r\}$. Prove that $\mathcal{O}$ is open in $\mathbb{R}^n$ by showing that its complement is closed in $\mathbb{R}^n$.

6. Let r be a positive number and define $F = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| = r\}$. Prove that F is closed in $\mathbb{R}^n$ by using the Componentwise Convergence Criterion together with the sum and product properties of convergent real sequences.

7. Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{w}$ be a point in $\mathbb{R}^n$. The *translate* of A by $\mathbf{w}$ is denoted by $\mathbf{w} + A$ and is defined by

$$\mathbf{w} + A \equiv \{\mathbf{w} + \mathbf{u} \mid \mathbf{u} \text{ in } A\}.$$

a. Show that A is open if and only if $\mathbf{w} + A$ is open.
b. Show that A is closed if and only if $\mathbf{w} + A$ is closed.

8. For r a positive number and $\mathbf{u}$ a point in $\mathbb{R}^n$, define $A = \mathcal{B}_r(\mathbf{u})$. Show that $\text{int}\, A = A$, that $\text{bd}\, A = \{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{u}, \mathbf{v}) = r\}$, and that $\text{ext}\, A = \{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{u}, \mathbf{v}) > r\}$.

9. Let A and B be subsets of $\mathbb{R}^n$ with $A \subseteq B$.
a. Prove that $\text{int}\, A \subseteq \text{int}\, B$.
b. Is it necessarily true that $\text{bd}\, A \subseteq \text{bd}\, B$?

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10. For each index i with $1 \leq i \leq n$, let F_i be a closed subset of $\mathbb{R}$. Prove that the Cartesian product

$$F_1 \times F_2 \times \cdots \times F_n$$

is a closed subset of $\mathbb{R}^n$.

11. For each index i with $1 \leq i \leq n$, let $\mathcal{O}_i$ be an open subset of $\mathbb{R}$. Prove that the Cartesian product

$$\mathcal{O}_1 \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_n$$

is an open subset of $\mathbb{R}^n$.

12. For a subset A of $\mathbb{R}^n$, the *closure* of A, denoted by cl A, is defined by

$$\text{cl } A = \text{int } A \cup \text{bd} A.$$

Prove that $A \subseteq \text{cl } A$ and that $A = \text{cl } A$ if and only if A is closed in $\mathbb{R}^n$.

13. Let A be a subset of $\mathbb{R}^n$.

a. Prove that int A is an open subset of $\mathbb{R}^n$.

b. Use (a) to show that ext A is also an open subset of $\mathbb{R}^n$.

c. Use (a) and (b), together with the decomposition (10.11), to show that bd A is a closed subset of $\mathbb{R}^n$.

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CHAPTER 11

CONTINUITY, COMPACTNESS, AND CONNECTEDNESS

11.1 CONTINUOUS FUNCTIONS AND MAPPINGS

Recall that a function $f : A \to \mathbb{R}$ whose domain A is a subset of $\mathbb{R}$ has been defined to be *continuous at the point* x in A provided that whenever a sequence $\{x_k\}$ in A converges to x, the image sequence $\{f(x_k)\}$ converges to $f(x)$. Moreover, the function $f : A \to \mathbb{R}$ has been defined to be *continuous* provided that it is continuous at each point in its domain. We studied such functions in Chapter 3. In the present chapter, we study more general functions of the form $F: A \to \mathbb{R}^m$, where A is a subset of $\mathbb{R}^n$ and m and n can be greater than 1. It turns out that it is sometimes useful to distinguish between the case where $m = 1$ and the case where $m > 1$; there are a number of particular results that hold only in the case where $m = 1$. To emphasize this distinction, we call general functions $F: A \to \mathbb{R}^m$ *mappings* and reserve the use of the word *function* for the case where $m = 1$—that is, where the range is $\mathbb{R}$.

Definition Let A be a subset of $\mathbb{R}^n$.

i. A mapping $F: A \to \mathbb{R}^m$ is said to be *continuous at the point* $\mathbf{u}$ in A provided that whenever a sequence $\{\mathbf{u}_k\}$ in A converges to $\mathbf{u}$, the image sequence $\{F(\mathbf{u}_k)\}$ converges to $F(\mathbf{u})$.

ii. A mapping $F: A \to \mathbb{R}^m$ is said to be *continuous* provided that it is continuous at every point in its domain.

Continuity of the Projection Functions

Proposition 11.1 For each index i with $1 \leq i \leq n$, the ith component projection function $p_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof

Let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Suppose that $\{\mathbf{u}_k\}$ is a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$. Then, by the Componentwise Convergence Criterion,

$$\lim_{k\to\infty} \{p_i(\mathbf{u}_k)\} = p_i(\mathbf{u}).$$

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Thus, the function $p_i : \mathbb{R}^n \to \mathbb{R}$ is continuous at the point $\mathbf{u}$. Since $\mathbf{u}$ is an arbitrarily chosen point in $\mathbb{R}^n$, the function $p_i : \mathbb{R}^n \to \mathbb{R}$ is continuous. ∎

Example 11.2 Define the functions $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^3 \to \mathbb{R}$, and $h : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x, \quad g(x, y, z) = y, \quad \text{and} \quad h(x, y, z) = z \quad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

Proposition 11.1 implies that these three functions are continuous. ∎

Continuity of Sums, Products, and Quotients of Functions

We will first establish a number of results for real-valued functions of several real variables, and then we will turn to the study of general mappings. We begin with the following extension of Theorem 3.4.

Theorem 11.3 Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$ and suppose that the functions $h : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both continuous at $\mathbf{u}$. For any real numbers α and β, the function

$$\alpha h + \beta g : A \to \mathbb{R}$$

is continuous at $\mathbf{u}$. Also, the product

$$h \cdot g : A \to \mathbb{R}$$

is continuous at $\mathbf{u}$. Moreover, if $g(\mathbf{v}) \neq 0$ for all $\mathbf{v}$ in A, then the quotient

$$\frac{h}{g} : A \to \mathbb{R}$$

is continuous at $\mathbf{u}$.

Proof

Let $\{\mathbf{u}_k\}$ be a sequence in the set A that converges to the point $\mathbf{u}$. Since the function $h : A \to \mathbb{R}$ is continuous at the point $\mathbf{u}$, the image sequence $\{h(\mathbf{u}_k)\}$ converges to $h(\mathbf{u})$. Similarly, the continuity of the function $g : A \to \mathbb{R}$ at the point $\mathbf{u}$ implies that the sequence $\{g(\mathbf{u}_k)\}$ converges to $g(\mathbf{u})$. From the sum, product, and quotient properties of convergent sequences of real numbers, it follows that

$$\lim_{k\to\infty} (\alpha h + \beta g)(\mathbf{u}_k) = (\alpha h + \beta g)(\mathbf{u}),$$
$$\lim_{k\to\infty} (hg)(\mathbf{u}_k) = (hg)(\mathbf{u}),$$

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and

$$\lim_{k\to\infty}\left(\frac{h}{g}\right)(\mathbf{u}_k)=\left(\frac{h}{g}\right)(\mathbf{u}).$$

These three sequential limits are precisely what is required to prove the theorem. ■

Example 11.4 Define the function $f:\mathbb{R}^3\to\mathbb{R}$ by

$$f(x,\ y,\ z)=xz+y^3 \qquad \text{for } (x,\ y,\ z) \text{ in } \mathbb{R}^3.$$

Since this function is obtained from products and sums of component projection functions, it follows from Proposition 11.1 and Theorem 11.3 that it is continuous. ■

Continuity of Compositions of Mappings

Given a mapping $F\colon A\to\mathbb{R}^m$, if B is a subset of the domain A, the image of the set B under the mapping $F\colon A\to\mathbb{R}^m$, denoted by $F(B)$, is defined by the formula

$$F(B)\equiv\{\mathbf{v} \text{ in } \mathbb{R}^m \mid \mathbf{v}=F(\mathbf{u}) \qquad \text{for some } \mathbf{u} \text{ in } B\}.$$

Theorem 11.5 Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$. Suppose that the mapping $G:A\to\mathbb{R}^m$ is continuous at the point $\mathbf{u}$. Let B be a subset of $\mathbb{R}^m$ with $G(A)\subseteq B$ and suppose that the mapping $H:B\to\mathbb{R}^k$ is continuous at the point $G(\mathbf{u})$. Then the composition

$$H\circ G:A\to\mathbb{R}^k$$

is continuous at $\mathbf{u}$.

Proof

Let $\{\mathbf{u}_k\}$ be a sequence in A that converges to the point $\mathbf{u}$. Since the mapping $G:A\to\mathbb{R}^m$ is continuous at $\mathbf{u}$, it follows that the image sequence $\{G(\mathbf{u}_k)\}$ converges to $G(\mathbf{u})$. But then $\{G(\mathbf{u}_k)\}$ is a sequence in B that converges to the point $G(\mathbf{u})$. The continuity of the mapping $H:B\to\mathbb{R}^k$ at the point $G(\mathbf{u})$ implies that the sequence $\{H(G(\mathbf{u}_k))\}$ converges to $H(G(\mathbf{u}))$; that is, the sequence $\{(H\circ G)(\mathbf{u}_k)\}$ converges to $(H\circ G)(\mathbf{u})$. ■

Example 11.6 Define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,\ y)=x^2y+e^{xy+1} \qquad \text{for } (x,\ y) \qquad \text{in } \mathbb{R}^2.$$

Then the function $f:\mathbb{R}^2\to\mathbb{R}$ is continuous. To see this, we observe that

$$f=p_1\cdot p_1\cdot p_2+h\circ(p_1\cdot p_2):\mathbb{R}^2\to\mathbb{R},$$

where $h:\mathbb{R}\to\mathbb{R}$ is defined by $h(x)=e^{x+1}$ for x in $\mathbb{R}$. Since products, sums, and compositions of continuous maps are again continuous, it follows that the function $f:\mathbb{R}^2\to\mathbb{R}$ is continuous. ■

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Example 11.7 Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(\mathbf{u}) = \|\mathbf{u}\| \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

Then the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. To see this, we observe that

$$f = h \circ (p_1 p_1 + \cdots + p_n p_n) : \mathbb{R}^n \to \mathbb{R},$$

where $h(x) = \sqrt{x}$ for $x \geq 0$. Since products, sums, and compositions of continuous maps are again continuous, it follows that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. ■

The Componentwise Continuity Criterion for Continuous Mappings

Definition Given a mapping $F: A \to \mathbb{R}^m$, where A is a subset of $\mathbb{R}^n$, and an index i with $1 \leq i \leq n$, we define the function $F_i : A \to \mathbb{R}$ to be the composition of $F: A \to \mathbb{R}^m$ with the ith component projection. We call the function $F_i : A \to \mathbb{R}$ the ith *component function* of the mapping $F: A \to \mathbb{R}^m$. Thus,

$$F(\mathbf{u}) = (F_1(\mathbf{u}), \ldots, F_m(\mathbf{u})) \qquad \text{for } \mathbf{u} \text{ in } A,$$

and the mapping $F: A \to \mathbb{R}^m$ is said to be represented by its component functions as

$$F = (F_1, \ldots, F_m) : A \to \mathbb{R}^m. \tag{11.1}$$

Example 11.8 Let $\mathcal{O}$ be the set of all nonzero points in $\mathbb{R}^n$. Define the mapping $F : \mathcal{O} \to \mathbb{R}^n$ by

$$F(\mathbf{u}) = \mathbf{u}/\|\mathbf{u}\|^2 \qquad \text{for } \mathbf{u} \text{ in } \mathcal{O}.$$

Then the representation of the mapping in component functions is

$$F(\mathbf{u}) = (u_1/\|\mathbf{u}\|^2, \ldots, u_n/\|\mathbf{u}\|^2) \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

In the case where $n = 3$, this component representation can be written as

$$F(x, y, z) = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right) \qquad \text{for } (x, y, z) \text{ in } \mathcal{O}.$$

■

Just as we have the Componentwise Convergence Criterion for the convergence of sequences in Euclidean space, we also have the following simple, useful criterion for the continuity of a mapping.

Theorem 11.9 The Componentwise Continuity Criterion Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$ and consider the mapping

$$F = (F_1, \ldots, F_m) : A \to \mathbb{R}^m.$$

Then the mapping $F : A \to \mathbb{R}^m$ is continuous at $\mathbf{u}$ if and only if each of its component functions $F_i : A \to \mathbb{R}$ is continuous at $\mathbf{u}$.

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Proof

This result follows immediately from the Componentwise Convergence Theorem since if $\{\mathbf{u}_k\}$ is a sequence in A that converges to the point $\mathbf{u}$, then the image sequence $\{F(\mathbf{u}_k)\}$ converges to $F(\mathbf{u})$ if and only if for each index i with $1 \leq i \leq n$ the sequence $\{F_i(\mathbf{u}_k)\}$ converges to $F_i(\mathbf{u})$. ■

The Componentwise Continuity Criterion provides the following extension of the first assertion of Theorem 11.3.

Corollary 11.10 Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$ and suppose that the mappings $H : A \to \mathbb{R}^m$ and $G : A \to \mathbb{R}^m$ are both continuous at the point $\mathbf{u}$. Then for any real numbers α and β, the mapping

$$\alpha H + \beta G : A \to \mathbb{R}^m$$

is also continuous at $\mathbf{u}$.

Proof

This result follows from the Componentwise Continuity Criterion and Theorem 11.3 if we observe that, for each index i with $1 \leq i \leq n$, the ith component function of the mapping $\alpha H + \beta G : A \to \mathbb{R}^m$ is the function $\alpha H_i + \beta G_i : A \to \mathbb{R}$. ■

The ϵ-δ Criterion for Continuity of Mappings

For real-valued functions of a single real variable, we defined the concept of continuity at a point in Section 3.1 in terms of convergent sequences. Then, in Section 3.5, we introduced the ϵ-δ criterion for continuity at a point and proved it equivalent to the definition in terms of convergent sequences: This was Theorem 3.20. The proof of Theorem 3.20 extends, almost word for word, to the case of mappings to provide a proof of the following theorem (Exercise 12).

Theorem 11.11 Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{u}$. Then the following two assertions about a mapping $F : A \to \mathbb{R}^m$ are equivalent:

i. The mapping $F : A \to \mathbb{R}^m$ is continuous at the point $\mathbf{u}$; that is, for a sequence $\{\mathbf{u}_k\}$ in A,

$$\lim_{k\to\infty} \operatorname{dist}(F(\mathbf{u}_k),\ F(\mathbf{u})) = 0 \qquad \text{if } \lim_{k\to\infty} \operatorname{dist}(\mathbf{u}_k, \mathbf{u}) = 0.$$

ii. For each positive number ϵ there is a positive number δ such that for a point $\mathbf{v}$ in A,

$$\operatorname{dist}(F(\mathbf{v}),\ F(\mathbf{u})) < \epsilon \qquad \text{if } \operatorname{dist}(\mathbf{v}, \mathbf{u}) < \delta.$$

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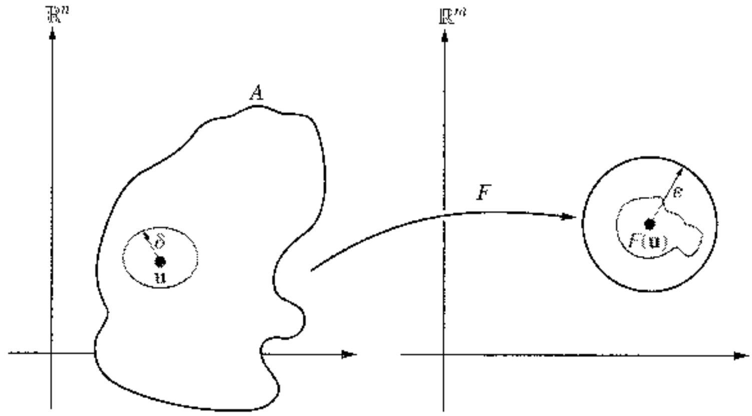

FIGURE 11.1 The ϵ-δ criterion for continuity at a point.

A Final Criterion for Continuity: The Openness of Inverse Images of Open Sets

The preceding characterization of continuity of a mapping at a point leads to the following useful characterization of continuity on the whole domain for mappings that have as their domains open subsets of $\mathbb{R}^n$.

Theorem 11.12 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and consider the mapping $F: \mathcal{O} \to \mathbb{R}^m$. Then the following assertions are equivalent:

i. The mapping $F: \mathcal{O} \to \mathbb{R}^m$ is continuous.
ii. $F^{-1}(V)$ is an open subset of $\mathbb{R}^n$ whenever V is an open subset of $\mathbb{R}^m$.

Proof

First, suppose that (i) holds. Let V be an open subset of $\mathbb{R}^m$. We wish to show that $F^{-1}(V)$ is open in $\mathbb{R}^n$, which by definition means that every point in $F^{-1}(V)$ is an interior point. Let $\mathbf{u}$ be a point in $F^{-1}(V)$. Then $F(\mathbf{u})$ belongs to V and V is open in $\mathbb{R}^m$, so there is some positive number ϵ such that $\mathcal{B}_\epsilon(F(\mathbf{u})) \subseteq V$. Since the mapping $F: \mathcal{O} \to \mathbb{R}^m$ is continuous at the point $\mathbf{u}$, it follows from the ϵ-δ characterization of continuity provided in Theorem 11.11 that we can select a positive number δ such that

$$\operatorname{dist}(F(\mathbf{v}), F(\mathbf{u})) < \epsilon \qquad \text{if } \mathbf{v} \text{ is in } \mathcal{O} \text{ and } \operatorname{dist}(\mathbf{v}, \mathbf{u}) < \delta. \tag{11.2}$$

However, by assumption, the set $\mathcal{O}$ is open in $\mathbb{R}^n$, so we can select a positive number r less than δ such that $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$. From (11.2), it follows that

$$F(\mathcal{B}_r(\mathbf{u})) \subseteq \mathcal{B}_\epsilon(F(\mathbf{u})) \subseteq V.$$

Thus, $\mathcal{B}_r(\mathbf{u}) \subseteq F^{-1}(V)$, so $\mathbf{u}$ is an interior point of $F^{-1}(V)$.

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To prove the converse, suppose that (ii) holds. Let $\mathbf{u}$ be a point in $\mathcal{O}$. To show that the mapping $F: \mathcal{O} \to \mathbb{R}^m$ is continuous at the point $\mathbf{u}$, we verify the ϵ-δ characterization of continuity asserted in Theorem 11.11. Let $\epsilon > 0$. Open balls are open subsets of $\mathbb{R}^m$. Thus, $\mathcal{B}_\epsilon(F(\mathbf{u}))$ is open in $\mathbb{R}^m$. From (ii) it follows that $F^{-1}(\mathcal{B}_\epsilon(F(\mathbf{u})))$ is open in $\mathbb{R}^n$. Thus, $\mathbf{u}$, which belongs to $F^{-1}(\mathcal{B}_\epsilon(F(\mathbf{u})))$, is an interior point of $F^{-1}(\mathcal{B}_\epsilon(F(\mathbf{u})))$, so we can choose a positive number δ with $\mathcal{B}_\delta(\mathbf{u}) \subseteq F^{-1}(\mathcal{B}_\epsilon(F(\mathbf{u})))$. This means that $F(\mathcal{B}_\delta(\mathbf{u})) \subseteq \mathcal{B}_\epsilon(F(\mathbf{u}))$. Thus, the mapping $F: \mathcal{O} \to \mathbb{R}^m$ satisfies the ϵ-δ characterization of continuity at the point $\mathbf{u}$. ■

Criteria for Sets Being Open and Being Closed

Corollary 11.13 Let the function $f: \mathbb{R}^n \to \mathbb{R}$ be continuous and let c be a real number. Then each of the sets

$$\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) < c\} \qquad \text{and} \qquad \{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) > c\}$$

is open in $\mathbb{R}^n$, while each of the sets

$$\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \leq c\} \qquad \text{and} \qquad \{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \geq c\}$$

is closed in $\mathbb{R}^n$.

Proof

By observing that the sets $\{v \text{ in } \mathbb{R} \mid v < c\}$ and $\{v \text{ in } \mathbb{R} \mid v > c\}$ are both open in $\mathbb{R}$, from the characterization of continuity asserted in Theorem 11.12 it follows that both $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) < c\}$ and $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) > c\}$ are open in $\mathbb{R}^n$. But observe that

$$\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \geq c\} = \mathbb{R}^n \setminus \{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) < c\}$$

and

$$\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \leq c\} = \mathbb{R}^n \setminus \{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) > c\},$$

so from the Complementing Characterization of openness and closedness, it follows that both $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \leq c\}$ and $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) \geq c\}$ are closed in $\mathbb{R}^n$. ■

Example 11.14 Define

$$\mathcal{O} = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid 2x + 3y + 4z > 0\}.$$

Then $\mathcal{O}$ is an open subset of $\mathbb{R}^3$. This follows from Corollary 11.13 if we observe that the function $f: \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = 2x + 3y + 4z \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3,$$

is continuous. ■

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Example 11.15 Fix positive numbers a and b with $a < b$ and define

$$\mathcal{O} = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid a < \|\mathbf{u}\| < b\}.$$

Then $\mathcal{O}$ is an open subset of $\mathbb{R}^n$. This follows from Corollary 11.13 if we first observe that since the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{u}) = \|\mathbf{u}\| \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n$$

is continuous, both $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| > a\}$ and $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| < b\}$ are open subsets of $\mathbb{R}$. Hence $\mathcal{O}$, being the intersection of these two sets, is also open in $\mathbb{R}^n$. ■

EXERCISES FOR SECTION 11.1

1. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \cos(x + y) + x^2y^2 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

2. Define $\mathcal{O} = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid (x, y, z) \neq (0, 0, 0)\}$ and define the function $f : \mathcal{O} \to \mathbb{R}$ by

$$f(x, y, z) = \frac{x}{x^2 + y^2 + z^2} \qquad \text{for } (x, y, z) \text{ in } \mathcal{O}.$$

Prove that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous.

3. Fix a point $\mathbf{v}$ in $\mathbb{R}^n$ and define the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous.

4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\mathbf{u}) > 0$ if the point $\mathbf{u}$ in $\mathbb{R}^n$ has at least one rational component. Prove that $f(\mathbf{u}) \geq 0$ for all points $\mathbf{u}$ in $\mathbb{R}^n$.

5. Use Corollary 11.13 to show that each of the following sets is open in $\mathbb{R}^2$:

a. $\{(x, y) \text{ in } \mathbb{R}^2 \mid y > 0\}$
b. $\{(x, y) \text{ in } \mathbb{R}^2 \mid x^2/5 + y^2/4 < 1\}$
c. $\{(x, y) \text{ in } \mathbb{R}^2 \mid y > x^2\}$
d. $\{(x, y) \text{ in } \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$

6. Suppose that the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are both continuous. Prove that the set $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid f(\mathbf{u}) = g(\mathbf{u}) = 0\}$ is closed in $\mathbb{R}^n$.

7. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous. If a and b are numbers with $a < b$, prove that the set

$$\{\mathbf{u} \text{ in } \mathcal{O} \mid a < f(\mathbf{u}) < b\}$$

is open in $\mathbb{R}^n$.

8. Show that the set $\{\mathbf{u} \text{ in } \mathbb{R}^n \mid u_n > 0\}$ is open in $\mathbb{R}^n$.

9. Use Corollary 11.13 to show that if $\mathbf{u}$ is a point in $\mathbb{R}^n$ and r is a positive number, then the set $\{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{u}, \mathbf{v}) \leq r\}$ is closed in $\mathbb{R}^n$.

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10. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous. Suppose that $\mathbf{u}$ is a point in $\mathcal{O}$ at which $f(\mathbf{u}) > 0$. Prove that there is an open ball $\mathcal{B}$ about $\mathbf{u}$ such that $f(\mathbf{v}) > f(\mathbf{u})/2$ for all $\mathbf{v}$ in $\mathcal{B}$.

11. Let A be a subset of $\mathbb{R}^n$. The *characteristic function* of the set A is the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \text{ is in } A \\ 0 & \text{if } \mathbf{u} \text{ is not in } A. \end{cases}$$

Prove that this characteristic function is continuous at each interior point of A and at each exterior point of A but fails to be continuous at each boundary point of A.

12. Prove Theorem 11.11. (*Hint:* Follow the proof of Theorem 3.20, replacing the absolute value of the difference of two numbers by the distance between two points.)

13. Give a direct proof of Corollary 11.10 without quoting the Componentwise Continuity Criterion.

11.2 SEQUENTIAL COMPACTNESS, EXTREME VALUES, AND UNIFORM CONTINUITY

In Section 2.4, we introduced the concept of sequential compactness for sets of real numbers. A set S of real numbers is defined to be sequentially compact provided that any sequence in S has a subsequence that converges to a point in S. We proved that a closed bounded interval is sequentially compact, a result we called the Sequential Compactness Theorem (Theorem 2.36).

In Section 3.2, we proved the Extreme Value Theorem (Theorem 3.9), which asserts that a continuous function defined on a closed bounded interval has both a minimum and a maximum functional value. In Section 3.4, we introduced the concept of uniform continuity and proved Theorem 3.17, which asserts that a continuous function defined on a closed bounded interval is uniformly continuous. In fact, an examination of the proofs of Theorems 3.9 and 3.17 reveals that the only property of a closed bounded interval used is that it is sequentially compact. We had no need to pursue this point in Chapter 3. However, it is useful to do so now in the context of functions of several variables. Our first aim in this section is to introduce the concept of sequential compactness for a subset of $\mathbb{R}^n$ and show that a subset of $\mathbb{R}^n$ is sequentially compact if and only if it is both closed and bounded. Our second aim is to prove that a continuous function defined on a sequentially compact subset of $\mathbb{R}^n$ (i) has a minimum and a maximum functional value and (ii) is uniformly continuous.

Sequentially Compact Subsets of $\mathbb{R}^n$

Given a sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ and $\{k_j\}$ a strictly increasing sequence of natural numbers, the sequence $\{\mathbf{x}_{k_j}\}$ is called a *subsequence* of $\{\mathbf{x}_k\}$. Observe that if the sequence $\{\mathbf{x}_k\}$ converges to the point $\mathbf{x}$ in $\mathbb{R}^n$, then each subsequence $\{\mathbf{x}_{k_j}\}$ also converges to $\mathbf{x}$ since this property has already been established for real sequences, and so if $\lim_{k\to\infty} \text{dist}(\mathbf{x}_k, \mathbf{x}) = 0$, then $\lim_{j\to\infty} \text{dist}(\mathbf{x}_{k_j}, \mathbf{x}) = 0$ also.

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Definition Let A be a subset of $\mathbb{R}^n$. Then A is said to be *sequentially compact* provided that every sequence in A has a subsequence that converges to a point in A.

The Sequential Compactness Theorem (Theorem 2.36) is the assertion that if a and b are real numbers such that $a \leq b$, then the closed bounded interval $[a, b]$ is sequentially compact. It is useful to characterize the sequentially compact subsets of $\mathbb{R}^n$. As a first step, we establish two conditions that are necessary in order for a subset of $\mathbb{R}^n$ to be sequentially compact.

Definition A subset A of $\mathbb{R}^n$ is said to be *bounded* provided that there is a number M such that

$$\|\mathbf{u}\| \leq M \qquad \text{for all points } \mathbf{u} \text{ in } A.$$

Theorem 11.16 A sequentially compact subset of $\mathbb{R}^n$ is bounded and closed in $\mathbb{R}^n$.

Proof
Let A be a sequentially compact subset of $\mathbb{R}^n$. First, we show that A is closed in $\mathbb{R}^n$. Let $\{\mathbf{u}_k\}$ be a sequence in A that converges to the point $\mathbf{u}$ in $\mathbb{R}^n$. Then every subsequence of $\{\mathbf{u}_k\}$ also converges to $\mathbf{u}$. By the definition of sequential compactness, some subsequence converges to a point in A. Thus, $\mathbf{u}$ belongs to A; hence A is closed.

To prove that A is bounded, we assume the contrary and derive a contradiction. Suppose that A is not bounded. Then for each natural number k, it is not true that

$$\|\mathbf{u}\| \leq k \qquad \text{for all points } \mathbf{u} \text{ in } A.$$

So we can select a point in A, which we label $\mathbf{u}_k$, such that $\|\mathbf{u}_k\| > k$. Since A is sequentially compact, a subsequence $\{\mathbf{u}_{k_j}\}$ of the sequence $\{\mathbf{u}_k\}$ converges to a point $\mathbf{u}$ that belongs to A. But then

$$\|\mathbf{u}_{k_j}\| > k_j \geq j \qquad \text{for all positive integers } j.$$

Thus, the real sequence $\{\|\mathbf{u}_{k_j}\|\}$ is unbounded but converges to the norm of $\mathbf{u}$. However, one of the first properties that we proved about convergent sequences of real numbers was that such sequences are bounded. Thus, we have a contradiction, and hence A must be bounded. ∎

It turns out that for a subset A of $\mathbb{R}^n$ to be sequentially compact, it is both necessary and sufficient that A be bounded and closed in $\mathbb{R}^n$. In order to prove sufficiency, the crucial result is the following theorem.

Theorem 11.17 Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof
We will prove the theorem by induction. The case where $n = 1$ is precisely the assertion of Theorem 2.33.

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Now suppose that n is a positive integer and that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. We must show that every bounded sequence in $\mathbb{R}^{n+1}$ has a convergent subsequence.

Let $\{\mathbf{u}_k\}$ be a bounded sequence in $\mathbb{R}^{n+1}$. Fix a positive integer k. Define x_k to be the last component of $\mathbf{u}_k$ and write

$$\mathbf{u}_k = (\mathbf{v}_k,\ x_k),$$

where $\mathbf{v}_k$ is the point in $\mathbb{R}^n$ whose ith component is equal to the ith component of $\mathbf{u}_k$ for each index i with $1 \leq i \leq n$. This defines two sequences: the sequence of real numbers $\{x_k\}$ and the sequence $\{\mathbf{v}_k\}$ in $\mathbb{R}^n$.

Now $\{\mathbf{v}_k\}$ is a bounded sequence in $\mathbb{R}^n$, and $\{x_k\}$ is a bounded sequence of real numbers. By the induction assumption, a subsequence of $\{\mathbf{v}_k\}$ converges to a point $\mathbf{v}$ in $\mathbb{R}^n$. The corresponding subsequence of $\{x_k\}$ itself has a further subsequence that converges to a real number x. From the Componentwise Convergence Criterion, it follows that the subsequence of $\{\mathbf{u}_k\}$ corresponding to this last subsequence converges to the point $\mathbf{u} = (\mathbf{v},\ x)$ in $\mathbb{R}^{n+1}$.

The Principle of Mathematical Induction implies that this theorem is true for every positive integer n. ■

Theorem 11.18 The Sequential Compactness Theorem A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is bounded and closed in $\mathbb{R}^n$.

Proof

First, according to Theorem 11.16, every sequentially compact subset of $\mathbb{R}^n$ is bounded and closed in $\mathbb{R}^n$.

To prove the converse, let A be a subset of $\mathbb{R}^n$ that is bounded and closed in $\mathbb{R}^n$. Suppose that $\{\mathbf{u}_k\}$ is a sequence in A. Since the sequence $\{\mathbf{u}_k\}$ is bounded, it follows from Theorem 11.17 that there is a subsequence $\{\mathbf{u}_{k_j}\}$ of $\{\mathbf{u}_k\}$ that converges to a point $\mathbf{u}$ in $\mathbb{R}^n$. But $\{\mathbf{u}_{k_j}\}$ is itself a sequence in A, and since A is closed, $\mathbf{u}$ belongs to the set A. Thus, A is sequentially compact. ■

The above Sequential Compactness Theorem is often referred to as the Bolzano–Weierstrass Theorem.

Recall that a subset $\mathbf{I}$ of $\mathbb{R}^n$ that is the Cartesian product of n closed bounded intervals in $\mathbb{R}$ is called a generalized rectangle.

Corollary 11.19 A generalized rectangle in $\mathbb{R}^n$ is sequentially compact.

Proof

Let $\mathbf{I}$ be a generalized rectangle. According to the preceding theorem, to show that $\mathbf{I}$ is sequentially compact we must show $\mathbf{I}$ is both bounded and closed in $\mathbb{R}^n$. By definition, a generalized rectangle is the Cartesian product of n closed bounded intervals of real numbers. First, we show that $\mathbf{I}$ is bounded. Since a closed bounded interval of real numbers is bounded, we can choose $M > 0$ such that for $1 \leq i \leq n$,

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if the real number u belongs to the ith edge of $\mathbf{I}$, then $|u| \leq M$. Thus,

$$\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_i^2 + \cdots + u_n^2} \leq \sqrt{n}M \qquad \text{for all } \mathbf{u} \text{ in } \mathbf{I},$$

and so $\mathbf{I}$ is bounded. On the other hand, we have shown that a closed bounded interval of real numbers is closed in $\mathbb{R}$ and therefore, by the Componentwise Convergence Criterion, $\mathbf{I}$ is closed in $\mathbb{R}^n$. ∎

We now turn to the study of continuous mappings that have as their domains sequentially compact subsets of $\mathbb{R}^n$.

Theorem 11.20 Let A be a subset of $\mathbb{R}^n$ and suppose that the mapping $F : A \to \mathbb{R}^m$ is continuous. If the domain A is sequentially compact, then the image $F(A)$ is also sequentially compact.

Proof
Let $\{\mathbf{u}_k\}$ be a sequence in $F(A)$. For each positive integer k, choose a point $\mathbf{v}_k$ in A with $\mathbf{u}_k = F(\mathbf{v}_k)$. Since A is sequentially compact, there is a subsequence $\{\mathbf{v}_{k_j}\}$ that converges to a point $\mathbf{v}$ in A. But the mapping $F : A \to \mathbb{R}^m$ is continuous at the point $\mathbf{v}$. Thus, $\{\mathbf{u}_{k_j}\} = \{F(\mathbf{v}_{k_j})\}$ converges to the point $F(\mathbf{v})$ in $F(A)$. Hence every sequence in $F(A)$ has a subsequence that converges to a point in $F(A)$. By definition, this means that $F(A)$ is sequentially compact. ∎

Lemma 11.21 Every nonempty sequentially compact subset of $\mathbb{R}$ has a smallest and a largest member.

Proof
Let A be a subset of $\mathbb{R}$ that is sequentially compact. According to Theorem 11.16, the set A is bounded and closed in $\mathbb{R}$. Since A is bounded by the Completeness Axiom for $\mathbb{R}$, A has a least upper bound. Denote the least upper bound of A by b. Then $x \leq b$ for all x in A. On the other hand, if k is any positive integer, then $b - 1/k$ is not an upper bound of A, so we can choose a point in A, which we label x_k, with $b - 1/k < x_k \leq b$. From the Comparison Lemma, it follows that the sequence $\{x_k\}$ converges to b. But A is a closed subset of $\mathbb{R}$, so b belongs to A. The number b is the largest member of A.

A similar proof, with the greatest lower bound replacing the least upper bound, shows the existence of a smallest member of A. ∎

Theorem 11.22 The Extreme Value Theorem Let A be a nonempty sequentially compact subset of $\mathbb{R}^n$ and suppose that the function $f : A \to \mathbb{R}$ is continuous. Then the function $f : A \to \mathbb{R}$ attains a smallest and a largest value.

Proof
From Theorem 11.20, it follows that the image $f(A)$ is sequentially compact. According to Lemma 11.21, $f(A)$ has both a smallest and a largest member. ∎

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Corollary 11.23 Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function defined on a generalized rectangle in $\mathbb{R}^n$. Then the function $f : \mathbf{I} \to \mathbb{R}$ attains a smallest and a largest value.

Proof

The proof follows immediately from the Extreme Value Theorem and Corollary 11.19, which asserts that a generalized rectangle is sequentially compact. ■

Definition A nonempty subset A of $\mathbb{R}^n$ is said to have the Extreme Value Property provided that every continuous function $A : \mathbb{R} \to \mathbb{R}$ has a maximum and a minimum functional value.

Theorem 11.22 is the assertion that a nonempty sequentially compact subset of $\mathbb{R}^n$ has the Extreme Value Property. In fact, this is the only type of subset of $\mathbb{R}^n$ that has this property.

Theorem 11.24 A nonempty subset of $\mathbb{R}^n$ has the Extreme Value Property if and only if it is sequentially compact.

Proof

Let A be a nonempty subset of $\mathbb{R}^n$. If A is sequentially compact, then, by Theorem 11.22, it has the Extreme Value Property. It remains to prove the converse. Suppose that A has the Extreme Value Property. To show that A is sequentially compact, by Theorem 11.18, it suffices to show that A is bounded and closed. First, we show that it is bounded. Indeed, define the continuous function $g : A \to \mathbb{R}$ by $g(\mathbf{u}) = \|\mathbf{u}\|$ for u in A. Since A has the Extreme Value Property, this function has a maximum functional value and therefore, in particular, it is bounded above. This means that the set A is bounded. It remains to show that the set A is closed. Indeed, let $\{\mathbf{v}_k\}$ be a sequence in A that converges to the point $\mathbf{v}$ in $\mathbb{R}^n$. We must show that $\mathbf{v}$ belongs to A. Define the continuous function $h : A \to \mathbb{R}$ by $h(\mathbf{u}) = \|\mathbf{u} - \mathbf{v}\|$ for $\mathbf{u}$ in A. Select $\mathbf{u}_*$ in A to be a minimizer for $h : A \to \mathbb{R}$. Since $\lim_{k\to\infty} \text{dist}(\mathbf{v}_k, \mathbf{v}) = 0$,

$$h(\mathbf{u}_*) = \inf h(A) = 0.$$

Thus, $\|\mathbf{u}_* - \mathbf{v}\| = 0$; so $\mathbf{v} = \mathbf{u}_*$ and therefore $\mathbf{v}$ belongs to A. ■

Uniform Continuity

Recall that in order to prove that a continuous function $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is integrable, we needed to prove that each such function is uniformly continuous. In the study of integration for functions of several real variables, a similar result is needed.

Moreover, the concept of uniform continuity for mappings between Euclidean spaces is also needed to establish a change of variables formula for integrals of functions of several variables.

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Definition Let A be a subset of $\mathbb{R}^n$. A mapping $F : A \to \mathbb{R}^m$ is said to be *uniformly continuous* provided that whenever $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ are sequences in A such that

$$\lim_{k\to\infty} \text{dist}(\mathbf{u}_k,\ \mathbf{v}_k) = 0,$$

then

$$\lim_{k\to\infty} \text{dist}(F(\mathbf{u}_k),\ F(\mathbf{v}_k)) = 0.$$

It is clear that a uniformly continuous mapping is continuous. Indeed, suppose that $F : A \to \mathbb{R}^m$ is uniformly continuous and let $\{\mathbf{u}_k\}$ be a sequence in A that converges to the point $\mathbf{u}$ in A. Set $\mathbf{v}_k \equiv \mathbf{u}$ for each index k. Observe that $\lim_{k\to\infty} \text{dist}(\mathbf{u}_k,\ \mathbf{v}_k) = 0$. By the definition of uniform convergence, $\lim_{k\to\infty} \text{dist}(F(\mathbf{u}_k),\ F(\mathbf{v}_k)) = 0$; that is, the image sequence $\{F(\mathbf{u}_k)\}$ converges to $F(\mathbf{u})$. Thus, $F : A \to \mathbb{R}^m$ is continuous. As we have already seen in the case of functions of a single real variable, a continuous mapping is not necessarily uniformly continuous. However, continuous mappings that have a domain that is sequentially compact are uniformly continuous. We leave the proof of this result as an exercise (Exercise 5) since the proof just requires a minor modification of the proof provided of Theorem 3.17, the special case of a function whose domain is a closed bounded interval of real numbers.

Theorem 11.25 Let A be a subset of $\mathbb{R}^n$ and suppose that the mapping $F : A \to \mathbb{R}^m$ is continuous. If the domain A is sequentially compact, then the mapping $F : A \to \mathbb{R}^m$ is uniformly continuous.

Corollary 11.26 A continuous mapping $f : \mathbf{I} \to \mathbb{R}$ defined on a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$ is uniformly continuous.

Proof

The proof follows immediately from the preceding theorem and Corollary 11.19, which asserts that a generalized rectangle is sequentially compact. ∎

Just as there is the ϵ-δ criterion for continuity of a mapping at a point that is equivalent to the sequential criterion for continuity at a point, we have the following ϵ-δ criterion for uniform continuity of a mapping. It is a direct extension of Theorem 3.22; we leave the proof as an exercise.

Theorem 11.27 For a mapping $\mathbf{F} : A \to \mathbb{R}^m$ defined on a subset A of $\mathbb{R}^n$, the following two assertions are equivalent:

i. The mapping $\mathbf{F} : A \to \mathbb{R}^n$ is uniformly continuous; that is, for two sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ in A,

$$\lim_{k\to\infty} \text{dist}(\mathbf{F}(\mathbf{u}_k), \mathbf{F}(\mathbf{v}_k)) = 0 \qquad \text{if} \qquad \lim_{k\to\infty} \text{dist}(\mathbf{u}_k, \mathbf{v}_k) = 0.$$

ii. For each positive number ϵ there is a positive number δ such that for $\mathbf{u}$, $\mathbf{v}$ in A,

$$\text{dist}(\mathbf{F}(\mathbf{u}), \mathbf{F}(\mathbf{v})) < \epsilon \qquad \text{if} \qquad \text{dist}(\mathbf{u}, \mathbf{v}) < \delta.$$

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EXERCISES FOR SECTION 11.2

1. Determine which of the following subsets of $\mathbb{R}$ is sequentially compact. Justify your conclusions.
   a. $\{x \text{ in } [0, 1] \mid x \text{ is rational}\}$
   b. $\{x \text{ in } \mathbb{R} \mid x^2 > x\}$
   c. $\{x \text{ in } \mathbb{R} \mid e^x - x^2 \leq 2\}$
2. Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Prove that the set

$$\{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{u}, \mathbf{v}) \leq r\}$$

   is sequentially compact.
3. Show that an open ball in $\mathbb{R}^n$ is bounded.
4. Can an open ball in $\mathbb{R}^n$ be sequentially compact?
5. Examine the proof of Theorem 3.17 and replace the absolute value of the difference of two points by the distance between the points to provide a proof of Theorem 11.25.
6. Let A be a subset of $\mathbb{R}^n$ and let the function $f : A \to \mathbb{R}$ be continuous.
   a. If A is bounded, is $f(A)$ bounded?
   b. If A is closed, is $f(A)$ closed?
7. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and that $f(\mathbf{u}) \geq \|\mathbf{u}\|$ for every point $\mathbf{u}$ in $\mathbb{R}^n$. Prove that $f^{-1}([0, 1])$ is sequentially compact.
8. Let A and B be sequentially compact subsets of $\mathbb{R}$. Define $K = \{(x, y) \text{ in } \mathbb{R}^2 \mid x \text{ in } A,\ y \text{ in } B\}$. Prove that K is sequentially compact.
9. Let A be a subset of $\mathbb{R}^n$ that is sequentially compact and let $\mathbf{v}$ be a point in $\mathbb{R}^n \backslash A$. Prove that there is a point $\mathbf{u}_0$ in the set A such that

$$\text{dist}(\mathbf{u}_0, \mathbf{v}) \leq \text{dist}(\mathbf{u}, \mathbf{v}) \qquad \text{for all points } \mathbf{u} \text{ in } A.$$

   Is this point unique?
10. A mapping $F: \mathbb{R}^n \to \mathbb{R}^m$ is said to be *Lipschitz* if there is a number C such that

$$\text{dist}(F(\mathbf{u}), F(\mathbf{v})) \leq C\text{dist}(\mathbf{u}, \mathbf{v}) \qquad \text{for all points } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathbb{R}^n.$$

    The number C is called a *Lipschitz constant* for the mapping. Show that a Lipschitz mapping is uniformly continuous.
11. Prove Theorem 11.27.

11.3 PATHWISE CONNECTEDNESS AND THE INTERMEDIATE VALUE THEOREM*

In Section 3.3, we defined a subset A of $\mathbb{R}$ to be convex provided that whenever the points u and v belong to A and $u < v$, then the whole interval $[u, v]$ is contained in A. We showed that a set of real numbers is convex if and only if it is an interval. We proved the Intermediate Value Theorem, as expressed in Theorem 3.14, which asserts that if I is an interval and the function $f : I \to \mathbb{R}$ is continuous, then its image $f(I)$ is also an interval.

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We devote this section to studying subsets A of $\mathbb{R}^n$ that we call *pathwise-connected* and proving that a continuous function whose domain is pathwise-connected has an interval as its image.

The following is the natural generalization of the concept of convexity to subsets of $\mathbb{R}^n$.

Definition A subset A of $\mathbb{R}^n$ is said to be *convex* provided that if $\mathbf{u}$ and $\mathbf{v}$ belong to A, then the segment $\{t\mathbf{u} + (1-t)\mathbf{v} \mid 0 \leq t \leq 1\}$ is contained in A.

Example 11.28 Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let r be a positive real number. Then the open ball $\mathcal{B}_r(\mathbf{u})$ is convex. To see this, let $\mathbf{v}$ and $\mathbf{w}$ be points in $\mathcal{B}_r(\mathbf{u})$ and let t be a real number such that $0 \leq t \leq 1$. We will show that $(1-t)\mathbf{v} + t\mathbf{w}$ belongs to $\mathcal{B}_r(\mathbf{u})$; that is,

$$\|(1-t)\mathbf{v} + t\mathbf{w} - \mathbf{u}\| < r.$$

Indeed, since

$$(1-t)\mathbf{v} + t\mathbf{w} - \mathbf{u} = (1-t)(\mathbf{v}-\mathbf{u}) + t(\mathbf{w}-\mathbf{u}),$$

the Triangle Inequality implies that

$$\begin{aligned}
\|(1-t)\mathbf{v} + t\mathbf{w} - \mathbf{u}\| &= \|(1-t)(\mathbf{v}-\mathbf{u}) + t(\mathbf{w}-\mathbf{u})\| \\
&\leq \|(1-t)(\mathbf{v}-\mathbf{u})\| + \|t(\mathbf{w}-\mathbf{u})\| \\
&= (1-t)\|(\mathbf{v}-\mathbf{u})\| + t\|(\mathbf{w}-\mathbf{u})\| \\
&< (1-t)r + tr \\
&= r.
\end{aligned}$$

■

Example 11.29 For $1 \leq i \leq n$, let I_i be an interval of real numbers. Then the Cartesian product

$$A \equiv I_1 \times I_2 \times \cdots \times I_n$$

is a convex subset of $\mathbb{R}^n$. Indeed, let $\mathbf{u}$ and $\mathbf{v}$ belong to A and let $0 \leq t \leq 1$. Then for $1 \leq i \leq n$, $p_i[t\mathbf{u} + (1-t)\mathbf{v}] = tp_i(\mathbf{u}) + (1-t)p_i(\mathbf{v})$ belongs to I_i since $p_i(\mathbf{u})$ and $p_i(\mathbf{u})$ belong to I_i and I_i is a convex set of real numbers. Thus, $t\mathbf{u} + (1-t)\mathbf{v}$ belongs to A. ■

While convexity is a natural extension of the concept of an interval of real numbers, there is a more general concept called *pathwise connectedness* that is useful. To define this we first need to introduce the concept of parametrized paths.

Definition Let a and b be real numbers with $a < b$. Then a continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ is called a *parametrized path*. The domain of $\gamma : [a, b] \to \mathbb{R}^n$ is called the *parameter space*, and the image of $\gamma : [a, b] \to \mathbb{R}^n$ is called a *path*.

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It is important to distinguish between a path and a parametrized path: A path is a *subset* of $\mathbb{R}^n$, whereas a parametrized path is a *mapping*. Furthermore, any path is the image of several different parametrized paths.

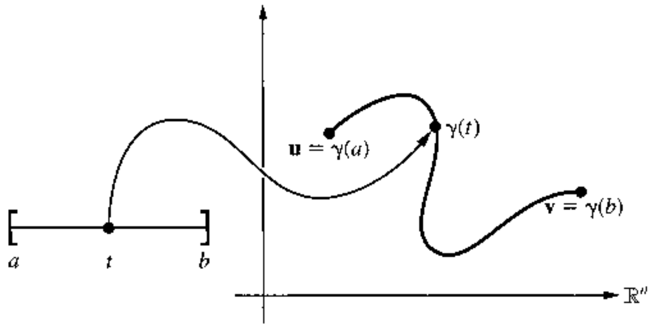

FIGURE 11.2 Parametrized paths.

Example 11.30 The upper half-circle

$$\{(x,\ y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 = 1,\ y \geq 0\}$$

is a path that is the image of the parametrized path $\gamma : [-1, 1] \to \mathbb{R}^2$ defined by $\gamma(t) = (t, \sqrt{1-t^2})$ for $-1 \leq t \leq 1$. This upper half-circle is also the image of the parametrized path $\gamma : [0, \pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (-\cos t,\ \sin t)$ for $0 \leq t \leq \pi$. ■

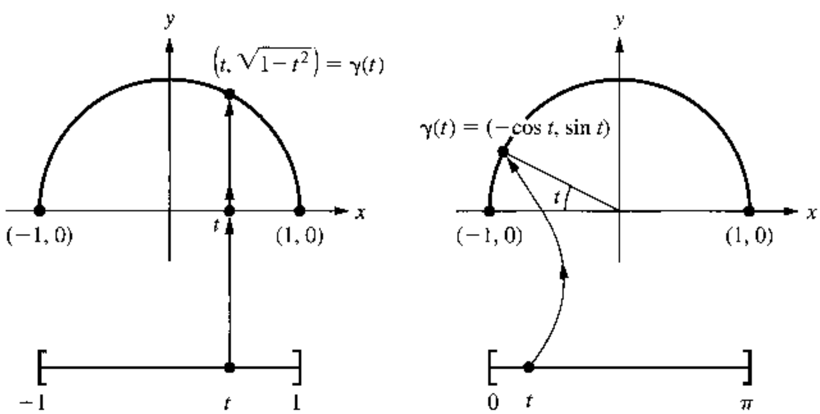

FIGURE 11.3 Two parametrizations of the upper half-circle.

Also note that every path is the image of a parametrized path having as its parameter space the interval $[0, 1]$. Indeed, if a path is the image of a parametrized path $\gamma : [a, b] \to \mathbb{R}^n$, then it is also the image of the parametrized path $\bar{\gamma} : [0, 1] \to \mathbb{R}^n$

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defined by

$$\bar{\gamma}(t) = \gamma[(1-t)a + tb] \qquad \text{for } 0 \leq t \leq 1.$$

Definition

i. Let A be a subset of $\mathbb{R}^n$ that contains the points $\mathbf{u}$ and $\mathbf{v}$. By a path in A joining $\mathbf{u}$ and $\mathbf{v}$ we mean the image of a parametrized path $\gamma : [a, b] \to \mathbb{R}^n$, with $\gamma(a) = \mathbf{u}$ and $\gamma(b) = \mathbf{v}$, such that the image of $\gamma : [a, b] \to \mathbb{R}^n$ is contained in A.

ii. A subset A of $\mathbb{R}^n$ is said to be *pathwise-connected* provided that every pair of points in A can be joined by a path in A.

Theorem 11.31 A subset of $\mathbb{R}$ is pathwise-connected if and only if it is an interval.

Proof

First, suppose that I is an interval. Let u and v be points in I. We can suppose that $u < v$. Since I is an interval, the closed interval $[u, v]$ is a subset of I, so we can define $[u, v]$ to be the parameter space and define $\gamma(t) = t$ for $u \leq t \leq v$ to obtain a path in I joining the points u and v.

To prove the converse, suppose that I is a pathwise-connected subset of $\mathbb{R}$. To verify that I is an interval, we select two points u and v in I with $u < v$. We must show that the closed interval $[u, v]$ is a subset of I. Since I is pathwise-connected, there is a parametrized path $\gamma[a, b] \to I$ with $\gamma(a) = u$ and $\gamma(b) = v$. Now the Intermediate Value Theorem asserts that $[\gamma(a), \gamma(b)] \subseteq \gamma([a, b])$, so $[u, v] \subseteq I$. ■

Of course, a convex subset A of $\mathbb{R}^n$ is pathwise-connected since if $\mathbf{u}$ and $\mathbf{v}$ belong to A, then

$$\gamma(t) = t\mathbf{u} + (1-t)\mathbf{v}, \qquad 0 \leq t \leq 1,$$

defines a path in A joining the points $\mathbf{u}$ to $\mathbf{v}$.

Example 11.32 Let D be a convex subset of $\mathbb{R}^2$ and suppose that the function $f : D \to \mathbb{R}$ is continuous. Then the graph of the function

$$G = \{(x, y, z) \mid (x, y) \qquad \text{in } D \text{ and } z = f(x, y)\}$$

is a pathwise-connected subset of $\mathbb{R}^3$. Indeed, choose two points $(\mathbf{u}, f(\mathbf{u}))$ and $(\mathbf{v}, f(\mathbf{v}))$ in the graph G, where $\mathbf{u}$ and $\mathbf{v}$ belong to D. Then

$$\gamma(t) = (t\mathbf{u} + (1-t)\mathbf{v}), f(t\mathbf{u} + (1-t)\mathbf{v}) \qquad \text{for } 0 \leq t \leq 1$$

defines a path in G joining $(\mathbf{u}, f(\mathbf{u}))$ and $(\mathbf{v}, f(\mathbf{v}))$. ■

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Example 11.33 The unit sphere S in $\mathbb{R}^3$ is defined to be the set

$$S = \{\mathbf{u} = (x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

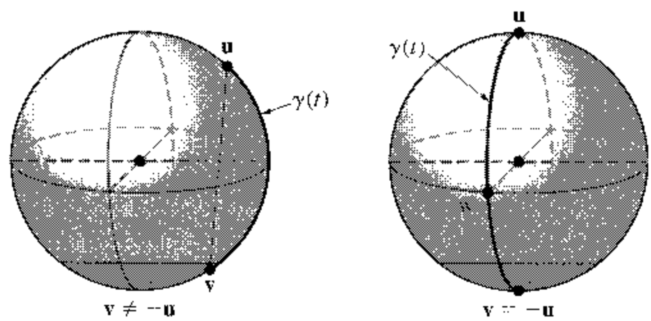

FIGURE 11.4 The sphere is pathwise-connected.

The unit sphere is pathwise-connected. To verify this, let $\mathbf{u}$ and $\mathbf{v}$ be points in S. If $\mathbf{u}$ and $\mathbf{v}$ are not antipodal points—that is, if $\mathbf{v} \neq -\mathbf{u}$—then we can check that $(1-t)\mathbf{u} + t\mathbf{v} \neq 0$ for $0 \leq t \leq 1$, so

$$\gamma(t) = \frac{(1-t)\mathbf{u} + t\mathbf{v}}{\|(1-t)\mathbf{u} + t\mathbf{v}\|}, \qquad \text{for } 0 \leq t \leq 1,$$

defines a parametrized path joining $\mathbf{u}$ and $\mathbf{v}$. It remains to consider the case where $\mathbf{u}$ and $\mathbf{v}$ are antipodal, that is, $\mathbf{v} = -\mathbf{u}$. In this case, choose a point $\mathbf{w}$ in S that is perpendicular to $\mathbf{u}$. We can check that $\cos t\mathbf{u} + \sin t\mathbf{w}$ lies in S for each real t, and so

$$\gamma(t) = \cos t\mathbf{u} + \sin t\mathbf{w} \qquad \text{for } 0 \leq t \leq \pi$$

defines a parametrized path $\gamma : [0, \pi] \to \mathbb{R}$ joining the points $\mathbf{u}$ and $-\mathbf{u}$. We leave as an exercise the verification that the image of each of these parametrized paths lies in S. ■

Theorem 11.34 Let A be a subset of $\mathbb{R}^n$ and suppose that the mapping $F: A \to \mathbb{R}^m$ is continuous. If A is pathwise-connected, then its image $F(A)$ is also pathwise-connected.

Proof

Let $\mathbf{u}$ and $\mathbf{v}$ be points in $F(A)$. We must find a path in $F(A)$ joining $\mathbf{u}$ and $\mathbf{v}$. Choose points $\mathbf{x}$ and $\mathbf{y}$ in A such that $F(\mathbf{x}) = \mathbf{u}$ and $F(\mathbf{y}) = \mathbf{v}$. Since by assumption the domain A is pathwise-connected, it follows that there is a parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = \mathbf{x}$, $\gamma(b) = \mathbf{y}$, and $\gamma([a, b]) \subseteq A$. Since the composition of continuous mappings is continuous, it follows that the composition $F \circ \gamma : [a, b] \to \mathbb{R}^m$ is a parametrized path in $F(A)$ joining $\mathbf{u}$ and $\mathbf{v}$. ■

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Theorem 11.35 Let A be a pathwise-connected subset of $\mathbb{R}^n$ and suppose that the function $f : A \to \mathbb{R}$ is continuous. Then its image $f(A)$ is an interval.

Proof

According to Theorem 11.34, $f(A)$ is a pathwise-connected subset of $\mathbb{R}$. It then follows from Theorem 11.31 that $f(A)$ is an interval. ∎

Definition A subset A of $\mathbb{R}^n$ is said to have the *Intermediate Value Property* provided that every continuous function $f : A \to \mathbb{R}$ has an interval as its image.

Theorem 11.35 asserts that every pathwise-connected subset of $\mathbb{R}^n$ has the Intermediate Value Property. It turns out that there are other subsets of $\mathbb{R}^n$ that also have the Intermediate Value Property. In the following section we introduce the concept of connectedness and show that a subset of $\mathbb{R}^n$ is connected if and only if it has the Intermediate Value Property.

EXERCISES FOR SECTION 11.3

1. Let I be an interval of real numbers and suppose that the function $f : I \to \mathbb{R}$ is continuous. The graph of this function is the subset of $\mathbb{R}^2$ defined by
$$G = \{(x, y) \text{ in } \mathbb{R}^2 \mid x \text{ in } I,\ y = f(x)\}.$$
Show that G is pathwise-connected. Is G convex?
2. Let A and B be pathwise-connected subsets of $\mathbb{R}^n$ whose intersection $A \cap B$ is nonempty. Prove that the union $A \cup B$ is also pathwise-connected.
3. Let a and b be positive real numbers. Use Exercises 1 and 2 to show that the ellipse
$$\{(x, y) \text{ in } \mathbb{R}^2 \mid x^2/a + y^2/b = 1\}$$
is pathwise-connected.
4. Let A and B be convex subsets of $\mathbb{R}^n$. Prove that the intersection $A \cap B$ is also convex. Is it true that the intersection of two pathwise-connected subsets of $\mathbb{R}^n$ is also pathwise-connected?
5. Show that the set $\{(x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^4 = 1\}$ is pathwise-connected.
6. Show that the set $S = \{(x, y) \text{ in } \mathbb{R}^2 \mid \text{either } x \text{ or } y \text{ is rational}\}$ is pathwise-connected.
7. Show that the two parametrized paths described in Example 11.33 have images in S.
8. Let $\mathbf{u}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Show that the closed ball $\{\mathbf{v} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{v}, \mathbf{u}) \leq r\}$ is convex.
9. Let $\mathbf{u}$ be a fixed point in $\mathbb{R}^n$ and let c be a fixed real number. Prove that each of the following three sets is convex:
$$\{\mathbf{v} \text{ in } \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{u}\rangle > c\}, \qquad \{\mathbf{v} \text{ in } \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{u}\rangle = c\}, \qquad \{\mathbf{v} \text{ in } \mathbb{R}^n \mid \langle \mathbf{v}, \mathbf{u}\rangle < c\}$$
10. Given a point $\mathbf{u}$ in $\mathbb{R}^n$ and a point $\mathbf{v}$ in $\mathbb{R}^m$, we define the point $(\mathbf{u}, \mathbf{v})$ to be the point in $\mathbb{R}^{n+m}$ whose first n components coincide with the components of $\mathbf{u}$ and whose last m components coincide with those of $\mathbf{v}$. Suppose that A is a subset of $\mathbb{R}^n$ and

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that $F: A \to \mathbb{R}^m$ is continuous. The graph G of this mapping is defined by

$$G = \{(\mathbf{u}, \mathbf{v}) \text{ in } \mathbb{R}^{n+m} \mid \mathbf{u} \text{ in } A,\ \mathbf{v} = F(\mathbf{u})\}.$$

Show that if A is pathwise-connected, then G is also pathwise-connected.

11. Use Exercises 2 and 10 to show that the set $\{(x, y, z) \text{ in } \mathbb{R}^3 \mid 2x^2 + y^2 + z^2 = 1\}$ is pathwise-connected.

11.4 CONNECTEDNESS AND THE INTERMEDIATE VALUE PROPERTY*

In the preceding section we proved that a pathwise-connected subset of $\mathbb{R}^n$ has the Intermediate Value Property; that is, every continuous function whose domain is pathwise-connected has an interval as its image. In this section we characterize all subsets of $\mathbb{R}^n$ that have the Intermediate Value Property.

Definition Let A be a subset of $\mathbb{R}^n$. Two open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n$ are said to *separate* the set A provided that the two sets $A \cap \mathcal{U}$ and $A \cap \mathcal{V}$ are nonempty, they are disjoint, and their union equals A; that is

$$A \cap \mathcal{U} \neq \emptyset,\ A \cap \mathcal{V} \neq \emptyset$$

and

$$(A \cap \mathcal{U}) \cap (A \cap \mathcal{V})\ \emptyset, \quad (A \cap \mathcal{U}) \cup (A \cap \mathcal{V}) = A.$$

Definition A subset A of $\mathbb{R}^n$ is said to be *connected* provided that there do not exist two open subsets of $\mathbb{R}^n$ that separate A.

The following theorem justifies the introduction of the concept of connectedness.

Theorem 11.36 A subset of $\mathbb{R}^n$ is connected if and only if it has the Intermediate Value Property.

Proof

First, suppose that A is a subset of $\mathbb{R}^n$ that is connected. We will show that A has the Intermediate Value Property. Indeed, let the function $f : A \to \mathbb{R}$ be continuous. To show that its image $f(A)$ is an interval, we suppose otherwise and derive a contradiction. If $f(A)$ is not an interval, then there are points $\mathbf{u}$ and $\mathbf{v}$ in A and a real number c such that

$$f(\mathbf{u}) < c < f(\mathbf{v}),$$

but c does not belong to the image $f(A)$. Define

$$A_1 = f^{-1}(-\infty, c) \qquad \text{and} \qquad A_2 = f^{-1}(c, \infty).$$

Observe that neither A_1 nor A_2 is empty since $\mathbf{u}$ belongs to A_1 and $\mathbf{v}$ belongs to A_2. The sets A_1 and A_2 are disjoint, and furthermore, $A_1 \cup A_2 = A$ since the number c is not in $f(A)$.

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We find two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n$, each of which is open in $\mathbb{R}^n$, that have the property that

$$A \cap \mathcal{U} = A_1 \qquad \text{and} \qquad A \cap \mathcal{V} = A_2.$$

Once this is done, from the above-mentioned properties of the sets A_1 and A_2 it follows that $\mathcal{U}$ and $\mathcal{V}$ separate A. This contradicts the assumption that A is connected.

Let $\mathbf{u}$ be a point in A_1. Since $f(\mathbf{u}) < c$, it follows from the continuity of $f : A \to \mathbb{R}$ and the ϵ-δ criterion for continuity provided in Theorem 11.11 that we can choose a positive number $r = r(\mathbf{u})$ such that $f(\mathbf{v}) < c$ if $\mathbf{v}$ is in $\mathcal{B}_r(\mathbf{u}) \cap A$. Define $\mathcal{U}$ to be the union of these open balls $\mathcal{B}_r(\mathbf{u})$ as $\mathbf{u}$ varies in A_1. Then $\mathcal{U}$ is open in $\mathbb{R}^n$ since it is the union of open subsets in $\mathbb{R}^n$, and it is clear that $A \cap \mathcal{U} = A_1$. Similarly, for each point $\mathbf{v}$ in A_2 we can choose an open ball whose intersection with A is contained in A_2. The union of such open balls defines an open subset $\mathcal{V}$ of $\mathbb{R}^n$ whose intersection with A equals A_2.

To prove the converse, suppose that every continuous function $f : A \to \mathbb{R}$ has the Intermediate Value Property. We will show that A is connected by assuming otherwise and deriving a contradiction. Suppose that A is not connected. Then there are two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n$, each of which is open in $\mathbb{R}^n$, that separate A. Define the function $f : A \to \mathbb{R}$ by

$$f(\mathbf{u}) = \begin{cases} 0 & \text{if } \mathbf{u} \text{ is in } \mathcal{U} \cap A \\ 1 & \text{if } \mathbf{u} \text{ is in } \mathcal{V} \cap A. \end{cases}$$

Then the function $f : A \to \mathbb{R}$ certainly fails to have the Intermediate Value Property since it attains exactly two functional values, 0 and 1. On the other hand, the function $f : A \to \mathbb{R}$ is continuous. Indeed, to verify continuity, just observe that since both $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}^n$, it follows from the definition of $f : A \to \mathbb{R}$ that for each point $\mathbf{u}$ in A there is an open ball $\mathcal{B}_r(\mathbf{u})$ such that $f : A \to \mathbb{R}$ is constant on $A \cap \mathcal{B}_r(\mathbf{u})$. Thus, this function is certainly continuous at the point $\mathbf{u}$. The existence of this continuous function whose image is not an interval contradicts the assumption that A has the Intermediate Value Property. Thus, A must be connected. ∎

Corollary 11.37 Every pathwise-connected subset of $\mathbb{R}^n$ is connected.

Proof

Let A be a pathwise-connected subset of $\mathbb{R}^n$. Theorem 11.35 asserts that A has the Intermediate Value Property, and so, by Theorem 11.36, A must be connected. ∎

Theorem 11.31 asserts that a subset of $\mathbb{R}$ is an interval if and only if it is pathwise-connected. In particular, by Corollary 11.37, each interval is connected. In fact, it is not difficult to show that each connected subset of $\mathbb{R}$ is an interval (Exercise 2). Thus, for a subset of $\mathbb{R}$, there is no distinction between being an interval, being pathwise-connected, and being connected. For this reason, there is no need to introduce the concepts of connectedness and pathwise connectedness in the study of real-valued functions of a single real variable.

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It is reasonable to ask about the distinction between connectedness and pathwise connectedness for subsets of $\mathbb{R}^n$ if $n > 1$. Corollary 11.37 asserts that every pathwise-connected subset of $\mathbb{R}^n$ is connected. The converse is not true; there are connected subsets of $\mathbb{R}^n$ that are not pathwise-connected.

Example 11.38 We will describe a subset of the plane $\mathbb{R}^2$ that is connected but not pathwise-connected. It is the union of two pathwise-connected sets. First, define $K = \{(x, y) \text{ in } \mathbb{R}^2 \mid x = 0, -1 \leq y \leq 1\}$. Observe that K is convex, so it is pathwise-connected. Now define $G = \{(x, y) \text{ in } \mathbb{R}^2 \mid 0 < x \leq 1, y = \sin 1/x\}$. Then G is also pathwise-connected since it is the graph of a continuous function whose domain is an interval. We define $A = K \cup G$.

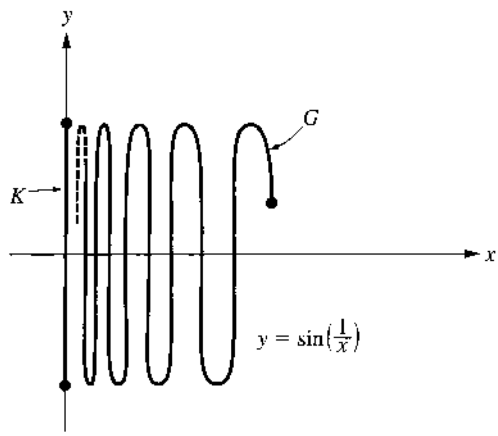

FIGURE 11.5 $A = K \cup G$ is connected but is not pathwise-connected.

It turns out that the set A is connected. But A is not pathwise-connected since it is not possible to find a path in the set A that joins a point in K to a point in G. The details of verifying these assertions are left as exercises (Exercises 6 and 7). ■

We will study connectedness more generally in the context of our study of metric spaces, which we will undertake in Chapter 12. In particular, in Section 12.5, we will prove that an open subset of $\mathbb{R}^n$ is connected if and only if it is pathwise-connected. So it is not so surprising that the preceding example of a subset of the plane that is connected but not pathwise-connected had to be a little wild.

We defined pathwise connectedness in terms of paths joining points. Intuitively, a path seems to be a familiar geometric object. As always, however, care is needed in framing our geometric intuition as a precise mathematical assertion. With respect to pathwise connectedness, the warning note is sounded by a famous example of G. Peano,[1]

[1] See details in the excellent book *Introduction to Topology and Modern Analysis* by George Simmons (New York: McGraw-Hill, 1963).

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who proved that the square $S = \{(x, y) \text{ in } \mathbb{R}^2 \mid |x| \leq 1, |y| \leq 1\}$ is a path; that is, he discovered a continuous mapping $\gamma : [0, 1] \to \mathbb{R}^2$ whose image is the square S.

EXERCISES FOR SECTION 11.4

1. Let $\mathbb{Q}$ be the set of rational numbers. Show that $\mathbb{Q}$ is not connected.

2. Show that a connected subset of $\mathbb{R}$ must be an interval.

3. Let A be a connected subset of $\mathbb{R}^3$. Suppose that the points $(0, 0, 1)$ and $(4, 3, 0)$ are in A.

a. Prove that there is a point in A whose second component is 2.

b. Prove that there is a point in A whose norm is 4.

4. Suppose that A is a subset of $\mathbb{R}^n$ that fails to be connected and let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of $\mathbb{R}^n$ that separate A. Suppose that B is a subset of A that is connected. Prove that either $B \subseteq \mathcal{U}$ or $B \subseteq \mathcal{V}$.

5. Let K be a sequentially compact subset of $\mathbb{R}^n$ and suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ that contains K. Prove that there is some positive number r such that for any point $\mathbf{u}$ in K, $\mathcal{B}_r(\mathbf{u}) \subseteq \mathcal{O}$.

6. Use Exercises 4 and 5 to show that the set A defined in Example 11.38 is connected.

7. Show that the set A defined in Example 11.38 is not pathwise-connected. [*Hint:* Let $\mathbf{u} = (0, 1)$ and $\mathbf{v} = (1, \sin 1)$. Suppose that there is a parametrized path $\gamma : [0, 1] \to \mathbb{R}^2$ in A joining $\mathbf{u}$ to $\mathbf{v}$. Define t_* to be the supremum of the points t in $[0, 1]$ such that γ maps the interval $[0, t]$ into K. Show that the second component of γ is not continuous at the point t_*.]

8. Which subsets of $\mathbb{R}$ are both sequentially compact and connected?

9. Let K be a sequentially compact subset of $\mathbb{R}^n$. Prove that K is not connected if and only if there are nonempty disjoint subsets A and B of K, with $A \cup B = K$ and a positive number ϵ such that $d(\mathbf{u}, \mathbf{v}) > \epsilon$ for all $\mathbf{u}$ in A and all $\mathbf{v}$ in B. Is the assumption of sequentially compactness necessary for the existence of such an ϵ?

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CHAPTER 12*

METRIC SPACES

The first nine chapters of this book were devoted to the study of real-valued functions of a single real variable. Chapters 10 and 11 dealt with the study of Euclidean spaces and mappings between these spaces. It was observed in the late nineteenth century that several concepts that are useful in this study could be isolated and then examined in the more abstract context of metric spaces. This direction of thought clarifies the basis of many of the arguments that we have used; more important, it permits us to understand those concepts that can be generalized for use in the study of diverse problems in mathematics. In the first section of this chapter, we will define a *metric space* and give a number of examples of metric spaces. We will also extend to metric spaces the concepts of openness and closedness that were introduced in Chapter 11 for Euclidean spaces. In Section 12.2, we will introduce the notion of a Cauchy sequence in a metric space and will call a metric space *complete* provided that every Cauchy sequence in the space converges to a point in that space. Examples of complete metric spaces will be considered. An important theorem called the Contraction Mapping Principle will be proved. In Section 12.3, we will review some of the results established earlier regarding the solutions of differential equations. Then we will use the Contraction Mapping Principle to prove a fundamental theorem about the solvability of a *nonlinear differential equation*. The final two sections will be devoted to extending the results on continuity, sequential compactness, and connectedness established in Chapter 11 in the context of Euclidean spaces.

12.1 OPEN SETS, CLOSED SETS, AND SEQUENTIAL CONVERGENCE

Definition A set X is called a *metric space* if for any two points p and q in X there is defined a real number $d(p,q)$, called the *distance* between p and q such that the following three properties are satisfied:

i. Nonnegativity:

$$d(p,q) > 0 \qquad \text{if } p \neq q;\, d(p,\ p) = 0.$$

ii. Symmetry:

$$d(p,q) = d(q,p).$$

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iii. The Triangle Inequality:

$$d(p, q) \leq d(p, w) + d(w, q) \qquad \text{for all } w \text{ in } X.$$

The above function $d : X \times X \to [0, \infty)$ is called a *metric* on X. It is possible to have more than one metric on a set. When we call X a metric space, we suppose that a fixed metric has been prescribed. Some of the most important metric spaces in the study of classical analysis are described in the following three theorems.

Theorem 12.1 For any two real numbers p and q, define

$$d(p, q) = |p - q|.$$

Then d is a metric on $\mathbb{R}$.

The above theorem is an immediate consequence of the properties of the absolute value and is a special case of the following theorem regarding Euclidean space.

Theorem 12.2 For any two points p and q in Euclidean n-space $\mathbb{R}^n$, define

$$d(p, q) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}.$$

Then d is a metric on $\mathbb{R}^n$.

Proof

The nonnegativity property follows from the observation that the sum of squares of real numbers is always nonnegative and is 0 if and only if all the numbers are 0. The symmetry property is clear. In Chapter 10, the Triangle Inequality was established for points in $\mathbb{R}^n$. ■

Theorem 12.3 Define $C([a, b], \mathbb{R})$ to be the set of all continuous functions $f : [a, b] \to \mathbb{R}$, and for any two functions f and g in $C([a, b], \mathbb{R})$, define

$$d(f, g) = \max\{|f(x) - g(x)| \mid x \text{ in } [a, b]\}.$$

Then d is a metric on $C([a, b], \mathbb{R})$.

Proof

First, observe that if f and g are in $C([a, b], \mathbb{R})$, then the function $|f - g| : [a, b] \to \mathbb{R}$ is continuous. According to the Extreme Value Theorem, the function $|f - g| : [a, b] \to \mathbb{R}$ has a maximum value. Thus, $d(f, g)$ is properly defined.

Choose a point x_0 in the interval $[a, b]$ with $d(f, g) = |f(x_0) - g(x_0)|$. Then

$$0 \leq |f(x) - g(x)| = |g(x) - f(x)| \leq |g(x_0) - f(x_0)| \qquad \text{for all } x \text{ in } [a, b],$$

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so $d(f, g) = d(g, f) \geq 0$ and $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all x in $[a, b]$. It remains to verify the Triangle Inequality. Let h be in $C([a, b], \mathbb{R})$. By the Triangle Inequality in $\mathbb{R}$ and the definition of the metric,

$$\begin{aligned}|f(x) - g(x)| &\leq |f(x) - h(x)| + |h(x) - g(x)| \\ &\leq d(f, h) + d(h, g) \qquad \text{for all } x \text{ in } [a, b].\end{aligned}$$

Thus, we obtain the Triangle Inequality, $d(f, g) \leq d(f, h) + d(h, g)$. ∎

It is useful to observe that, for any two functions f and g in $C([a, b], \mathbb{R})$ and any positive number r,

$$d(f, g) < r$$

if and only if

$$f(x) - r < g(x) < f(x) + r \qquad \text{for all } x \text{ in } [a, b].$$

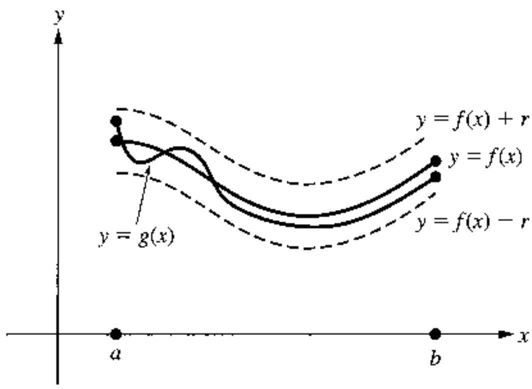

FIGURE 12.1 $d(f, g) < r$.

Theorem 12.4 Let X be any set. For any two points p and q in X, define

$$d(p, q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases}$$

This defines a metric on X that is called the *discrete metric*.

Proof

The nonnegativity and symmetry properties and the Triangle Inequality follow immediately from the definition. ∎

It is quite clear that the discrete metric on $\mathbb{R}^n$ and the Euclidean metric on $\mathbb{R}^n$ are distinct metrics.

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Every subset Y of a metric space X is a metric space in its own right, where given two points p and q in Y, the distance between p and q is the same as the distance between these points when they are considered points in the set X. Thus, every subset of $\mathbb{R}$, every subset of Euclidean n-space $\mathbb{R}^n$, and every subset of $C([a, b], \mathbb{R})$ is a metric space. When we consider a subset Y of a metric space X a metric space with the metric inherited from the metric on X, we refer to the *subspace* Y of the metric space X.

Definition Let X be a metric space.

i. For a point p in X and a positive number r, the set

$$\mathcal{B}_r(p) \equiv \{q \text{ in } X \mid d(q, p) < r\}$$

is called the *open ball* about p in X of radius r.

ii. Given a subset A of X, a point p in A is called an *interior point* of A if some open ball about p in X is contained in A. The set of all interior points of A is called the *interior* of A and is denoted by int A.

iii. A subset $\mathcal{O}$ of X is called *open* in X if every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$.

Observe that the definitions we have given of open ball, interior point, and openness for a general metric space generalize the corresponding concepts we have already considered for Euclidean n-space.

Example 12.5 Consider the metric space $C([0, 1], \mathbb{R})$. Given a function f in $C([0, 1], \mathbb{R})$ and a positive number r, we see that the open ball about f in $C([0, 1], \mathbb{R})$ of radius r consists of those continuous functions $g : [0, 1] \to \mathbb{R}$ such that

$$f(x) - r < g(x) < f(x) + r \qquad \text{for all } x \text{ in } [0, 1].$$

■

Example 12.6 Let X be any set considered a metric space with the discrete metric. Given a point p in X and a positive number r, it follows directly from the definition of the discrete metric that the open ball about p in X of radius r consists of the whole set X if $r \geq 1$ and consists of the single point p if $r < 1$. ■

Example 12.7 Let h and g be in $C([0, 1], \mathbb{R})$, with $h(x) < g(x)$ for all x in $[0, 1]$. Define $\mathcal{O} = \{f \text{ in } C([0, 1], \mathbb{R}) \mid h(x) < f(x) < g(x) \text{ for all } x \text{ in } [0, 1]\}$. Then $\mathcal{O}$ is open. Indeed, let f be in $\mathcal{O}$. Let r_1 and r_2 be the minimum values of the functions $f - h : [0, 1] \to \mathbb{R}$ and $g - f : [0, 1] \to \mathbb{R}$, respectively, and then let r be the smaller of the positive numbers r_1 and r_2. Then we see that $\mathcal{B}_r(f) \subseteq \mathcal{O}$. Thus, $\mathcal{O}$ is open. ■

Proposition 12.8 Let X be a metric space. Then every open ball in X is open in X.

Proof

Let p be a point in X and r be a positive number. Consider the open ball $\mathcal{B}_r(p)$. We will show that every point in $\mathcal{B}_r(p)$ is an interior point of $\mathcal{B}_r(p)$. Let q be a point

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in $\mathcal{B}_r(p)$. Define $R = r - d(p, q)$ and observe that R is positive. We claim that $\mathcal{B}_R(q) \subseteq \mathcal{B}_r(p)$. Indeed, by the Triangle Inequality, if x is an element of X, then

$$d(x, p) \leq d(x, q) + d(q, p),$$

so that if $d(x, q) < R = r - d(q, p)$, then $d(x, p) < r$. Thus, $\mathcal{B}_R(q) \subseteq \mathcal{B}_r(p)$, so q is an interior point of $\mathcal{B}_r(p)$. ■

For a set X, by a *sequence* in X, we mean a function $f : \mathbb{N} \to X$. It is customary to denote sequences by symbols such as $\{p_k\}$, $\{q_k\}$, and so on. If X is a metric space, we define the concept of convergence of a sequence as follows.

Definition Let X be a metric space. A sequence $\{p_k\}$ in X is said to *converge* to a point p in X provided that for each positive number ϵ there is a natural number N such that

$$d(p_k, p) < \epsilon \qquad \text{if } k \geq N.$$

We see that a sequence $\{p_k\}$ in X converges to the point p in X if and only if the real sequence $\{d(p_k, p)\}$ converges to 0. We call p the *limit* of the sequence $\{p_k\}$. Observe that a sequence can have at most one limit since if $\{p_k\}$ converges to p and also to p', then according to the Triangle Inequality,

$$0 \leq d(p, p') \leq d(p, p_k) + d(p_k, p') \qquad \text{for all indices } k.$$

From the Comparison Lemma for convergent real sequences it follows that $d(p, p') = 0$, so $p' = p$.

Given a sequence $\{p_k\}$ in X and $\{k_j\}$ a strictly increasing sequence of positive integers, the sequence $\{p_{k_j}\}$ is called a *subsequence* of $\{p_k\}$. Observe that if $\{p_k\}$ converges to p, then each subsequence of $\{p_{k_j}\}$ also converges to p since, by our results for real sequences, if $\lim_{k\to\infty} d(p_k, p) = 0$, then $\lim_{j\to\infty} d(p_{k_j}, p) = 0$.

Of course, in the Euclidean space $\mathbb{R}^n$, the above definition of convergence coincides with the concept of convergence described in Section 10.2. The Componentwise Convergence Criterion directly reduces the property of convergence of a sequence in $\mathbb{R}^n$ to the convergence of each of the component sequences.

Example 12.9 If X is any set that we consider a metric space with the discrete metric, then a sequence $\{p_k\}$ in X converges to the point p in X if and only if there is some index N such that $p_k = p$ for all $k \geq N$. This follows from the observation that $d(p_k, p) < 1$ if and only if $p_k = p$. ■

Example 12.10 Consider the metric space $C([a, b], \mathbb{R})$. Let $\{f_n\}$ be a sequence of functions in $C([a, b], \mathbb{R})$ and let f be a fixed function in $C([a, b], \mathbb{R})$. In Chapter 9 we described two types of convergence for sequences of functions. The sequence of functions $\{f_k : [a, b] \to \mathbb{R}\}$ was defined to *converge pointwise* to the function $f : [a, b] \to \mathbb{R}$ provided that for each point x in $[a, b]$ the sequence of real

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numbers $\{f_k(x)\}$ converges to the number $f(x)$. Also, the sequence of functions $\{f_k : [a, b] \to \mathbb{R}\}$ was defined to *converge uniformly* to the function $f : [a, b] \to \mathbb{R}$ provided that for each $\epsilon > 0$ there is an index N such that

$$f(x) - \epsilon < f_k(x) < f(x) + \epsilon \qquad \text{for all } x \text{ in } [a, b] \text{ and all } k \geq N.$$

It is easy to see that this inequality holds if and only if

$$d(f_k, f) < \epsilon \qquad \text{for all } k \geq N.$$

Thus, the sequence of functions converges uniformly if and only if it converges as a sequence in the metric space $C([a, b], \mathbb{R})$. For this reason, this metric is often referred to as the *uniform metric* on $C([a, b], \mathbb{R})$. ▪

Definition Let X be a metric space. Then a subset C of X is said to be *closed* in X if whenever $\{p_k\}$ is a sequence in C that converges to a point p in X, then p belongs to C.

Again, the concept of closedness defined above is a generalization of the concept we have defined in Euclidean n-space. We considered examples of closed subsets of $\mathbb{R}^n$ in Section 10.3.

Example 12.11 In the metric space $C([a, b], \mathbb{R})$, consider the set A consisting of all functions whose functional values are nonnegative. Then A is a closed subset of $C([a, b], \mathbb{R})$. This follows from the observation that uniform convergence implies pointwise convergence and the fact that the set of nonnegative real numbers is a closed subset of $\mathbb{R}$. ▪

For any two sets A and B, the *complement* of A in B, denoted by $B\backslash A$, is defined by

$$B\backslash A \equiv \{p \in B \mid p \notin A\}.$$

If B is any set and $\{A_s \mid s \in S\}$ is a collection of sets, then from the definitions of union, intersection, and complement, it follows that

$$B\backslash \bigcap_{s\in S} A_s = \bigcup_{s\in S}(B\backslash A_s) \qquad \text{and} \qquad B\backslash \bigcup_{s\in S} A_s = \bigcap_{s\in S}(B\backslash A_s).$$

These two formulas are frequently referred to as *deMorgan's Laws.*

Theorem 12.12 The Complementing Characterization Let X be a metric space and let A be a subset of X. Then A is open in X if and only if its complement in X is closed in X.

Proof

First, suppose that A is open in X. Then every point in A is an interior point of A, so a sequence in $X\backslash A$ cannot converge to a point in A. It follows that a sequence in

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$X\backslash A$ that converges to a point in X must converge to a point in $X\backslash A$. Thus, $X\backslash A$ is closed.

We now prove the converse. Suppose that $X\backslash A$ is closed. We must show that every point in A is an interior point of A. Let p be a point in A. Suppose that p is not an interior point of A. Let k be a positive integer. Then $\mathcal{B}_{1/k}(p)$ is not a subset of A, so we can choose a point in $X\backslash A$, which we label p_k, such that $d(p_k, p) < 1/k$. Thus, $\{p_k\}$ converges to p. But $X\backslash A$ is closed, so p is an element of $X\backslash A$. This contradiction shows that p is an interior point of A. ■

Theorem 12.13 Let X be a metric space.

i. The union of a collection of open subsets of X is open in X.
ii. The intersection of a collection of closed subsets of X is closed in X.

Proof of (i)
Suppose that $\mathcal{O} = \bigcup_{s\in S} \mathcal{O}_s$, where each $\mathcal{O}_s$ is open in X. Let p be a point in $\mathcal{O}$. We must show that p is an interior point of $\mathcal{O}$. But p belongs to some $\mathcal{O}_s$, so since $\mathcal{O}_s$ is open in X, there is an open ball about p in X that is contained in $\mathcal{B}_r(q)$. Thus, $\mathcal{B}_r(p)$ is also contained in $\mathcal{O}$, so p is an interior point of $\mathcal{O}$. ■

Proof of (ii)
Suppose that $C = \bigcap_{s\in S} C_s$, where each C_s is closed in X. Then $X\backslash C = X\backslash \bigcap_{s\in S} C_s = \bigcup_{s\in S} (X\backslash C_s)$. From (i) and the Complementing Characterization, it follows that C is closed in X. ■

As we have already noted in the context of Euclidean spaces, the intersection of a collection of open sets need not be open, nor need the union of a collection of closed sets be closed. However, for finite collections of sets, we have the following theorem.

Theorem 12.14 Let X be a metric space.

i. The intersection of a finite collection of open subsets of X is open in X.
ii. The union of a finite collection of closed subsets of X is closed in X.

Proof of (i)
Suppose $\mathcal{O} = \bigcap_{i=1}^k \mathcal{O}_i$ for some positive integer k, where each $\mathcal{O}_i$ is open in X. Let p be an element of $\mathcal{O}$. If $1 \leq i \leq k$, p is an element of $\mathcal{O}_i$ and $\mathcal{O}_i$ is open in X, so there exists a positive number r_i with $\mathcal{B}_{r_i}(p) \subseteq \mathcal{O}_i$. Define $r = \min\{r_1, \ldots, r_k\}$. Then r is positive, and $\mathcal{B}_r(p) \subseteq \bigcap_{i=1}^k \mathcal{O}_i = \mathcal{O}$. Thus, p is an interior point of $\mathcal{O}$. Therefore every point in $\mathcal{O}$ is an interior point of $\mathcal{O}$ and so $\mathcal{O}$ is open in X. ■

Proof of (ii)
Suppose that $C = \bigcup_{i=1}^k C_i$ for some positive integer k, where each C_i is closed in X. Observe that $X\backslash C = \bigcap_{i=1}^k (X\backslash C_i)$. From part (i) and the Complementing Characterization, it follows that C is closed in X. ■

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EXERCISES FOR SECTION 12.1

1. Let $A = \{f \text{ in } C([0, 1], \mathbb{R}) \mid f(x) \geq 0 \text{ for all } x \text{ in } [0, 1]\}$. Prove that A is a closed subset of $C([0, 1], \mathbb{R})$, but that A is not open in $C([0, 1], \mathbb{R})$.

2. Let $X = C([0, 1], \mathbb{R})$. Find $d(f, g)$ for each of the following pairs of functions:
   a. $f(x) = x$ and $g(x) = \cos x$ for x in $[0, 1]$
   b. $f(x) = 4x^3$ and $g(x) = 6x^2 - 3x$ for x in $[0, 1]$

3. Let $\{f_k\}$ be the sequence in $C([0, 1], \mathbb{R})$ defined by

$$f_k(x) = (1 - x)x^k \qquad \text{for } x \text{ in } [0, 1] \text{ and each positive integer } k.$$

   Prove that the sequence converges pointwise to the function whose constant value is 0. Is the sequence $\{f_k\}$ a convergent sequence in the metric space $C([0, 1], \mathbb{R})$?

4. For each positive integer k, define the function $f_k : [0, 1] \to \mathbb{R}$ by $f_k(x) = \cos(x/k)$ for x in $[0, 1]$. Show that the sequence $\{f_k\}$ converges in the metric space $C([0, 1], \mathbb{R})$.

5. Suppose that X is a metric space that contains the point p and r is a positive number. Prove that the set $\{q \text{ in } X \mid d(p, q) \leq r\}$ is closed in X.

6. Verify the assertions made in Example 12.10.

7. Verify the assertions made in Example 12.11.

8. For any two points in the plane $\mathbb{R}^2$, define

$$d^*(p, q) = |p_1 - q_1| + |p_2 - q_2|.$$

   a. Show that d^* defines a metric on $\mathbb{R}^2$.
   b. Compare an open ball about $(0, 0)$ in this metric with an open ball about $(0, 0)$ in the Euclidean metric.
   c. Show that a sequence in $\mathbb{R}^2$ converges with respect to the above metric if and only if it converges with respect to the Euclidean metric.

9. Let X be any set considered a metric space with the discrete metric. With this metric, show that every subset of X is both open and closed in X.

10. For a metric space X and a positive number r, can one have $\mathcal{B}_r(p) = \mathcal{B}_r(q)$ and yet $p \neq q$? Can this happen in $\mathbb{R}^n$ with the Euclidean metric?

11. For any two functions f and g in $C([a, b], \mathbb{R})$, define

$$d^*(f, g) = \int_a^b |f(x) - g(x)|\, dx.$$

   a. Prove that this defines a metric on $C([a, b], \mathbb{R})$.
   b. Prove the following inequality relating this metric and the uniform metric:

$$d^*(f, g) \leq (b - a)\, d(f, g).$$

   c. Compare the concepts of convergence of a sequence of functions in this metric and in the uniform metric.

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12.2 COMPLETENESS AND THE CONTRACTION MAPPING PRINCIPLE

Definition Let X be a metric space. A sequence $\{p_k\}$ in X is said to be a *Cauchy sequence* provided that for each positive number ϵ there is a natural number N such that

$$d(p_k, p_\ell) < \epsilon \qquad \text{if } k \geq N \text{ and } \ell \geq N.$$

Observe that the preceding concept is a direct generalization of the concept of a Cauchy sequence of real numbers, which we considered in Section 9.1.

Proposition 12.15 Every convergent sequence is a Cauchy sequence.

Proof
Let X be a metric space. Suppose that $\{p_k\}$ is a sequence in X that converges to the point p in X. Let $\epsilon > 0$. We can choose a positive integer N such that $d(p_k, p) < \epsilon/2$ if $k \geq N$. From the Triangle Inequality it follows that

$$d(p_k, p_\ell) \leq d(p_k, p) + d(p_\ell, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{if } k \geq N \text{ and } \ell \geq N.$$

Thus, the sequence $\{p_k\}$ is Cauchy. ∎

In Section 9.1, we proved the Cauchy Convergence Criterion, which asserts that a sequence of real numbers converges if and only if it is a Cauchy sequence. The point of this equivalence is that it often happens that we have a sequence of real numbers that we wish to prove converges, but there is not sufficient information to identify the proposed limit. In such a case the Cauchy Convergence Criterion is useful since it is a criterion that is intrinsic to the sequence itself and requires no knowledge of the proposed limit. It will be useful to discover other metric spaces in which every Cauchy sequence converges to a point in the space.

Example 12.16 Consider a sequence $\{\mathbf{u}_k\}$ in Euclidean space $\mathbb{R}^n$. As in the case of sequences of real numbers, we claim that this sequence is a Cauchy sequence if and only if it converges to a point $\mathbf{u}$ in $\mathbb{R}^n$. Since in any metric space every convergent sequence is a Cauchy sequence, to verify this assertion we need only show that if $\{\mathbf{u}_k\}$ is a Cauchy sequence, then it converges. But observe that if $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are points in $\mathbb{R}^n$, then

$$|u_i - v_i| \leq \|\mathbf{u} - \mathbf{v}\| \qquad \text{for each index } i \text{ with } 1 \leq i \leq n.$$

Thus, if $\{\mathbf{u}_k\}$ is a Cauchy sequence in $\mathbb{R}^n$ and $1 \leq i \leq n$, then the sequence of ith components is Cauchy, and hence, by the Cauchy Convergence Criterion, this sequence of ith components converges to some number u_i. Define $\mathbf{u}$ to be the point in $\mathbb{R}^n$ whose ith component is u_i. Then from the Componentwise Convergence Criterion it follows that the sequence $\{\mathbf{u}_k\}$ converges to the point $\mathbf{u}$. ∎

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Example 12.17 Consider a sequence $\{f_k\}$ in $C([a, b], \mathbb{R})$. From the definition of the metric on $C([a, b], \mathbb{R})$, it follows that the sequence $\{f_k\}$ is a Cauchy sequence in $C([a, b], \mathbb{R})$ if and only if for each positive number ϵ there is a natural number N such that

$$|f_k(x) - f_\ell(x)| < \epsilon \qquad \text{for all } x \text{ in } [a, b] \text{ if } k \geq N \text{ and } \ell \geq N.$$

This is a concept that we considered in Section 9.2: We refer to a sequence that has this property as *uniformly Cauchy*. In fact, in that section we established the Weierstrass Uniform Convergence Criterion, which asserts that a sequence in $C([a, b], \mathbb{R})$ is uniformly Cauchy if and only if it converges uniformly to a continuous function $f : [a, b] \to \mathbb{R}$. ∎

Definition A metric space X is said to be *complete* provided that every Cauchy sequence in X converges to a point in X.

The discussion that preceded this definition can be conveniently summarized in the statement of the following theorem.

Theorem 12.18 The metric spaces $\mathbb{R}$, $\mathbb{R}^n$, and $C([a, b], \mathbb{R})$ are complete.

A subspace of a complete metric space is not necessarily complete. For instance, if X is the subspace of $\mathbb{R}$ consisting of the interval $(0, 2)$, then X is not complete. To see this, observe that the sequence $\{1/k\}$ is a Cauchy sequence in X that does not converge to a point in X since it converges to the point 0, which is not in X. However, there is the following criterion for deciding which subspaces of a complete metric space are also complete.

Theorem 12.19 Let X be a complete metric space and Y be a subspace of X. Then Y is a complete metric space if and only if Y is a closed subset of X.

Proof
First, suppose that Y is a closed subset of X. Let $\{p_k\}$ be a Cauchy sequence in Y. Then $\{p_k\}$ is a Cauchy sequence in X, and since X is complete, $\{p_k\}$ converges to a point p in X. But Y is a closed subset of X, so p belongs to Y. Thus, Y is complete.

To prove the converse, suppose that Y is a complete metric space. Let $\{p_k\}$ be a sequence in Y that converges to the point p in X. From Proposition 12.15 it follows that $\{p_k\}$ is a Cauchy sequence. But Y is complete, so $\{p_k\}$ converges to a point in Y. Since a sequence can converge to at most one point, p belongs to Y. Thus, Y is a closed subset of X. ∎

Corollary 12.20 Every closed subset of $\mathbb{R}$, of $\mathbb{R}^n$, and of $C([a, b], \mathbb{R})$ is a complete metric space.

Proof
The result follows from Theorems 12.18 and 12.19. ∎

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Definition Let X and Y be metric spaces.

i. A mapping $T : X \to Y$ is said to be a *Lipschitz mapping* provided that there is some nonnegative number c, called a *Lipschitz constant* for the mapping, such that

$$d(T(p), T(q)) \leq cd(p, q) \qquad \text{for all points } p \text{ and } q \text{ in } X.$$

ii. A Lipschitz mapping $T : X \to Y$ that has a Lipschitz constant less than 1 is called a *contraction*.

Example 12.21 Let I be an open interval in $\mathbb{R}$ and suppose that the function $f : I \to \mathbb{R}$ is differentiable. We claim that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with a Lipschitz constant c if and only if

$$|f'(x)| \leq c \qquad \text{for all } x \text{ in } I.$$

To verify this, first suppose that this inequality holds. Then if u and v are points in I with $u \neq v$, it follows from the Mean Value Theorem that there is some point z between u and v such that

$$f(u) - f(v) = f'(z)[u - v],$$

so $|f(u) - f(v)| \leq c|u - v|$. The converse follows from the very definition of a derivative as the limit of difference quotients. ■

There is a useful generalization of the above example to mappings between Euclidean spaces. Since the extension uses the concept of a partial derivative, we will postpone the discussion of this extension until we have established the appropriate extension of the Mean Value Theorem for such mappings.

Definition Let X be a metric space. A point p in X is called a *fixed point* for the mapping $T : X \to X$ provided that

$$T(p) = p.$$

We are interested in finding assumptions on a mapping that ensure the existence of fixed points of the mapping. Of course, a mapping may or may not have fixed points. For instance, the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x + 1$ certainly has no fixed points.

For real-valued functions of a single real variable, a fixed point of the function corresponds to a point where the graph of the function intersects the line $y = x$. This observation provides the geometric insight for the following example.

Example 12.22 Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that the image $f([a, b])$ is contained in $[a, b]$. Then $f : [a, b] \to \mathbb{R}$ has a fixed point. This follows from the Intermediate Value Theorem when we observe that if we define $g(x) = f(x) - x$ for x in $[a, b]$, then $g(a) \geq 0$ and $g(b) \leq 0$, so $g(x_0) = 0$ for some x_0 in $[a, b]$, which means that $f(x_0) = x_0$. ■

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The above result generalizes to mappings between Euclidean spaces as follows: If K is a subset of $\mathbb{R}^n$ that is closed, bounded, and convex and the mapping $T : K \to K$ is continuous, then $T : K \to K$ has a fixed point. This is called Brouwer's Fixed-Point Theorem. Unfortunately, its proof lies outside the scope of this book.[1]

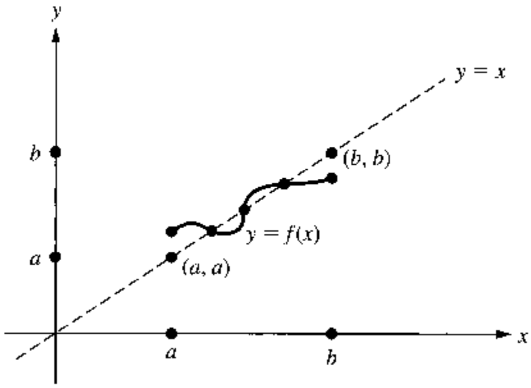

FIGURE 12.2 x is fixed by f if and only if $(x, f(x))$ crosses the diagonal.

We now prove a theorem called the Contraction Mapping Principle, which has many important applications. In addition to being useful, this theorem is particularly interesting because its proof requires merely the definition of a complete metric space and two results that we established in Chapters 1 and 2. The first is that if c is a real number, then

$$\lim_{k\to\infty} c^k = 0 \qquad \text{if } |c| < 1;$$

the second is the Geometric Sum Formula.

Geometric Sum Formula

$$\sum_{k=0}^{n} c^k = \frac{1 - c^{n+1}}{1 - c} \qquad \text{if } c \neq 1.$$

Theorem 12.23 The Contraction Mapping Principle Let X be a complete metric space and suppose that the mapping $T : X \to X$ is a contraction. Then the mapping $T : X \to X$ has exactly one fixed point.

Proof

Let c be a number with $0 \leq c < 1$ that is a Lipschitz constant for the mapping $T : X \to X$; that is,

$$d(T(p), T(q)) \leq c\,d(p, q) \qquad \text{for all points } p \text{ and } q \text{ in } X.$$

[1] A proof of this theorem can be found in John Milnor's elegant book *Topology from the Differentiable Viewpoint* (Charlottesville: University Press of Virginia, 1965).

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Select some point in X and label it p_0. Now define the sequence $\{p_k\}$ inductively by setting $p_1 = T(p_0)$; then if k is an index such that p_k is defined, set $p_{k+1} = T(p_k)$. This sequence is properly defined since $T(X)$ is a subset of X. We will show that the sequence $\{p_k\}$ converges to a fixed point of the mapping $T : X \to X$.

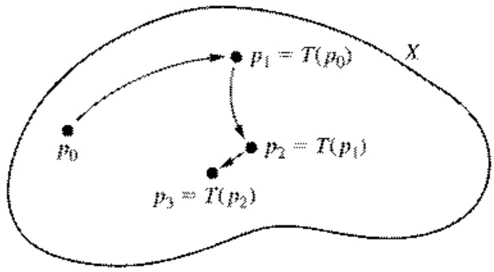

FIGURE 12.3 x_{k+1} is the image under f of its predecessor x_k.

First, observe that from the definition of the sequence and the definition of the Lipschitz constant c, it follows that

$$d(p_2, p_1) = d(T(p_1), T(p_0)) \leq cd(T(p_0), p_0),$$

and that

$$d(p_{k+1}, p_k) = d(T(p_k), T(p_{k-1})) \leq cd(p_k, p_{k-1}) \qquad \text{if } k \geq 2.$$

Using these two inequalities, an induction argument implies that

$$d(p_{k+1}, p_k) \leq c^k d(T(p_0), p_0) \qquad \text{for every index } k.$$

Hence, if m and k are indices with $m > k$, from the Triangle Inequality and the Geometric Sum Formula it follows that

$$\begin{aligned} d(p_m, p_k) &\leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-2}) + \cdots + d(p_{k+1}, p_k) \\ &\leq [c^{m-1} + c^{m-2} + \cdots + c^k] d(T(p_0), p_0) \\ &= c^k[1 + c + \cdots + c^{m-1-k}] d(T(p_0), p_0) \\ &= c^k \frac{[1 - c^{m-k}]}{1 - c} d(T(p_0), p_0). \end{aligned}$$

Since $c \geq 0$,

$$d(p_m, p_k) \leq \frac{c^k}{1 - c} d(T(p_0), p_0) \qquad \text{if } m > k. \tag{12.1}$$

But, since $0 \leq c < 1$, $\lim_{k\to\infty} c^k = 0$, and hence from (12.1) we conclude that $\{p_k\}$ is a Cauchy sequence.

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By assumption, the metric space X is complete. Thus, there is a point p in X to which the sequence $\{p_k\}$ converges. But

$$d(T(p_k), T(p)) \le cd(p_k, p) \qquad \text{for every index } k, \tag{12.2}$$

so from the Comparison Lemma for real sequences it follows that the image sequence $\{T(p_k)\}$ converges to the point $T(p)$. However, since $T(p_k) = p_{k+1}$ for each k, the sequence $\{T(p_k)\}$ is a subsequence of the sequence $\{p_k\}$, so $T(p) = p$. Thus, the mapping $T : X \to X$ has at least one fixed point.

It remains to check that there is only one fixed point. But if p and q are points in X such that $T(p) = p$ and $T(q) = q$, then

$$0 \le d(p, q) = d(T(p), T(q)) \le cd(p, q),$$

so since $0 \le c < 1$, we must have $d(p, q) = 0$; that is, $p = q$. Thus, there is exactly one fixed point. ■

The above proof of the Contraction Mapping Principle actually proves much more than the mere *existence* of a unique fixed point. *It provides an iterative method for approximating the fixed point.* Indeed, under the assumptions of the above theorem, what has been proved is not only that the mapping $T : X \to X$ has exactly one fixed point p_* but also that if p_0 is any point in X, then the sequence $\{p_k\}$ is defined recursively by setting $p_1 = T(p_0)$ and, if k is an index such that p_k is defined, defining $p_{k+1} = T(p_k)$ converges to p_*. Moreover, if c is a number with $0 \le c < 1$ that is a Lipschitz constant for the mapping $T : X \to X$, the following error bounds hold:

$$d(p_*, p_k) \le \frac{c^k}{1-c} d(T(p_0), p_0) \qquad \text{for every index } k. \tag{12.3}$$

EXERCISES FOR SECTION 12.2

1. Show that none of the following mappings $f : X \to X$ have a fixed point and explain why the Contraction Mapping Principle is not contradicted:

a. $X = (0, 1) \subseteq \mathbb{R}$ and $f(x) = x/2$ for x in X

b. $X = \mathbb{R}$ and $f(x) = x + 1$ for x in X

c. $X = \{(x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and $f(x, y) = (-y, x)$ for (x, y) in X

2. Fix α a positive real number, let $X = [0, 1] \subseteq \mathbb{R}$, and define $f : X \to \mathbb{R}$ by

$$f(x) = \alpha x(1 - x) \qquad \text{for } x \text{ in } X.$$

a. For what values of α does the mapping $f : X \to \mathbb{R}$ have the property that $f(X) \subseteq X$?

b. For what values of α does the mapping $f : X \to \mathbb{R}$ have the property that $f(X) \subseteq X$ and $f : X \to X$ is a contraction?

3. Define the function $f : [1, \infty) \to \mathbb{R}$ by

$$f(x) = 1 + \sqrt{x} \qquad \text{for } x \ge 1.$$

Show that this function has exactly one fixed point.

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4. Fill in the details of Example 12.21.
5. Suppose that $p : \mathbb{R} \to \mathbb{R}$ is a polynomial. Show that $p : \mathbb{R} \to \mathbb{R}$ is Lipschitz if and only if the degree of the polynomial is less than 2.
6. Suppose that both of the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are Lipschitz. Is the product of these functions Lipschitz?
7. Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Prove that $f : [a, b] \to \mathbb{R}$ is Lipschitz if and only if $f : (a, b) \to \mathbb{R}$ is Lipschitz.
8. **a.** Define $f(x) = \sqrt{x}$ for $x \geq 0$. Show that the function $f : [0, \infty) \to \mathbb{R}$ is continuous but is not Lipschitz.
   b. Define $f(x) = |x|$ for all real numbers x. Show that the function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz but not differentiable.
9. For each positive integer k, define $f_k(x) = x^k$ for $0 \leq x \leq 1$. Is the sequence $\{f_k : [0, 1] \to \mathbb{R}\}$ a Cauchy sequence in the metric space $C([0, 1], \mathbb{R})$?
10. For each positive integer k, define $f_k(x) = e^{x/k}$ for $0 \leq x \leq 1$. Is the sequence $\{f_k : [0, 1] \to \mathbb{R}\}$ a Cauchy sequence in the metric space $C([0, 1], \mathbb{R})$?
11. For each positive integer k, define $f_k(x) = \cos(x/k)$ for $0 \leq x \leq 1$. Is the sequence $\{f_k : [0, 1] \to \mathbb{R}\}$ a Cauchy sequence in the metric space $C([0, 1], \mathbb{R})$?
12. Verify that the inequality (12.1) implies that the sequence $\{p_k\}$ is a Cauchy sequence.
13. Verify that the inequality (12.1) implies inequality (12.3).
14. Let X be a metric space. Suppose that $\{p_k\}$ is a sequence in X with the property that

$$d(p_{k+1}, p_k) \leq 1/k \qquad \text{for all natural numbers } k.$$

    Is the sequence a Cauchy sequence? (*Hint:* Let $X = \mathbb{R}$ and let $p_k = \sum_{i=1}^{k} 1/i$ for each index k.)
15. Let U be an open ball in $\mathbb{R}^n$. By explicitly finding a Cauchy sequence in U that does not converge to a point in U, show that U is not complete.
16. Let X be a subset of $\mathbb{R}^n$ and suppose that the mapping $T : X \to \mathbb{R}^m$ is Lipschitz. Prove that $T(X)$ is bounded if X is bounded. Is this result true if the mapping is only assumed to be continuous?
17. Let X be a complete metric space containing the point p_0 and let r be a positive real number. Define $K = \{p \text{ in } X \mid d(p, p_0) \leq r\}$. Suppose that $T : K \to X$ is Lipschitz with Lipschitz constant c. Suppose also that $cr + d(T(p_0), p_0) \leq r$. Prove that $T(K) \subseteq K$ and that $T : K \to K$ has a fixed point.

12.3 THE EXISTENCE THEOREM FOR NONLINEAR DIFFERENTIAL EQUATIONS

The aim of this section is to use the Contraction Mapping Principle to prove an important theorem regarding the existence of solutions of certain nonlinear differential equations. We begin by recalling some results about differential equations that we established earlier. In Section 4.5 we first encountered the following question regarding the solvability of a differential equation.

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Suppose that I is an open interval of real numbers that contains the point x_0. Then given a number y_0 and a function $h : I \to \mathbb{R}$, does there exist a differentiable function $f : I \to \mathbb{R}$ that is a solution of the following differential equation?

$$\begin{cases} f'(x) = h(x) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases} \tag{12.4}$$

In general, there may not be any solutions of this equation. For instance, in Chapter 4 we observed that if the function $h : I \to \mathbb{R}$ is a nonconstant step function, then there is no solution of equation (12.4). However, if the function $h : I \to \mathbb{R}$ is continuous, then, in Chapter 7, we established the following variant of the Second Fundamental Theorem of Calculus (Differentiating Integrals).

Theorem 12.24 Let I be an open interval of real numbers that contains the point x_0, let y_0 be a real number and suppose that the function $h : I \to \mathbb{R}$ is continuous. Then there is exactly one differentiable function $f : I \to \mathbb{R}$ that is a solution of the differential equation (12.4), and it is given by the formula

$$f(x) = y_0 + \int_{x_0}^{x} h(t)\,dt \qquad \text{for all } x \text{ in } I. \tag{12.5}$$

Formula (12.5) represents the solution of equation (12.4) as an integral. Thus, for each point x in I, the actual value of $f(x)$ is the limit of Riemann sums. It is, of course, a properly defined function; that is, it associates a definite value with each point x in I. Sometimes, of course, it is possible to simplify this representation, using a technique such as integration by parts or integration by substitution, and thus to represent the solution in terms of more familiar functions. However, it is often not possible to represent the solution explicitly in terms of the "elementary functions" x^k, $\sin x$, $\ln x$, and so on. In such a case, it is necessary to use an approximation technique, such as one of those described in Section 7.4, in order to obtain more precise information about the actual functional values of the solution of the differential equation (12.4).

Example 12.25 Consider the differential equation

$$\begin{cases} f'(x) = 1/(1+x^4) & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 1. \end{cases}$$

Define $h(x) = 1/(1+x^4)$ for each real number x to obtain a continuous function $h : \mathbb{R} \to \mathbb{R}$. Theorem 12.24 implies that this differential equation has a unique solution $f : \mathbb{R} \to \mathbb{R}$ defined by the formula

$$f(x) = 1 + \int_0^x \frac{1}{1+t^4}\,dt \qquad \text{for all } x \text{ in } \mathbb{R}.$$

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We point out that one cannot explicitly evaluate the functional values of this solution by inspection. For instance,

$$f(2) = 1 + \int_0^2 \frac{1}{1+t^4}\, dt,$$

and it is necessary to use some approximation technique, such as one of those described in Section 7.4, in order to approximate $f(2)$. ■

For certain choices of the function $h : I \to \mathbb{R}$, we might be able to recall a differentiable function $g : I \to \mathbb{R}$ having the property that

$$g'(x) = h(x) \qquad \text{for all } x \text{ in } I.$$

When we can recall such a function, the solution of the differential equation (12.4) is given by

$$f(x) = y_0 - g(x_0) + g(x) \qquad \text{for all } x \text{ in } I.$$

Example 12.26 Consider the differential equation

$$\begin{cases} f'(x) = 1/(1+x^2) & \text{for all } x \text{ in } \mathbb{R} \\ f(1) = 2. \end{cases}$$

Recall that we proved that the arctangent function has the property that

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Therefore, the unique solution of the above differential equation is given by the formula

$$f(x) = 2 - \arctan 1 + \arctan x = 2 - \frac{\pi}{4} + \arctan x \qquad \text{for all } x \text{ in } \mathbb{R}.$$ ■

In Section 7.2, we considered a differential equation that was more general than the differential equation (12.4). Specifically, we introduced an additional parameter b and sought a differentiable function $f : I \to \mathbb{R}$ that is a solution of the equation

$$\begin{cases} f'(x) = bf(x) + h(x) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases} \tag{12.6}$$

In fact, there is a trick that reduces this equation to the type we have just considered. The trick[2] is to multiply the first equation in (12.6) by e^{-bx} and see that the equation then becomes

$$\begin{cases} d/dx[e^{-bx} f(x)] = e^{-bx} h(x) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases} \tag{12.7}$$

[2] The trick is called *multiplication by an integrating factor.*

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If the function $h : I \to \mathbb{R}$ is continuous, it follows from Theorem 12.24 that there is a unique solution of (12.6) given by the formula

$$f(x) = e^{b(x-x_0)}y_0 + \int_{x_0}^{x} e^{b(x-t)}h(t)\,dt \qquad \text{for all } x \text{ in } I. \tag{12.8}$$

Example 12.27 Consider the differential equation

$$\begin{cases} f'(x) = 2f(x) + x & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 1. \end{cases}$$

Since the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x$ for all x continuous, it follows from the above discussion that this differential equation has a unique solution $f : \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(x) = e^{2x} + \int_{0}^{x} e^{2(x-t)}t\,dt \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Integrating by parts and using the First Fundamental Theorem of Calculus (Integrating Derivatives), the last formula reduces to

$$f(x) = \frac{5}{4}e^{2x} - \frac{x}{2} - \frac{1}{4} \qquad \text{for all } x \text{ in } \mathbb{R}.$$ ■

We now consider much more general differential equations. Suppose that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0) and that the function $g : \mathcal{O} \to \mathbb{R}$ is continuous. The problem is to find an open interval I that contains the point x_0 and a differentiable function $f : I \to \mathbb{R}$ such that

$$\begin{cases} f'(x) = g(x, f(x)) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases} \tag{12.9}$$

This differential equation contains both the differential equation (12.4) and equation (12.6) as particular cases. Indeed, defining $\mathcal{O} = \{(x, y) \mid x \text{ in } I, y \text{ in } \mathbb{R}\}$, in the case where $g(x, y) = h(x)$ for (x, y) in $\mathcal{O}$, the differential equation (12.9) reduces to the differential equation (12.4), whereas in the case where $g(x, y) = by + h(x)$ for (x, y) in $\mathcal{O}$, the differential equation (12.9) reduces to the differential equation (12.6).

For a general continuous function $g : \mathcal{O} \to \mathbb{R}$ the study of equation (12.9) can be quite delicate. First of all, as the next example illustrates, there may be more than one solution.

Example 12.28 Let $\mathcal{O} = \mathbb{R}^2$, let $(x_0, y_0) = (0, 0)$, and define $g(x, y) = 3y^{2/3}$. Then equation (12.9) becomes

$$\begin{cases} f'(x) = 3[f(x)]^{2/3} & \text{for all } x \text{ in } I \\ f(0) = 0. \end{cases} \tag{12.10}$$

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It is clear that the function $f:\mathbb{R}\to\mathbb{R}$ that is identically 0 is a solution of equation (12.10). But there is another solution. It is not difficult to check that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}0 & \text{if } x<0\\ x^3 & \text{if } x\geq 0\end{cases}$$

is also a solution of the differential equation (12.10). ∎

Another difficulty that can arise in the study of equation (12.9) is that the interval I on which the solution is defined may be small. This is illustrated in the following example.

Example 12.29 Let $\mathcal{O}=\mathbb{R}^2$, let $(x_0, y_0)=(0,0)$, and define $g(x,y)=1+y^2$. Then equation (12.9) becomes

$$\begin{cases}f'(x)=1+[f(x)]^2 & \text{for all } x \text{ in } I\\ f(0)=0.\end{cases} \tag{12.11}$$

We claim that there is a unique solution of this differential equation on the interval $I=(-\pi/2,\pi/2)$ and that there is no solution on any larger interval. To analyze the differential equation (12.11), first suppose that I is an interval containing the point 0 and that the differentiable function $f:I\to\mathbb{R}$ is a solution of (12.11). Then

$$\frac{f'(x)}{1+[f(x)]^2}=1 \qquad \text{for all } x \text{ in } I,$$

which, in view of the Chain Rule and the formula for the derivative of the arctangent function, means that

$$\frac{d}{dx}[\arctan f(x)-x]=0 \qquad \text{for all } x \text{ in } I.$$

Since $\arctan f(0)=\arctan 0=0$, it follows from the Identity Criterion that

$$\arctan f(x)=x \qquad \text{for all } x \text{ in } I.$$

But the image of the arctangent function is the interval $(-\pi/2,\pi/2)$, so it follows that $I\subseteq(-\pi/2,\pi/2)$ and that $f(x)=\tan x$ for all x in I.

This argument shows that if there is an interval containing the point 0 and a differentiable function $f:I\to\mathbb{R}$ that is a solution of the differential equation (12.11), then $I\subseteq(-\pi/2,\pi/2)$ and $f(x)=\tan x$ for all x in I. A straightforward computation of derivatives shows that if we define $I=(-\pi/2,\pi/2)$ and $f:I\to\mathbb{R}$ by $f(x)=\tan x$, then this function is a solution of the differential equation (12.11). ∎

The purpose of the above example is to show that even when $g:\mathbb{R}^2\to\mathbb{R}$ is as simple a function as a second-degree polynomial in y, there are restrictions on the size of the neighborhood I of x_0 on which there is a solution of (12.11) (Exercise 6).

We now turn to an analysis of the differential equation (12.9) for fairly general functions $g:\mathcal{O}\to\mathbb{R}$. Once more, the Second Fundamental Theorem of Calculus

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(Differentiating Integrals) is essential to our analysis. The following lemma establishes the equivalence between the differential equation and an associated integral equation.

Lemma 12.30 The Equivalence Lemma Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0) and suppose that the function $g : \mathcal{O} \to \mathbb{R}$ is continuous. Let I be a neighborhood of the point x_0 and suppose that the function $f : I \to \mathbb{R}$ has the property that

$$(x, f(x)) \text{ is in } \mathcal{O} \qquad \text{for all } x \text{ in } I.$$

Then the following two assertions are equivalent:

i. The function $f : I \to \mathbb{R}$ is differentiable and is a solution of the differential equation

$$\begin{cases} f'(x) = g(x, f(x)) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases} \tag{12.12}$$

ii. The function $f : I \to \mathbb{R}$ is continuous and is a solution of the integral equation

$$f(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \qquad \text{for all } x \text{ in } I. \tag{12.13}$$

Proof

Define the function $h : I \to \mathbb{R}$ by

$$h(x) = g(x, f(x)) \qquad \text{for all } x \text{ in } I$$

and observe that the function $h : I \to \mathbb{R}$ is continuous since it is the composition of continuous functions.

First, suppose that (i) holds. By the very definition of the function $h : I \to \mathbb{R}$, the differential equation (12.12) can be written as

$$\begin{cases} f'(x) = h(x) & \text{for all } x \text{ in } I \\ f(x_0) = y_0. \end{cases}$$

Theorem 12.24 implies that

$$f(x) = y_0 + \int_{x_0}^{x} h(t)\, dt = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \qquad \text{for all } x \text{ in } I,$$

so (ii) holds.

Conversely, if (ii) holds, then clearly $f(x_0) = y_0$. Moreover, the Second Fundamental Theorem of Calculus (Differentiating Integrals) implies that

$$\frac{d}{dx}\left[\int_{x_0}^{x} g(t, f(t))\, dt\right] = g(x, f(x)) \qquad \text{for all } x \text{ in } I.$$

From the integral equation (12.13), we conclude that the function $f : I \to \mathbb{R}$ is a differentiable function that is a solution of the differential equation (12.12). ∎

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Theorem 12.31 The Existence Theorem Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0). Suppose that the function $g : \mathcal{O} \to \mathbb{R}^2$ is continuous and that there is a positive number M such that

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \qquad \text{for all points } (x, y_1) \text{ and } (x, y_2) \text{ in } \mathcal{O}. \tag{12.14}$$

Then there is an open interval I containing the point x_0 such that the differential equation

$$\begin{cases} f'(x) = g(x, f(x)) & \text{for all } x \text{ in } I \\ f(x_0) = y_0 \end{cases} \tag{12.15}$$

has exactly one solution.

Proof

For ℓ a positive number, define I_ℓ to be the closed interval $[x_0 - \ell, x_0 + \ell]$. We will show that ℓ can be chosen so that there is exactly one continuous function $f : I_\ell \to \mathbb{R}$ with

$$f(x) = y_0 + \int_{x_0}^{x} g(s, f(s))\, ds \qquad \text{for all } x \text{ in } I_\ell. \tag{12.16}$$

Once such an ℓ is chosen, it follows from the Equivalence Lemma that there is exactly one solution of the differential equation (12.15) on the interval $I = (x_0 - \ell, x_0 + \ell)$.

Since $\mathcal{O}$ is open, we can choose positive numbers a and b such that the rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is contained in $\mathcal{O}$. Now for each positive number ℓ with $\ell \leq a$, define X_ℓ to be the subspace of the metric space $C(I_\ell, \mathbb{R})$ consisting of those continuous functions $f : I_\ell \to \mathbb{R}$ with the property that

$$|f(x) - y_0| \leq b \qquad \text{for all } x \text{ in } I_\ell.$$

Observe that X_ℓ consists of the continuous functions $f : I_\ell \to \mathbb{R}$ whose graphs lie in the rectangle $I_\ell \times [y_0 - b, y_0 + b]$.

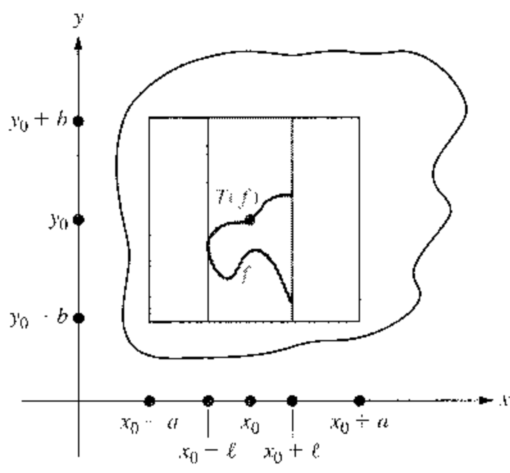

FIGURE 12.4 Choosing ℓ so that $T(X_\ell)$ is contained in X_ℓ.

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For a function f in X_ℓ, define the function $T(f)$ in $C(I_\ell, \mathbb{R})$ by

$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \qquad \text{for all } x \text{ in } I_\ell.$$

Observe that a solution of the integral equation (12.16) is simply a fixed point of the mapping $T : X_\ell \to C(I_\ell, \mathbb{R})$.

The strategy of the proof is as follows: Since $C(I_\ell, \mathbb{R})$ is a complete metric space and X_ℓ is a closed subset of $C(I_\ell, \mathbb{R})$, it follows from Corollary 12.20 that X_ℓ is also a complete metric space. We will show that if ℓ is chosen sufficiently small, then

$$T(X_\ell) \subseteq X_\ell \text{ and } T : X_\ell \to X_\ell \text{ is a contraction.} \tag{12.17}$$

Hence it will follow from the Contraction Mapping Principle that the mapping $T : X_\ell \to X_\ell$ has a unique fixed point.

In order to choose ℓ so that $T(X_\ell) \subseteq X_\ell$, we first choose any positive number K such that

$$|g(x, y)| \le K \qquad \text{for all points } (x, y) \text{ in the rectangle } R.$$

The Extreme Value Theorem permits us to make such a choice. Then, if f is any function in X_ℓ and the point x belongs to I_ℓ,

$$|T(f)(x) - y_0| = \left| \int_{x_0}^{x} g(t, f(t))\, dt \right| \le \ell K,$$

so

$$T(X_\ell) \subseteq X_\ell \text{ provided that } \ell K \le b. \tag{12.18}$$

Now observe that if f_1 and f_2 are any two functions in X_ℓ and the point x is in I_ℓ, then, by assumption (12.14),

$$|g(x, f_1(x)) - g(x, f_2(x))| \le M|f_1(x) - f_2(x)| \le M d(f_1, f_2);$$

hence, using the linearity and monotonicity properties of the integral, it follows that

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &= \left| \int_{x_0}^{x} [g(t, f_1(t)) - g(t, f_2(t))]\, dt \right| \\ &\le \left| \int_{x_0}^{x} M d(f_1, f_2)\, dt \right| \\ &\le |x - x_0| M d(f_1, f_2) \\ &\le \ell M d(f_1, f_2). \end{aligned}$$

This last inequality, together with inequality (12.18), implies that for $0 \le \ell \le a$,

$$T : X_\ell \to X_\ell \text{ is a contraction provided that } \ell K \le b \text{ and } \ell M < 1. \tag{12.19}$$

Hence we can apply the Contraction Mapping Principle to find the unique fixed point of T, that is, the unique solution of the integral equation (12.16) and also of the differential equation (12.15). ∎

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EXERCISES FOR SECTION 12.3

1. For each of the following differential equations, use Theorem 12.24 to write an integral formula for the solution and, if possible, write the solution in terms of elementary functions:

a. $$\begin{cases} f'(x) = x\cos x & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 1 \end{cases}$$

b. $$\begin{cases} f'(x) = 1 + x^3 & \text{for all } x \text{ in } \mathbb{R} \\ f(1) = 4 \end{cases}$$

c. $$\begin{cases} f'(x) = e^{x^2} & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 0 \end{cases}$$

2. For each of the following differential equations, use formula (12.8) to write an integral formula for the solution and, if possible, write the solution in terms of elementary functions:

a. $$\begin{cases} f'(x) = f(x) + 1 & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 1 \end{cases}$$

b. $$\begin{cases} f'(x) = -f(x) + 2 + x & \text{for all } x \text{ in } \mathbb{R} \\ f(0) = 1 \end{cases}$$

c. $$\begin{cases} f'(x) = 2f(x) + e^x & \text{for all } x \text{ in } \mathbb{R} \\ f(1) = 0 \end{cases}$$

3. Follow the analysis in Example 12.29 to explicitly find the maximal interval I about 0 on which we can solve the following differential equations:

a. $$\begin{cases} f'(x) = 1 - [f(x)]^2 & \text{for all } x \text{ in } I \\ f(0) = 0 \end{cases}$$

b. $$\begin{cases} f'(x) = xf(x) & \text{for all } x \text{ in } I \\ f(0) = 1 \end{cases}$$

c. $$\begin{cases} f'(x) = [f(x)]^2 & \text{for all } x \text{ in } I \\ f(0) = -1 \end{cases}$$

4. Verify the details of Example 12.28.

5. Verify the details of Example 12.29.

6. Let ϵ be a positive number. Consider the differential equation

$$\begin{cases} f'(x) = (1/\epsilon)(1 + (f(x))^2) & \text{for all } x \text{ in } I \\ f(0) = 0. \end{cases}$$

Find the length of the maximal interval on which this differential equation has a solution.

7. Verify the integral inequalities that were asserted in the proof of the Existence Theorem.

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8. Follow the proof of the Existence Theorem to prove the following: *Let a, b, M, and K be positive real numbers, let (x_0, y_0) be a point in the plane $\mathbb{R}^2$, and let R be the closed rectangle $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Suppose that the function $g : R \to \mathbb{R}$ is continuous and that $aM < 1$ and $aK \leq b$, where*

$$|g(x, y)| \leq K \qquad \text{for all } (x, y) \text{ in } R$$

and

$$|g(x, y_1) - g(x, y_2)| \leq M|y_1 - y_2| \qquad \text{for all } (x, y_1), (x, y_2) \text{ in } R.$$

Then there is a unique solution of the differential equation

$$\begin{cases} f'(x) = g(x, f(x)) & \text{for all } x \text{ in } (x_0 - a, x_0 + a) \\ f(x_0) = y_0. \end{cases}$$

9. Use Exercise 8 to determine a value of r such that the differential equation

$$\begin{cases} f'(x) = \sin(xf(x)) & \text{for all } x \text{ in } (-r, r) \\ f(0) = 1 \end{cases}$$

has a unique solution.

10. Under the assumptions of Exercise 8, show that if $f_0(x) = y_0$ for all x in $[x_0 - a, x_0 + a]$ and the sequence of functions $\{f_k : (x_0 - a, x_0 + a) \to \mathbb{R}\}$ is defined recursively by the formula

$$f_{k+1}(x) = y_0 + \int_{x_0}^{x} g(s, f_k(s))\, ds \qquad \text{for all } x \text{ in } (x_0 - a, x_0 + a),$$

then the sequence $\{f_k : (x_0 - a, x_0 + a) \to \mathbb{R}\}$ converges uniformly to the solution of the differential equation.

11. Under the assumptions of Exercise 8, let $\{f_k\}$ be as defined in Exercise 10. Show that

$$|f_k(x) - f(x)| \leq \frac{(aM)^k}{1 - aM}\, ak$$

for all x in $(x_0 - a, x_0 + a)$ and all natural numbers k, where $f : I \to \mathbb{R}$ is a solution of equation (12.9). [*Hint:* Use the estimate (12.3).]

12.4 CONTINUOUS MAPPINGS BETWEEN METRIC SPACES

We now turn to the study of functions $f : X \to Y$, where X and Y are general metric spaces. In this generality, functions are often referred to as mappings. As we will see, many of the results for continuous mappings between Euclidean spaces carry over, almost word for word, to mappings between general metric spaces.

Given a mapping $f : X \to Y$, if A is a subset of X, we define

$$f(A) = \{q \text{ in } Y \mid q = f(p) \qquad \text{for some point } p \text{ in } A\}$$

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and call $f(A)$ the *image* of the set A under the mapping $f : X \to Y$. Also, if B is a subset of Y, we define

$$f^{-1}(B) = \{p \text{ in } X \mid f(p) \text{ in } B\}$$

and call $f^{-1}(B)$ the *preimage* of the set B under the mapping $f : X \to Y$.

Since both X and Y are metric spaces, the concept of convergence of a sequence is defined in both X and Y. This leads us to the following natural concept of continuity.

Definition Let X and Y be metric spaces.

i. A mapping $f : X \to Y$ is said to be *continuous at the point* p in X provided that whenever a sequence $\{p_k\}$ in X converges to p, the image sequence $\{f(p_k)\}$ converges to $f(p)$.

ii. A mapping $f : X \to Y$ is said to be *continuous* provided that it is continuous at every point in X.

The above definition extends the definition of continuity that we gave in Chapter 3 for real-valued functions of a real variable and in Chapter 11 for mappings between Euclidean spaces.

For mappings $f : X \to Y$ between general metric spaces, it does not make sense to consider sums and products of mappings since there may be no addition or multiplication defined on the set Y. However, for real-valued mappings—that is, for mappings from a metric space into $\mathbb{R}$—there is a direct extension of Theorem 11.3 concerning the continuity of sums, products, and quotients of continuous mappings.

Given a metric space X and real-valued mappings $f, g : X \to \mathbb{R}$, we define the *product* $fg : X \to \mathbb{R}$ and the *sum* $f + g : X \to \mathbb{R}$ by

$$(fg)(p) \equiv f(p)g(p) \qquad \text{and} \qquad (f+g)(p) \equiv f(p) + g(p) \qquad \text{for all } p \text{ in } X.$$

If $g(p) \neq 0$ for all p in X, we define the *quotient* $f/g : X \to \mathbb{R}$ by

$$\left(\frac{f}{g}\right)(p) \equiv \frac{f(p)}{g(p)} \qquad \text{for all } p \text{ in } X.$$

Theorem 12.32 Let X be a metric space and p be a point in X. Suppose that the mappings $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are both continuous at the point p. Then for any real numbers α and β, the function

$$\alpha f + \beta g : X \to \mathbb{R}$$

is continuous at p. Also, the product

$$f \cdot g : X \to \mathbb{R}$$

is continuous at p. Moreover, if $g(q) \neq 0$ for all q in X, then the quotient

$$\frac{f}{g} : X \to \mathbb{R}$$

is continuous at p.

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Proof
Let $\{p_k\}$ be a sequence in X that converges to p. It follows from the definition of continuity that the real sequences $\{f(p_k)\}$ and $\{g(p_k)\}$ converge to $f(p)$ and $g(p)$, respectively. By the sum, product, and quotient properties of convergent sequences of real numbers,

$$\lim_{k\to\infty}(f+g)(p_k) = (f+g)(p),$$
$$\lim_{k\to\infty}(fg)(p_k) = (fg)(p),$$

and

$$\lim_{k\to\infty}\left(\frac{f}{g}\right)(p_k) = \left(\frac{f}{g}\right)(p).$$

These three equalities prove the theorem. ∎

We also have the following theorem concerning the composition of continuous mappings, which generalizes Theorem 11.5.

Theorem 12.33 Let X, Y, and Z be metric spaces and let p be a point in X. Suppose that the mapping $f: X \to Y$ is continuous at p and the mapping $g: Y \to Z$ is continuous at $f(p)$. Then the composition $g \circ f: X \to Z$ is continuous at p.

Proof
Let $\{p_k\}$ be a sequence in X that converges to p. Then $\{f(p_k)\}$ converges to $f(p)$ since the mapping $f: X \to Y$ is continuous at p. Thus, the sequence $\{g(f(p_k))\}$ converges to $\{g(f(p))\}$ since $\{f(p_k)\}$ is a sequence in Y that converges to $f(p)$ and the mapping $g: Y \to Z$ is continuous at $f(p)$. Hence $\{(g \circ f)(p_k)\}$ converges to $(g \circ f)(p)$. ∎

For real-valued functions of a single real variable, we established the ϵ-δ criterion for continuity of a function at a point that is equivalent to the sequential convergence definition: For mappings between Euclidean spaces, this criterion was established in Theorem 11.27. In fact, as we now show, this ϵ-δ criterion for continuity of a mapping at a point holds in general metric spaces.

Theorem 12.34 Let X and Y be metric spaces, let p be a point in X, and consider the mapping $f: X \to Y$. Then the following two assertions are equivalent:

i. The mapping $f: X \to Y$ is continuous at the point p.
ii. For each positive number ϵ there is a positive number δ such that

$$d(f(p), f(q)) < \epsilon \qquad \text{for each point } q \text{ in } X \text{ with } d(p, q) < \delta. \tag{12.20}$$

Proof
First, suppose that $f: X \to Y$ is continuous at p. To verify (12.20), we suppose the contrary and derive a contradiction. If (12.20) does not hold, then there is some

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$\epsilon_0 > 0$ such that for no positive number δ is it true that $f(\mathcal{B}_\delta(p)) \subseteq \mathcal{B}_{\epsilon_0}(f(p))$. In particular, if k is a positive integer, then it is not true that $f(\mathcal{B}_{1/k}(p)) \subseteq \mathcal{B}_{\epsilon_0}(f(p))$. This means that there is a point in X, which we label p_k, such that $d(p, p_k) < 1/k$ while $d(f(p), f(p_k)) \geq \epsilon_0$. This defines a sequence $\{p_k\}$ in X that converges to p, whose image sequence $\{f(p_k)\}$ does not converge to $f(p)$. This contradicts the continuity of the mapping $f : X \to Y$ at the point p. Thus, (12.20) holds.

To prove the converse, suppose that (12.20) holds. Let $\{p_k\}$ be a sequence in X that converges to p. We must show that $\{f(p_k)\}$ converges to $f(p)$. Let $\epsilon > 0$. According to (12.20), we can choose a positive number δ such that $f(\mathcal{B}_\delta(p)) \subseteq \mathcal{B}_\epsilon(f(p))$. Moreover, since the sequence $\{p_k\}$ converges to p, we can select an index N such that p_k is in $\mathcal{B}_\delta(p)$ if $k \geq N$; hence $f(p_k)$ is in $\mathcal{B}_\epsilon(f(p))$ if $k \geq N$. Thus, the sequence $\{f(p_k)\}$ converges to $f(p)$. By definition, this means that $f : X \to Y$ is continuous at the point p. ∎

Finally, we use the preceding characterization of continuity of a mapping at a point to provide the following criterion for a mapping to be continuous on all of its domain.

Theorem 12.35 Let X and Y be metric spaces and consider the mapping $f : X \to Y$. Then the following assertions are equivalent:

i. The mapping $f : X \to Y$ is continuous.
ii. $f^{-1}(V)$ is open in X whenever V is open in Y.

Proof

First, suppose that (i) holds; that is, the mapping $f : X \to Y$ is continuous. Let V be an open subset of Y. We wish to show that $f^{-1}(V)$ is open in X. Let p be a point in $f^{-1}(V)$; we must show that p is an interior point of $f^{-1}(V)$. But $f(p)$ is a point in V, which is open in Y, so there is some positive number r with $\mathcal{B}_r(f(p)) \subseteq V$. Since $f : X \to Y$ is continuous at the point p, it follows from Theorem 12.34 that we can select a positive number δ with $f(\mathcal{B}_\delta(p)) \subseteq \mathcal{B}_r(f(p)) \subseteq V$. Thus, $\mathcal{B}_\delta(p) \subseteq f^{-1}(V)$. So p is an interior point of $f^{-1}(V)$. Since p was arbitrarily chosen, every point in $f^{-1}(V)$ is an interior point. By definition, this means that $f^{-1}(V)$ is open.

To prove the converse, suppose that (ii) holds. Let p be a point in X. To show that $f : X \to Y$ is continuous at p, we use the ϵ-δ characterization of continuity asserted in Theorem 12.34. Let $\epsilon > 0$. According to Proposition 12.8, $\mathcal{B}_\epsilon(f(p))$ is open in Y. From (ii) it follows that $f^{-1}(\mathcal{B}_\epsilon(f(p))$ is open in X. Thus, the point p in $f^{-1}(\mathcal{B}_\epsilon(f(p))$ is an interior point of $f^{-1}(\mathcal{B}_\epsilon(f(p))$, so we can choose a positive number δ with $\mathcal{B}_\delta(p) \subseteq f^{-1}(\mathcal{B}_\epsilon(f(p))$. This means that $f(\mathcal{B}_\delta(p)) \subseteq \mathcal{B}_\epsilon(f(p))$. Thus, the function $f : X \to Y$ satisfies the ϵ-δ characterization of continuity at the point p. ∎

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EXERCISES FOR SECTION 12.4

1. Let X be a metric space. Prove the following inequality:

$$|d(p,x) - d(x,q)| \le d(p,q) \qquad \text{for all points } p, q, \text{ and } x \text{ in } X.$$

2. Let X be a metric space and p_* be a point in X. Define the mapping $f : X \to \mathbb{R}$ by

$$f(p) = d(p, p_*) \qquad \text{for all } p \text{ in } X.$$

Use Exercise 1 to prove that $f : X \to \mathbb{R}$ is continuous.

3. Let X be a metric space and let the mapping $f : X \to \mathbb{R}$ be continuous. Let p be a point in X at which $f(p) > 0$. Use Theorem 12.34 to show that there is some positive number r such that $f(q) > 0$ for all q in $\mathcal{B}_r(p)$.

4. Given a mapping $f : X \to Y$ and a subset Z of X, the *restriction* of $f : X \to Y$ to Z is the function $g : Z \to Y$ defined by $g(x) = f(x)$ for x in Z. Though $f : X \to Y$ and $g : Z \to Y$ are distinct functions when $X \neq Z$, it is customary to denote $g : Z \to Y$ by $f : Z \to Y$. Give an example of a function $f : X \to Y$, a subspace Z of X, and a point p in Z such that $f : X \to Y$ is not continuous at p while $f : Z \to Y$ is continuous at p.

5. Let X and Y be metric spaces and let p be a point in X. Show that a mapping $f : X \to Y$ is continuous at p if and only if there is an open ball $\mathcal{B}_r(p)$ about p in X such that the restriction $f : \mathcal{B}_r(p) \to Y$ is continuous at p. Compare this result with that for Exercise 4.

6. Suppose that X is a metric space and that the mapping $f : \mathbb{R} \to X$ is continuous. Let C be a closed subset of X and let $f(x)$ belong to C if x is rational. Prove that $f(\mathbb{R}) \subseteq C$.

7. Let X and Y be metric spaces and suppose that the mapping $f : X \to Y$ is continuous. Is it true that if $\mathcal{O}$ is open in X, then $f(\mathcal{O})$ is open in Y?

8. Let X and Y be metric spaces. Prove that $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y.

9. Let X be a metric space. A subset D of X is said to be *dense* in X if every point in X is the limit of a sequence in D. Formulate the Weierstrass Approximation Theorem as an assertion of denseness in the metric space $C([a,b], \mathbb{R})$.

10. Let $X = C([a,b], \mathbb{R})$ and define the function $\psi : X \to \mathbb{R}$ by

$$\psi(f) = \int_a^b f(x)\,dx \qquad \text{for each } f \text{ in } X.$$

a. Prove that

$$|\psi(f) - \psi(g)| \le (b-a)d(f,g) \qquad \text{for all } f \text{ and } g \text{ in } X.$$

b. Use the above inequality to verify that $\psi : X \to \mathbb{R}$ is continuous.

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12.5 SEQUENTIAL COMPACTNESS AND CONNECTEDNESS

We have already defined what it means for a subset of Euclidean space $\mathbb{R}^n$ to be sequentially compact, and we have established Theorem 11.22, the Extreme Value Theorem for continuous real-valued functions on sequentially compact subsets of $\mathbb{R}^n$. In Section 11.4, we defined what it means for a subset of Euclidean space to be connected, we described certain criteria for verifying connectedness, and we extended the Intermediate Value Theorem to continuous real-valued functions on connected subsets of Euclidean n-space. It turns out that there are straightforward extensions of the concepts of sequential compactness and connectedness for general metric spaces, and we consider these in the present section.

Definition A metric space X is said to be *sequentially compact* provided that every sequence in X has a subsequence that converges to a point in X.

This extends the definition of sequential compactness that we gave in Section 11.2 when X is a subspace of $\mathbb{R}^n$. The Extreme Value Theorem generalizes to continuous real-valued functions on sequentially compact metric spaces and, as we will now see, the proof of the general result is almost the same as the proof for sequentially compact subspaces of Euclidean space.

Proposition 12.36 Let X and Y be metric spaces. Suppose that the mapping $f : X \to Y$ is continuous. If X is sequentially compact, then $f(X)$ is also sequentially compact.

Proof
Let $\{p_k\}$ be a sequence in $f(X)$. For each natural number k, let q_k be a point in X such that $p_k = f(q_k)$. Since X is sequentially compact, there is a subsequence $\{q_{k_j}\}$ in X that converges to some point q. The mapping $f : X \to Y$ is continuous at the point q. Thus, the image sequence $\{p_{k_j}\} = \{f(q_{k_j})\}$ converges to the point $f(q)$ in $f(X)$. Hence every sequence in $f(X)$ has a subsequence that converges to a point in $f(X)$. By definition, this means that $f(X)$ is sequentially compact. ■

Theorem 12.37 The Extreme Value Theorem Let X be a nonempty sequentially compact metric space and let the function $f : X \to \mathbb{R}$ be continuous. Then $f : X \to \mathbb{R}$ attains both a minimum and a maximum value on X.

Proof
From the preceding proposition it follows that $f(X)$ is sequentially compact. However, Lemma 11.21 asserts that a nonempty sequentially compact subset of $\mathbb{R}$ has both a largest and a smallest member. ■

For X a subspace of $\mathbb{R}^n$, Theorem 11.18, the Sequential Compactness Theorem, is the assertion that X is sequentially compact if and only if X is a closed bounded subset

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of $\mathbb{R}^n$. For general metric spaces there is no such simple characterization of sequential compactness. In particular, as the following example shows, it is not true that every closed bounded subset of $C([a, b], \mathbb{R})$ is sequentially compact.

Example 12.38 Let K be the subset of $C([0, 1], \mathbb{R})$ consisting of all continuous functions $f : [0, 1] \to \mathbb{R}$ such that

$$|f(x)| \leq 1 \qquad \text{for all } x \text{ in } [0, 1].$$

We leave it as an exercise for the reader to verify that K is a closed bounded subset of $C([0, 1], \mathbb{R})$. However, K is not a sequentially compact metric space. To establish this assertion, it is necessary to find a sequence in K that has the property that no subsequence converges uniformly to a function in K. For each positive integer k define the function $f_k : [0, 1] \to \mathbb{R}$ by

$$f_k(x) = x^k \qquad \text{for } x \text{ in } [0, 1].$$

Then define the function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since $\lim_{k\to\infty} x^k = 0$ if $0 \leq x < 1$, it follows that the sequence $\{f_k : [0, 1] \to \mathbb{R}\}$ converges pointwise to the function $f : [0, 1] \to \mathbb{R}$. Hence every subsequence of $\{f_k : [0, 1] \to \mathbb{R}\}$ also converges pointwise to the function $f : [0, 1] \to \mathbb{R}$. But the function $f : [0, 1] \to \mathbb{R}$ is not continuous and hence is not in K. Thus, there is no subsequence of $\{f_k : [0, 1] \to \mathbb{R}\}$ that converges in the metric space $C([0, 1], \mathbb{R})$ (that is, converges uniformly) to a function in K. ∎

There is a theorem, called the Arzela–Ascoli Theorem, that characterizes the sequentially compact subspaces of $C([0, 1], \mathbb{R})$. Unfortunately, this theorem lies outside the scope of this book.[3]

Definition A metric space X is said to have the Intermediate Value Property provided that every continuous function $f : X \to \mathbb{R}$ has an interval as its image.

Theorem 11.36 is the assertion that a subset of $\mathbb{R}^n$ has the Intermediate Value Property if and only if it is connected. We wish to characterize the general metric spaces that have the Intermediate Value Property. This amounts to discovering the appropriate concept of connectedness for a general metric space.

Definition Let X be a metric space and let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of X. Then $\mathcal{U}$ and $\mathcal{V}$ are said to *separate* X provided that the two sets $\mathcal{U}$ and $\mathcal{V}$ are nonempty, they are

[3] The excellent book *Introduction to Topology and Modern Analysis* by George F. Simmons (New York: McGraw-Hill, 1963) presents a very clear discussion of this and of related matters.

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disjoint, and their union equals X; that is

$$\mathcal{U} \neq \emptyset, \mathcal{V} \neq \emptyset, \mathcal{U} \cap \mathcal{V} = \emptyset \qquad \text{and} \qquad \mathcal{U} \cup \mathcal{V} = X.$$

Definition A metric space X is said to be *connected* provided that there do not exist two open subsets of X that separate X.

The following theorem justifies the introduction of the concept of connectedness.

Theorem 12.39 A metric space X is connected if and only if it has the Intermediate Value Property.

Proof
First, suppose that the metric space X is connected. We will show that X has the Intermediate Value Property. Indeed, let $f : X \to \mathbb{R}$ be continuous. To show that the image $f(X)$ is an interval, we suppose otherwise and derive a contradiction. Indeed, if $f(X)$ is not an interval, then there is a real number c and points u and v in X such that

$$f(u) < c < f(v),$$

but c is not in $f(X)$. Define

$$X_1 = f^{-1}(-\infty, c) \qquad \text{and} \qquad X_2 = f^{-1}(c, \infty).$$

Since both $(-\infty, c)$ and (c, ∞) are open subsets of $\mathbb{R}$ and the function $f : X \to \mathbb{R}$ is continuous, it follows from Theorem 12.35 that both X_1 and X_2 are open subsets of X. Also, observe that neither X_1 nor X_2 is empty since u belongs to X_1 and v belongs to X_2. Moreover, $X = X_1 \cup X_2$ since c is not in $f(X)$. Finally, it is clear that X_1 and X_2 are disjoint. Thus, X_1 and X_2 separate the metric space X, which contradicts the assumption that X is connected. This contradiction shows that X has the Intermediate Value Property.

To prove the converse, suppose that every continuous function $f : X \to \mathbb{R}$ has the Intermediate Value Property. We will show that X is connected by assuming otherwise and deriving a contradiction. Suppose that X is not connected. Then there is a pair of open subsets $\mathcal{U}$ and $\mathcal{V}$ of X that separate X. Define the function $f : X \to \mathbb{R}$ by

$$f(p) = \begin{cases} 0 & \text{if } p \in \mathcal{U} \\ 1 & \text{if } p \in \mathcal{V}. \end{cases}$$

Then the function $f : X \to \mathbb{R}$ certainly fails to have the Intermediate Value Property since it attains exactly two functional values, namely, 0 and 1. We also claim that the function $f : X \to \mathbb{R}$ is continuous. To see this, according to Theorem 12.35, it is sufficient to check that the inverse images under $f : X \to \mathbb{R}$ of open subsets of $\mathbb{R}$ are open in X. But from the definition of the function $f : X \to \mathbb{R}$, it is clear that the inverse image of any subset of $\mathbb{R}$ is $\emptyset$, X, $\mathcal{U}$, or $\mathcal{V}$, and hence it certainly is open. The existence of this continuous function whose image is not an interval contradicts the assumption that X has the Intermediate Value Property. Thus, X must be connected. ■

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The definition that we have given of a connected metric space is formally different from the definition of a connected subset of $\mathbb{R}^n$ given in Section 11.4. However, the definitions are equivalent. This follows from the fact that each of them is equivalent to the Intermediate Value Property. However, it also follows by a direct argument. To understand this direct argument, we finish this chapter with a brief discussion of the relative concepts of open and closed.

Suppose that Y is a metric space and that X is a subspace of Y. Consider a subset A of X. Then, of course, A is also a subset of Y. Now, considering A as a subset of X, we can ask whether A is open in the metric space X. On the other hand, considering A as a subset of Y, we can ask whether A is open in the metric space Y. In general, the answers to these two questions are different. As the following example shows, the concept of openness is *relative;* it depends on the designation of an ambient metric space.

Example 12.40 Let $Y = \mathbb{R}$ and $X = [0, 1]$. Then $A = (1/2, 1]$ is not an open subset of the metric space Y since every open ball about 1 in Y contains points not in $(1/2, 1]$. On the other hand, $(1/2, 1]$ is open in the metric space X since $\{x \text{ in } X \mid d(x, 1) < r\} = (r, 1] \subseteq (1/2, 1]$ if $0 < r < 1/2$. ■

The following theorem provides a description, in the case when the metric space X is a subspace of the metric space Y, of the open subsets of X in terms of the open subsets of Y.

Theorem 12.41 Let X be a subspace of the metric space Y and let A be a subset of X. Then A is open in X if and only if $A = X \cap \mathcal{O}$, where $\mathcal{O}$ is open in Y.

Proof

First, suppose that A is open in X. Let p be a point in A. Then there is an open ball about p in A that is contained in X; that is, there is a positive number $r = r(p)$ such that

$$\{q \text{ in } X \mid d(p, q) < r\} \subseteq A. \tag{12.21}$$

Now the open ball about the point p, in Y, of radius r is an open subset of Y, and hence the union of such open balls, as p varies in A, is also an open subset of Y; that is,

$$\mathcal{O} = \bigcup_{p \in A} \{q \text{ in } Y \mid d(q, p) < r\}$$

is an open subset of Y. From the inclusion (12.21) it is clear that $A = X \cap \mathcal{O}$.

To prove the converse, suppose that $A = X \cap \mathcal{O}$, where $\mathcal{O}$ is an open subset of Y. Let p be a point in A. Then p also belongs to $\mathcal{O}$ and is therefore an interior point, relative to Y, of $\mathcal{O}$. Thus, we can select a positive number r with $\{q \text{ in } Y \mid d(q, p) < r\} \subseteq \mathcal{O}$. Thus, $\{q \text{ in } X \mid d(q, p) < r\} \subseteq X \cap \mathcal{O} = A$. Hence p is in the interior, relative to X, of A. Thus, every point in A is an interior point, relative to X, of A. By definition, this means that A is open in X. ■

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It follows immediately from the above theorem that the concept of connectedness introduced for subsets of Euclidean n-space in Section 11.4 is consistent with the general definition of connectedness given in the present section.

As the following example illustrates, the concept of closedness is also a relative concept.

Example 12.42 Let $Y = \mathbb{R}$ and $X = (0, 2]$. Consider the set $A = (0, 1]$. Then A is not a closed subset of the metric space Y since the sequence $\{1/2k\}$ is a sequence in A that converges to a point in Y that does not lie in A. However, A is a closed subset of the metric space X since it is clear that if a sequence in A converges to a point in X, then that point must be in A. ■

Using the Complementing Criterion and Theorem 12.41, we can show that if the metric space X is a subspace of the metric space Y, then a subset A of X is closed in X if and only if there is a closed subset C of Y such that $A = X \cap C$ (Exercise 8).

EXERCISES FOR SECTION 12.5

1. Let X be a sequentially compact metric space. Prove that the subspace K of X is sequentially compact if and only if K is a closed subset of X.
2. Let X be a metric space and let $K \subseteq X$ be a sequentially compact subspace. Let p be a point in $X \backslash K$. Prove that there is a point q_0 in K that is closest to p of all points in K in the sense that
$$d(p, q_0) \le d(p, q) \qquad \text{for all points } q \text{ in } K.$$
[*Hint:* Define the function $f: K \to \mathbb{R}$ by $f(q) = d(q, p)$ for all q in K. Show that $f: K \to \mathbb{R}$ is continuous.] Is this point unique?
3. Show that if X is a metric space and $\mathcal{U}$ and $\mathcal{V}$ are open subsets of X that separate X, then both $\mathcal{U}$ and $\mathcal{V}$ are also closed in X.
4. Prove that a metric space X is disconnected if and only if there is a subset D of X that is both open and closed in X, with $D \ne \emptyset$ and $D \ne X$.
5. Let X be a sequentially compact metric space. Prove that X is disconnected if and only if there are nonempty subsets A and B of X and a positive number ϵ, with $A \cap B = \emptyset$, $A \cup B = X$, and
$$d(p, q) > \epsilon \qquad \text{for all } p \text{ in } A \text{ and } q \text{ in } B.$$
Is sequentially compactness necessary?
6. Let X be a metric space. For each positive integer k, let F_k be a nonempty sequentially compact subspace of X and suppose that $F_{k+1} \subseteq F_k$. Prove that the intersection $\bigcap_{k=1}^{\infty} F_k$ is nonempty. (*Hint:* For each positive integer k, choose a point p_k in F_k. A subsequence of the sequence $\{p_k\}$ converges to a point p in X. Where does p lie?)
7. Use Theorem 12.41 to prove directly that for a subspace X of $\mathbb{R}^n$ the general definition of connectedness given in the present section coincides with that already given in Section 11.4.

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8. Using the Complementing Criterion and Theorem 12.41, show that if the metric space X is a subspace of the metric space Y, then a subset A of X is closed in X if and only if there is a closed subset C of Y such that $A = X \cap C$.

9. A metric space X is defined to be pathwise-connected provided that for any two points p and q in X, there is a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

a. Prove that a pathwise-connected metric space has the Intermediate Value Property. (*Hint:* Follow the corresponding proof in Section 11.3.)

b. Prove that a pathwise-connected metric space is connected.

10. We have defined what it means for a subset X of $\mathbb{R}^n$ to be convex.

a. Define what it means for a subset X of $C([a, b], \mathbb{R})$ to be convex.

b. Prove that convex subsets of $C([a, b], \mathbb{R})$ are pathwise-connected.

c. Prove that open balls in $C([a, b], \mathbb{R})$ are convex.

d. Prove that open balls in $C([a, b], \mathbb{R})$ are connected.

11. Suppose that X is a set consisting of more than one point, considered a metric space with the discrete metric. Show that X is not connected.

12. Let X be a sequentially compact metric space. Define $C(X, \mathbb{R})$ to be the set of all continuous functions $f : X \to \mathbb{R}$, and for two functions f and g in $C(X, \mathbb{R})$, define

$$d(f, g) = \max \{|f(p) - g(p)| \mid p \text{ in } X\}.$$

Prove that d defines a metric on $C(X, \mathbb{R})$.

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CHAPTER 13

DIFFERENTIATING FUNCTIONS OF SEVERAL VARIABLES

13.1 LIMITS

For I an open interval of real numbers, a function $f : I \to \mathbb{R}$ has been defined to be *differentiable at the point x_* in I* provided that the limit

$$\lim_{x \to x_*} \frac{f(x) - f(x_*)}{x - x_*} \tag{13.1}$$

exists. This limit, denoted by $f'(x_*)$, is called the *derivative* of the function $f : I \to \mathbb{R}$ at the point x_*. Moreover, if the function $f : I \to \mathbb{R}$ is differentiable at every point in I, then the function $f : I \to \mathbb{R}$ is said to be *differentiable* and the function $f' : I \to \mathbb{R}$ is called the *derivative* of the function $f : I \to \mathbb{R}$. The study of the relationship between a function and its derivative is one of the important topics in the analysis of real-valued functions of a single real variable.

In the present chapter we will turn to the study of differentiation for real-valued functions of several real variables, and we will extend to the case of functions of several variables many of the results we obtained earlier for functions of a single variable. We begin our study in this section by extending the definitions of the limit point of a set and of the limit of a function to the case of several variables.

Given a subset A of $\mathbb{R}^n$ and a point $\mathbf{x}_*$ in $\mathbb{R}^n$, recall that $A \backslash \{\mathbf{x}_*\}$ denotes the set $\{\mathbf{x} \text{ in } A \mid \mathbf{x} \neq \mathbf{x}_*\}$. In particular, $A \backslash \{\mathbf{x}_*\} = A$ if the point $\mathbf{x}_*$ does not belong to A.

Definition Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}_*$ be a point in $\mathbb{R}^n$. Then $\mathbf{x}_*$ is said to be a *limit point* of the set A provided there is a sequence in $A \backslash \{\mathbf{x}_*\}$ that converges to $\mathbf{x}_*$.

Definition Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A. Given a function $f : A \to \mathbb{R}$ and a real number ℓ, we write

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(x) = \ell \tag{13.2}$$

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provided that whenever $\{\mathbf{x}_k\}$ is a sequence in $A \backslash \{\mathbf{x}_*\}$ that converges to $\mathbf{x}_*$, the image sequence $\{f(\mathbf{x}_k)\}$ converges to ℓ.

We read (13.2) as "The limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{x}_*$ equals ℓ."

As in the case of real-valued functions of a single real variable, it is easy to see that if A is a subset of $\mathbb{R}^n$ and $\mathbf{x}_*$ is a limit point of A that also belongs to A, then the function $f : A \to \mathbb{R}$ is continuous at $\mathbf{x}_*$ if and only if

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = f(\mathbf{x}_*). \tag{13.3}$$

However, in Section 11.1 we studied continuous functions of several variables, and so, in view of the equivalence of continuity and (13.3), we already have at our disposal a number of ways of analyzing limits.

Example 13.1 Consider the point $(1, 2)$ in the plane $\mathbb{R}^2$. Then

$$\lim_{(x,y)\to(1,2)} [x^2 y + e^{xy+1}] = 2 + e^3.$$

This follows from the fact that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 y + e^{xy+1} \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2$$

has already been shown to be continuous. ■

Example 13.2 Let $\mathbf{x}_*$ be a point in $\mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_*} \|\mathbf{x}\| = \|\mathbf{x}_*\|.$$

This follows from the fact that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \|\mathbf{x}\| \qquad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

has already been shown to be continuous. ■

The following theorem follows directly from the definition of a limit by using the sum, product, and quotient properties of convergent sequences of real numbers. We leave the proof as an exercise.

Theorem 13.3 Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A. Suppose that the functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ and the real numbers ℓ_1 and ℓ_2 have the property that

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = \ell_1 \qquad \text{and} \qquad \lim_{\mathbf{x} \to \mathbf{x}_*} g(\mathbf{x}) = \ell_2.$$

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Then

$$\lim_{\mathbf{x}\to\mathbf{x}_*}[f(\mathbf{x})+g(\mathbf{x})]=\ell_1+\ell_2,$$

$$\lim_{\mathbf{x}\to\mathbf{x}_*}[f(\mathbf{x})g(\mathbf{x})]=\ell_1\ell_2,$$

and, if $g(\mathbf{x})\neq 0$ for all $\mathbf{x}$ in A and $\ell_2\neq 0$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_*}[f(\mathbf{x})/g(\mathbf{x})]=\ell_1/\ell_2.$$

One of the interesting but subtle features of many problems in mathematical analysis is the necessity of studying limits of quotients of the form

$$\lim_{\mathbf{x}\to\mathbf{x}_*}\frac{f(\mathbf{x})}{g(\mathbf{x})}$$

in the case where both

$$\lim_{\mathbf{x}\to\mathbf{x}_*}f(\mathbf{x})=0 \qquad\text{and}\qquad \lim_{\mathbf{x}\to\mathbf{x}_*}g(\mathbf{x})=0.$$

Such limits occur frequently. For instance, the limit (13.1), which is the very definition of the derivative for a function of a single real variable, is of this form provided that the function $f: I\to\mathbb{R}$ is continuous at the point x_0. Limits of this form also occur prominently in the study of functions of several real variables. To properly understand how to treat such limits, it is necessary to study the differentiation of functions of several real variables. We will study this in the succeeding sections of this chapter. For now, let us consider some examples of this type of limit.

Example 13.4 Let $(x_0,\ y_0)$ be a point in the plane $\mathbb{R}^2$. Consider the limit

$$\lim_{(x,y)\to(x_0,y_0)}\frac{xy}{x^2+y^2}. \tag{13.4}$$

Define $f(x,\ y)\ =\ xy/(x^2+y^2)$ if $(x,\ y)\ \neq\ (0,\ 0)$. Since polynomials are continuous,

$$\lim_{(x,y)\to(x_0,y_0)}xy=x_0y_0 \qquad\text{and}\qquad \lim_{(x,y)\to(x_0,y_0)}x^2+y^2=x_0^2+y_0^2,$$

so if $(x_0,\ y_0)\neq(0,\ 0)$, it follows that

$$\lim_{(x,y)\to(x_0,y_0)}\frac{xy}{x^2+y^2}=\frac{x_0y_0}{x_0^2+y_0^2}.$$

But the limit (13.4) does not exist if $(x_0,\ y_0)=(0,\ 0)$. To see this, first observe that the sequence $\{(1/k,\ 1/k)\}$ converges to the point $(0,\ 0)$, and since $f(1/k,1/k)=1/2$ for each natural number k, it follows that the image sequence $\{f(1/k,1/k)\}$ converges to $1/2$. On the other hand, the sequence $\{(1/k,\ 0)\}$ also converges to the point $(0,\ 0)$, and since $f(1/k,0)=0$ for each natural number k, it follows that the image sequence $\{f(1/k,0)\}$ converges to 0. Thus, the limit (13.4) does not exist if $(x_0,\ y_0)=(0,\ 0)$. ■

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Example 13.5 Consider the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}. \tag{13.5}$$

Define $f(x, y) = x^3/(x^2+y^2)$ if $(x, y) \neq (0, 0)$. By observing that the sequence $\{(0, 1/k)\}$ converges to $(0, 0)$ and that $f(0, 1/k) = 0$ for each index k, we see that the only possible value of the limit is 0. To verify that the limit is indeed 0, it is necessary to make some estimates of the size of $f(x, y)$. Indeed, if $x \neq 0$, then

$$\left|\frac{x^3}{x^2+y^2}\right| \leq \left|\frac{x^3}{x^2}\right| = |x|,$$

and therefore,

$$\left|\frac{x^3}{x^2+y^2}\right| \leq |x| \qquad \text{if } (x, y) \neq (0, 0)$$

since this estimate also clearly holds if $x = 0$ and $y \neq 0$. Now suppose that the sequence $\{(x_k, y_k)\}$ converges to $(0, 0)$ with each $(x_k, y_k) \neq (0, 0)$. Then the sequence $\{x_k\}$ converges to 0, so from the preceding estimate and the Comparison Lemma for convergent sequences, it follows that the image sequence $\{f(x_k, y_k)\}$ converges to 0. Thus,

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

■

Example 13.6 We claim that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1. \tag{13.6}$$

To verify this, let the sequence $\{(x_k, y_k)\}$ converge to $(0, 0)$ with each $(x_k, y_k) \neq (0, 0)$. Define $t_k = x_k^2 + y_k^2$ for each k and observe that $\{t_k\}$ is a sequence of nonzero real numbers that converges to 0. Since $\sin 0 = 0$, the derivative of the sine is the cosine, and $\cos 0 = 1$, it follows that

$$\lim_{k\to\infty} \frac{\sin\left(x_k^2+y_k^2\right)}{x_k^2+y_k^2} = \lim_{k\to\infty} \frac{\sin(t_k) - \sin 0}{t_k - 0} = 1.$$

This proves (13.6). ■

Theorem 11.11 provides an equivalence between the definition of continuity of a mapping at a point and the ϵ-δ criterion for continuity at a point. We have the following similar equivalent ϵ-δ criterion for limits, whose proof we leave as an exercise.

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Theorem 13.7 Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}_*$ be a limit point of A. For a function $f : A \to \mathbb{R}$ and a real number ℓ, the following two assertions are equivalent:

i. $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = \ell$;

that is, whenever $\{\mathbf{x}_k\}$ is a sequence in $A\backslash\{\mathbf{x}_*\}$,

$$\lim_{k\to\infty} f(\mathbf{x}_k) = \ell \qquad \text{if } \lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_*.$$

ii. For each positive number ϵ there is a positive number δ such that

$$|f(\mathbf{x}) - \ell| < \epsilon \qquad \text{if } \mathbf{x} \text{ is in } A\backslash\{\mathbf{x}_*\} \text{ and } \operatorname{dist}(\mathbf{x}, \mathbf{x}_*) < \delta.$$

EXERCISES FOR SECTION 13.1

1. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^2 + y^4} = 0.$$

2. Analyze the following limits:

a. $\lim_{h\to 0} \dfrac{e^h - 1}{h}$ **b.** $\lim_{t\to 0} \dfrac{\sin t^2}{t}$

3. Analyze the following limits:

a. $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y^2}$

b. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x+y+z}{x^2+y^2+z^2}$

c. $\lim_{(x,y)\to(0,0)} \dfrac{e^{x^2+y^2} - 1}{x^2+y^2}$

4. Let m and n be natural numbers. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^n y^m}{x^2 + y^2}$$

exists if and only if $m + n > 2$.

5. Give an example of a subset A of $\mathbb{R}$ and a point x in A that is not a limit point of the set A.

6. Let A be a subset of $\mathbb{R}^n$ and let $\mathbf{x}$ be a point in $\mathbb{R}^n$. Show that $\mathbf{x}$ is a limit point of A if and only if every open ball about $\mathbf{x}$ contains a point of A that is not equal to $\mathbf{x}$.

7. Let A be a subset of $\mathbb{R}^n$ and let the point $\mathbf{x}_*$ in $\mathbb{R}^n$ be a limit point of A. Suppose that the function $g : A \to \mathbb{R}$ is bounded; that is, there is a number c such that

$$|g(\mathbf{x})| \le c \qquad \text{for all } \mathbf{x} \text{ in } A.$$

Prove that if $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x}) = 0$, then $\lim_{\mathbf{x}\to\mathbf{x}_*}[g(\mathbf{x})f(\mathbf{x})] = 0$.

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8. Let A be a subset of $\mathbb{R}^n$. The function $f:\mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is in } A \\ 0 & \text{if } \mathbf{x} \text{ is not in } A \end{cases}$$

is called the *characteristic function* of A. Let $\mathbf{x}_*$ be a point in $\mathbb{R}^n$. Show that $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x})$ exists if and only if $\mathbf{x}_*$ is either an interior point or an exterior point of the set A.

9. Give an example of a function $f:\mathbb{R}^n \to \mathbb{R}$ such that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$ but that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})/\|\mathbf{x}\| \neq 0$.

10. For a function $f:\mathbb{R}^n \to \mathbb{R}$ and a natural number m, show that

$$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})/\|\mathbf{x}\|^{m+1} = 0 \text{ implies that } \lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})/\|\mathbf{x}\|^{m} = 0,$$

but that the converse does not hold.

11. Let A be a subset of $\mathbb{R}^n$ and suppose that $\mathbf{0}$ is a limit point of A. Suppose that the function $f:A \to \mathbb{R}$ has the property that there is a positive number c such that

$$f(\mathbf{x}) \geq c\|\mathbf{x}\|^2 \qquad \text{for all } \mathbf{x} \text{ in } A$$

and that the function $g:A \to \mathbb{R}$ has the property that

$$\lim_{\mathbf{x}\to\mathbf{0}} g(\mathbf{x})/\|\mathbf{x}\|^2 = 0.$$

Prove that there is a positive number r such that

$$f(\mathbf{x}) - g(\mathbf{x}) \geq (c/2)\|\mathbf{x}\|^2 \qquad \text{for all } \mathbf{x} \text{ in } A \text{ with } 0 < \|\mathbf{x}\| < r.$$

12. Let A be a subset of $\mathbb{R}^n$. Show that A is closed if and only if it contains all of its limit points.

13. Let A and B be subsets of $\mathbb{R}^n$ with $A \subseteq B$. Suppose that the point $\mathbf{x}_*$ in $\mathbb{R}^n$ is a limit point of A. Given a function $f:B \to \mathbb{R}$, we define its *restriction* to the set A to be the function $\bar{f}:A \to \mathbb{R}$ defined by $\bar{f}(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in A. Find an example of a function for which $\lim_{\mathbf{x}\to\mathbf{x}_*} \bar{f}(\mathbf{x})$ exists but $\lim_{\mathbf{x}\to\mathbf{x}_*} f(\mathbf{x})$ does not exist. (*Note:* Frequently, for notational simplicity, the function $\bar{f}:A \to \mathbb{R}$ is simply denoted by $f:A \to \mathbb{R}$, but then a new notational device needs to be invented to distinguish the two limits.)

13.2 PARTIAL DERIVATIVES

Consider a real-valued function of several real variables $f:\mathbb{R}^n \to \mathbb{R}$, together with two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Suppose that we want to compare $f(\mathbf{u})$ with $f(\mathbf{v})$. When $n = 1$ and the function $f:\mathbb{R} \to \mathbb{R}$ is differentiable, we can use the Mean Value Theorem to compare these two values. When $n > 1$, the following restriction procedure is natural. Look at the parametrized segment from $\mathbf{u}$ to $\mathbf{v}$—that is, the parametrized path $\gamma:[0,1] \to \mathbb{R}$ defined by

$$\gamma(t) = \mathbf{u} + t(\mathbf{v} - \mathbf{u}) = t\mathbf{v} + (1-t)\mathbf{u} \qquad \text{for } 0 \leq t \leq 1.$$

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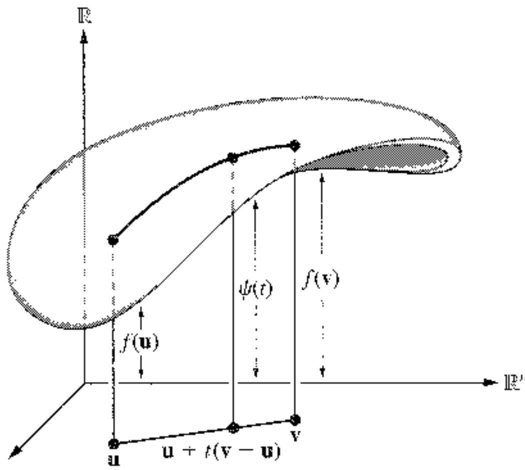

FIGURE 13.1 $f(\mathbf{v}) - f(\mathbf{u}) = \psi(1) - \psi(0)$.

Then consider the composition of the function $f : \mathbb{R}^n \to \mathbb{R}$ with this parametrized path, which is the function $\psi : [0, 1] \to \mathbb{R}$ defined by

$$\psi(t) = f(\mathbf{u} + t(\mathbf{v} - \mathbf{u})) \qquad \text{for } 0 \le t \le 1. \tag{13.7}$$

Then $\psi(0) = f(\mathbf{u})$ and $\psi(1) = f(\mathbf{v})$. Thus, to compare $f(\mathbf{u})$ with $f(\mathbf{v})$ is to compare $\psi(0)$ with $\psi(1)$. If we can determine that $\psi : [0, 1] \to \mathbb{R}$ is continuous and that $\psi : (0, 1) \to \mathbb{R}$ is differentiable, then we can apply the Mean Value Theorem for functions of a single variable to compare $f(\mathbf{u})$ with $f(\mathbf{v})$. Thus, it is necessary to investigate the properties of the function $f : \mathbb{R}^n \to \mathbb{R}$ that will allow us to conclude that the above auxiliary function $\psi : [0, 1] \to \mathbb{R}$ is continuous, to conclude that $\psi : (0, 1) \to \mathbb{R}$ is differentiable, and to compute $\psi' : (0, 1) \to \mathbb{R}$.

We can regard the above function $\psi : [0, 1] \to \mathbb{R}$ as being the restriction of the function $f : \mathbb{R}^n \to \mathbb{R}$ to the line segment between the points $\mathbf{u}$ and $\mathbf{v}$, together with the placing of a coordinate system on this line segment. In the case where $n = 2$, the graph of $\psi : [0, 1] \to \mathbb{R}$ is obtained by intersecting the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ with the plane that is parallel to the z-axis and contains the segment joining $\mathbf{u}$ and $\mathbf{v}$. For this reason, we refer to the function $\psi : [0, 1] \to \mathbb{R}$ as a *section* of the function $f : \mathbb{R}^n \to \mathbb{R}$.

In order to analyze the differentiability of the function $\psi : (0, 1) \to \mathbb{R}$ at the point t_0, we change variables by setting $\mathbf{x} = \mathbf{u} + t_0(\mathbf{v} - \mathbf{u})$, $\mathbf{p} = \mathbf{v} - \mathbf{u}$, and $s = t - t_0$; then

$$\frac{\psi(t) - \psi(t_0)}{t - t_0} = \frac{f(\mathbf{x} + s\mathbf{p}) - f(\mathbf{x})}{s},$$

and therefore,

$$\psi'(t_0) = \lim_{s \to 0} \frac{f(\mathbf{x} + s\mathbf{p}) - f(\mathbf{x})}{s} \tag{13.8}$$

provided that the limit exists. The strategy of looking at sections of a function, together with formula (13.8), motivates the introduction of the following concept of a partial derivative.

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Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and let i be an index with $1 \leq i \leq n$. A function $f : \mathcal{O} \to \mathbb{R}$ is said to have a *partial derivative with respect to its ith component at the point* $\mathbf{x}$ provided that

$$\lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t}$$

exists. If this limit exists, we denote its value by $\partial f / \partial x_i\,(\mathbf{x})$ and call it the *partial derivative of* $f : \mathcal{O} \to \mathbb{R}$ *with respect to the ith component at the point* $\mathbf{x}$.

The geometric meaning of $\partial f / \partial x_i\,(\mathbf{x})$ is as follows: Choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$ and consider the section defined by

$$\psi(t) = f(\mathbf{x} + t\mathbf{e}_i) \qquad \text{for } |t| < r.$$

Then $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its ith component at the point $\mathbf{x}$ precisely when there is a tangent line to the graph of this section at the point on the graph corresponding to $t = 0$, at which point the slope of this tangent is the number

$$\psi'(0) = \frac{\partial f}{\partial x_i}\,(\mathbf{x}).$$

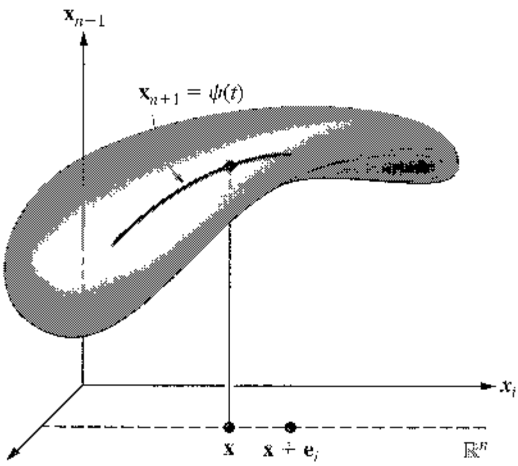

FIGURE 13.2 $\psi(t) = f(\mathbf{x} + t\mathbf{e}_i)$.

Thus, the existence of $\partial f / \partial x_i(\mathbf{x})$ is equivalent to the differentiability of a function of a single real variable, so we can immediately use the single-variable differentiation results to obtain addition, product, and quotient rules for partial derivatives.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Then the function $f : \mathcal{O} \to \mathbb{R}$ is said to have *first-order partial derivatives* provided that for each index i with $1 \leq i \leq n$, the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its ith component at every point in $\mathcal{O}$.

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It is clear that for each index i with $1 \le i \le n$, the ith component projection function

$$p_i : \mathbb{R}^n \to \mathbb{R}$$

has first-order partial derivatives and that if j is an index with $1 \le j \le n$ and $\mathbf{x}$ is any point in $\mathbb{R}^n$, then

$$\frac{\partial p_i}{\partial x_j}(\mathbf{x}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Hence sums, products, and suitable quotients of projection functions have first-order partial derivatives. This means that quotients of polynomials in the component variables, if their denominators are nonzero, define functions having first-order partial derivatives.

Naturally, if points in $\mathbb{R}^n$ have their components described in a notation without indices, we make a corresponding change in the notation for partial derivatives. So, for instance, if a function $f : \mathbb{R}^3 \to \mathbb{R}$ is given and points in $\mathbb{R}^3$ are written as (x, y, z), then $\partial f/\partial y\,(x_0, y_0, z_0)$ denotes the partial derivative of $f : \mathbb{R}^3 \to \mathbb{R}$ with respect to the second component at the point (x_0, y_0, z_0). Likewise, if a function $f : \mathbb{R}^4 \to \mathbb{R}$ is given and points in $\mathbb{R}^4$ are written as (u, v, w, t), then $\partial f/\partial t\,(u_0, v_0, w_0, t_0)$ denotes the partial derivative of $f : \mathbb{R}^4 \to \mathbb{R}$ with respect to the fourth component at the point (u_0, v_0, w_0, t_0).

Example 13.8 Define

$$f(x, y, z) = xyz + e^{xy^2} \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

Let us formally check that the function $f : \mathbb{R}^3 \to \mathbb{R}$ has a partial derivative with respect to its second component at each point in $\mathbb{R}^3$. Choose a point (x_0, y_0, z_0) in $\mathbb{R}^3$. Since

$$(x_0, y_0, z_0) + t\mathbf{e}_2 = (x_0, y_0 + t, z_0) \qquad \text{for each real number } t,$$

we must show that

$$\lim_{t \to 0} \frac{f(x_0, y_0 + t, z_0) - f(x_0, y_0, z_0)}{t}$$

exists. This is precisely the same as fixing the components x_0 and z_0, defining $g(y) = x_0 y z_0 + e^{x_0 y^2}$ for y in $\mathbb{R}$, and calculating $g'(y_0)$. Hence

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = g'(y_0) = x_0 z_0 + 2y_0 x_0 e^{x_0 y_0^2}. \qquad \blacksquare$$

The above example illustrates a general method for actually calculating specific first-order partial derivatives.

Proposition 4.5 asserts that if I is an open interval of real numbers and the function $f : I \to \mathbb{R}$ is differentiable at the point x in I, then the function $f : I \to \mathbb{R}$ is also continuous at the point x. This assertion is usually expressed as "differentiability implies

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continuity." The proof of this assertion is quite straightforward: If $h \neq 0$ and the point $x+h$ belongs to I, write the difference $f(x+h) - f(x)$ as

$$f(x+h) - f(x) = \frac{f(x+h) - f(x)}{h} \cdot h,$$

so that from the product rule for limits,

$$\lim_{h \to 0} [f(x+h) - f(x)] = f'(x) \cdot 0 = 0.$$

Thus, the function $f : I \to \mathbb{R}$ is continuous at the point x.

There is an important difference between functions of a single real variable and functions of several real variables. *For $n > 1$, a function $f : \mathbb{R}^n \to \mathbb{R}$ that has first-order partial derivatives need not be continuous.* The following example shows what can occur.

Example 13.9 Define

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For $(x, y) \neq (0, 0)$, there is a neighborhood of (x, y) on which the restriction of $f : \mathbb{R}^2 \to \mathbb{R}$ is a quotient of polynomials whose denominator does not vanish. Thus, $\partial f / \partial x(x, y)$ and $\partial f / \partial y(x, y)$ exist; moreover, a short computation yields

$$\frac{\partial f}{\partial x}(x, y) = \frac{y^3 - x^2 y}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = \frac{x^3 - y^2 x}{(x^2 + y^2)^2}.$$

On the other hand, at $(x, y) = (0, 0) = \mathbf{0}$, we observe that for each number t,

$$f(\mathbf{0} + t\mathbf{e}_1) = f(t, 0) = 0,$$

so that

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(\mathbf{0} + t\mathbf{e}_1) - f(\mathbf{0})}{t} = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = 0.$$

A similar calculation shows that $\partial f / \partial y(0, 0) = 0$. Thus, the function f has first-order partial derivatives at every point in the plane $\mathbb{R}^2$. Yet this function is not continuous at the point $\mathbf{0}$. To see this, observe that the sequence $\{(1/k, 1/k)\}$ converges to $(0, 0)$ and that $f(1/k, 1/k) = 1/2$ for each index k, so the image sequence $\{f(1/k, 1/k)\}$ converges to $1/2$. But $1/2 \neq f(0, 0)$. Thus, the function f is not continuous at the point $(0, 0)$. ■

But things are not as bad as the above example makes them seem. We will show that if $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives, then, if we assume in addition that $\partial f / \partial x_i : \mathcal{O} \to \mathbb{R}$ is continuous for each index i with $1 \leq i \leq n$, the basic results of the single-variable theory carry over. Since this additional assumption will play an important part later, it is useful to name it.

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Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. Then a function $f : \mathcal{O} \to \mathbb{R}$ is said to be *continuously differentiable* provided that it has first-order partial derivatives such that

$$\text{each partial derivative } \frac{\partial f}{\partial x_i} : \mathcal{O} \to \mathbb{R} \text{ is continuous for } 1 \leq i \leq n.$$

By calculating each of the first-order partial derivatives of the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined in Example 13.8, we see that this function is continuously differentiable. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined in Example 13.9 is not continuously differentiable since neither of its two first-order partial derivatives is continuous at the point $(0, 0)$ (Exercise 3).

We now turn to second-order partial derivatives. Given an open subset $\mathcal{O}$ of $\mathbb{R}^n$ and an index i with $1 \leq i \leq n$, if the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its ith component at each point in $\mathcal{O}$, then the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ is defined and we can ask whether this new function itself has first-order partial derivatives. Fix an index j with $1 \leq j \leq n$. If the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its jth component at the point $\mathbf{x}$ in $\mathcal{O}$, we use

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) \text{ to denote } \frac{\partial}{\partial x_j}\left[\frac{\partial f}{\partial x_i}\right](\mathbf{x}).$$

Naturally, when $n = 2$ or 3 and points are labeled without subscripts, we use a more suggestive notation for second partial derivatives, for example,

$$\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y \partial z}, \text{ and so on.}$$

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and consider a function $f : \mathcal{O} \to \mathbb{R}$:

i. The function $f : \mathcal{O} \to \mathbb{R}$ is said to have *second-order partial derivatives* provided that it has first-order partial derivatives and that, for each index i with $1 \leq i \leq n$, the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ also has first-order partial derivatives.

ii. The function $f : \mathcal{O} \to \mathbb{R}$ is said to have *continuous second-order partial derivatives* provided that it has second-order partial derivatives and that, for each pair of indices i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, the function $\partial^2 f/\partial x_i \partial x_j : \mathcal{O} \to \mathbb{R}$ is continuous.

It turns out that every continuously differentiable function is continuous and that, as a consequence, every function with continuous second-order partial derivatives is continuously differentiable. However, in order to prove this first assertion, it is necessary first to prove a Mean Value Theorem for functions of several variables, which we will do in the next section.

We close this section with a useful result about the equality of certain second-order partial derivatives.

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Theorem 13.10 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ has continuous second-order partial derivatives. For any two indices i and j with $1\le i\le n$ and $1\le j\le n$ and any point $\mathbf{x}$ in $\mathcal{O}$,

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(\mathbf{x})=\frac{\partial^2 f}{\partial x_j\partial x_i}(\mathbf{x}). \tag{13.9}$$

In order to prove this theorem, it will clarify matters if we first isolate the following lemma.

Lemma 13.11 Let $\mathcal{U}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0) and suppose that the function $f:\mathcal{U}\to\mathbb{R}$ has second-order partial derivatives. Then there are points (x_1, y_1) and (x_2, y_2) in $\mathcal{U}$ at which

$$\frac{\partial^2 f}{\partial x\partial y}(x_1, y_1)=\frac{\partial^2 f}{\partial y\partial x}(x_2, y_2).$$

Proof

Since $\mathcal{U}$ is open, we can choose a positive number r such that if we define the intervals of real numbers I and J by $I=(x_0-2r, x_0+2r)$ and $J=(y_0-2r, y_0+2r)$, then the rectangle $I\times J$ is contained in $\mathcal{U}$. The idea of the proof is to express

$$f(x_0+r, y_0+r)-f(x_0+r, y_0)-f(x_0, y_0+r)+f(x_0, y_0)$$

as a difference in two different ways: first as the difference

$$[f(x_0+r, y_0+r)-f(x_0+r, y_0)]-[f(x_0, y_0+r)-f(x_0, y_0)] \tag{13.10}$$

and then as the difference

$$[f(x_0+r, y_0+r)-f(x_0, y_0+r)]-[f(x_0+r, y_0)-f(x_0, y_0)]. \tag{13.11}$$

Then we use the Mean Value Theorem for functions of a single real variable to express (13.10) and (13.11) as second-order partial derivatives of the function $f:\mathcal{U}\to\mathbb{R}$.

First we analyze the difference (13.10). Define the auxiliary function $\varphi:I\to\mathbb{R}$ by

$$\varphi(x)=f(x, y_0+r)-f(x, y_0)\qquad\text{for } x \text{ in } I.$$

Since $f:\mathcal{U}\to\mathbb{R}$ has a partial derivative with respect to its first component, the function $\varphi:I\to\mathbb{R}$ is differentiable. Thus, we can apply the Mean Value Theorem to the restriction of the function $\varphi:I\to\mathbb{R}$ to the closed interval $[x_0, x_0+r]$ to select a point x_1 in the open interval (x_0, x_0+r) such that

$$\frac{\varphi(x_0+r)-\varphi(x_0)}{r}=\varphi'(x_1);$$

that is,

$$\frac{\varphi(x_0+r)-\varphi(x_0)}{r}=\frac{\partial f}{\partial x}(x_1, y_0+r)-\frac{\partial f}{\partial x}(x_1, y_0). \tag{13.12}$$

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With this point x_1 fixed, define another auxiliary function $\alpha : J \to \mathbb{R}$ by

$$\alpha(y) = \frac{\partial f}{\partial x}(x_1, y) \qquad \text{for } y \text{ in } J.$$

We can apply the Mean Value Theorem to the restriction of the function $\alpha : J \to \mathbb{R}$ to the closed interval $[y_0, y_0 + r]$ to select a point y_1 in the open interval $(y_0, y_0 + r)$ such that

$$\frac{\alpha(y_0 + r) - \alpha(y_0)}{r} = \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1). \tag{13.13}$$

From (13.12) and (13.13), we obtain

$$\varphi(x_0 + r) - \varphi(x_0) = r^2 \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1). \tag{13.14}$$

However, $\varphi(x_0 + r) - \varphi(x_0)$ equals the difference (13.10), and hence we have

$$\begin{aligned} &[f(x_0 + r, y_0 + r) - f(x_0 + r, y_0)] - [f(x_0, y_0 + r) - f(x_0, y_0)] \\ &\quad = r^2 \frac{\partial^2 f}{\partial y \partial x}(x_1, y_1). \end{aligned} \tag{13.15}$$

In order to analyze the difference (13.11), we now repeat the same argument applied to the auxiliary function $\psi : J \to \mathbb{R}$ defined by

$$\psi(y) = f(x_0 + r, y) - f(x_0, y) \qquad \text{for } y \text{ in } J.$$

From this it will follow that we can select a point (x_2, y_2) in the rectangle $I \times J$ such that

$$\begin{aligned} &[f(x_0 + r, y_0 + r) - f(x_0, y_0 + r)] - [f(x_0 + r, y_0) - f(x_0, y_0)] \\ &\quad = r^2 \frac{\partial^2 f}{\partial x \partial y}(x_2, y_2). \end{aligned} \tag{13.16}$$

From the equality of the left-hand sides of (13.15) and (13.16) follows the equality of the right-hand sides, so the lemma is proved. ■

Proof of Theorem 13.10

We prove the theorem when $n = 2$ and leave the general case to the reader (Exercise 16). Let (x_0, y_0) be a point in $\mathcal{O}$. Choose a positive number r such that the open ball $\mathcal{B}_r(x_0, y_0)$ is contained in $\mathcal{O}$. Let k be a natural number. Then we can apply the lemma with $\mathcal{U} = \mathcal{B}_{r/k}(x_0, y_0)$ and select points (x_k, y_k) and (u_k, v_k) in $\mathcal{B}_{r/k}(x_0, y_0)$ at which

$$\frac{\partial^2 f}{\partial x \partial y}(x_k, y_k) = \frac{\partial^2 f}{\partial y \partial x}(u_k, v_k). \tag{13.17}$$

But, by assumption, the function $\partial^2 f / \partial x \partial y : \mathcal{O} \to \mathbb{R}$ is continuous at (x_0, y_0), as is the function $\partial^2 f / \partial y \partial x : \mathcal{O} \to \mathbb{R}$. Since the sequences $\{(x_k, y_k)\}$ and $\{(u_k, v_k)\}$ both converge to the point (x_0, y_0), it follows that

$$\lim_{k \to \infty} \left[\frac{\partial^2 f}{\partial x \partial y}(x_k, y_k) \right] = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$$

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and

$$\lim_{k\to\infty}\left[\frac{\partial^2 f}{\partial y\partial x}(u_k,\ v_k)\right]=\frac{\partial^2 f}{\partial y\partial x}(x_0,\ y_0).$$

In view of (13.17), we conclude that

$$\frac{\partial^2 f}{\partial x\partial y}(x_0,\ y_0)=\frac{\partial^2 f}{\partial y\partial x}(x_0,\ y_0).$$ ■

Observe that in Lemma 13.11 we required only that the function $f:\mathcal{O}\to\mathbb{R}$ have second-order partial derivatives. On the other hand, in Theorem 13.10, we required that the second-order partial derivatives be continuous. This extra assumption is necessary. The following is an example of a function $f:\mathcal{O}\to\mathbb{R}$ that has second-order partial derivatives, and yet we do not have equality of $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ at all points.

Example 13.12 Define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,\ y)=\begin{cases} xy(x^2-y^2)/(x^2+y^2) & \text{if } (x,\ y)\neq(0,\ 0)\\ 0 & \text{if } (x,\ y)=(0,\ 0).\end{cases}$$

Calculations, which we leave to the reader (Exercise 13), show that the function $f:\mathbb{R}^2\to\mathbb{R}$ has second-order partial derivatives but that

$$\frac{\partial^2 f}{\partial y\partial x}(0,\ 0)=-1 \qquad \text{while} \qquad \frac{\partial^2 f}{\partial x\partial y}(0,\ 0)=1.$$ ■

Remark on Notation

There are other notations in common use for partial derivatives. For instance, when the function $f:\mathbb{R}^3\to\mathbb{R}$ has first-order partial derivatives, $\partial f/\partial x(x,\ y,\ z)$ is often denoted by $f_x(x,\ y,\ z)$. Also, when the function $f:\mathbb{R}^3\to\mathbb{R}$ has second-order partial derivatives,

$$f_{xx}(x,\ y,\ z),\qquad \frac{\partial^2 f}{\partial x\partial x}(x,\ y,\ z),\qquad \text{and}\qquad \frac{\partial^2 f}{\partial x^2}(x,\ y,\ z)$$

are used to denote the same quantity. Moreover, for $\mathcal{O}$ an open subset of $\mathbb{R}^n$, a continuously differentiable function $f:\mathcal{O}\to\mathbb{R}$ is often said to be C^1, and if $f:\mathcal{O}\to\mathbb{R}$ has continuous second-order partial derivatives, it is said to be C^2.

EXERCISES FOR SECTION 13.2

1. Calculate the first-order partial derivatives of the following functions:
   a. $f(x,\ y,\ z)=x+yz+xy+x\sin(yz)$ for $(x,\ y,\ z)$ in $\mathbb{R}^3$
   b. $f(x,\ y,\ z)=\sin(x^2y^2)/(1+x^2+y^3)$ for $(x,\ y,\ z)$ in $\mathbb{R}^3$
   c. $f(x,\ y,\ z)=\sqrt{1+\cos^2(xy)}$ for $(x,\ y,\ z)$ in $\mathbb{R}^3$
2. Prove that the function defined in Example 13.8 is continuously differentiable.
3. For the function $f:\mathbb{R}^2\to\mathbb{R}$ defined in Example 13.9, show that neither the function $\partial f/\partial x:\mathbb{R}^2\to\mathbb{R}$ nor the function $\partial f/\partial y:\mathbb{R}^2\to\mathbb{R}$ is continuous at the point $(0,\ 0)$.

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4. Suppose that the function $g : \mathbb{R}^2 \to \mathbb{R}$ has the property that

$$|g(x, y)| \leq x^2 + y^2 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that $g : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives with respect to both x and y at the point $(0, 0)$.

5. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives and that

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is constant, that is, that there is some number c such that

$$f(x, y) = c \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

(*Hint:* First show that the restriction of $f : \mathbb{R}^2 \to \mathbb{R}$ to a line parallel to one of the coordinate axes is constant.)

6. Define

$$g(x, y) = \begin{cases} x^2 y^4/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives. Is the function $g : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable?

7. (An Extension of the Chain Rule) Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to the ith component at the point $\mathbf{x}$. Let I be an open interval in $\mathbb{R}$ with $f(\mathcal{O}) \subseteq I$ and let the function $g : I \to \mathbb{R}$ have a derivative at the point $f(\mathbf{x})$. Prove that the composition $g \circ f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to the ith component at the point $\mathbf{x}$ and that

$$\frac{\partial}{\partial x_i}(g \circ f)(\mathbf{x}) = g'(f(\mathbf{x}))\frac{\partial f}{\partial x_i}(\mathbf{x}).$$

8. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable. Use Exercise 7 to calculate the first-order partial derivatives of the following functions:

a. The function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(x, y) = g(xy^2 + 1) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

b. The function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(u, v) = g(4u + 7v) \qquad \text{for } (u, v) \text{ in } \mathbb{R}^2.$$

c. The function $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$h(t, s) = g(t - s) \qquad \text{for } (t, s) \text{ in } \mathbb{R}^2.$$

9. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ has a second derivative. Calculate the second-order partial derivatives of the functions defined in Exercise 8.

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10. Suppose that the functions $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ have second-order partial derivatives. Define

$$f(x, y) = \varphi(x - y) + \psi(x + y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that

$$\frac{\partial^2 f}{\partial x^2}(x, y) - \frac{\partial^2 f}{\partial y^2}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

11. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* provided that it has second-order partial derivatives and

$$\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Which of the following functions is harmonic?

a. $f(x, y) = e^x \cos y$ for (x, y) in $\mathbb{R}^2$

b. $g(x, y) = x^2 - y^2$ for (x, y) in $\mathbb{R}^2$

c. $h(x, y) = x^2 - y^3$ for (x, y) in $\mathbb{R}^2$

12. Given a pair of functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$, it is often useful to know whether there exists some continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x, y) = \phi(x, y) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x, y) = \psi(x, y) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Such a function $f : \mathbb{R}^2 \to \mathbb{R}$ is called a *potential function* for the pair of functions (ϕ, ψ).

a. Show that if a potential function exists for the pair (ϕ, ψ), then this potential is uniquely determined up to an additive constant—that is, the difference of any two potentials is constant.

b. Show that if there is a potential function for the pair of continuously differentiable functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$, then

$$\frac{\partial \psi}{\partial x}(x, y) = \frac{\partial \phi}{\partial y}(x, y) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

13. Consider the function defined in Example 13.12.

a. Show that

$$\frac{\partial f}{\partial y}(x, 0) = x \qquad \text{for all } x \text{ in } \mathbb{R}$$

and that

$$\frac{\partial f}{\partial x}(0, y) = -y \qquad \text{for all } y \text{ in } \mathbb{R}.$$

b. From part (a), conclude that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1.$$

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14. Let $\mathcal{U}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0). Prove that there is a positive number r such that (x, y) belongs to $\mathcal{U}$ if $|x - x_0| < 2r$ and $|y - y_0| < 2r$.

15. Verify equation (13.16) by following the argument used in the proof of Lemma 13.11 to establish (13.15).

16. Let the function $f : \mathcal{O} \to \mathbb{R}$ satisfy the assumptions of Theorem 13.10 and let i and j be indices with $1 \le i \le n$ and $1 \le j \le n$. Choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Define

$$g(s, t) = f(\mathbf{x} + t\mathbf{e}_i + s\mathbf{e}_j) \qquad \text{for } s^2 + t^2 < r^2.$$

a. Verify that $g : \mathcal{B}_r(0, 0) \to \mathbb{R}$ has continuous second-order partial derivatives and that

$$\frac{\partial^2 g}{\partial s \partial t}(0, 0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) \qquad \text{and} \qquad \frac{\partial^2 g}{\partial t \partial s}(0, 0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}).$$

b. Use (a) to show that the general case of Theorem 13.10 follows from the case that $n = 2$.

17. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second-order partial derivatives and let (x_0, y_0) be a point in $\mathbb{R}^2$. Prove that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |h| < \delta$ and $0 < |k| < \delta$, then

$$\left| \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) - f(x_0, y_0 + k) + f(x_0, y_0)}{hk} - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon.$$

(*Hint:* Follow the idea of the proof of Lemma 13.11.)

13.3 THE MEAN VALUE THEOREM AND DIRECTIONAL DERIVATIVES

In the study of real-valued functions of a single variable, a prominent role is played by the Mean Value Theorem. For convenient reference, we restate this theorem, which was proved in Section 4.3.

Theorem 13.13 The Mean Value Theorem Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous and that its restriction to the open interval (a, b) is differentiable. Then there is a point c in the open interval (a, b) at which

$$f(b) - f(a) = f'(c)(b - a).$$

One of the goals of this section is to extend this theorem to the case of functions of several real variables. As we noted at the beginning of Section 13.2, the strategy for making this extension is to try to reduce the general case to the single-variable case by analyzing the restriction of the function to segments in the domain.

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Lemma 13.14 The Mean Value Lemma Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let i be an index with $1 \leq i \leq n$. Suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has a partial derivative with respect to its ith component at each point in $\mathcal{O}$. Let $\mathbf{x}$ be a point in $\mathcal{O}$ and let a be a real number such that the segment between the points $\mathbf{x}$ and $\mathbf{x} + a\mathbf{e}_i$ lies in $\mathcal{O}$. Then there is a number θ with $0 < \theta < 1$ such that

$$f(\mathbf{x} + a\mathbf{e}_i) - f(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x} + \theta a\mathbf{e}_i)a. \tag{13.18}$$

Proof

Since $\mathcal{O}$ is open in $\mathbb{R}^n$, we can select an open interval of real numbers I that contains the numbers 0 and a such that for each t in I the point $\mathbf{x} + t\mathbf{e}_i$ belongs to $\mathcal{O}$. Define the function $\phi : I \to \mathbb{R}$ by $\phi(t) = f(\mathbf{x} + t\mathbf{e}_i)$ for each t in I. Then the partial differentiability of the function $f : \mathcal{O} \to \mathbb{R}$ with respect to its ith component implies that at each point t in I,

$$\phi'(t) = \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{e}_i).$$

It follows that the function $\phi : I \to \mathbb{R}$ is differentiable. Thus, we can apply the Mean Value Theorem for functions of a single variable to the restriction of the function $\phi : I \to \mathbb{R}$ to the closed interval $[0, a]$ to obtain a point θ with $0 < \theta < 1$ such that

$$\phi(a) - \phi(0) = \phi'(\theta a)a,$$

which, in view of the definition of the function $\phi : I \to \mathbb{R}$ and the calculation of $\phi'(t)$, can be rewritten as (13.18). ■

Proposition 13.15 The Mean Value Proposition Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ and let r be a positive number. Suppose that the function $f : \mathcal{B}_r(\mathbf{x}) \to \mathbb{R}$ has first-order partial derivatives. Then if the point $\mathbf{x} + \mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$, there are points $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ in $\mathcal{B}_r(\mathbf{x})$ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i), \tag{13.19}$$

and

$$\|\mathbf{x} - \mathbf{z}_i\| < \|\mathbf{h}\| \qquad \text{for each index } i \text{ with } 1 \leq i \leq n.$$

Proof

We prove the result with $n = 3$. From this, it will be clear that the general result is also true. The trick is to expand the difference $f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})$. We have

$$\begin{aligned} f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) &= f(x_1 + h_1, x_2 + h_2, x_3 + h_3) - f(x_1, x_2, x_3) \\ &= f(x_1 + h_1, x_2 + h_2, x_3 + h_3) - f(x_1 + h_1, x_2 + h_2, x_3) \\ &\quad + f(x_1 + h_1, x_2 + h_2, x_3) - f(x_1 + h_1, x_2, x_3) \\ &\quad + f(x_1 + h_1, x_2, x_3) - f(x_1, x_2, x_3). \end{aligned}$$

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We apply the Mean Value Lemma to each of these differences to find numbers θ_1, θ_2, and θ_3 in the open interval $(0, 1)$ with

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) &= \frac{\partial f}{\partial x_3}(x_1+h_1,\, x_2+h_2,\, x_3+\theta_3 h_3)h_3 \\ &+ \frac{\partial f}{\partial x_2}(x_1+h_1,\, x_2+\theta_2 h_2,\, x_3)h_2 \\ &+ \frac{\partial f}{\partial x_1}(x_1+\theta_1 h_1,\, x_2,\, x_3)h_1. \end{aligned}$$

Setting $\mathbf{z}_3 = (x_1+h_1,\, x_2+h_2,\, x_3+\theta_3 h_3)$, $\mathbf{z}_2 = (x_1+h_1,\, x_2+\theta_2 h_2,\, x_3)$, and $\mathbf{z}_1 = (x_1+\theta_1 h_1,\, x_2,\, x_3)$, the result follows. ∎

For $\mathcal{O}$ an open subset of $\mathbb{R}^n$ containing the point $\mathbf{x}$, a function $f:\mathcal{O}\to\mathbb{R}$ has been defined to have a partial derivative with respect to the ith component at the point $\mathbf{x}$ provided that the limit

$$\lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{e}_i)-f(\mathbf{x})}{t}$$

exists. We now turn to an analysis of this limit when the point $\mathbf{e}_i$ is replaced by a general nonzero point $\mathbf{p}$ in $\mathbb{R}^n$.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$. Consider a function $f:\mathcal{O}\to\mathbb{R}$ and a nonzero point $\mathbf{p}$ in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{p})-f(\mathbf{x})}{t}$$

exists, we call this limit the directional derivative[1] of the function $f:\mathcal{O}\to\mathbb{R}$ in the direction $\mathbf{p}$ at the point $\mathbf{x}$ and denote it by

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}).$$

Observe that if $\mathbf{p} = \mathbf{e}_i$, then

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

Theorem 13.16 The Directional Derivative Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ is continuously differentiable. Then for each point $\mathbf{x}$ in $\mathcal{O}$ and each nonzero point $\mathbf{p}$ in $\mathbb{R}^n$, the function $f:\mathcal{O}\to\mathbb{R}$ has a directional derivative in the direction $\mathbf{p}$ at the point $\mathbf{x}$ that is given by the formula

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x}). \tag{13.20}$$

[1] The terminology *directional derivative* is standard but is somewhat misleading since the directional derivative depends not only on the direction of $\mathbf{p}$ but also on its length.

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Proof

Since $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Then from the Mean Value Proposition, we see that if t is any number with $|t|\|\mathbf{p}\| < r$, there are n points $\mathbf{z}_1, \ldots, \mathbf{z}_n$ such that

$$f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x}) = \sum_{i=1}^{n} tp_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) \tag{13.21}$$

and

$$\|\mathbf{z}_i - \mathbf{x}\| \leq |t|\|\mathbf{p}\| \qquad \text{for each index } i \text{ with } 1 \leq i \leq n. \tag{13.22}$$

We can rewrite (13.21) as

$$\frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{t} = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) \qquad \text{for } t \neq 0. \tag{13.23}$$

Since $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ is continuous at the point $\mathbf{x}$ for each index i with $1 \leq i \leq n$, it follows from (13.22) and (13.23) that

$$\lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{t} = \lim_{t\to 0} \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{z}_i) = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

This proves formula (13.20). ■

In view of formula (13.20) we introduce the following definition.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $\mathbf{x}$. We define the *gradient* of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$, denoted by $\nabla f(\mathbf{x})$, to be the point in $\mathbb{R}^n$ given by

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}),\ \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots,\ \frac{\partial f}{\partial x_n}(\mathbf{x})\right).$$

Through the identification of points in $\mathbb{R}^n$ with vectors, $\nabla f(\mathbf{x})$ is often referred to as the *gradient vector* or *derivative vector*. Using the scalar product and the gradient, formula (13.20) can be compactly written as

$$\left.\frac{d}{dt}[f(\mathbf{x}+t\mathbf{p})]\right|_{t=0} = \frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p}\rangle. \tag{13.24}$$

It is also useful to observe a slight extension of (13.24): Replacing the point $\mathbf{x}$ with the point $\mathbf{x} + t\mathbf{p}$, it follows that

$$\frac{d}{dt}[f(\mathbf{x}+t\mathbf{p})] = \langle \nabla f(\mathbf{x}+t\mathbf{p}), \mathbf{p}\rangle \qquad \text{for } 0 \leq t \leq 1, \tag{13.25}$$

provided that the segment between $\mathbf{x}$ and $\mathbf{x}+t\mathbf{p}$ lies in $\mathcal{O}$.

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Theorem 13.17 The Mean Value Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. If the segment joining the points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ lies in $\mathcal{O}$, then there is a number θ with $0 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x}+\theta\mathbf{h}), \mathbf{h} \rangle. \tag{13.26}$$

Proof

Since $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, we can select an open interval of real numbers I, which contains the numbers 0 and 1, such that $\mathbf{x} + t\mathbf{h}$ belongs to $\mathcal{O}$ for each t in I. Define

$$\phi(t) = f(\mathbf{x}+t\mathbf{h}) \qquad \text{for each } t \text{ in } I.$$

Using the slight generalization of the Directional Derivative Theorem stated as formula (13.25), we see that

$$\phi'(t) = \langle \nabla f(\mathbf{x}+t\mathbf{h}), \mathbf{h} \rangle \qquad \text{for each } t \text{ in } I. \tag{13.27}$$

Thus, we can apply the Mean Value Theorem for functions of a single real variable to the restriction of the function $\phi : I \to \mathbb{R}$ to the closed interval $[0, 1]$ in order to select a number θ with $0 < \theta < 1$ such that

$$\phi(1) - \phi(0) = \phi'(\theta).$$

Using (13.27) and the definition of $\phi : [0, 1] \to \mathbb{R}$, it is clear that this formula is a restatement of (13.26). ∎

In the case where $\mathbf{p}$ is a point in $\mathbb{R}^n$ of norm 1, a directional derivative in the direction $\mathbf{p}$ can be interpreted as a *rate of change*. To see this, let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then if the point $\mathbf{p}$ is of norm 1 and t is a positive real number,

$$t = \|t\mathbf{p}\|,$$

so if t is positive and sufficiently small,

$$\frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{t} = \frac{f(\mathbf{x}+t\mathbf{p}) - f(\mathbf{x})}{\|t\mathbf{p}\|}.$$

In view of this, if the norm of $\mathbf{p}$ is 1, it is reasonable to call $\partial f/\partial \mathbf{p}(\mathbf{x})$ the rate of change of the function $f : \mathcal{O} \to \mathbb{R}$ in the direction $\mathbf{p}$ at the point $\mathbf{x}$.

Corollary 13.18 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. If $\nabla f(\mathbf{x}) \neq \mathbf{0}$, then the direction of norm 1 at the point $\mathbf{x}$ in which the function $f : \mathcal{O} \to \mathbb{R}$ is increasing the fastest is the direction $\mathbf{p}_0$ defined by

$$\mathbf{p}_0 = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}. \tag{13.28}$$

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Proof

Using formula (13.24) and the Cauchy–Schwarz Inequality, it follows that if $\mathbf{p}$ is any point in $\mathbb{R}^n$ of norm 1, then

$$\left|\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x})\right| = |\langle \nabla f(\mathbf{x}), \mathbf{p}\rangle| \le \|\nabla f(\mathbf{x})\| \cdot \|\mathbf{p}\| = \|\nabla f(\mathbf{x})\|. \tag{13.29}$$

On the other hand, if $\mathbf{p}_0$ is defined by (13.28), then $\mathbf{p}_0$ has norm 1, and using (13.24), it follows that

$$\frac{\partial f}{\partial \mathbf{p}_0}(\mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{p}_0\rangle = \left\langle \nabla f(\mathbf{x}), \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}\right\rangle = \|\nabla f(\mathbf{x})\|.$$

This calculation, together with inequality (13.29), implies that if $\mathbf{p}$ has norm 1, then

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) \le \frac{\partial f}{\partial \mathbf{p}_0}(\mathbf{x}).$$ ■

Example 13.19 Define

$$f(x, y) = e^{x^2-y^2} \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. A short calculation shows that

$$\frac{\partial f}{\partial x}(1, 1) = 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(1, 1) = -2.$$

Thus, $\nabla f(1, 1) = (2, -2)$, so the direction in which the function $f : \mathbb{R}^2 \to \mathbb{R}$ is increasing the fastest at the point $(1, 1)$ is given by the vector $(1/\sqrt{2}, -1/\sqrt{2})$. ■

In Section 13.2, we mentioned that a continuously differentiable function is continuous. We can now prove this assertion.

Theorem 13.20 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then the function $f : \mathcal{O} \to \mathbb{R}$ is continuous.

Proof

Let $\mathbf{x}$ be a point in $\mathcal{O}$. We need to show that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous at $\mathbf{x}$. We directly apply the sequential definition of continuity. First, since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can select a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$.

Let $\{\mathbf{x}_k\}$ be a sequence in $\mathcal{B}_r(\mathbf{x})$ that converges to $\mathbf{x}$. For each natural number k, set $\mathbf{h}_k = \mathbf{x}_k - \mathbf{x}$ and apply the Mean Value Theorem to select a number θ_k with $0 < \theta_k < 1$ such that

$$f(\mathbf{x}_k) - f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h}_k) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x} + \theta_k \mathbf{h}_k), \mathbf{h}_k\rangle. \tag{13.30}$$

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Now observe that

$$\lim_{k\to\infty} \mathbf{h}_k = \mathbf{0} \qquad \text{and} \qquad \lim_{k\to\infty} [\mathbf{x} + \theta_k \mathbf{h}_k] = \mathbf{x}.$$

Since $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, it follows that

$$\lim_{k\to\infty} \nabla f(\mathbf{x} + \theta_k \mathbf{h}_k) = \nabla f(\mathbf{x}).$$

Thus, since (13.30) holds for every index k, we conclude that

$$\lim_{k\to\infty} [f(\mathbf{x}_k) - f(\mathbf{x})] = \langle \nabla f(\mathbf{x}), \mathbf{0} \rangle = 0,$$

which means that the image sequence $\{f(\mathbf{x}_k)\}$ converges to $f(\mathbf{x})$. ∎

Corollary 13.21 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Then the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable.

Proof

For each index i with $1 \leq i \leq n$, the function $\partial f / \partial x_i : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, and hence, by Theorem 13.20, it is continuous. This is precisely what it means for the function $f : \mathbb{R}^n \to \mathbb{R}$ to be continuously differentiable. ∎

EXERCISES FOR SECTION 13.3

1. For each of the following functions, find the derivative vector $\nabla f(\mathbf{x})$ for those points $\mathbf{x}$ in $\mathbb{R}^n$ where it is defined:
   a. $f(\mathbf{x}) = e^{\|\mathbf{x}\|^2}$ for $\mathbf{x}$ in $\mathbb{R}^n$
   b. $f(x, y) = \sin(xy)/\sqrt{x^2 + y^2 + 1}$ for (x, y) in $\mathbb{R}^2$
   c. $f(\mathbf{x}) = 1/\|\mathbf{x}\|^2$ for $\mathbf{x}$ in $\mathcal{O} = \{\mathbf{x} \text{ in } \mathbb{R}^3 \mid \mathbf{x} \neq \mathbf{0}\}$
2. Assume that the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. Find a formula for $\nabla(fg)(\mathbf{x})$ in terms of $\nabla f(\mathbf{x})$ and $\nabla g(\mathbf{x})$.
3. Suppose that the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable. Find a formula for $\nabla(g \circ f)(\mathbf{x})$ in terms of $\nabla f(\mathbf{x})$ and $g'(f(\mathbf{x}))$.
4. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has first-order partial derivatives and that the point $\mathbf{x}$ in $\mathbb{R}^n$ is a local minimizer for $f : \mathbb{R}^n \to \mathbb{R}$, meaning that there is a positive number r such that

$$f(\mathbf{x} + \mathbf{h}) \geq f(\mathbf{x}) \qquad \text{if } \operatorname{dist}(\mathbf{x}, \mathbf{x} + \mathbf{h}) < r.$$

   Prove that $\nabla f(\mathbf{x}) = \mathbf{0}$.
5. Show that in the case where $\mathbf{h} = a\mathbf{e}_i$, the mean value formula (13.26) is the same as the mean value formula (13.18).
6. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz + x^2 + y^2 \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

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The Mean Value Theorem implies that there is a number θ with $0 < \theta < 1$ for which

$$f(1,1,1) - f(0,0,0) = \frac{\partial f}{\partial x}(\theta,\theta,\theta) + \frac{\partial f}{\partial y}(\theta,\theta,\theta) + \frac{\partial f}{\partial z}(\theta,\theta,\theta).$$

Find such a value of θ.

7. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives and that $f(0, 0) = 1$, while

$$\frac{\partial f}{\partial x}(x, y) = 2 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x, y) = 3 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that

$$f(x, y) = 1 + 2x + 3y \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

8. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let $\mathbf{x}$ be a point in $\mathbb{R}^n$. For $\mathbf{p}$ a nonzero point in $\mathbb{R}^n$ and α a nonzero real number, show that

$$\frac{\partial f}{\partial(\alpha\mathbf{p})}(\mathbf{x}) = \alpha\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}).$$

9. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} (x/|y|)\sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

a. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is not continuous at the point $(0, 0)$.

b. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at the point $(0, 0)$.

c. Prove that if c is any number, then there is a vector $\mathbf{p}$ of norm 1 such that

$$\frac{\partial f}{\partial \mathbf{p}}(0, 0) = c.$$

d. Does (c) contradict Corollary 13.18?

10. Consider the following assertions for a function $f : \mathbb{R}^2 \to \mathbb{R}$:

a. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

b. The function $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at each point in $\mathbb{R}^2$.

c. The function $f : \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives at each point in $\mathbb{R}^2$.

Explain the implications among these assertions.

11. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Define $K = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$.

a. Prove that there is a point $\mathbf{x}$ in K at which the function $f : K \to \mathbb{R}$ attains a smallest value.

b. Now suppose also that if $\mathbf{p}$ is any point in $\mathbb{R}^n$ of norm 1, then $\langle \nabla f(\mathbf{p}), \mathbf{p}\rangle > 0$. Show that the minimizer $\mathbf{x}$ in part (a) has norm less than 1.

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CHAPTER 14

LOCAL APPROXIMATION OF REAL-VALUED FUNCTIONS

14.1 FIRST-ORDER APPROXIMATION, TANGENT PLANES, AND AFFINE FUNCTIONS

Let $\mathcal{O}$ be an open subset of Euclidean space $\mathbb{R}^n$ and suppose that we wish to analyze the behavior of the real-valued function $f : \mathcal{O} \to \mathbb{R}$ in a neighborhood of the point $\mathbf{x}$ in $\mathcal{O}$. (For instance, we might want to establish that the point $\mathbf{x}$ is a local maximizer or a local minimizer for the function.) One strategy for doing this is to choose another function $g : \mathcal{O} \to \mathbb{R}$ that is simpler than the given function $f : \mathcal{O} \to \mathbb{R}$ and is a good approximation of $f : \mathcal{O} \to \mathbb{R}$ near the point $\mathbf{x}$. Then we can see what properties the function $f : \mathcal{O} \to \mathbb{R}$ inherits from the simpler function $g : \mathcal{O} \to \mathbb{R}$.

In this first section, we define what is meant by an *affine function* and show that, for a continuously differentiable function $f : \mathcal{O} \to \mathbb{R}$, at each point $\mathbf{x}$ in $\mathcal{O}$ there is an affine function that is a first-order approximation. For functions of two variables, the graph of this affine approximation is the tangent plane. The remaining two sections of this chapter are concerned with functions $f : \mathcal{O} \to \mathbb{R}$ that have continuous second-order partial derivatives. In Section 14.2, the second-order partial derivatives of $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$ are organized into an $n \times n$ matrix, called the *Hessian matrix*, and a formula for second-order directional derivatives is established. We also associate a quadratic function with each $n \times n$ matrix and obtain estimates for the values attained by quadratic functions. In Section 14.3, we show how to find a function $g : \mathcal{O} \to \mathbb{R}$ that is the sum of an affine function and a quadratic function and is a second-order approximation to $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$. This permits us to provide a criterion for deciding when a point $\mathbf{x}$ is a local maximizer or minimizer for the function $f : \mathcal{O} \to \mathbb{R}$.

Suppose that I is an open interval of real numbers and that the function $f : I \to \mathbb{R}$ is differentiable. By definition, this means that if x is a point in I, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x).$$

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If we rewrite the difference

$$\frac{f(x+h)-f(x)}{h}-f'(x)=\frac{f(x+h)-[f(x)+f'(x)h]}{h},$$

the above definition of a derivative can be rewritten as

$$\lim_{h\to 0}\frac{f(x+h)-[f(x)+f'(x)h]}{h}=0. \tag{14.1}$$

In Chapter 8, we studied Taylor polynomials and established error estimates for the difference between values of a function and its Taylor polynomials at a point. The Lagrange Remainder Theorem, which we proved in Chapter 8, implies that if k is a natural number and the function $f: I \to \mathbb{R}$ has continuous derivatives up to order $k+1$, then for a point x in I and a perturbation $x+h$ that also belongs to I, there is a number $\theta, 0<\theta<1$, such that

$$f(x+h)-\left[f(x)+f'(x)h+\cdots+\left(\frac{1}{k}!\right)f^k(x)h^k\right]=\frac{f^{k+1}(x+\theta h)}{(k+1)!}\cdot h^{k+1}$$

and therefore,

$$\lim_{h\to 0}\frac{f(x+h)-[f(x)+f'(x)h+\cdots+(1/k!)f^k(x)h^k]}{h^k}=0. \tag{14.2}$$

In this chapter, we wish to establish results analogous to the approximation formulas (14.1) and, for $k=2$, (14.2) for functions of *several* real variables. It is useful to introduce the following definition.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$. For a positive integer k, two functions $f:\mathcal{O}\to\mathbb{R}$ and $g:\mathcal{O}\to\mathbb{R}$ are said to be *kth-order approximations* of one another at the point $\mathbf{x}$ provided that

$$\lim_{\mathbf{h}\to 0}\frac{f(\mathbf{x}+\mathbf{h})-g(\mathbf{x}+\mathbf{h})}{\|\mathbf{h}\|^k}=0. \tag{14.3}$$

Example 14.1 Define $f(h)=e^h$ for each number h. Then $f(0)=f'(0)=f''(0)=1$. From formula (14.2), at the point $x=0$,

$$\lim_{h\to 0}\frac{e^h-[1+h]}{h}=0,$$

while

$$\lim_{h\to 0}\frac{e^h-[1+h+(1/2)h^2]}{h^2}=0.$$

Thus, the first-degree Taylor polynomial $p_1(h)=1+h$ is a first-order approximation of f at $x=0$, while the second-degree Taylor polynomial $p_2(h)=1+h+(1/2)h^2$ is a second-order approximation of f at $x=0$. ■

The following theorem provides an extension to functions of several variables of the approximation formula (14.1).

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Theorem 14.2 The First-Order Approximation Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Let $\mathbf{x}$ be a point in $\mathcal{O}$. Then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle]}{\|\mathbf{h}\|} = 0. \tag{14.4}$$

Proof

Since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Fix a nonzero point $\mathbf{h}$ in $\mathbb{R}^n$ with $\|\mathbf{h}\| < r$. Then the point $\mathbf{x}+\mathbf{h}$ belongs to $\mathcal{B}_r(\mathbf{x})$ and so, by the Mean Value Theorem, we can select a number θ with $0 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \langle \nabla f(\mathbf{x} + \theta\mathbf{h}), \mathbf{h}\rangle.$$

Thus,

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle = \langle \nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x}), \mathbf{h}\rangle,$$

so that, using the Cauchy–Schwarz Inequality, we obtain the estimate

$$|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle| \leq \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\| \cdot \|\mathbf{h}\|.$$

Dividing this estimate by $\|\mathbf{h}\|$, we obtain

$$\frac{|f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle]|}{\|\mathbf{h}\|} \leq \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\|. \tag{14.5}$$

But the function $f : \mathcal{O} \to \mathbb{R}$ has been assumed to be continuously differentiable, so

$$\lim_{\mathbf{h}\to\mathbf{0}} \|\nabla f(\mathbf{x}+\theta\mathbf{h}) - \nabla f(\mathbf{x})\| = 0,$$

and thus (14.4) follows from the estimate (14.5). ■

For a continuously differentiable function $f : \mathcal{O} \to \mathbb{R}$ whose domain $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and a point (x_0, y_0) in $\mathcal{O}$, if we denote a general point in $\mathcal{O}$ by (x, y) and set $\mathbf{h} = (x - x_0, y - y_0)$, it is clear that $\mathbf{h}$ approaches 0 if and only if (x, y) approaches (x_0, y_0) and that $\|\mathbf{h}\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. Hence the approximation property (14.4) can be rewritten as

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - [f(x_0, y_0) + \partial f/\partial x(x_0, y_0)(x - x_0) + \partial f/\partial y(x_0, y_0)(y - y_0)]}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{14.6}$$

This last formula has a geometric interpretation involving the existence of a tangent plane. To describe this, we state the following definition.

Definition Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous at the point (x_0, y_0) in $\mathcal{O}$. By the *tangent plane* to the graph of

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$f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$, we mean the graph of a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$\psi(x, y) = a + b(x - x_0) + c(y - y_0) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2,$$

where a, b, and c are real numbers, which has the property that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \psi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0. \tag{14.7}$$

A continuous function of two variables $f : \mathcal{O} \to \mathbb{R}$ can have directional derivatives in all directions at the point (x_0, y_0) in $\mathcal{O}$ without having a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ (Exercises 17, 18, and 19); such examples occur because the definition of tangent plane requires that the limit (14.7) exist independently of the way in which the point (x, y) approaches (x_0, y_0). However, for continuously differentiable functions, the approximation property (14.6) is exactly what is required in order to prove the following corollary.

Corollary 14.3 Suppose that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ that contains point (x_0, y_0) and that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then there is a tangent plane to the graph of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$. This tangent plane is the graph of the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined for (x, y) in $\mathbb{R}^2$ by

$$\psi(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0). \tag{14.8}$$

Proof

For a general point (x, y) in $\mathcal{O}$, set

$$h = (x, y) - (x_0, y_0)$$

and observe that

$$\langle \nabla f(x_0, y_0), \mathbf{h} \rangle = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Since $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, the First-Order Approximation Theorem implies that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \psi(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0;$$

that is, the graph of the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is the tangent plane to the graph of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$. ■

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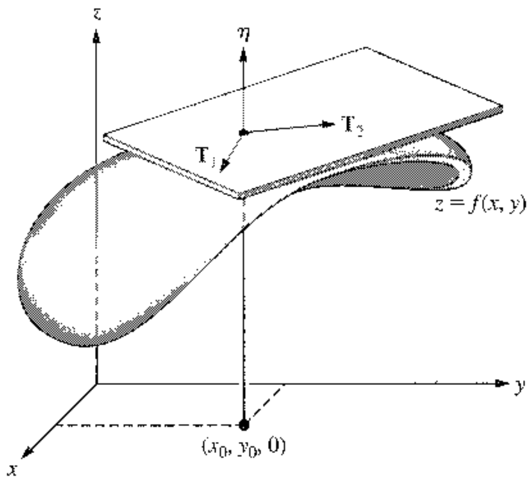

FIGURE 14.1 The tangent plane to the graph at the point (x_0, y_0, z_0).

We can reason geometrically to see why the tangent plane described in the preceding corollary is necessarily described by equation (14.8).[1] Indeed, suppose that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and consider the function $f : \mathcal{O} \to \mathbb{R}$. At the point (x_0, y_0) in $\mathcal{O}$, we look for a plane that is tangent to the graph of $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0, y_0, f(x_0, y_0))$. If the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at (x_0, y_0), then from the definition of a partial derivative and the meaning, in the case of functions of a single real variable, of the derivative as the slope of the tangent line, it follows that the vectors

$$\mathbf{T}_1 = \left(1, 0, \frac{\partial f}{\partial x}(x_0, y_0)\right) \qquad \text{and} \qquad \mathbf{T}_2 = \left(0, 1, \frac{\partial f}{\partial y}(x_0, y_0)\right)$$

should be parallel to the proposed tangent plane. Thus, the proposed tangent plane should have a cross-product

$$\eta = \mathbf{T}_1 \times \mathbf{T}_2 = (-\partial f/\partial x(x_0, y_0), -\partial f/\partial y(x_0, y_0), 1) \tag{14.9}$$

as a normal vector. The plane that passes through the point $(x_0, y_0, f(x_0, y_0))$ and is normal to η consists of all points (x, y, z) in $\mathbb{R}^3$ that satisfy the equation

$$\langle \eta, (x - x_0, y - y_0, z - f(x_0, y_0)) \rangle = 0,$$

and it is clear that this means that the point (x, y, z) in $\mathbb{R}^3$ lies on the graph of the function defined by equation (14.8).

The First-Order Approximation Theorem is also useful from another, less geometric, perspective. It enables us to approximate rather complicated functions by simpler ones and to assert precisely the manner in which the functions are close to one another.

[1] Appendix B, on linear algebra, includes a discussion of the properties of planes in $\mathbb{R}^3$ and the cross-product of two vectors in $\mathbb{R}^3$.

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Of course, the simplest type of function is a constant function. The next two simplest types of functions are linear functions and affine functions, which are defined as follows.

Definition A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *affine* if it is defined by

$$g(\mathbf{u}) = c + \sum_{i=1}^{n} a_i u_i \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n,$$

where c and the a_i's are prescribed numbers. If $c = 0$, the function is called *linear*.

Corollary 14.4 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Then there is an affine function that is a first-order approximation of f at the point $\mathbf{x}$, namely, the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$g(\mathbf{u}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{u} - \mathbf{x} \rangle \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

Proof

Observe that the function $g : \mathbb{R}^n \to \mathbb{R}$ is affine and that

$$g(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h} \rangle \qquad \text{for } \mathbf{x} + \mathbf{h} \text{ in } \mathbb{R}^n.$$

The First-Order Approximation Theorem asserts that the functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are first-order approximations of one another at the point $\mathbf{x}$. ■

Example 14.5 Define

$$f(x, y) = \sin(x - y - y^2) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Computing partial derivatives at the point $(0, 0)$, we find that the affine function that is a first-order approximation of f at the point $(0, 0)$ is defined by

$$\psi(x, y) = x - y \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Computing partial derivatives at the point $(\pi, 0)$, we find that the affine function that is a first-order approximation of f at the point $(\pi, 0)$ is given by

$$\psi(x, y) = \pi - x + y \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$ ■

EXERCISES FOR SECTION 14.1

1. Define

$$f(x, y) = e^{2x+4y+1} \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Find the equation of the tangent plane to the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0, 0, e)$.

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2. Define

$$f(x, y) = x^2 - xy + 2y^2 + x \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

At what points on the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ is the tangent plane parallel to the xy plane?

3. Let a, b, and c be positive numbers. The set of points (x, y, z) in $\mathbb{R}^3$ such that

$$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$$

is called a *hyperboloid*. Find the equation of the tangent plane to this hyperboloid at a point (x_0, y_0, z_0) on the hyperboloid with z_0 positive.

4. Define

$$f(x, y) = 2y + x^2 + xy \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Find the affine function that is a first-order approximation to the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0, 0)$.

5. Define

$$f(x, y) = e^{\sin(x-y)} \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Find the affine function that is a first-order approximation to the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0, 0)$.

6. Define

$$f(x, y, z) = x^2 + y^2 + z \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

Find the affine function that is a first-order approximation to the function $f : \mathbb{R}^3 \to \mathbb{R}$ at the point $(0, 0, 0)$.

7. Fix a point $\mathbf{x}$ in $\mathbb{R}^n$. Let $\mathbf{c}$ be a point in $\mathbb{R}^n$ and define the function $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(\mathbf{u}) = \langle \mathbf{c}, \mathbf{u} - \mathbf{x} \rangle \qquad \text{for } \mathbf{u} \text{ in } \mathbb{R}^n.$$

a. Show that the function $\psi : \mathbb{R}^n \to \mathbb{R}$ is affine.

b. Now show that given any nonconstant affine function $\psi : \mathbb{R}^n \to \mathbb{R}$, it is possible to choose points $\mathbf{x}$ and $\mathbf{c}$ in $\mathbb{R}^n$ so that $\psi : \mathbb{R}^n \to \mathbb{R}$ has the above form.

8. Suppose that the functions $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are both continuously differentiable. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = xg(x, y) + yh(x, y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Find the affine function that is a first-order approximation to $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$.

9. Suppose that the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Find necessary and sufficient conditions for these functions to be first-order approximations of each other at the point $(0, 0)$.

10. Let a, b, and c be real numbers. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{ax^2 + bxy + y^2}{\sqrt{x^2 + y^2}} = 0.$$

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11. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x+2y)-2x-2y}{\sqrt{x^2+y^2}} = 0.$$

12. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{(1+2x+y^2)^{3/2}-1-3x}{\sqrt{x^2+y^2}} = 0.$$

13. Let a be a real number. Prove that if

$$\lim_{(x,y)\to(0,0)} \frac{ax}{\sqrt{x^2+y^2}} = 0,$$

then $a = 0$.

14. Let $a,\ b$, and c be real numbers. Prove that if

$$\lim_{(x,y)\to(0,0)} \frac{c+ax+by}{\sqrt{x^2+y^2}} = 0,$$

then $c = a = b = 0$.

15. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Let a and b be any real numbers. Prove that

$$\lim_{(x,y)\to(0,0)} [f(x,\ y) - (f(0,\ 0) + ax + by)] = 0.$$

Is it true that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,\ y) - [f(0,\ 0) + ax + by]}{\sqrt{x^2+y^2}} = 0?$$

16. Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuous at the point $(x_0,\ y_0)$ in $\mathcal{O}$. For numbers a, b, and c, define

$$\psi(x,\ y) = a + b(x - x_0) + c(y - y_0) \qquad \text{for } (x,\ y) \text{ in } \mathbb{R}^2.$$

Assume that the graph of $\psi : \mathbb{R}^2 \to \mathbb{R}$ is tangent to the graph of $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0,\ y_0,\ f(x_0,\ y_0))$.

a. Show that $a = f(x_0,\ y_0)$.

b. Show that $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $(x_0,\ y_0)$ and that $b = \partial f/\partial x(x_0,\ y_0)$ and $c = \partial f/\partial y(x_0,\ y_0)$.

c. Use (a) and (b) to show that there can be only one tangent plane.

17. Define

$$f(x,\ y) = \begin{cases} \sin(y^2/x)\cdot\sqrt{x^2+y^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

a. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the point $(0,\ 0)$ and has directional derivatives in every direction at $(0,\ 0)$.

b. Show that there is no plane that is tangent to the graph of $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,\ 0,\ f(0,\ 0))$.

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18. Suppose that the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ has a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has directional derivatives in all directions at the point (x_0, y_0).

14.2 QUADRATIC FUNCTIONS, HESSIAN MATRICES, AND SECOND DERIVATIVES*

For a function of several variables, it often is necessary to determine the points at which the function attains maximum and minimum values. For functions of a single variable, we developed the second-derivative test in Chapter 4. We wish to discover the appropriate correspondent of this test for functions of several variables. We will do this in Section 14.3. In preparation for this, in the present section we will find a formula for the second derivative of sections of functions of several variables and provide estimates of the values attained by quadratic functions.

Recall that by an $n \times n$ *matrix* we mean a rectangular array of real numbers consisting of n rows and n columns. If such an $n \times n$ matrix is denoted by $\mathbf{A}$, we write

$$\mathbf{A} = [a_{ij}],$$

where for each pair of indices i and j with $1 \le i \le n$ and $1 \le j \le n$, a_{ij} denotes the number in the ith row and jth column of the matrix $\mathbf{A}$; we call a_{ij} the ijth entry of the matrix.

Definition Given any $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ and a point $\mathbf{x}$ in $\mathbb{R}^n$, by the symbol $\mathbf{Ax}$ we denote the point in $\mathbb{R}^n$ that for each index i with $1 \le i \le n$ has an ith component equal to the inner product of the ith row of $\mathbf{A}$ and $\mathbf{x}$. Thus,

$$\mathbf{Ax} \equiv \mathbf{y},$$

where

$$y_i \equiv \sum_{j=1}^{n} a_{ij}x_j \qquad \text{for each index } i \text{ with } 1 \le i \le n.$$

Since the point $\mathbf{Ax}$ has the ith component given by the inner product of $\mathbf{x}$ and the ith row of the matrix $\mathbf{A}$, if we denote by $\mathbf{A}_i$ the ith row of the matrix $\mathbf{A}$, then

$$\mathbf{Ax} = (\langle \mathbf{A}_1, \mathbf{x}\rangle, \ldots, \langle \mathbf{A}_i, \mathbf{x}\rangle, \ldots, \langle \mathbf{A}_n, \mathbf{x}\rangle).$$

Definition Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q(\mathbf{x}) \equiv \langle \mathbf{Ax}, \mathbf{x}\rangle \qquad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

is called the quadratic function associated with the matrix $\mathbf{A}$.

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Observe that

$$Q(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_jx_i \qquad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n,$$

so $Q(\mathbf{x})$ is a linear combination of x_jx_i's; hence the name *quadratic function.*

Example 14.6 The 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ has associated with it the quadratic function $Q : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$Q(x,\ y) = ax^2 + 2bxy + cy^2 \qquad \text{for } (x,\ y) \text{ in } \mathbb{R}^2. \qquad \blacksquare$$

Example 14.7 Consider the 3×3 matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Associated with the matrix $\mathbf{A}$ is the quadratic function $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$Q(x,\ y,\ z) = \lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2 \qquad \text{for } (x,\ y,\ z) \text{ in } \mathbb{R}^3. \qquad \blacksquare$$

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has second-order partial derivatives. The Hessian matrix of the function $f : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$ in $\mathcal{O}$, denoted by

$$\nabla^2 f(\mathbf{x}),$$

is defined to be the $n \times n$ matrix that for each pair of indices i and j, with $1 \leq i \leq n$ and $1 \leq j \leq n$, has the ijth entry defined by

$$(\nabla^2 f(\mathbf{x}))_{ij} \equiv \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}).$$

Observe that for each index i with $1 \leq i \leq n$, the ith row of the Hessian matrix $\nabla^2 f(\mathbf{x})$ is the gradient of the function $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$ at the point $\mathbf{x}$. Also observe that in view of the equality of cross–partial derivatives asserted in Theorem 13.10 it follows that the Hessian matrix $\nabla^2 f(\mathbf{x})$ is *symmetric*; that is, the ijth entry equals the jith entry provided that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives.

Example 14.8 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^2$ containing the point $(x_0,\ y_0)$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has second-order partial derivatives. Then the Hessian matrix of f at $(x_0,\ y_0)$ is given by

$$\nabla^2 f(x_0, y_0) = \begin{bmatrix} \partial^2 f/\partial x\partial x(x_0,\ y_0) & \partial^2 f/\partial y\partial x(x_0,\ y_0) \\ \partial^2 f/\partial x\partial y(x_0,\ y_0) & \partial^2 f/\partial y\partial y(x_0,\ y_0) \end{bmatrix}. \qquad \blacksquare$$

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Example 14.9 Define

$$f(x, y) = x^2y + y^2 + 1 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

A short computation of second-order derivatives shows that at the point (x_0, y_0) in $\mathbb{R}^2$,

$$\nabla^2 f(x_0, y_0) = \begin{bmatrix} 2y_0 & 2x_0 \\ 2x_0 & 2 \end{bmatrix}.$$

In particular, at the point $(3, 2)$,

$$\nabla^2 f(1, 1) = \begin{bmatrix} 4 & 6 \\ 6 & 2 \end{bmatrix}. \qquad \blacksquare$$

Example 14.10 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has second-order partial derivatives. Then the Hessian matrix of f at the point (x_0, y_0, z_0) in $\mathcal{O}$ is given by

$$\nabla^2 f(x_0, y_0, z_0)$$
$$= \begin{bmatrix} \partial^2 f/\partial x\partial x(x_0, y_0, z_0) & \partial^2 f/\partial y\partial x(x_0, y_0, z_0) & \partial^2 f/\partial z\partial x(x_0, y_0, z_0) \\ \partial^2 f/\partial x\partial y(x_0, y_0, z_0) & \partial^2 f/\partial y\partial y(x_0, y_0, z_0) & \partial^2 f/\partial z\partial y(x_0, y_0, z_0) \\ \partial^2 f/\partial x\partial z(x_0, y_0, z_0) & \partial^2 f/\partial y\partial z(x_0, y_0, z_0) & \partial^2 f/\partial z\partial z(x_0, y_0, z_0) \end{bmatrix}. \qquad \blacksquare$$

Example 14.11 Define

$$f(x, y, z) = \sin(xyz) + e^{x+y} + z^2 \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

A short computation of second-order derivatives shows that at the point $(0, 0, 0)$,

$$\nabla^2 f(0, 0, 0) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \qquad \blacksquare$$

The following theorem explains how the Hessian matrix enters into second-derivative calculations of the sections of a function of several variables.

Theorem 14.12 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Choose a positive number r such that the open ball about $\mathbf{x}$, $\mathcal{B}_r(\mathbf{x})$, is contained in $\mathcal{O}$. Then if $\|\mathbf{h}\| < r$ and $|t| \leq 1$,

$$\frac{d}{dt}[f(\mathbf{x} + t\mathbf{h})] = \langle \nabla f(\mathbf{x} + t\mathbf{h}), \mathbf{h} \rangle \tag{14.10}$$

and

$$\frac{d^2}{dt^2}[f(\mathbf{x} + t\mathbf{h})] = \langle \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}, \mathbf{h} \rangle. \tag{14.11}$$

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Proof

Let I be an open interval of real numbers that contains the points 0 and 1 and is such that the point $\mathbf{x} + t\mathbf{h}$ belongs to $\mathcal{O}$ if t belongs to I. Define

$$\phi(t) = f(\mathbf{x} + t\mathbf{h}) \qquad \text{for } t \text{ in } I.$$

The Directional Derivative Theorem implies that if t is in I, then

$$\phi'(t) = \frac{d}{dt}[f(\mathbf{x} + t\mathbf{h})] = \langle \nabla f(\mathbf{x} + t\mathbf{h}), \mathbf{h}\rangle = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h}).$$

However, for each index i with $1 \le i \le n$, we can again apply the Directional Derivative Theorem to the partial derivative $\partial f/\partial x_i : \mathcal{O} \to \mathbb{R}$, and hence by differentiating each side of the preceding equality, we see that

$$\begin{aligned}
\phi''(t) = \frac{d}{dt}[\phi'(t)] &= \sum_{i=1}^{n} h_i \frac{d}{dt}\left[\frac{\partial f}{\partial x_i}(\mathbf{x} + t\mathbf{h})\right] \\
&= \sum_{i=1}^{n} h_i \left\langle \nabla\left[\frac{\partial f}{\partial x_i}\right](\mathbf{x} + t\mathbf{h}), \mathbf{h}\right\rangle \\
&= \sum_{i=1}^{n} \left\langle \nabla\left[\frac{\partial f}{\partial x_i}\right](\mathbf{x} + t\mathbf{h}), \mathbf{h}\right\rangle h_i \\
&= \langle \nabla^2 f(\mathbf{x} + t\mathbf{h})\mathbf{h}, \mathbf{h}\rangle.
\end{aligned}$$

■

The preceding formula (14.11) will be useful in the next section for establishing a second-derivative criterion for extreme points of a function of several variables. But we will also need some estimates of the sizes of the values attained by quadratic functions. The remainder of this section is devoted to obtaining these estimates.

Definition The norm of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, denoted by $\|\mathbf{A}\|$, is defined by

$$\|\mathbf{A}\| \equiv \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}^2}.$$

Observe that if we define the point $\mathbf{A}_i$ in $\mathbb{R}^n$ to be the ith row of the $n \times n$ matrix $\mathbf{A}$, then the square of the norm of $\mathbf{A}$ can be written as

$$\|\mathbf{A}\|^2 = \|\mathbf{A}_1\|^2 + \|\mathbf{A}_2\|^2 + \cdots + \|\mathbf{A}_n\|^2.$$

The above definition of the norm of a matrix is introduced because with this definition of the norm, we have the following useful variant of the Cauchy–Schwarz Inequality.

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Theorem 14.13 A Generalized Cauchy–Schwarz Inequality Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then

$$\|\mathbf{Au}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{u}\|. \tag{14.12}$$

Proof
Squaring both sides of (14.12), it is clear that this inequality holds if and only if

$$\|\mathbf{Au}\|^2 \leq \|\mathbf{A}\|^2\|\mathbf{u}\|^2. \tag{14.13}$$

If for each index i with $1 \leq i \leq n$ we let the point $\mathbf{A}_i$ in $\mathbb{R}^n$ be the ith row of $\mathbf{A}$, then

$$\mathbf{Au} = (\langle \mathbf{A}_1, \mathbf{u}\rangle, \ldots, \langle \mathbf{A}_n, \mathbf{u}\rangle).$$

Thus, by the standard Cauchy–Schwarz Inequality,

$$\begin{aligned}\|\mathbf{Au}\|^2 &= (\langle \mathbf{A}_1, \mathbf{u}\rangle)^2 + \cdots + (\langle \mathbf{A}_n, \mathbf{u}\rangle)^2 \\ &\leq \|\mathbf{A}_1\|^2\|\mathbf{u}\|^2 + \cdots + \|\mathbf{A}_n\|^2\|\mathbf{u}\|^2 \\ &= (\|\mathbf{A}_1\|^2 + \cdots + \|\mathbf{A}_n\|^2)\|\mathbf{u}\|^2 \\ &= \|\mathbf{A}\|^2\|\mathbf{u}\|^2.\end{aligned}$$

We have verified inequality (14.13), and hence also inequality (14.12). ■

Corollary 14.14 Let $\mathbf{A}$ be an $n \times n$ matrix, let $Q : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function associated with $\mathbf{A}$, and let $\mathbf{u}$ be a point in $\mathbb{R}^n$. Then

$$|Q(\mathbf{u})| \leq \|\mathbf{A}\|\|\mathbf{u}\|^2. \tag{14.14}$$

Proof
By definition,

$$|Q(\mathbf{u})| = |\langle \mathbf{Au}, \mathbf{u}\rangle|.$$

Thus, if we first use the standard Cauchy–Schwarz Inequality and then the Generalized Cauchy–Schwarz Inequality, it follows that

$$\begin{aligned}|Q(\mathbf{u})| &= |\langle \mathbf{Au}, \mathbf{u}\rangle| \\ &\leq \|\mathbf{Au}\| \cdot \|\mathbf{u}\| \\ &\leq \|\mathbf{A}\|\|\mathbf{u}\|^2.\end{aligned}$$

■

Definition An $n \times n$ matrix $\mathbf{A}$ is said to be positive definite provided that

$$\langle \mathbf{Au}, \mathbf{u}\rangle > 0 \qquad \text{for all nonzero points } \mathbf{u} \text{ in } \mathbb{R}^n$$

and is said to be negative definite provided that

$$\langle \mathbf{Au}, \mathbf{u}\rangle < 0 \qquad \text{for all nonzero points } \mathbf{u} \text{ in } \mathbb{R}^n.$$

It is possible to give precise criteria in terms of the entries of a matrix that determine when it is positive definite or negative definite. The simplest case is the 2×2 case.

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Proposition 14.15 The 2×2 symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is positive definite if and only if

$$a > 0 \qquad \text{and} \qquad ac - b^2 > 0.$$

Proof

Observe that the quadratic function $Q : \mathbb{R}^2 \to \mathbb{R}$ associated with the matrix $\mathbf{A}$ has the form

$$Q(x, y) = ax^2 + 2bxy + cy^2 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

For points (x, y) with $y \neq 0$, set $t = x/y$ and $p(t) = at^2 + 2bt + c$. Observe that

$$Q(x, y) = y^2[a(x/y)^2 + 2b(x/y) + c] = y^2 p(t).$$

The second-degree polynomial $p(t)$ is positive for all t if and only if $a > 0$ and $ac - b^2 > 0$. In the case where $y = 0$, observe that $a > 0$ if and only if $Q(x, 0) = ax^2 > 0$ for all $x \neq 0$. ∎

A similar argument shows that the matrix $\mathbf{A}$ in Proposition 14.15 is negative definite if and only if

$$a < 0 \qquad \text{and} \qquad ac - b^2 > 0.$$

Proposition 14.16 Let $\mathbf{A}$ be an $n \times n$ positive definite matrix. Then there is a positive number c such that

$$Q(\mathbf{u}) = \langle \mathbf{Au}, \mathbf{u} \rangle \geq c\|\mathbf{u}\|^2 \qquad \text{for all } \mathbf{u} \text{ in } \mathbb{R}^n.$$

Proof

Since the quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ is the sum of products of continuous functions, namely, the component projection functions, it is continuous. On the other hand, the unit sphere $S = \{\mathbf{u} \text{ in } \mathbb{R}^n \mid \|\mathbf{u}\| = 1\}$ is a closed bounded subset of $\mathbb{R}^n$. According to the Sequential Compactness Theorem, Theorem 11.18, the unit sphere is therefore sequentially compact. Thus, by Extreme Value Theorem, Theorem 11.22, there is a point in S that is a minimizer for the restriction of the quadratic function to S. Define c to be the value of the quadratic function at this minimizer. Observe that c is positive, since we have assumed that the matrix $\mathbf{A}$ is positive definite, and that

$$Q(\mathbf{u}) \geq c \qquad \text{for all points } \mathbf{u} \text{ in } S. \tag{14.15}$$

Now, for all points $\mathbf{u}$ in $\mathbb{R}^n$ and all real numbers λ, $\mathbf{A}(\lambda\mathbf{u}) = \lambda\mathbf{Au}$, so

$$Q(\lambda\mathbf{u}) = \lambda^2 Q(\mathbf{u}). \tag{14.16}$$

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Moreover, note that if $\mathbf{u}$ is any nonzero point in $\mathbb{R}^n$, then

$$Q(\mathbf{u}) = Q\left(\|\mathbf{u}\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right).$$

From equality (14.16), it follows that

$$Q(\mathbf{u}) = \|\mathbf{u}\|^2 Q\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right).$$

But $\mathbf{u}/\|\mathbf{u}\|$ is a point in S, so by inequality (14.15),

$$Q(\mathbf{u}) \geq c\|\mathbf{u}\|^2.$$

It is clear that this inequality also holds if $\mathbf{u} = \mathbf{0}$. ■

EXERCISES FOR SECTION 14.2

1. Define

$$f(x, y) = e^{xy} + x^2 + 2xy \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

a. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = f(2t, 3t)$ for t in $\mathbb{R}$. Calculate $\phi''(0)$ directly.

b. Find the Hessian matrix of the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $(0, 0)$ and use formula (14.11) to calculate

$$\phi''(0) = \frac{d^2}{dt^2}[f(2t, 3t)]\bigg|_{t=0}.$$

2. Let $\mathcal{O} = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid xyz > -1\}$ and define $g : \mathcal{O} \to \mathbb{R}$ by

$$g(x, y, z) = \sqrt{1 + xyz} \qquad \text{for } (x, y, z) \text{ in } \mathcal{O}.$$

a. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = g(3t, 1 - t, t)$ for t in $\mathbb{R}$. Calculate $\phi''(0)$ directly.

b. Find the Hessian matrix of the function $g : \mathcal{O} \to \mathbb{R}$ at the point $(0, 1, 0)$ and use formula (14.11) to calculate

$$\phi''(0) = \frac{d^2}{dt^2}[g(3t, 1 - t, t)]\bigg|_{t=0}.$$

3. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second-order partial derivatives, and at the origin $(0, 0)$ suppose that

$$\frac{\partial f}{\partial x}(0, 0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0, 0) = 0.$$

Let $\mathbf{h}$ be a nonzero point in the plane $\mathbb{R}^2$ and suppose that

$$\langle \nabla^2 f(0, 0)\mathbf{h}, \mathbf{h}\rangle > 0.$$

Use the single-variable theory to prove that there is some positive number r such that

$$f(t\mathbf{h}) > f(0, 0) \qquad \text{if } 0 < |t| < r.$$

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4. In Exercise 3, suppose in fact that

$$\langle \nabla^2 f(0,\ 0)\mathbf{h},\ \mathbf{h}\rangle > 0 \qquad \text{for every nonzero point } \mathbf{h} \text{ in } \mathbb{R}^2.$$

Explain why this is not sufficient to directly conclude that the origin is a local minimizer of the function $f: \mathbb{R}^2 \to \mathbb{R}$.

5. Let a, b, and c be real numbers with $a \neq 0$, and define $p(t) = at^2 + 2bt + c$.
a. Show that $p(t) > 0$ for every number t if and only if $a > 0$ and $ac - b^2 > 0$.
b. Show that $p(t) < 0$ for every number t if and only if $a < 0$ and $ac - b^2 > 0$.

6. Suppose that $\mathbf{A}$ is a 3×3 symmetric matrix that is positive definite. Show that each of the following four properties holds, and from each of them obtain information about the entries of the matrix $\mathbf{A}$.
a. $\langle \mathbf{Ae}_1,\ \mathbf{e}_1\rangle > 0$, $\langle \mathbf{Ae}_2,\ \mathbf{e}_2\rangle > 0$, and $\langle \mathbf{Ae}_3,\ \mathbf{e}_3\rangle > 0$
b. $\langle \mathbf{Au},\ \mathbf{u}\rangle > 0$ for all nonzero $\mathbf{u} = (h,\ k,\ 0)$; h, k in $\mathbb{R}$
c. $\langle \mathbf{Au},\ \mathbf{u}\rangle > 0$ for all nonzero $\mathbf{u} = (0,\ h,\ k)$; h, k in $\mathbb{R}$
d. $\langle \mathbf{Au},\ \mathbf{u}\rangle > 0$ for all nonzero $\mathbf{u} = (h,\ 0,\ k)$; h, k in $\mathbb{R}$

7. For each of the following quadratic functions, find a 2×2 matrix with which it is associated.
a. $h(x,\ y) = x^2 - y^2$ for $(x,\ y)$ in $\mathbb{R}^2$
b. $g(x,\ y) = x^2 + 8xy + y^2$ for $(x,\ y)$ in $\mathbb{R}^2$

8. By making a suitable choice of the matrix $\mathbf{A}$, show that the Generalized Cauchy–Schwarz Inequality contains the standard Cauchy–Schwarz Inequality as a special case.

9. Find a 3×3 matrix associated with the quadratic function $Q: \mathbb{R}^3 \to \mathbb{R}$ defined by

$$Q(x,\ y,\ z) = x^2 - y^2 + 3xy + yz - z^2 \qquad \text{for } (x,\ y,\ z) \text{ in } \mathbb{R}^3.$$

10. Define the function $Q: \mathbb{R} \to \mathbb{R}$ by $Q(x) = x^4$. Observe that

$$Q(x) > 0 \qquad \text{for all } x \neq 0.$$

Show that there is no positive number c such that

$$Q(x) \geq cx^2 \qquad \text{for all } x \neq 0.$$

Explain why this does not contradict Proposition 14.16.

14.3 SECOND-ORDER APPROXIMATION AND THE SECOND-DERIVATIVE TEST*

Definition Let A be a subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and consider the function $f: A \to \mathbb{R}$:

i. The point $\mathbf{x}$ is called a local maximizer for the function $f: A \to \mathbb{R}$ provided that there is some positive number r such that

$$f(\mathbf{x} + \mathbf{h}) \leq f(\mathbf{x}) \qquad \text{if } \mathbf{x} + \mathbf{h} \text{ is in } A \text{ and } \|\mathbf{h}\| < r.$$

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ii. The point $\mathbf{x}$ is called a local minimizer for the function $f : A \to \mathbb{R}$ provided that there is some positive number r such that

$$f(\mathbf{x}+\mathbf{h}) \geq f(\mathbf{x}) \qquad \text{if } \mathbf{x}+\mathbf{h} \text{ is in } A \text{ and } \|\mathbf{h}\| < r.$$

iii. The point $\mathbf{x}$ is called a local extreme point for the function $f : A \to \mathbb{R}$ provided that it is either a local minimizer or a local maximizer for $f : A \to \mathbb{R}$.

We immediately find the following necessary condition for a point to be a local extreme point for a function.

Proposition 14.17 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives. If the point $\mathbf{x}$ is a local extreme point for the function $f : \mathcal{O} \to \mathbb{R}$, then

$$\nabla f(\mathbf{x}) = \mathbf{0}. \tag{14.17}$$

Proof

Since $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Fix an index i with $1 \leq i \leq n$ and define the function $\phi : (-r, r) \to \mathbb{R}$ by

$$\phi(t) = f(\mathbf{x}+t\mathbf{e}_i) \qquad \text{for } |t| < r.$$

Then the point 0 is an extreme point of the function $\phi : (-r, r) \to \mathbb{R}$, so

$$\phi'(0) = \frac{\partial f}{\partial x_i}(\mathbf{x}) = 0.$$

But this holds for each index i with $1 \leq i \leq n$, which means that (14.17) holds. ■

Observe that in order to search for local extreme points, we must first find the points $\mathbf{x}$ in $\mathcal{O}$ at which

$$\nabla f(\mathbf{x}) = \mathbf{0}. \tag{14.18}$$

However, equation (14.18) is a system of n scalar equations in n real unknowns. Unless the function $f : \mathcal{O} \to \mathbb{R}$ has a very simple form, it is not possible to find explicit solutions of (14.18). This should not be so surprising since in fact even for a differentiable function of a single variable $f : \mathbb{R} \to \mathbb{R}$, unless $f : \mathbb{R} \to \mathbb{R}$ is very simple, it is not possible to explicitly find all the numbers x that are solutions of the equation

$$f'(x) = 0.$$

Example 14.18 Consider the functions $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f(x, y) &= x^2 + y^2 && \text{for } (x, y) \text{ in } \mathbb{R}^2 \\ g(x, y) &= -x^2 - y^2 && \text{for } (x, y) \text{ in } \mathbb{R}^2 \\ h(x, y) &= x^2 - y^2 && \text{for } (x, y) \text{ in } \mathbb{R}^2. \end{aligned}$$

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Each of these functions is continuously differentiable, and we observe that for each of them the origin $(0, 0)$ is the only point in the plane where the gradient vector is zero. We see that the point $(0, 0)$ is a local minimizer for f and a local maximizer for g but fails to be a local extreme point for h. ■

In the above example, it is only because the functions were so simple that we were able to determine their behavior near the point $(0, 0)$. In order to analyze more complicated functions, we need to organize the set of second-order partial derivatives in a manner that will allow us to formulate a Second-Derivative Test for functions of several variables. We will see that the right way to do this is to arrange the second-order derivatives in the Hessian matrix and then to examine the quadratic function associated with this matrix.

In Chapter 8, we considered the approximation of a function of a single variable by a Taylor polynomial and obtained estimates for the difference between the function and its polynomial approximation. The Lagrange Remainder Theorem provides such estimates, and we need the following special case of this theorem.

Theorem 14.19 Let I be an open interval of real numbers and suppose that the function $f : I \to \mathbb{R}$ has a second derivative. Then for each pair of points x and $x + h$ in the interval I, there is a number θ with $0 < \theta < 1$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x+\theta h)h^2. \tag{14.19}$$

From this result for functions of a single variable and the derivative calculations for functions of several variables we obtained in Section 14.1, we obtain the following theorem.

Theorem 14.20 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Then for each pair of points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ in $\mathcal{O}$ with the property that the segment between these points also lies in $\mathcal{O}$, there is a number θ with $0 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + \frac{1}{2}\langle \nabla^2 f(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\rangle. \tag{14.20}$$

Proof

Choose I to be an open interval of real numbers containing both 0 and 1 such that $\mathbf{x} + t\mathbf{h}$ belongs to $\mathcal{O}$ if t is in I. Then define the function $\psi : I \to \mathbb{R}$ by

$$\psi(t) = f(\mathbf{x}+t\mathbf{h}) \qquad \text{for } t \text{ in } I.$$

Observe that Theorem 14.12 implies that the function $\psi : I \to \mathbb{R}$ has a second derivative and that we have the following formulas for the first and second derivatives:

$$\psi'(t) = \langle \nabla f(\mathbf{x}+t\mathbf{h}), \mathbf{h}\rangle \qquad \text{and} \qquad \psi''(t) = \langle \nabla^2 f(\mathbf{x}+t\mathbf{h})\mathbf{h}, \mathbf{h}\rangle \qquad \text{for } t \text{ in } I. \tag{14.21}$$

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We now apply Theorem 14.19 to the function $\psi : I \to \mathbb{R}$ with $x = 0$ and $h = 1$ to choose a number θ with $0 < \theta < 1$ such that

$$\psi(1) = \psi(0) + \psi'(0) + \frac{1}{2}\psi''(\theta), \tag{14.22}$$

an equality that, after substituting the values of $\psi(1)$ and $\psi(0)$ and using the above formulas for $\psi'(0)$ and $\psi''(\theta)$, is seen to be precisely formula (14.20). ■

Theorem 14.21 The Second-Order Approximation Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + 1/2\langle \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h}\rangle]}{\|\mathbf{h}\|^2} = 0. \tag{14.23}$$

Proof

Since the point $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. It is convenient to define

$$R(\mathbf{h}) = f(\mathbf{x}+\mathbf{h}) - \left[f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + \frac{1}{2}\langle \nabla^2 f(\mathbf{x})\mathbf{h}, \mathbf{h}\rangle\right] \qquad \text{for } \|\mathbf{h}\| < r.$$

We must show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R(\mathbf{h})}{\|\mathbf{h}\|^2} = 0. \tag{14.24}$$

Fix the point $\mathbf{h}$ in $\mathbb{R}^n$ with $0 < \|\mathbf{h}\| < r$. Using Theorem 14.20, we can choose a number θ with $1 < \theta < 1$ such that

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle + \frac{1}{2}\langle \nabla^2 f(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\rangle,$$

so that

$$\begin{aligned} R(\mathbf{h}) &= \frac{1}{2}\langle \nabla^2 f(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \mathbf{h}\rangle - \frac{1}{2}\langle \nabla^2 f(\mathbf{x}+\mathbf{h})\mathbf{h}, \mathbf{h}\rangle \\ &= \frac{1}{2}\langle [\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x}+\mathbf{h})]\mathbf{h}, \mathbf{h}\rangle. \end{aligned} \tag{14.25}$$

We can use this formula and the Generalized Cauchy–Schwarz Inequality to obtain the estimate

$$|R(\mathbf{h})| \le \frac{1}{2}\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x})\|\,\|\mathbf{h}\|^2.$$

Dividing this estimate by $\|\mathbf{h}\|^2$, we obtain

$$\frac{|R(\mathbf{h})|}{\|\mathbf{h}\|^2} \le \frac{1}{2}\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h}) - \nabla^2 f(\mathbf{x})\|. \tag{14.26}$$

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But the function $f:\mathcal{O}\to\mathbb{R}$ has been assumed to have continuous second-order partial derivatives, so

$$\lim_{\mathbf{h}\to\mathbf{0}}\|\nabla^2 f(\mathbf{x}+\theta\mathbf{h})-\nabla^2 f(\mathbf{x})\|=\mathbf{0},$$

and hence (14.24) follows from the estimate (14.26). ■

Theorem 14.22 The Second-Derivative Test Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ has continuous second-order partial derivatives. Assume that

$$\nabla f(\mathbf{x})=0.$$

i. If the Hessian matrix $\nabla^2 f(\mathbf{x})$ is positive definite, then the point $\mathbf{x}$ is a strict local minimizer of the function $f:\mathcal{O}\to\mathbb{R}$.
ii. If the Hessian matrix $\nabla^2 f(\mathbf{x})$ is negative definite, then the point $\mathbf{x}$ is a strict local maximizer of the function $f:\mathcal{O}\to\mathbb{R}$.

Proof
We need only consider case (i) since case (ii) follows from (i) if we replace f with $-f$. So suppose that the Hessian matrix $\nabla^2 f(\mathbf{x})$ is positive definite. Since the point $\mathbf{x}$ is an interior point of $\mathcal{O}$, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$.

The strategy of the proof is to write the difference $f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})$ as

$$f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})=Q(\mathbf{h})+R(\mathbf{h}),\tag{14.27}$$

where $Q:\mathbb{R}^n\to\mathbb{R}$ is a positive definite quadratic function and

$$\lim_{\mathbf{h}\to\mathbf{0}}\frac{R(\mathbf{h})}{\|\mathbf{h}\|^2}=0.\tag{14.28}$$

Indeed, if we define

$$R(\mathbf{h})=f(\mathbf{x}+\mathbf{h})-\left[f(\mathbf{x})+\langle\nabla f(\mathbf{x}),\mathbf{h}\rangle+\frac{1}{2}\langle\nabla^2 f(\mathbf{x})\mathbf{h},\mathbf{h}\rangle\right]\qquad\text{for }\|\mathbf{h}\|<r,\tag{14.29}$$

then the Second-Order Approximation Theorem asserts that (14.28) holds. Moreover, if we define $Q:\mathbb{R}^n\to\mathbb{R}$ to be the quadratic function associated with one-half the Hessian matrix $\nabla^2 f(\mathbf{x})$, then this quadratic function is positive definite. Finally, since $\nabla f(\mathbf{x})=\mathbf{0}$, we can rewrite (14.29) to obtain (14.27).

Since the quadratic function $Q:\mathbb{R}^n\to\mathbb{R}$ is positive definite, we can use Proposition 14.16 to choose a positive number c such that

$$Q(\mathbf{h})\geq c\|\mathbf{h}\|^2\qquad\text{for all }\mathbf{h}\text{ in }\mathbb{R}^n.$$

On the other hand, using (14.28), it follows that we can choose a positive number δ less than r such that

$$\frac{|R(\mathbf{h})|}{\|\mathbf{h}\|^2}<\frac{c}{2}\qquad\text{if }0<\|\mathbf{h}\|<\delta.$$

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Combining these two estimates, it follows from (14.27) that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = Q(\mathbf{h}) + R(\mathbf{h}) \geq c\|\mathbf{h}\|^2 + R(\mathbf{h}) > \frac{c}{2}\|\mathbf{h}\|^2 \qquad \text{if } 0 < \|\mathbf{h}\| < \delta,$$

so the point $\mathbf{x}$ is a strict local minimizer of the function $f : \mathcal{O} \to \mathbb{R}$. ■

In Proposition 14.15, we characterized the positive definite, symmetric 2×2 matrices; hence it is interesting to record the following special case of the Second-Derivative Test.

Corollary 14.23 Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ that contains the point (x_0, y_0) and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives. Assume that

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Suppose also that

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

and that

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) \cdot \frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left[\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right]^2 > 0.$$

Then the point (x_0, y_0) is a strict local minimizer for the function $f : \mathcal{O} \to \mathbb{R}$.

EXERCISES FOR SECTION 14.3

1. Analyze the local extrema of the following functions:
   a. $f(x, y) = e^{x^2-4y+y^2}$ for (x, y) in $\mathbb{R}^2$
   b. $g(x, y, z) = e^{x^2-4y+y^2} + z^2$ for (x, y, z) in $\mathbb{R}^3$
   c. $f(x, y) = (x^2 + y^2)e^{x^2+y^2}$ for (x, y) in $\mathbb{R}^2$
   d. $f(x, y) = x^3y^2(6 - x - y)$ for (x, y) in $\mathbb{R}^2$ with $x > 0$ and $y > 0$
2. Find necessary and sufficient conditions for a 2×2 symmetric matrix to be negative definite. Use this information to state and prove a sufficient condition for a point to be a local maximizer for a function of two variables.
3. Define K to be the closed rectangle $\{(x, y) \text{ in } \mathbb{R}^2 \mid -1 \leq x \leq 1, -\pi \leq y \leq \pi\}$ and define the function $f : K \to \mathbb{R}$ by $f(x, y) = xe^{-x}\cos y$ for (x, y) in K. Find the largest and smallest functional values of the function $f : K \to \mathbb{R}$. (*Hint:* Analyze the behavior on the boundary of K separately.)
4. Show that the point $(-1, 1)$ is the minimizer of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = (2x + 3y)^2 + (x + y - 1)^2 + (x + 2y - 2)^2 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

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5. Define

$$f(x, y) = x^2 + y^2 - 3xy \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Explain why the function $f : \mathbb{R}^2 \to \mathbb{R}$ has no local extreme points.

6. Analyze the local extreme points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \cos(x + y) + \sin(x + y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

7. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ at which $\nabla f(\mathbf{x}) = 0$. Assume also that there are points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ at which

$$\langle \nabla^2 f(\mathbf{x})\mathbf{u}, \mathbf{u}\rangle > 0 \qquad \text{and} \qquad \langle \nabla^2 f(\mathbf{x})\mathbf{v}, \mathbf{v}\rangle < 0.$$

Show that the point $\mathbf{x}$ is neither a local maximum nor a local minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$.

8. Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives. Let $\mathbf{x}$ be a point in $\mathbb{R}^n$ at which $\nabla f(\mathbf{x}) = 0$ and such that all the entries of the Hessian matrix $\nabla^2 f(\mathbf{x})$ are also 0. By giving specific examples, show that it is possible for the point $\mathbf{x}$ to be a local maximum, a local minimum, or neither.

9. Show that if the function $f : \mathbb{R}^n \to \mathbb{R}$ has continuous second-order partial derivatives and if at the point $\mathbf{x}$ in $\mathbb{R}^n$, $\nabla f(\mathbf{x}) = 0$ with $\nabla^2 f(\mathbf{x})$ positive definite, then there are positive numbers c and δ such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \geq c\|\mathbf{h}\|^2 \qquad \text{if } \|\mathbf{h}\| < \delta.$$

10. Complete the proof of the Second-Derivative Test by proving that if the Hessian is negative definite at a point, then the point is a strict local maximum.

11. Calculate the following limits by applying the First-Order or the Second-Order Approximation Theorem:

a. $$\lim_{(x,y)\to(0,0)} \frac{\sin(x + xy - y) - (x + y)}{\sqrt{x^2 + y^2}}$$

b. $$\lim_{(s,t)\to(0,0)} \frac{s^2 t}{\sqrt{s^2 + t^2}}$$

c. $$\lim_{(x,y)\to(0,0)} \frac{e^{x-y} - 1 - x + y}{x^2 + y^2}$$

d. $$\lim_{(x,y)\to(0,0)} \frac{\cos(x - y + xy) - (1 - 1/2x^2 + xy - 1/2y^2)}{x^2 + y^2}$$

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CHAPTER
15

APPROXIMATING NONLINEAR MAPPINGS BY LINEAR MAPPINGS

15.1 LINEAR MAPPINGS AND MATRICES

We now turn to the study of nonlinear mappings $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and n and m are positive integers. In order to study such general mappings, we return to a strategy that we have often used before: We approximate such mappings by mappings of a much simpler form. Here, we consider linear mappings the simpler ones.

In this first section, we consider linear mappings and the correspondence between linear mappings and matrices. In particular, we establish the equivalence between the invertibility of a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ and the invertibility of the $n \times n$ matrix with which it is associated. Section 15.2 will be devoted to describing the way in which nonlinear mappings can be approximated by linear mappings; the crucial concepts are the derivative matrix and the differential. In Section 15.3, we will study a generalization of the Chain Rule, established in Chapter 4 for differentiating the composition of functions, to a formula for the differential of the composition of general nonlinear mappings.

Definition A mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ is said to be *linear* provided that for each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and each pair of numbers α and β,

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v}). \tag{15.1}$$

Example 15.1 For each index i such that $1 \le i \le n$, the ith component projection function

$$p_i: \mathbb{R}^n \to \mathbb{R}$$

is linear. This follows directly from the definitions of addition and scalar multiplication since for each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and each pair of numbers α and β,

$$p_i(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha u_i + \beta v_i = \alpha p_i(\mathbf{u}) + \beta p_i(\mathbf{v}). \qquad \blacksquare$$

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Proposition 15.2 For a point $\mathbf{a}$ in $\mathbb{R}^n$, define the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$ by

$$\mathbf{T}(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n. \tag{15.2}$$

Then the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$ is linear. Moreover, for each linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$, there is a unique point $\mathbf{a}$ in $\mathbb{R}^n$ such that $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$ is defined by formula (15.2).

Proof

Given a point $\mathbf{a}$ in $\mathbb{R}^n$, as we have previously noted, the scalar product has the property that for each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and each pair of numbers α and β,

$$\langle \mathbf{a}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle = \alpha\langle \mathbf{a}, \mathbf{u} \rangle + \beta\langle \mathbf{a}, \mathbf{v} \rangle.$$

This is exactly what it means for the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$ to be linear.

Now suppose that $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$ is any linear mapping. For each index i such that $1 \le i \le n$, define $a_i = \mathbf{T}(\mathbf{e}_i)$ and then define the point $\mathbf{a}$ in $\mathbb{R}^n$ by $\mathbf{a} = (a_1, \ldots, a_n)$. Then, by the linearity of $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned}
\mathbf{T}(\mathbf{x}) &= \mathbf{T}(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\
&= x_1\mathbf{T}(\mathbf{e}_1) + \cdots + x_n\mathbf{T}(\mathbf{e}_n) \\
&= x_1 a_1 + \cdots + x_n a_n \\
&= \langle \mathbf{a}, \mathbf{x} \rangle \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.
\end{aligned}$$
■

Observe that for an index i such that $1 \le i \le n$, the ith component projection function $p_i: \mathbb{R}^n \to \mathbb{R}$ is defined by formula (15.2), where $\mathbf{a} = \mathbf{e}_i$, the vector in $\mathbb{R}^n$ whose ith component is 1 and other components are 0.

Example 15.3 A linear transformation $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}$ is completely determined by specifying three numbers a, b, and c and defining

$$\mathbf{T}(x, y, z) = ax + by + cz \qquad \text{for all } (x, y, z) \text{ in } \mathbb{R}^3.$$
■

For linear mappings $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$, where m is now an integer greater than 1, there is a natural generalization of Proposition 15.2. To understand what this generalization is, for a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ and an index i such that $1 \le i \le m$, recall that the function

$$\mathrm{T}_i = p_i \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}$$

is called the ith *component function* of the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ and that

$$\mathbf{T}(\mathbf{x}) = (\mathrm{T}_1(\mathbf{x}), \ldots, \mathrm{T}_m(\mathbf{x})) \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

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Proposition 15.4 A mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if each of its component functions is linear.

Proof
For each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and each pair of numbers α and β, by the very definition of equality in $\mathbb{R}^m$,

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v}) \tag{15.3}$$

if and only if for each index i such that $1 \leq i \leq m$,

$$p_i(\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v})) = p_i(\alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v})). \tag{15.4}$$

But we have already observed that the component projection functions are linear, so for each index i such that $1 \leq i \leq m$,

$$p_i(\alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v})) = \alpha p_i(\mathbf{T}(\mathbf{u})) + \beta p_i(\mathbf{T}(\mathbf{v})).$$

Thus, the identity (15.3) holds if and only if for each index i such that $1 \leq i \leq m$,

$$\mathrm{T}_i(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathrm{T}_i(\mathbf{u}) + \beta\mathrm{T}_i(\mathbf{v}). \tag{15.5}$$

The equivalence of (15.3) and (15.5) is exactly what is required in order to prove the proposition. ∎

Example 15.5 A mapping $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$\mathbf{T}(x, y) = (g(x, y), h(x, y)) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2,$$

where $g:\mathbb{R}^2 \to \mathbb{R}$ and $h:\mathbb{R}^2 \to \mathbb{R}$ are the component functions. Proposition 15.4 implies that the mapping $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ is linear if and only if the functions $g:\mathbb{R}^2 \to \mathbb{R}$ and $h:\mathbb{R}^2 \to \mathbb{R}$ are linear. From the characterization of linear functions just established in Proposition 15.2, it follows that $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ is linear if and only if there are numbers a, b, c, and d for which

$$\mathbf{T}(x, y) = (ax + by, cx + dy) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$ ∎

Let m and n be positive integers. Recall that by an $m \times n$ *matrix* we mean a rectangular array of real numbers consisting of m rows and n columns. If such an $m \times n$ matrix is denoted by $\mathbf{A}$, we write

$$\mathbf{A} = [a_{ij}],$$

where for each pair of indices i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, a_{ij} denotes the number in the ith row and jth column of the matrix $\mathbf{A}$; we call a_{ij} the ijth *entry* of the matrix and sometimes, for convenience, denote it by $(\mathbf{A})_{ij}$.

Definition For an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and a point $\mathbf{x}$ in $\mathbb{R}^n$, by the symbol $\mathbf{Ax}$ we denote the point in $\mathbb{R}^m$ that, for each index i such that $1 \leq i \leq m$, has an ith component

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equal to the scalar product of the ith row of $\mathbf{A}$ and $\mathbf{x}$. Thus,

$$\mathbf{Ax} \equiv \mathbf{y},$$

where

$$y_i \equiv \sum_{j=1}^{n} a_{ij}x_j \qquad \text{for each index } i \text{ such that } 1 \leq i \leq m.$$

The fundamental correspondence between matrices and linear mappings is described in the next theorem.

Theorem 15.6 For an $m \times n$ matrix $\mathbf{A}$, define the mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ by

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n. \tag{15.6}$$

Then the mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ is linear. Moreover, for each linear mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$, there is a unique $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ such that $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ is defined by (15.6); for each pair of indices i and j such that $1 \leq i \leq m$, $1 \leq j \leq n$, the ijth entry of the matrix $\mathbf{A}$ is determined by the formula

$$a_{ij} = \langle \mathbf{T}(\mathbf{e}_j), \mathbf{e}_i \rangle. \tag{15.7}$$

Proof

Since a mapping is linear if and only if each of its component functions is linear, to verify that formula (15.6) defines a linear mapping, we must show that for each index i such that $1 \leq i \leq m$, the component function $p_i \circ \mathbf{T}:\mathbb{R}^n \to \mathbb{R}$ is linear. But, by the very definition of $\mathbf{Ax}$,

$$(p_i \circ \mathbf{T})(\mathbf{x}) = \langle \mathbf{A}_i, \mathbf{x} \rangle \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n,$$

where $\mathbf{A}_i$ is the ith row of $\mathbf{A}$, and so, by Proposition 15.2, each component function is linear.

Now suppose that $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ is any linear mapping. Then each component function is linear, and hence, by Proposition 15.2, for each index i such that $1 \leq i \leq m$ we can select a point $\mathbf{A}_i$ in $\mathbb{R}^n$ such that the ith component function is defined by

$$(p_i \circ \mathbf{T})(\mathbf{x}) = \langle \mathbf{A}_i, \mathbf{x} \rangle \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n. \tag{15.8}$$

Define $\mathbf{A}$ to be the $m \times n$ matrix whose ith row is $\mathbf{A}_i$. Then (15.8) is equivalent to (15.6).

Observe that if $\mathbf{A} = [a_{ij}]$, then for each pair of indices i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, the ijth entry of the matrix $\mathbf{A}$ is the jth component of the ith row; that is, $a_{ij} = \langle \mathbf{A}_i, \mathbf{e}_j \rangle$. Hence, letting $\mathbf{x} = \mathbf{e}_j$ in (15.8), we have

$$a_{ij} = \langle \mathbf{A}_i, \mathbf{e}_j \rangle = (p_i \circ \mathbf{T})(\mathbf{e}_j) = \langle \mathbf{T}(\mathbf{e}_j), \mathbf{e}_i \rangle. \qquad \blacksquare$$

Definition For a linear mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$, the $m \times n$ matrix $\mathbf{A}$ such that

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

is said to be the matrix associated with the mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$.

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Example 15.7 Suppose that the mapping $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^3$ is linear and that

$$\mathbf{T}(1, 0) = (-4, 1/2, 1) \qquad \text{and} \qquad \mathbf{T}(0, 1) = (0, 6, 10).$$

From formula (15.7), it follows that the 3×2 matrix associated with this mapping is the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 1/2 & 6 \\ 1 & 10 \end{bmatrix}.$$

Thus, by formula (15.6) we see that

$$\mathbf{T}(x, y) = (-4x, x/2 + 6y, x + 10y) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$ ■

Example 15.8 Define the mapping $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathbf{T}(x, y, z) = (x, z) \qquad \text{for all } (x, y, z) \text{ in } \mathbb{R}^3.$$

Then this mapping is linear and is associated with the 2×3 matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ ■

Example 15.9 For a number θ, define the 2×2 matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping associated with the matrix $\mathbf{A}$. This linear mapping rotates points in the plane by θ radians about the origin. To see why this is so, we note that if (x, y) is a point in $\mathbb{R}^2$ written in polar coordinates as $(r\cos\phi, r\sin\phi)$, then a direct calculation, using just the addition properties of the sine and cosine, shows that its image $\mathbf{T}(x, y)$ is the point $(r\cos(\theta + \phi), r\sin(\theta + \phi))$.

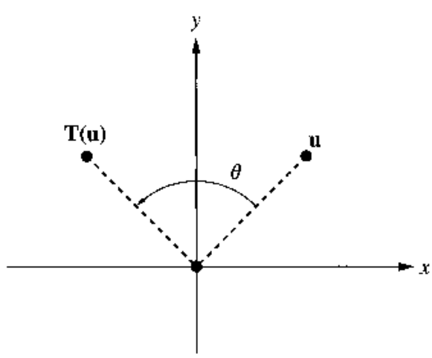

FIGURE 15.1 The rotation about the origin by θ radians. ■

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Example 15.10 In the plane $\mathbb{R}^2$, consider the line $\ell = \{(x, y) \text{ in } \mathbb{R}^2 \mid y = x\}$. For a point (x, y) in $\mathbb{R}^2$, define $\mathbf{T}(x, y)$ to be the point on the line ℓ that is closest to (x, y). This defines a mapping $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ that is linear. To see why it is linear, note that from the geometric significance of the scalar product we have the following formula: Let $(x_0, y_0) = (1/\sqrt{2}, 1/\sqrt{2})$, so that (x_0, y_0) is a point of length 1 on ℓ. From a simple trigonometric argument using Proposition 10.3, it follows that

$$\begin{aligned}\mathbf{T}(x, y) &= \langle (x, y), (x_0, y_0)\rangle (x_0, y_0) \\ &= \left(\frac{x}{2} + \frac{y}{2}, \frac{x}{2} + \frac{y}{2}\right) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.\end{aligned}$$

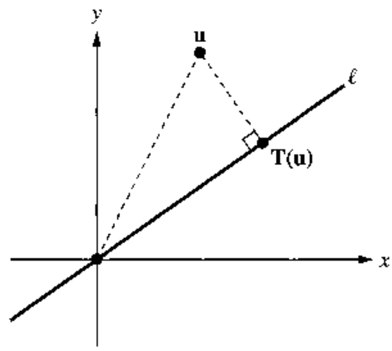

FIGURE 15.2 The closest point on the diagonal. ■

For two linear mappings $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{S}:\mathbb{R}^n \to \mathbb{R}^m$ and two numbers α and β, define the mapping $\alpha\mathbf{T} + \beta\mathbf{S}:\mathbb{R}^n \to \mathbb{R}^m$ by

$$(\alpha\mathbf{T} + \beta\mathbf{S})(\mathbf{x}) \equiv \alpha\mathbf{T}(\mathbf{x}) + \beta\mathbf{S}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Such a mapping is called a *linear combination* of the mappings $\mathbf{T}$ and $\mathbf{S}$. It immediately follows that a linear combination of linear mappings is again a linear mapping.

Definition For two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ and two numbers α and β, the matrix $\alpha\mathbf{A} + \beta\mathbf{B}$ is the $m \times n$ matrix whose ijth entry $(\alpha\mathbf{A} + \beta\mathbf{B})_{ij}$ is defined by

$$(\alpha\mathbf{A} + \beta\mathbf{B})_{ij} \equiv \alpha a_{ij} + \beta b_{ij}.$$

This definition of the linear combination of matrices was constructed so that the matrix associated with the linear combinations of two linear mappings $\mathbf{T}$ and $\mathbf{S}$ is the same linear combination of the matrices associated with $\mathbf{T}$ and $\mathbf{S}$, respectively. We formulate this as the next proposition; the proof follows directly from the linearity of the projection functions and formula (15.8).

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Proposition 15.11 Suppose that the linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ is associated with the $m \times n$ matrix $\mathbf{A}$ and that the linear mapping $\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^m$ is associated with the $m \times n$ matrix $\mathbf{B}$. Then for each pair of numbers α and β, the linear mapping $\alpha\mathbf{T} + \beta\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^m$ is associated with the $m \times n$ matrix $\alpha\mathbf{A} + \beta\mathbf{B}$.

Certain linear mappings can be composed. Specifically, given a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ and a linear mapping $\mathbf{S}: \mathbb{R}^m \to \mathbb{R}^k$, the composition $\mathbf{S} \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}^k$ is defined by

$$(\mathbf{S} \circ \mathbf{T})(\mathbf{x}) \equiv \mathbf{S}(\mathbf{T}(\mathbf{x})) \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

It is not difficult to check that the composition is again a linear mapping.

Definition For positive integers n, m, and k, let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix and let $\mathbf{B} = [b_{ij}]$ be a $k \times m$ matrix. Then the product matrix $\mathbf{BA}$ is defined to be the $k \times n$ matrix whose ijth entry $(\mathbf{BA})_{ij}$ is defined by

$$(\mathbf{BA})_{ij} \equiv \sum_{\ell=1}^{m} b_{i\ell} a_{\ell j}. \tag{15.9}$$

Formula (15.9) asserts that the ijth entry of $\mathbf{BA}$ is the inner product of the ith row of $\mathbf{B}$ and the jth column of $\mathbf{A}$. This definition of a matrix product is made so that the matrix associated with the composition of linear mappings will be the product of the matrices associated with the mappings that make up the composition. We formulate this as the next proposition; here again, the proof follows from the linearity of the projections and from (15.8).

Proposition 15.12 For natural numbers n, m, and k, suppose that the linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ is associated with the $m \times n$ matrix $\mathbf{A}$ and that the linear mapping $\mathbf{S}: \mathbb{R}^m \to \mathbb{R}^k$ is associated with the $k \times m$ matrix $\mathbf{B}$. Then the composite mapping $\mathbf{S} \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}^k$ is associated with the $k \times n$ product matrix $\mathbf{BA}$.

Corollary 15.13 For natural numbers n, m, k, and ℓ, let $\mathbf{A}$ be an $m \times n$ matrix, let $\mathbf{B}$ be a $k \times m$ matrix, and let $\mathbf{C}$ be an $\ell \times k$ matrix. Then

$$\mathbf{C}(\mathbf{BA}) = (\mathbf{CB})\mathbf{A}. \tag{15.10}$$

Proof

Let the linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^m$ be associated with the $m \times n$ matrix $\mathbf{A}$, let the linear mapping $\mathbf{S}: \mathbb{R}^m \to \mathbb{R}^k$ be associated with the $k \times m$ matrix $\mathbf{B}$, and let the linear mapping $\mathbf{L}: \mathbb{R}^k \to \mathbb{R}^\ell$ be associated with the $\ell \times k$ matrix $\mathbf{C}$. Proposition 15.12 implies that the mapping $\mathbf{S} \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}^k$ is associated with the matrix $\mathbf{BA}$ and that the mapping $\mathbf{L} \circ (\mathbf{S} \circ \mathbf{T}): \mathbb{R}^n \to \mathbb{R}^\ell$ is associated with the matrix $\mathbf{C}(\mathbf{BA})$. Similarly, the mapping $(\mathbf{L} \circ \mathbf{S}) \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}^\ell$ is associated with the matrix $(\mathbf{CB})\mathbf{A}$. But for

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each point $\mathbf{x}$ in $\mathbb{R}^n$,

$$((\mathbf{L} \circ \mathbf{S}) \circ \mathbf{T})(\mathbf{x}) = \mathbf{L}(\mathbf{S}(\mathbf{T}(\mathbf{x}))) = (\mathbf{L} \circ (\mathbf{S} \circ \mathbf{T}))(\mathbf{x}),$$

which simply means that the mappings $(\mathbf{L} \circ \mathbf{S}) \circ \mathbf{T} : \mathbb{R}^n \to \mathbb{R}^\ell$ and $\mathbf{L} \circ (\mathbf{S} \circ \mathbf{T}) : \mathbb{R}^n \to \mathbb{R}^\ell$ are equal. Thus, their associated matrices are equal; that is, (15.10) holds. ∎

The property of matrix multiplication stated in formula (15.10) is called the *associative property* of matrix multiplication. For two matrices $\mathbf{A}$ and $\mathbf{B}$, the product matrix $\mathbf{BA}$ is defined only when the number of columns in $\mathbf{B}$ equals the number of rows in $\mathbf{A}$. Thus, $\mathbf{AB}$ need not be defined when $\mathbf{BA}$ is defined. Moreover, when $n = m = k$, so that both $\mathbf{BA}$ and $\mathbf{AB}$ are properly defined $n \times n$ matrices, in general it is *not* true that $\mathbf{AB} = \mathbf{BA}$. In the language of algebra, this means that matrix multiplication is not commutative.

Example 15.14 Define the 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix}, \quad \text{whereas} \quad \mathbf{BA} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}.$$ ∎

Recall that given any two sets A and B and a mapping $F : A \to B$, the *image* of this mapping, denoted by $F(A)$, is defined to be the subset of B consisting of all points b in B for which there is a point a in A such that $b = F(a)$. This mapping is said to be *one-to-one* provided that for each point b in $F(A)$ there is exactly one point a in A such that $F(a) = b$. Moreover, the mapping $F : A \to B$ is said to be *onto* provided that its image $F(A)$ equals B. Finally, the mapping $F : A \to B$ is said to be *invertible* provided that it is both one-to-one and onto. Given an invertible mapping $F : A \to B$, for each point b in B define $F^{-1}(b)$ to be the unique point a in A such that $F(a) = b$. This defines the mapping $F^{-1} : B \to A$, which is called the *inverse mapping* of the mapping $F : A \to B$.

An invertible linear mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ has an inverse that also is linear. To verify this, we must show that for each pair of points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and each pair of numbers α and β,

$$\mathbf{T}^{-1}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}^{-1}(\mathbf{u}) + \beta\mathbf{T}^{-1}(\mathbf{v}),$$

which, by the very definition of inverse, means that

$$\mathbf{T}(\alpha\mathbf{T}^{-1}(\mathbf{u}) + \beta\mathbf{T}^{-1}(\mathbf{v})) = \alpha\mathbf{u} + \beta\mathbf{v}.$$

This last equality, however, follows from the linearity of the mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ and the definition of inverse.

The *identity mapping* on $\mathbb{R}^n$, which is denoted by $\mathbf{Id} : \mathbb{R}^n \to \mathbb{R}^n$, is the mapping that maps each point to itself; that is, $\mathbf{Id}(\mathbf{x}) \equiv \mathbf{x}$ for all points $\mathbf{x}$ in $\mathbb{R}^n$. The $n \times n$ matrix associated with the identity mapping is the matrix that has 1's on the diagonal and 0's elsewhere; it is denoted by $\mathbf{I}_n$ and is called the *identity matrix*.

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Just as there is a concept of invertibility for mappings, there is also a concept of invertibility for $n \times n$ matrices.

Definition An $n \times n$ matrix $\mathbf{A}$ is said to be invertible provided there is an $n \times n$ matrix $\mathbf{B}$ having the property that

$$\mathbf{AB} = \mathbf{I}_n \qquad \text{and} \qquad \mathbf{BA} = \mathbf{I}_n.$$

There is only one matrix $\mathbf{B}$ that has the above property (Exercise 11); it is denoted by $\mathbf{A}^{-1}$ and is called the *inverse matrix* of the matrix $\mathbf{A}$.

We have the following important equivalence between the invertibility of a linear mapping and the invertibility of the matrix with which it is associated.

Theorem 15.15 For a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ associated with the $n \times n$ matrix $\mathbf{A}$, the following two assertions are equivalent:

i. The mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible mapping.
ii. The matrix $\mathbf{A}$ is an invertible matrix.

Proof
First, suppose that the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible mapping. The inverse mapping $\mathbf{T}^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is also linear. Let the inverse mapping $\mathbf{T}^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ be associated with the $n \times n$ matrix $\mathbf{B}$.

Now the identity mapping $\mathbf{Id}: \mathbb{R}^n \to \mathbb{R}^n$ is associated with the identity matrix $\mathbf{I}_n$, and the matrix associated with a composition of two mappings is the product of the matrices associated with the mappings that make up the composite. Thus, since

$$\mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{Id}: \mathbb{R}^n \to \mathbb{R}^n \qquad \text{and} \qquad \mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{Id}: \mathbb{R}^n \to \mathbb{R}^n,$$

it follows that

$$\mathbf{AB} = \mathbf{I}_n \qquad \text{and} \qquad \mathbf{BA} = \mathbf{I}_n,$$

so the matrix $\mathbf{A}$ is invertible and its inverse matrix is $\mathbf{B}$.

To show that the invertibility of the matrix $\mathbf{A}$ implies the invertibility of the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$, suppose that the matrix $\mathbf{A}$ is invertible. Define $\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^n$ to be the linear mapping associated with the inverse matrix $\mathbf{A}^{-1}$. Now the matrix associated with the composition $\mathbf{S} \circ \mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$, which means that

$$(\mathbf{S} \circ \mathbf{T})(\mathbf{x}) = \mathbf{I}_n\mathbf{x} = \mathbf{x} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n;$$

that is,

$$\mathbf{S}(\mathbf{T}(\mathbf{x})) = \mathbf{x} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n. \tag{15.11}$$

Similarly, since $\mathbf{AA}^{-1} = \mathbf{I}_n$,

$$\mathbf{T}(\mathbf{S}(\mathbf{x})) = \mathbf{x} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n. \tag{15.12}$$

The identities (15.11) and (15.12) imply that the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is invertible (Exercise 12) and that its inverse mapping is $\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^n$. ∎

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Though Theorem 15.15 asserts the equivalence between the invertibility of a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ and the invertibility of its associated $n \times n$ matrix, it does not suggest how we can determine whether such a mapping is indeed invertible. For each $n \times n$ matrix $\mathbf{A}$, there is defined a number $\det \mathbf{A}$ called the *determinant* of $\mathbf{A}$. The determinant plays a central role in algebra, geometry, and analysis. One of its properties is that an $n \times n$ matrix is invertible if and only if its determinant is nonzero.

For an $n \times n$ matrix $\mathbf{A}$ and a pair of indices i and j such that $1 \leq i \leq n$ and $1 \leq j \leq n$, the *ijth minor* of $\mathbf{A}$, denoted by $\mathbf{A}^{ij}$, is the $(n-1) \times (n-1)$ matrix obtained by removing the ith row and the jth column from the matrix $\mathbf{A}$.

The determinant of a 1×1 matrix is simply the value of its single entry. Suppose that k is a positive integer and that the determinant has been defined for all $k \times k$ matrices. Then if $\mathbf{A}$ is a $(k+1) \times (k+1)$ matrix, the determinant of $\mathbf{A}$ is defined as follows:

$$\det \mathbf{A} \equiv \sum_{j=1}^{k-1} (-1)^{1+j} a_{1j} \det \mathbf{A}^{1j}. \tag{15.13}$$

By the Principle of Mathematical Induction, the determinant is defined for all square matrices.

Example 15.16 For a 2×2 matrix $\mathbf{A} = [a_{ij}]$, the determinant of $\mathbf{A}$ is given by

$$\begin{aligned} \det \mathbf{A} &= a_{11} \det \mathbf{A}^{11} - a_{12} \det \mathbf{A}^{12} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

■

Example 15.17 For a 3×3 matrix $\mathbf{A} = [a_{ij}]$, the determinant of $\mathbf{A}$ is given by

$$\begin{aligned} \det \mathbf{A} &= a_{11} \det \mathbf{A}^{11} - a_{12} \det \mathbf{A}^{12} + a_{13} \det \mathbf{A}^{13} \\ &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}[a_{22}a_{33} - a_{23}a_{32}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]. \end{aligned}$$

■

It would take us too far afield to provide a full discussion of the determinant for general $n \times n$ matrices. Appendix B, on linear algebra, includes a complete discussion of the determinant of 3×3 matrices and its relation to the cross-product and scalar product of two vectors in $\mathbb{R}^3$ and to the computation of volume.[1]

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[1] A clear exposition of elementary linear algebra is presented in the book by David C. Lay, *Linear Algebra and Its Applications* (Boston: Addison Wesley, 2002). A more advanced exposition can be found in the book by Peter D. Lax, *Linear Algebra* (New York: John Wiley, 1996). In particular, the proof of Theorem 15.18 can be found in Lax's book; the 3×3 case is proved in Appendix B.

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Theorem 15.18 Cramer's Rule An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$. Moreover, if $\det \mathbf{A} \neq 0$, then there is the following formula for the entries of the inverse matrix $\mathbf{A}^{-1}$:

$$(\mathbf{A}^{-1})_{ij} = \frac{1}{\det \mathbf{A}}((-1)^{i+j} \det \mathbf{A}^{ji}).$$

Example 15.19 For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, Cramer's Rule implies that if

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

then the matrix $\mathbf{A}$ has an inverse $\mathbf{A}^{-1}$ given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

In this case, direct matrix multiplication shows that this formula does in fact define the inverse matrix. ■

Cramer's Rule, together with the equivalence of the invertibility of a mapping and the invertibility of its associated matrix, gives the following corollary.

Corollary 15.20 For a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ that is associated with the $n \times n$ matrix $\mathbf{A}$, the following three assertions are equivalent:

i. $\det \mathbf{A} \neq 0$.
ii. The matrix $\mathbf{A}$ is an invertible matrix.
iii. The mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible mapping.

A general continuous mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$ can be one-to-one without being onto, and it can be onto without being one-to-one. However, a *linear* mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto. The proof for the case where $n = 3$ is presented in Appendix B. This property of linear maps between Euclidean spaces of the same dimension is necessary to prove the implication (ii) implies (i) of the following theorem.

Theorem 15.21 For an $n \times n$ matrix $\mathbf{A}$, the following two assertions are equivalent:

i. The matrix $\mathbf{A}$ is invertible.
ii. There is a positive number c such that

$$\|\mathbf{Ah}\| \geq c\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n.$$

Proof

First suppose that the matrix $\mathbf{A}$ is invertible. Then observe that for each point $\mathbf{h}$ in $\mathbb{R}^n$,

$$\mathbf{h} = \mathbf{A}^{-1}(\mathbf{Ah}).$$

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and hence by the Generalized Cauchy–Schwarz Inequality,

$$\|\mathbf{h}\| = \|\mathbf{A}^{-1}(\mathbf{A}\mathbf{h})\| \leq \|\mathbf{A}^{-1}\|\,\|\mathbf{A}\mathbf{h}\|.$$

Thus, (ii) holds where $c = 1/\|\mathbf{A}^{-1}\|$.

Conversely, suppose that (ii) holds. Then, in particular, we see that for a point $\mathbf{h}$ in $\mathbb{R}^n$, if $\mathbf{A}\mathbf{h} = 0$, then $\mathbf{h} = 0$. Let $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^n$ be the linear mapping associated with the matrix $\mathbf{A}$. Since for two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{T}(\mathbf{u}-\mathbf{v}) = \mathbf{T}(\mathbf{u}) - \mathbf{T}(\mathbf{v})$, setting $\mathbf{h} = \mathbf{u} - \mathbf{v}$, we see that if $\mathbf{T}(\mathbf{u}) = \mathbf{T}(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$; that is, the linear mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^n$ is one-to-one. By the remarks that preceded this theorem, the linear mapping $\mathbf{T}:\mathbb{R}^n \to \mathbb{R}^n$ is invertible. Thus, by Theorem 15.15, the matrix $\mathbf{A}$ that represents it is also invertible. ∎

To each $n \times n$ matrix $\mathbf{A}$ there corresponds another $n \times n$ matrix, called the *transpose matrix,* whose properties are closely related to those of $\mathbf{A}$. It is defined as follows.

Definition For an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, the transpose matrix $\mathbf{A}^{\mathbf{T}}$ is defined to be the $n \times n$ matrix that, for indices i and j such that $1 \leq i \leq n$ and $1 \leq j \leq n$, has ijth entry equal to a_{ji}.

Example 15.22 For 3×3 matrices, the definition of transpose means that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$ ∎

For any $n \times n$ matrix $\mathbf{A}$, $\det \mathbf{A} = \det \mathbf{A}^{\mathrm{T}}$. In the cases where $n = 2$ and $n = 3$, the proof is a straightforward calculation. Using Corollary 15.20 and the equality of the determinant of a matrix with the determinant of its transpose, we obtain the following important theorem.

Theorem 15.23 An $n \times n$ matrix is invertible if and only if its transpose is invertible.

EXERCISES FOR SECTION 15.1

1. Which of the following mappings $\mathbf{F}:\mathbb{R}^2 \to \mathbb{R}^2$ is linear?

a. $\mathbf{F}(x, y) = (-y, e^x)$ for (x, y) in $\mathbb{R}^2$

b. $\mathbf{F}(x, y) = (x - y^2, 2y)$ for (x, y) in $\mathbb{R}^2$

c. $\mathbf{F}(x, y) = 17(x, y)$ for (x, y) in $\mathbb{R}^2$

2. Define

$$\mathbf{T}(x, y) = (x + y, x - y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

a. Prove directly that the mapping $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one and onto.

b. Use Corollary 15.20 to show that the mapping $\mathbf{T}:\mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one and onto.

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3. Define

$$\mathbf{T}(x, y, z) = (x + z, 2x - 4y + 3z, y + 6z) \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

Prove that the mapping $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ is invertible.

4. Show that there is no linear mapping $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^2$ having the property that

$$\mathbf{T}(1, 1) = (4, 0) \qquad \text{and} \qquad \mathbf{T}(-2, -2) = (0, 1).$$

5. Find a linear transformation $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ that has the property that

$$\mathbf{T}(1, 1, 1) = (0, 2, 0), \quad \mathbf{T}(1, 1, -1) = (1, 2, 0), \quad \text{and} \quad \mathbf{T}(2, 0, 0) = (1, 1, 1).$$

[*Hint:* Use linearity to determine $\mathbf{T}(\mathbf{e}_i)$ for $i = 1, 2, 3$.]

6. For a point (x, y) in the plane $\mathbb{R}^2$, define $\mathbf{T}(x, y)$ to be the point on the line $\ell = \{(x, y) \text{ in } \mathbb{R}^2 \mid y = 2x\}$ that is closest to (x, y). Show that the mapping $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^2$ is linear and find the 2×2 matrix associated with this mapping.

7. For a point (x, y, z) in $\mathbb{R}^3$, define $\mathbf{T}(x, y, z)$ to be the point on the plane $P = \{(x, y, z) \text{ in } \mathbb{R}^2 \mid x + y + z = 0\}$ that is closest to (x, y, z). Show that the mapping $\mathbf{T}: \mathbb{R}^3 \to \mathbb{R}^3$ is linear and find the 3×3 matrix associated with this mapping.

8. Define $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Find all 2×2 matrices $\mathbf{B}$ with the property that

$$\mathbf{AB} = \mathbf{BA}.$$

9. For a number θ, define the rotation matrix 2×2 matrix $\mathbf{A}_\theta$ by

$$\mathbf{A}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

By a direct calculation of the matrix product, show that for any θ and ϕ,

$$\mathbf{A}_\theta \mathbf{A}_\phi = \mathbf{A}_{\theta+\phi}.$$

Thus, the product of two rotation matrices is again a rotation matrix. Use Proposition 15.12 to explain why this product formula is expected.

10. Find the 2×2 matrix associated with the mapping in the plane that rotates points 90° counterclockwise about the origin.

11. Let $\mathbf{A}$ be an $n \times n$ matrix and suppose that $\mathbf{B}$ and $\mathbf{B}'$ are two $n \times n$ matrices with the property that

$$\mathbf{AB} = \mathbf{I}_n = \mathbf{B}'\mathbf{A}.$$

Show that $\mathbf{B} = \mathbf{B}'$ by verifying that

$$\mathbf{B} = \mathbf{I}_n\mathbf{B} = (\mathbf{B}'\mathbf{A})\mathbf{B} = \mathbf{B}'(\mathbf{AB}) = \mathbf{B}'\mathbf{I}_n = \mathbf{B}'.$$

12. Suppose that the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ has the property that there is another mapping $\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathbf{T}(\mathbf{S}(\mathbf{x})) = \mathbf{S}(\mathbf{T}(\mathbf{x})) = \mathbf{x} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Prove that $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is invertible and that its inverse is the mapping $\mathbf{S}: \mathbb{R}^n \to \mathbb{R}^n$.

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13. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that for each pair of indices i and j such that $1 \leq i, j \leq n$,

$$\langle \mathbf{A}\mathbf{e}_i, \mathbf{e}_j \rangle = \langle \mathbf{e}_i, \mathbf{A}^{\mathrm{T}}\mathbf{e}_j \rangle.$$

Use this and the linearity of the scalar product to show that for any two points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$,

$$\langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^{\mathrm{T}}\mathbf{v} \rangle.$$

15.2 THE DERIVATIVE MATRIX AND THE DIFFERENTIAL

In this section, we consider nonlinear mappings between Euclidean spaces and the manner in which they can be approximated by linear mappings. The following definition singles out the classes of mappings that we will consider.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and consider a mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ represented in component functions as $\mathbf{F} = (F_1, \ldots, F_m)$.

i. The mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is said to have first-order partial derivatives at the point $\mathbf{x}$ in $\mathcal{O}$ provided that for each index i such that $1 \leq i \leq m$, the component function $F_i : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives at $\mathbf{x}$.
ii. Moreover, the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is said to have first-order partial derivatives provided that it has first partial derivatives at every point in $\mathcal{O}$.
iii. Finally, the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is said to be continuously differentiable provided that each component function is continuously differentiable.

Example 15.24 Define the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y) = (x^2 + e^{xy}, \sin(xy), y + 1) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Then $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^3$ is continuously differentiable since it is clear that each of its three component functions is continuously differentiable. ■

Proposition 15.25 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Then the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuous.

Proof
By definition, each of the component functions of the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. It follows from Theorem 13.20 that each component function is continuous. Consequently, from the Componentwise Continuity Theorem (Theorem 11.9), we conclude that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is itself continuous. ■

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Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$ in $\mathcal{O}$. The derivative matrix of $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ at the point $\mathbf{x}$ is defined to be the $m \times n$ matrix $\mathbf{DF}(\mathbf{x})$, which, for each index i such that $1 \le i \le m$, has an ith row equal to $\nabla F_i(\mathbf{x})$. Thus, the ijth entry of this derivative matrix is given by the formula

$$(\mathbf{DF}(\mathbf{x}))_{ij} \equiv \frac{\partial \mathrm{F}_i}{\partial x_j}(\mathbf{x}).$$

Example 15.26 If the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ has first-order partial derivatives and has component representation

$$\mathbf{F}(x, y) = (u(x, y), v(x, y)) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2,$$

then at the point (x_0, y_0) the derivative matrix is

$$\mathbf{DF}(x_0, y_0) = \begin{bmatrix} \partial u/\partial x(x_0, y_0) & \partial u/\partial y(x_0, y_0) \\ \partial v/\partial x(x_0, y_0) & \partial v/\partial y(x_0, y_0) \end{bmatrix}.$$ ■

Example 15.27 Suppose that the function $f: \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives. Then at the point (x_0, y_0) in $\mathbb{R}^2$, the derivative matrix is

$$\nabla f(x_0, y_0) = [\partial f/\partial x(x_0, y_0), \quad \partial f/\partial y(x_0, y_0)],$$

which is the 1×2 matrix corresponding to the gradient vector. ■

Example 15.28 Suppose that the mapping $\mathbf{F}: \mathbb{R} \to \mathbb{R}^3$ has first-order partial derivatives and has the component representation

$$\mathbf{F}(t) = (x(t), y(t), z(t)) \qquad \text{for } t \text{ in } \mathbb{R}.$$

Then at the point t_0 in $\mathbb{R}$, the derivative matrix is the 3×1 matrix given by

$$\mathbf{DF}(t_0) = \begin{bmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{bmatrix}.$$ ■

Theorem 15.29 The Mean Value Theorem for General Mappings Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Suppose that the points $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ are in $\mathcal{O}$ and that the segment joining these points also lies in $\mathcal{O}$. Then there are numbers $\theta_1, \theta_2, \ldots, \theta_m$ in the open interval $(0, 1)$ such that

$$\mathrm{F}_i(\mathbf{x} + \mathbf{h}) - \mathrm{F}_i(\mathbf{x}) = \langle \nabla \mathrm{F}_i(\mathbf{x} + \theta_i \mathbf{h}), \mathbf{h} \rangle \qquad \text{for } 1 \le i \le m; \tag{15.14}$$

that is,

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{h}, \tag{15.15}$$

where $\mathbf{A}$ is the $m \times n$ matrix whose ith row is $\nabla \mathrm{F}_i(\mathbf{x} + \theta_i \mathbf{h})$.

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Proof

Just apply the Mean Value Theorem for real-valued functions to each of the continuously differentiable component functions and we obtain formula (15.14). Formula (15.15) is simply a rewriting of (15.14). ∎

A natural question to ask is whether in (15.14) we can choose all the θ_i's equal. In that case, (15.15) would become

$$\mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x}) = \mathbf{DF}(\mathbf{x}+\theta\mathbf{h})\mathbf{h}, \tag{15.16}$$

and this would be a symbol-by-symbol extension of the Mean Value Theorem for real-valued functions. However, formula (15.16) is false. In general, it is not true that there is a number θ in the interval $(0, 1)$ such that (15.16) holds. The following example illustrates what can occur.

Example 15.30 Define the mapping $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(x, y) = (x^2, y^3) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Set $\psi(x, y) = x^2$ and $\phi(x, y) = y^3$ for (x, y) in $\mathbb{R}^2$. Take the point $\mathbf{x}$ to be the origin $(0, 0)$ and take $\mathbf{h}$ to be the point $(1, 1)$. The above extension of the Mean Value Theorem implies that there are numbers θ_1 and θ_2 in the interval $(0, 1)$ such that

$$\begin{aligned} \psi(1, 1) - \psi(0, 0) &= \langle \nabla\psi(\theta_1, \theta_1), (1, 1)\rangle \\ \phi(1, 1) - \phi(0, 0) &= \langle \nabla\phi(\theta_2, \theta_2), (1, 1)\rangle. \end{aligned} \tag{15.17}$$

But $\nabla\psi(x, y) = (2x, 0)$ and $\nabla\phi(x, y) = (0, 3y^2)$ for (x, y) in $\mathbb{R}^2$; hence, substituting in (15.17), we see that

$$1 = 2\theta_1 \qquad \text{and} \qquad 1 = 3\theta_2^2.$$

Thus, $\theta_1 = 1/2$ and $\theta_2 = \sqrt{1/3}$. Therefore, we certainly cannot find $\theta_1 = \theta_2$ so that (15.16) holds. ∎

Recall the First-Order Approximation Theorem for scalar-valued functions, which asserts that if $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, then at each point $\mathbf{x}$ in $\mathcal{O}$,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{x}+\mathbf{h}) - [f(\mathbf{h}) + \langle \nabla f(\mathbf{x}), \mathbf{h}\rangle]}{\|\mathbf{h}\|} = 0. \tag{15.18}$$

The following is an extension of this result to general mappings.

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Theorem 15.31 First-Order Approximation Theorem for Mappings Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}]\|}{\|\mathbf{h}\|} = 0. \tag{15.19}$$

Proof

Since $\mathcal{O}$ is open, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. For a point $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, define

$$\mathbf{R}(\mathbf{h}) = \mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}].$$

We must show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0. \tag{15.20}$$

But if we represent the mappings $\mathbf{F}$ and $\mathbf{R}$ as $\mathbf{F} = (\mathrm{F}_1, \ldots, \mathrm{F}_m)$ and $\mathbf{R} = (\mathrm{R}_1, \ldots, \mathrm{R}_m)$, then it is clear that for each index i such that $1 \le i \le m$,

$$\mathrm{R}_i(\mathbf{h}) = \mathrm{F}_i(\mathbf{x}+\mathbf{h}) - [\mathrm{F}_i(\mathbf{x}) + \langle \nabla\mathrm{F}_i(\mathbf{x}), \mathbf{h}\rangle] \qquad \text{for } \|\mathbf{h}\| < r.$$

Since the function $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable, the First-Order Approximation Theorem for real-valued functions implies that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathrm{R}_i(\mathbf{h})}{\|\mathbf{h}\|} = 0.$$

Since

$$\frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = \left(\sum_{i=1}^{m} \left[\frac{\mathrm{R}_i(\mathbf{h})}{\|\mathbf{h}\|}\right]^2\right)^{1/2} \qquad \text{for } 0 < \|\mathbf{h}\| < r,$$

it follows that (15.20) holds. ∎

For a function $f: I \to \mathbb{R}$, where I is an open interval, if at the point x in I there is a number a such that

$$\lim_{h\to 0} \frac{f(x+h) - [f(x) + ah]}{h} = 0,$$

then since if $h \neq 0$ and $x + h$ is in I,

$$\frac{f(x+h) - [f(x) + ah]}{h} = \frac{f(x+h) - f(x)}{h} - a,$$

it follows that $f: I \to \mathbb{R}$ is differentiable at x and that $f'(x) = a$. This property generalizes to mappings as follows.

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Theorem 15.32 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and consider a mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$. Suppose that $\mathbf{A}$ is an $m \times n$ matrix with the property that

$$\lim_{\mathbf{h}\to 0} \frac{\|\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]\|}{\|\mathbf{h}\|} = 0. \tag{15.21}$$

Then the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$ and

$$\mathbf{A} = \mathbf{DF}(\mathbf{x}).$$

Proof

Represent the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ in component functions as $\mathbf{F} = (\mathrm{F}_1, \ldots, \mathrm{F}_m)$ and set $a_{ij} = (\mathbf{A})_{ij}$. We must show that for each pair of indices i and j such that $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$a_{ij} = \frac{\partial \mathrm{F}_i}{\partial x_j}(\mathbf{x}).$$

For each index i such that $1 \leq i \leq m$, define $\mathbf{A}_i$ to be the ith row of the matrix $\mathbf{A}$.

Since $\mathcal{O}$ is open, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{x})$ is contained in $\mathcal{O}$. Now observe that if $1 \leq i \leq m$ and $\|\mathbf{h}\| < r$, then

$$\mathrm{F}_i(\mathbf{x}+\mathbf{h}) - [\mathrm{F}_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h}\rangle] = p_i(\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]),$$

so that

$$|\mathrm{F}_i(\mathbf{x}+\mathbf{h}) - [\mathrm{F}_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h}\rangle]| \leq \|\mathbf{F}(\mathbf{x}+\mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{A}\mathbf{h}]\|.$$

From (15.21) it follows that

$$\lim_{\mathbf{h}\to 0} \frac{\mathrm{F}_i(\mathbf{x}+\mathbf{h}) - [\mathrm{F}_i(\mathbf{x}) + \langle \mathbf{A}_i, \mathbf{h}\rangle]}{\|\mathbf{h}\|} = 0.$$

In particular, for an index j such that $1 \leq j \leq n$,

$$\lim_{t\to 0} \frac{\mathrm{F}_i(\mathbf{x}+t\mathbf{e}_j) - [\mathrm{F}_i(\mathbf{x}) + \langle \mathbf{A}_i, t\mathbf{e}_j\rangle]}{\|t\mathbf{e}_j\|} = 0. \tag{15.22}$$

However, $\|t\mathbf{e}_j\| = |t|$, so (15.22) is equivalent to

$$\lim_{t\to 0} \frac{\mathrm{F}_i(\mathbf{x}+t\mathbf{e}_j) - \mathrm{F}_i(\mathbf{x})}{t} = \langle \mathbf{A}_i, \mathbf{e}_j\rangle,$$

thus proving that $\mathbf{F}: \mathcal{O} \to \mathbb{R}^n$ has first partial derivatives at $\mathbf{x}$ and

$$a_{ij} = \langle \mathbf{A}_i, \mathbf{e}_j\rangle = \frac{\partial \mathrm{F}_i}{\partial x_j}(\mathbf{x}) \qquad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq n. \qquad \blacksquare$$

The above theorem implies that for a continuously differentiable mapping, the derivative matrix is the only matrix having the first-order approximation property (15.19).

In view of the correspondence described in Section 15.1 between $m \times n$ matrices and linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, it is useful to introduce the following correspondent of the derivative matrix.

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Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ has first-order partial derivatives at the point $\mathbf{x}$. The linear mapping

$$\mathbf{dF}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$$

defined by

$$\mathbf{dF}(\mathbf{x})(\mathbf{h}) \equiv \mathbf{DF}(\mathbf{x})\mathbf{h} \qquad \text{for all } \mathbf{h} \text{ in } \mathbb{R}^n$$

is called the *differential* of the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ at the point $\mathbf{x}$.

We record as a theorem the content of Corollary 15.20 in the case where the linear mapping is the differential of a nonlinear mapping at a point.

Theorem 15.33 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the point $\mathbf{x}$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^n$ has first-order partial derivatives at $\mathbf{x}$. Then the following three assertions are equivalent:

i. $\det \mathbf{DF}(\mathbf{x}) \neq 0$.
ii. The derivative matrix $\mathbf{DF}(\mathbf{x})$ is an invertible $n \times n$ matrix.
iii. The differential $\mathbf{dF}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear mapping.

The First-Order Approximation Theorem is a precise assertion of the manner in which

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) \approx \mathbf{F}(\mathbf{x}) + \mathbf{dF}(\mathbf{x})(\mathbf{h})$$

when $\mathbf{h}$ is sufficiently close to $\mathbf{0}$. The general theme of the next two chapters will be expressed in descriptions of the properties that a continuously differentiable mapping inherits from the properties of its differential at a point.

EXERCISES FOR SECTION 15.2

1. Define

$$\mathbf{F}(x, y) = (e^{xy} + 2x, \ y^2 + \sin(x - y)) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Find the derivative matrix of the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ at the points $(0, 0)$ and $(\pi, 0)$.

2. Define

$$\mathbf{F}(x, y, z) = (xyz, \ x^2 + yz, \ 1 + 3x) \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3.$$

Find the derivative matrix of the mapping $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ at the points $(1, 2, 3)$, $(0, 1, 0)$, and $(-1, 4, 0)$.

3. Suppose that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that the derivative matrix $\mathbf{DF}(\mathbf{x})$ at each point $\mathbf{x}$ in $\mathbb{R}^n$ has all its entries equal to 0. Prove that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m$ is constant; that is, there is some point $\mathbf{c}$ in $\mathbb{R}^m$ such that

$$\mathbf{F}(\mathbf{x}) = \mathbf{c} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^n.$$

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4. Suppose that $\mathbf{A}$ is an $m \times n$ matrix. Define the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m$ by

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Prove that $\mathbf{DF}(\mathbf{x}) = \mathbf{A}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

5. Suppose that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that there is a fixed $m \times n$ matrix $\mathbf{A}$ so that

$$\mathbf{DF}(\mathbf{x}) = \mathbf{A} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Prove that there is some $\mathbf{c}$ in $\mathbb{R}^m$ so that

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c} \qquad \text{for every } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Restate this result for the case when $n = m = 1$.

6. Define the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(x, y) = (x^2 - y^2, 2xy) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

a. Find the points (x_0, y_0) in $\mathbb{R}^2$ at which the derivative matrix $\mathbf{DF}(x_0, y_0)$ is invertible.

b. Find the points (x_0, y_0) in $\mathbb{R}^2$ at which the differential $\mathbf{dF}(x_0, y_0): \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear mapping.

7. Give a proof of the First-Order Approximation Theorem based on the Mean Value Theorem.

8. Suppose that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Suppose also that $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and that the derivative matrix $\mathbf{DF}(\mathbf{0})$ has the property that there is some positive number c such that

$$\|\mathbf{DF}(\mathbf{0})\mathbf{h}\| \geq c\|\mathbf{h}\| \qquad \text{for all } \mathbf{h} \text{ in } \mathbb{R}^n.$$

Prove that there is some positive number r such that

$$\|\mathbf{F}(\mathbf{h})\| \geq c/2\|\mathbf{h}\| \qquad \text{if } \|\mathbf{h}\| \leq r.$$

9. Suppose that the continuously differentiable mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is represented in component functions as

$$\mathbf{F}(x, y) = (\psi(x, y), \varphi(x, y)) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Define the function $g: \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x, y) = \frac{1}{2}[(\psi(x, y))^2 + (\varphi(x, y))^2] \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

a. Show that

$$\mathbf{D}g(x_0, y_0) = [\mathbf{DF}(x_0, y_0)]^{\mathbf{T}}\, \mathbf{F}(x_0, y_0).$$

b. Use (a) to prove that if (x_0, y_0) is a minimizer of the function $g: \mathbb{R}^2 \to \mathbb{R}$ and the matrix $\mathbf{DF}(x_0, y_0)$ is invertible, then

$$\mathbf{F}(x_0, y_0) = \mathbf{0}.$$

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15.3 THE CHAIN RULE

From the Chain Rule for real-valued functions of a single variable, it follows that if $\mathcal{O}$ and $\mathcal{U}$ are open sets of real numbers and the functions $f:\mathcal{O} \to \mathbb{R}$ and $g:\mathcal{U} \to \mathbb{R}$ are continuously differentiable, with $f(\mathcal{O})$ contained in $\mathcal{U}$, then the composite function

$$g \circ f : \mathcal{O} \to \mathbb{R}$$

is also continuously differentiable and, moreover, for each point x in $\mathcal{O}$,

$$(g \circ f)'(x) = g'(f(x))f'(x). \tag{15.23}$$

The Chain Rule carries over to compositions of general continuously differentiable mappings in which the derivative matrix replaces the derivative and matrix multiplication replaces scalar multiplication. The general Chain Rule follows from the following special case of the composition of a mapping with a real-valued function.

In order to clearly state the Chain Rule, it is helpful to use the following notation: For an open subset $\mathcal{U}$ of $\mathbb{R}^m$ and a function $g:\mathcal{U} \to \mathbb{R}$ that has first-order partial derivatives, at each point $\mathbf{p}$ in $\mathcal{U}$ and for each index i such that $1 \leq i \leq m$, we define

$$D_i g(\mathbf{p}) \equiv \lim_{t \to 0} \frac{f(\mathbf{p} + t\mathbf{e}_i) - g(\mathbf{p})}{t}.$$

This notation has the advantage that the partial derivative with respect to the ith component is denoted by a symbol independent of the notation being used for the points in the domain. Moreover, there is the formula

$$\nabla g(\mathbf{p}) = (D_1 g(\mathbf{p}), \ldots, D_m g(\mathbf{p})) \qquad \text{for each point } \mathbf{p} \text{ in } \mathcal{U}.$$

Theorem 15.34 The Chain Rule Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}:\mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Suppose also that $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ and that the function $g:\mathcal{U} \to \mathbb{R}$ is continuously differentiable. Finally, suppose that $\mathbf{F}(\mathcal{O})$ is contained in $\mathcal{U}$. Then the composition $g \circ \mathbf{F}:\mathcal{O} \to \mathbb{R}$ is also continuously differentiable. Moreover, for each point $\mathbf{x}$ in $\mathcal{O}$ and each index i such that $1 \leq i \leq n$,

$$\frac{\partial}{\partial x_i}(g \circ \mathbf{F})(\mathbf{x}) = \sum_{j=1}^{m} D_j g(\mathbf{F}(\mathbf{x}))\frac{\partial \mathbf{F}_j}{\partial x_i}(\mathbf{x}); \tag{15.24}$$

that is,

$$\nabla(g \circ \mathbf{F})(\mathbf{x}) = \nabla g(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x}). \tag{15.25}$$

Proof

Let $\mathbf{x}$ be a point in $\mathcal{O}$. Since $\mathcal{O}$ is open, we can select a positive number r such that the open ball $B_r(\mathbf{x})$ is contained in $\mathcal{O}$. Moreover, since the mapping $\mathbf{F}:\mathcal{O} \to \mathbb{R}^m$ is continuous and $\mathcal{U}$ is an open subset of $\mathbb{R}^m$, we can also suppose that the segment joining the points $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x}+\mathbf{h})$ lies in $\mathcal{U}$ if $\|\mathbf{h}\| < r$. For each $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, define

$$\mathbf{R}(\mathbf{h}) = \mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x}) - \mathbf{DF}(\mathbf{x})\mathbf{h}.$$

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According to the First-Order Approximation Theorem for Mappings,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{R}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0, \tag{15.26}$$

and, by the definition of $\mathbf{R}(\mathbf{h})$, if $\|\mathbf{h}\| < r$,

$$\mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x}) = \mathbf{DF}(\mathbf{x})\mathbf{h} + \mathbf{R}(\mathbf{h}). \tag{15.27}$$

Now for each $\mathbf{h}$ in $\mathbb{R}^n$ such that $\|\mathbf{h}\| < r$, we can apply the Mean Value Theorem to the function $g : \mathcal{U} \to \mathbb{R}$ on the segment joining the points $\mathbf{F}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x}+\mathbf{h})$ in order to select a point on this segment, which we label $\mathbf{v}(\mathbf{h})$, at which

$$g(\mathbf{F}(\mathbf{x}+\mathbf{h})) - g(\mathbf{F}(\mathbf{x})) = \langle \nabla g(\mathbf{v}(\mathbf{h})),\ \mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x})\rangle.$$

Substituting (15.27) gives

$$(g\circ\mathbf{F})(\mathbf{x}+\mathbf{h}) - (g\circ\mathbf{F})(\mathbf{x}) = \langle \nabla g(\mathbf{v}(\mathbf{h})),\ \mathbf{DF}(\mathbf{x})\mathbf{h}\rangle + \langle \nabla g(\mathbf{v}(\mathbf{h})),\ \mathbf{R}(\mathbf{h})\rangle. \tag{15.28}$$

Observe that the continuity of $\mathbf{F} : \mathcal{O} \to \mathbb{R}^m$ implies that

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{v}(\mathbf{h}) = \mathbf{F}(\mathbf{x}). \tag{15.29}$$

We now verify (15.24). Fix an index i, with $1 \le i \le n$. For a number t such that $0 < |t| < r$, if we define $\mathbf{h} = t\mathbf{e}_i$, then from (15.28) we obtain

$$\frac{(g\circ\mathbf{F})(\mathbf{x}+t\mathbf{e}_i) - (g\circ\mathbf{F})(\mathbf{x})}{t} = \langle \nabla g(\mathbf{v}(t\mathbf{e}_i)),\ \mathbf{DF}(\mathbf{x})\mathbf{e}_i\rangle + \left\langle \nabla g(\mathbf{v}(t\mathbf{e}_i)),\ \frac{\mathbf{R}(t\mathbf{e}_i)}{t}\right\rangle.$$

From this equality, by using (15.26) and (15.29), it follows that

$$\frac{\partial}{\partial x_i}(g\circ\mathbf{F})(\mathbf{x}) = \langle \nabla g(\mathbf{F}(\mathbf{x})),\ \mathbf{DF}(\mathbf{x})\mathbf{e}_i\rangle. \tag{15.30}$$

But

$$\mathbf{DF}(\mathbf{x})\mathbf{e}_i = \left(\frac{\partial \mathrm{F}_1}{\partial x_i}(\mathbf{x}), \ldots, \frac{\partial \mathrm{F}_m}{\partial x_i}(\mathbf{x})\right),$$

so the scalar equation (15.30) is exactly equation (15.24). In particular, this shows that the function $g\circ\mathbf{F} : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives, and then, because of the continuity with respect to $\mathbf{x}$ of the right-hand side of formula (15.24), that $g\circ\mathbf{F} : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. To conclude the proof, simply observe that (15.25) is a rewriting of (15.24) in matrix notation. ∎

We now examine some of the forms of the Chain Rule that most commonly occur.

Example 15.35 Suppose that the functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Suppose also that $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$ and that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Finally, suppose that $(\psi(x, y), \varphi(x, y))$ is in $\mathcal{O}$ for all (x, y) in $\mathbb{R}^2$. Then

$$\begin{aligned}\frac{\partial}{\partial x}(f(\psi(x, y), \varphi(x, y))) &= D_1 f(\psi(x, y), \varphi(x, y))\frac{\partial\psi}{\partial x}(x, y)\\ &\quad + D_2 f(\psi(x, y), \varphi(x, y))\frac{\partial\varphi}{\partial x}(x, y)\end{aligned}$$

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and

$$\frac{\partial}{\partial y}(f(\psi(x, y), \varphi(x, y))) = D_1 f(\psi(x, y), \varphi(x, y))\frac{\partial \psi}{\partial y}(x, y)$$
$$+ D_2 f(\psi(x, y), \varphi(x, y))\frac{\partial \varphi}{\partial y}(x, y). \qquad \blacksquare$$

Example 15.36 Let the function $g : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable and define the function $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = g(x^2, 2x, 1 - x)$ for x in $\mathbb{R}$. Then for each x in $\mathbb{R}$,

$$\psi'(x) = D_1 g(x^2, 2x, 1 - x)(2x)$$
$$+ D_2 g(x^2, 2x, 1 - x)(2) + D_3 g(x^2, 2x, 1 - x)(-1). \qquad \blacksquare$$

Example 15.37 Suppose that each of the functions $u : \mathbb{R}^2 \to \mathbb{R}$, $v : \mathbb{R}^2 \to \mathbb{R}$, and $w : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and that the function $g : \mathbb{R}^3 \to \mathbb{R}$ is also continuously differentiable. Then for each point (x, y) in the plane $\mathbb{R}^2$,

$$\frac{\partial}{\partial x}(g(u(x, y), v(x, y), w(x, y))) = D_1 g(u(x, y), v(x, y), w(x, y))\frac{\partial u}{\partial x}(x, y)$$
$$+ D_2 g(u(x, y), v(x, y), w(x, y))\frac{\partial v}{\partial x}(x, y)$$
$$+ D_3 g(u(x, y), v(x, y), w(x, y))\frac{\partial w}{\partial x}(x, y).$$

$\blacksquare$

Remark on Notation

In books in which there are calculations involving partial derivatives, the reader will find a large variety of notation. For example, the second derivative formula in Example 15.35 is often abbreviated as

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \psi}\frac{\partial \psi}{\partial y} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial y}. \tag{15.31}$$

Similarly, the derivative formula in Example 15.37 is often abbreviated as

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x}. \tag{15.32}$$

As another common instance of terse but useful notational devices, we note that if the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and the function $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(r, \theta) = f(r\cos\theta, r\sin\theta) \qquad \text{for } (r, \theta) \text{ in } \mathbb{R}^2,$$

then according to the Chain Rule, for each point (r, θ) in $\mathbb{R}^2$,

$$\frac{\partial g}{\partial r}(r, \theta) = D_1 f(r\cos\theta, r\sin\theta)\cos\theta + D_2 f(r\cos\theta, r\sin\theta)\sin\theta.$$

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The last formula is frequently abbreviated as

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{15.33}$$

One must carefully interpret this formula in order to understand that it signifies the same thing as its predecessor.

Abbreviated formulas such as (15.31), (15.32), and (15.33) are very useful in compressing long equations and in shortening various calculations. But such formulas are not precise because there is no indication of where the derivatives are to be evaluated and there is ambiguity about what the variables are. Extra care is needed when using them.

When we analyze functions of two or three variables, especially when computing higher derivatives, it is notationally useful to denote

$$D_1 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial x}(\mathbf{p}), \qquad D_2 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial y}(\mathbf{p}), \qquad \text{and} \qquad D_3 g(\mathbf{p}) \text{ by } \frac{\partial g}{\partial z}(\mathbf{p}),$$

even when x, y, and z have not been explicitly introduced as notation for the component variables. In the following example, we use this notational convention.

Example 15.38 A function $u : \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic provided it has continuous second-order partial derivatives that satisfy the identity

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Suppose that the function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Define

$$v(x, y) = u(x^2 - y^2, 2xy) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

Then it turns out that the function $v : \mathbb{R}^2 \to \mathbb{R}$ is also harmonic. To verify this, we must show that

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

However, for (x, y) in $\mathbb{R}^2$,

$$\frac{\partial v}{\partial x}(x, y) = \frac{\partial u}{\partial x}(x^2 - y^2, 2xy)2x + \frac{\partial u}{\partial y}(x^2 - y^2, 2xy)2y,$$

so

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2}(x, y) = {} & \frac{\partial^2 u}{\partial x^2}(x^2 - y^2, 2xy)4x^2 + \frac{\partial u}{\partial x}(x^2 - y^2, 2xy)2 \\ & + \frac{\partial^2 u}{\partial x \partial y}(x^2 - y^2, 2xy)8xy + \frac{\partial^2 u}{\partial y^2}(x^2 - y^2, 2xy)4y^2.\end{aligned}$$

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We carry out a similar computation for $\partial^2 v/\partial y^2(x, y)$, and since

$$\frac{\partial^2 u}{\partial x^2}(x^2 - y^2, 2xy) + \frac{\partial^2 u}{\partial y^2}(x^2 - y^2, 2xy) = 0 \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2,$$

a calculation that is left as Exercise 4 shows that

$$\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial y^2}(x, y) = 0 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2. \qquad \blacksquare$$

The special case of the Chain Rule that we have just proved leads to the proof of the general case.

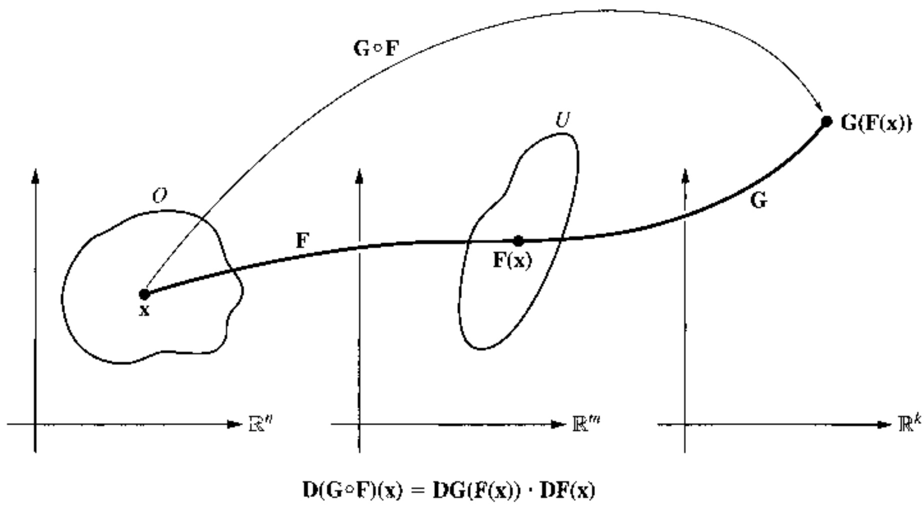

FIGURE 15.3 The composition of the mapping $\mathbf{F}$ with the mapping $\mathbf{G}$.

Theorem 15.39 The Chain Rule for General Mappings Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^m$ is continuously differentiable. Suppose also that $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ and that the mapping $\mathbf{G}: \mathcal{U} \to \mathbb{R}^k$ is continuously differentiable. Finally, suppose that $\mathbf{F}(\mathcal{O})$ is contained in $\mathcal{U}$. Then the composite mapping $\mathbf{G} \circ \mathbf{F}: \mathcal{O} \to \mathbb{R}^k$ is also continuously differentiable. Moreover, for each point $\mathbf{x}$ in $\mathcal{O}$,

$$\mathbf{D}(\mathbf{G} \circ \mathbf{F})(\mathbf{x}) = \mathbf{D}\mathbf{G}(\mathbf{F}(\mathbf{x})) \cdot \mathbf{D}\mathbf{F}(\mathbf{x}). \tag{15.34}$$

Proof

Represent the mapping $\mathbf{G}$ in component functions by $\mathbf{G} = (G_1, \ldots, G_k)$. Then observe that the composition $\mathbf{G} \circ \mathbf{F}: \mathcal{O} \to \mathbb{R}^k$ is represented in component functions by $\mathbf{G} \circ \mathbf{F} = (G_1 \circ \mathbf{F}, G_2 \circ \mathbf{F}, \ldots, G_k \circ \mathbf{F})$. For an index j such that $1 \le j \le k$, and that the component function $G_j : \mathcal{U} \to \mathbb{R}$ is continuously differentiable it follows from Theorem 15.34 that for all points $\mathbf{x}$ in $\mathcal{O}$,

$$\nabla(G_j \circ \mathbf{F})(\mathbf{x}) = \nabla G_j(\mathbf{F}(\mathbf{x}))\mathbf{D}\mathbf{F}(\mathbf{x}).$$

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This formula is an assertion of the equality of the jth rows of each of the matrices in formula (15.34) for $1 \leq j \leq k$. Thus, the matrix formula (15.34) holds. Therefore, the composition $\mathbf{G} \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}^k$ has first-order partial derivatives at each point, and from the continuity of the entries on the right-hand side of (15.34) we conclude that the composition is continuously differentiable. ∎

EXERCISES FOR SECTION 15.3

1. Suppose that the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable. Define the function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(s, t) = \psi(s^2 t, s) \qquad \text{for } (s, t) \text{ in } \mathbb{R}^2.$$

Find $\partial g/\partial s(s, t)$ and $\partial g/\partial t(s, t)$.

2. Suppose that the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Define the function $\eta : \mathbb{R}^3 \to \mathbb{R}$ by

$$\eta(u, v, w) = (3u + 2v)h(u^2, v^2, uvw) \qquad \text{for } (u, v, w) \text{ in } \mathbb{R}^3.$$

Find $D_1\eta(u, v, w)$, $D_2\eta(u, v, w)$, and $D_3\eta(u, v, w)$.

3. Suppose that the functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ have continuous second-order partial derivatives. Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(s, t) = g(s - t) + h(s + t) \qquad \text{for } (s, t) \text{ in } \mathbb{R}^2.$$

Prove that

$$\frac{\partial^2 u}{\partial t^2}(s, t) - \frac{\partial^2 u}{\partial s^2}(s, t) = 0 \qquad \text{for all } (s, t) \text{ in } \mathbb{R}^2.$$

4. Carry out the calculations needed in Example 15.38 in order to verify that the function $v : \mathbb{R}^2 \to \mathbb{R}$ is harmonic.

5. Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and let the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^2$ be represented by $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ for (x, y) in $\mathcal{O}$. Then the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^2$ is called a *Cauchy–Riemann mapping* provided that each of the functions $u : \mathcal{O} \to \mathbb{R}$ and $v : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives and

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) \qquad \text{for all } (x, y) \text{ in } \mathcal{O}.$$

Prove that if the function $w : \mathbb{R}^2 \to \mathbb{R}$ is harmonic and the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^2$ is a Cauchy–Riemann mapping, then the function $w \circ \mathbf{F} : \mathcal{O} \to \mathbb{R}$ is also harmonic.

6. Suppose that the function $w : \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Use Exercise 5 to show that each of the following functions is also harmonic:
   a. $v : \mathbb{R}^2 \to \mathbb{R}$ defined by $v(x, y) = w(e^x \cos y, e^x \sin y)$ for (x, y) in $\mathbb{R}^2$
   b. $v : \mathbb{R}^2 \to \mathbb{R}$ defined by $v(x, y) = w(x^2 - y^2, 2xy)$ for (x, y) in $\mathbb{R}^2$
   c. $v : \mathcal{O} \to \mathbb{R}$ defined by $v(x, y) = w(x/(x^2 + y^2), -y/(x^2 + y^2))$ for (x, y) in $\mathcal{O}$, where $\mathcal{O} = \{(x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 > 0\}$

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7. Suppose that the function $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic. Let a, b, c, and d be real numbers such that

$$a^2 + b^2 = 1, \quad c^2 + d^2 = 1, \qquad \text{and} \qquad ac + bd = 0.$$

Define the function $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$v(x, y) = u(ax + by, cx + dy) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that the function $v : \mathbb{R}^2 \to \mathbb{R}$ is also harmonic.

8. Suppose that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ have continuous second-order partial derivatives. Also suppose that there is a number λ such that

$$f''(x) = \lambda f(x) \qquad \text{and} \qquad g''(x) = \lambda g(x) \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = f(x)g(y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that

$$\frac{\partial^2 u}{\partial x^2}(x, y) - \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \qquad \text{for every } (x, y) \text{ in } \mathbb{R}^2.$$

9. Let $\mathcal{O} = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 + y^2 + z^2 > 0\}$ and define the function $u : \mathcal{O} \to \mathbb{R}$ by

$$u(\mathbf{p}) = \frac{1}{\|\mathbf{p}\|} \qquad \text{for } \mathbf{p} \text{ in } \mathcal{O}.$$

Prove that

$$\frac{\partial^2 u}{\partial x^2}(x, y, z) + \frac{\partial^2 u}{\partial y^2}(x, y, z) + \frac{\partial^2 u}{\partial z^2}(x, y, z) = 0 \qquad \text{for every } (x, y, z) \text{ in } \mathcal{O}.$$

10. Suppose that the functions $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable. Express the following two limits in terms of partial derivatives of these functions:

a. $\displaystyle \lim_{t \to 0} \frac{f(g(1+t, 2), h(1+t, 2)) - f(g(1, 2), h(1, 2))}{t}$

b. $\displaystyle \lim_{t \to 0} \frac{f(g(1, 2) + t, h(1, 2)) - f(g(1, 2), h(1, 2))}{t}$

11. Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. For points $\mathbf{x}$ and $\mathbf{p}$ in $\mathbb{R}^n$, the Directional Derivative Theorem asserts that if $\psi(t) = g(\mathbf{x} + t\mathbf{p})$ for t in $\mathbb{R}$, then

$$\psi'(t) = \langle \nabla g(\mathbf{x} + t\mathbf{p}), \mathbf{p} \rangle \qquad \text{for every } t \text{ in } \mathbb{R}.$$

Show that this formula is a special case of the Chain Rule.

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CHAPTER

16

IMAGES AND INVERSES: THE INVERSE FUNCTION THEOREM

16.1 FUNCTIONS OF A SINGLE VARIABLE AND MAPS IN THE PLANE

Suppose that $\mathcal{O}$ is an open subset of Euclidean space $\mathbb{R}^n$ and that the nonlinear mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^n$ is continuously differentiable. At a point x_* in $\mathcal{O}$, suppose that the differential

$$\mathbf{dF}(\mathrm{x}_*) : \mathbb{R}^n \to \mathbb{R}^n \text{ is one-to-one and onto;}$$

this is equivalent to the assumption that the determinant of the derivative matrix is nonzero; that is,

$$\det \left[\frac{\partial F_i}{\partial x_j}(\mathrm{x}_*) \right] \neq 0.$$

Then it turns out that the nonlinear mapping inherits "local invertibility" near the point x_*, in the precise sense that there is an open subset U of $\mathbb{R}^n$ containing the point x_* and an open subset V of $\mathbb{R}^n$ containing its image $\mathbf{F}(\mathrm{x}_*)$ such that

$$\mathbf{F} : U \to V \text{ is one-to-one and onto}$$

and the inverse $\mathbf{F}^{-1} : V \to U$ is also continuously differentiable. This is the Inverse Function Theorem. The proof of this theorem provides the opportunity to introduce a number of ideas that are new and of independent interest. This chapter is devoted to describing these ideas, proving this theorem, and considering examples of this theorem.

In this first section we prove the Inverse Function Theorem for functions of a single variable and state the theorem for maps in the plane; several examples are considered. In Section 16.2, we will introduce the concept of stability of a nonlinear mapping. A linear mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ is stable if and only if it is invertible, and stability in an open set containing the point x_* is the important analytic property that a continuously differentiable nonlinear mapping inherits from the invertibility of its differential $\mathbf{dF}(\mathrm{x}_*) : \mathbb{R}^n \to \mathbb{R}^n$. In Section 16.3, we will first introduce a Minimization Principle to

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show that a continuously differentiable stable mapping maps open sets to open sets. We will then state, prove, and discuss the General Inverse Function Theorem.

As motivation for the subject of the present chapter, we begin by proving the following theorem about real-valued functions of a single real variable.

Theorem 16.1 Let $\mathcal{O}$ be an open subset of $\mathbb{R}$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ is continuously differentiable. Let x_0 be a point in $\mathcal{O}$ at which

$$f'(x_0)\neq 0. \tag{16.1}$$

Then there is an open interval I containing the point x_0 and an open interval J containing its image $f(x_0)$ such that the function

$$f:I\to J \text{ is one-to-one and onto.} \tag{16.2}$$

Moreover, the inverse function $f^{-1}:J\to I$ is also continuously differentiable, and for a point y in J, if x is the point in I at which $f(x)=y$, then

$$(f^{-1})'(y)=\frac{1}{f'(x)}. \tag{16.3}$$

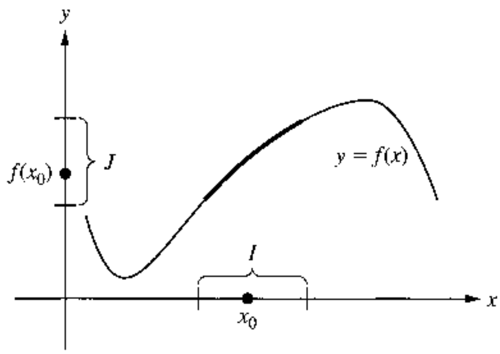

FIGURE 16.1 The inverse function of a function of a single variable.

Proof

We suppose that $f'(x_0)>0$. Since x_0 is an interior point of $\mathcal{O}$ and the function $f':\mathcal{O}\to\mathbb{R}$ is continuous, we can select a positive number r such that the closed interval $[x_0-r,\ x_0+r]$ is contained in $\mathcal{O}$ and $f'(x)>0$ for all points x in the interval $[x_0-r,\ x_0+r]$. The Mean Value Theorem implies that the function $f:[x_0-r,\ x_0+r]\to\mathbb{R}$ is strictly increasing. In particular, $f:[x_0-r,\ x_0+r]\to\mathbb{R}$ is one-to-one. Moreover, by the Intermediate Value Theorem, if the point y lies strictly between $f(x_0-r)$ and $f(x_0+r)$, then there is some point x in the open interval $(x_0-r,\ x_0+r)$ with $f(x)=y$.

Define $I=(x_0-r,\ x_0+r)$ and $J=(f(x_0-r),\ f(x_0+r))$. Then the function $f:I\to J$ is one-to-one and onto. The differentiability of the inverse and formula (16.3) have already been proved in Theorem 4.11. ∎

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The aim of this chapter is to describe a circle of ideas related to the way in which we can extend Theorem 16.1 to a result for mappings $\mathbf{F}: \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of Euclidean space $\mathbb{R}^n$. Before we describe the extension to maps in the plane, it is useful to introduce the concept of neighborhood of a point that generalizes the concept of an open ball about a point.

Definition An open subset of Euclidean space $\mathbb{R}^n$ that contains the point $\mathbf{x}$ is called a *neighborhood* of the point $\mathbf{x}$.

For mappings of an open subset of the plane $\mathbb{R}^2$ into $\mathbb{R}^2$, we state the following theorem.

Theorem 16.2 The Inverse Function Theorem in the Plane Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^2$ is continuously differentiable. Let (x_0, y_0) be a point in $\mathcal{O}$ at which the derivative matrix

$$\mathbf{DF}(x_0, y_0) \text{ is invertible.} \tag{16.4}$$

Then there is a neighborhood U of the point (x_0, y_0) and a neighborhood V of its image $\mathbf{F}(x_0, y_0)$ such that

$$\mathbf{F}: U \to V \text{ is one-to-one and onto.} \tag{16.5}$$

Moreover, the inverse mapping $\mathbf{F}^{-1}: V \to U$ is also continuously differentiable, and for a point (u, v) in V, if (x, y) is the point in U at which $\mathbf{F}(x, y) = (u, v)$, then the derivative matrix of the inverse mapping at the point (u, v) is given by the formula

$$\mathbf{DF}^{-1}(u, v) = [\mathbf{DF}(x, y)]^{-1}. \tag{16.6}$$

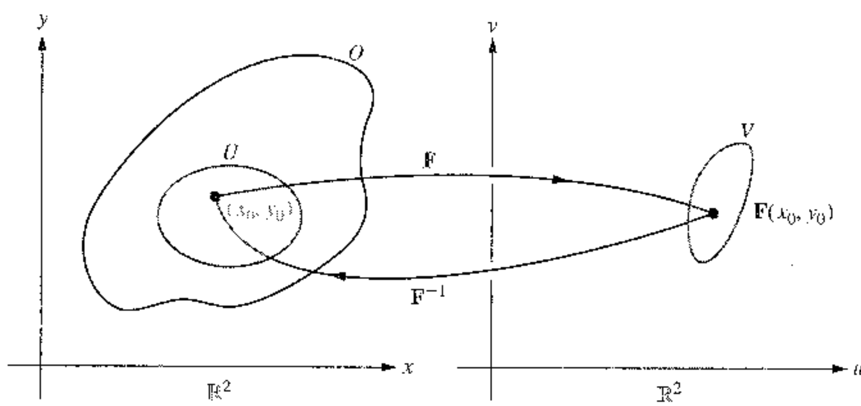

FIGURE 16.2 A mapping in the plane and its inverse mapping.

Observe that in the proof of Theorem 16.1, we used the Intermediate Value Theorem, a result that does not easily generalize to mappings whose image lies in the plane $\mathbb{R}^2$.

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The proof of the Inverse Function Theorem in the Plane has to be quite different. In the next two sections, we will discuss certain ideas that are of independent interest and that we will use in Section 16.3 to prove the General Inverse Function Theorem, a result that contains Theorem 16.2 as a special case. We devote the rest of this section to a discussion of the Inverse Function Theorem in the Plane.

As discussed in Section 15.1, an $n \times n$ matrix is *invertible* if and only if its determinant is nonzero, and when the matrix is invertible, there is a formula called Cramer's Rule for the inverse matrix. For 2×2 matrices, Cramer's Rule is clear by inspection. Indeed, for a 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

if

$$\det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

then a straightforward multiplication of matrices establishes the validity of the following formula for the inverse of $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

In particular, for the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^2$ in the statement of the Inverse Function Theorem in the Plane, assumption (16.4) holds if and only if

$$\det \mathbf{DF}(x_0, y_0) \neq 0.$$

If the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^2$ is represented in component functions as

$$\mathbf{F}(x, y) = (\psi(x, y), \phi(x, y)) \qquad \text{for } (x, y) \text{ in } \mathcal{O},$$

then

$$\mathbf{DF}(x, y) = \begin{bmatrix} \partial\psi/\partial x(x, y) & \partial\psi/\partial y(x, y) \\ \partial\phi/\partial x(x, y) & \partial\phi/\partial y(x, y) \end{bmatrix},$$

so assumption (16.4) is equivalent to the assumption that

$$\frac{\partial\psi}{\partial x}(x_0, y_0)\frac{\partial\phi}{\partial y}(x_0, y_0) - \frac{\partial\psi}{\partial y}(x_0, y_0)\frac{\partial\phi}{\partial x}(x_0, y_0) \neq 0. \tag{16.7}$$

The above explicit formula for the inverse of a 2×2 matrix permits us to use formula (16.6) to compute the partial derivatives of the component functions of the inverse mapping $\mathbf{F}^{-1}: V \to U$. Indeed, write the inverse mapping $\mathbf{F}^{-1}: V \to U$ in component functions as

$$\mathbf{F}^{-1}(u, v) = (g(u, v), h(u, v)) \qquad \text{for } (u, v) \text{ in } V,$$

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so that

$$\mathbf{DF}^{-1}(u, v) = \begin{bmatrix} \partial g/\partial u(u, v) & \partial g/\partial v(u, v) \\ \partial h/\partial u(u, v) & \partial h/\partial v(u, v) \end{bmatrix}.$$

For a point (u, v) in V, let (x, y) be the point in U at which

$$u = \psi(x, y) \qquad \text{and} \qquad v = \phi(x, y).$$

For notational convenience,[1] set

$$\mathrm{J}(x, y) \equiv \det \mathbf{DF}(x, y) = \frac{\partial \psi}{\partial x}(x, y)\frac{\partial \phi}{\partial y}(x, y) - \frac{\partial \psi}{\partial y}(x, y)\frac{\partial \phi}{\partial x}(x, y).$$

Then, using the above computation of the inverse of a 2×2 matrix, it follows that formula (16.6) is equivalent to

$$\begin{aligned} \frac{\partial g}{\partial u}(u, v) &= \frac{1}{\mathrm{J}(x, y)} \cdot \frac{\partial \phi}{\partial y}(x, y); \\ \frac{\partial g}{\partial v}(u, v) &= -\frac{1}{\mathrm{J}(x, y)} \cdot \frac{\partial \psi}{\partial y}(x, y); \\ \frac{\partial h}{\partial u}(u, v) &= -\frac{1}{\mathrm{J}(x, y)} \cdot \frac{\partial \phi}{\partial x}(x, y); \\ \frac{\partial h}{\partial v}(u, v) &= \frac{1}{\mathrm{J}(x, y)} \cdot \frac{\partial \psi}{\partial x}(x, y). \end{aligned} \tag{16.8}$$

Example 16.3 For a point (x, y) in the plane $\mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^{x-y} + x^2y + x(y-1)^4,\ 1 + x^2 + x^4 + (xy)^5).$$

Again, in this example, the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable since each of its component functions is continuously differentiable. At the point $(x_0, y_0) = (1, 1)$, a short computation of partial derivatives shows that

$$\mathbf{DF}(1, 1) = \begin{bmatrix} 3 & 0 \\ 11 & 5 \end{bmatrix}.$$

The determinant of $\mathbf{DF}(1, 1)$ is nonzero. We can use the Inverse Function Theorem to conclude that there are neighborhoods U of the point $(1, 1)$ and V of its image $(2, 4)$ such that the mapping $\mathbf{F}: U \to V$ is one-to-one and onto and that the inverse mapping $\mathbf{F}^{-1}: V \to U$ is also continuously differentiable. Moreover, if the inverse is represented in components as $\mathbf{F}^{-1}(u, v) = (g(u, v), h(u, v))$, then from (16.8) it follows that

$$\frac{\partial g}{\partial u}(2, 4) = \frac{1}{3}, \qquad \frac{\partial g}{\partial v}(2, 4) = 0, \qquad \frac{\partial h}{\partial u}(2, 4) = \frac{-11}{15}, \qquad \text{and} \qquad \frac{\partial h}{\partial v}(2, 4) = \frac{1}{5}.$$

■

[1] We use the letter J here because the determinant of the derivative matrix at a point is sometimes referred to as the *Jacobian determinant*. The derivative matrix is often called the *Jacobian matrix*.

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Example 16.4 For a point $(x,\ y)$ in the plane $\mathbb{R}^2$, define

$$\mathbf{F}(x,\ y) = (x^2 - y^2,\ 2xy).$$

This mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is also continuously differentiable since each of its component functions is obviously continuously differentiable. First, consider a nonzero point $(x_0,\ y_0)$ in $\mathbb{R}^2$. We have

$$\mathbf{DF}(x_0,\ y_0) = \begin{bmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{bmatrix},$$

so $\det \mathbf{DF}(x_0,\ y_0) = 4(x_0^2 + y_0^2) \neq 0$. Once more applying the Inverse Function Theorem, it follows that there are neighborhoods U of the point $(x_0,\ y_0)$ and V of its image $(x_0^2 - y_0^2,\ 2x_0y_0)$ such that the mapping $\mathbf{F}: U \to V$ is one-to-one and onto and has an inverse $\mathbf{F}^{-1}: V \to U$ that also is continuously differentiable. If the inverse is represented in component functions as $\mathbf{F}^{-1}(u,\ v) = (g(u,\ v),\ h(u,\ v))$, then if we set $(u_0,\ v_0) = (x_0^2 - y_0^2,\ 2x_0y_0)$, from (16.8) it follows that

$$\frac{\partial g}{\partial u}(u_0,\ v_0) = \frac{x_0}{2(x_0^2 + y_0^2)}, \qquad \frac{\partial g}{\partial v}(u_0,\ v_0) = \frac{y_0}{2(x_0^2 + y_0^2)},$$
$$\frac{\partial h}{\partial u}(u_0,\ v_0) = \frac{-y_0}{2(x_0^2 + y_0^2)}, \qquad \frac{\partial h}{\partial v}(u_0,\ v_0) = \frac{x_0}{(x_0^2 + y_0^2)}.$$

Now consider the point $(x_0,\ y_0) = (0, 0)$. The assumptions of the Inverse Function Theorem certainly fail at this point since

$$\mathbf{DF}(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, the conclusion of the Inverse Function Theorem also fails at this point because if we observe that $\mathbf{F}(x,\ y) = \mathbf{F}(-x,\ -y)$ at all points $(x,\ y)$ in the plane, there is no neighborhood of the point $(0, 0)$ on which the mapping is one-to-one. ■

Example 16.5 Define $\mathbf{F}(x,\ y) = (\cos(x + y^2),\ \sin(x + y^2))$ for $(x,\ y)$ in $\mathbb{R}^2$. Then the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. But there is no point at which the conclusion of the Inverse Function Theorem is true. To see this, observe that if the point $(u,\ v)$ lies in the image of $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$, then $u^2 + v^2 = 1$. Thus, the image lies on the circle of radius 1 centered at the origin. In particular, the image does not contain an open subset of the plane, so there certainly are no open sets U and V in the plane such that $\mathbf{F}: U \to V$ is one-to-one and onto. ■

It is useful to note that the conclusion of the Inverse Function Theorem in the Plane can be refined slightly. In the conclusion of the theorem, it is asserted that the sets U and V are neighborhoods of the point $(x_0,\ y_0)$ and of its image $\mathbf{F}(x_0,\ y_0)$, respectively. In fact, we can choose V to be an open ball about the point $\mathbf{F}(x_0,\ y_0)$. To see why this is so, first recall that a continuously differentiable mapping is continuous and that, by Theorem 11.12, the inverse image of an open set by a continuous map is again an

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open set. Now since V is open, we can choose a positive number r such that the open ball $\mathcal{B}_r(\mathbf{F}(x_0, y_0))$ is contained in V. Denote $\mathcal{B}_r(\mathbf{F}(x_0, y_0))$ by V' and define $U' = \mathbf{F}^{-1}(V') \cap U$. Then U' is a neighborhood of (x_0, y_0). Thus, replacing U by U' and V by V', U' is a neighborhood of (x_0, y_0) and V' is a neighborhood of $\mathbf{F}(x_0, y_0)$, and the mapping $\mathbf{F}: U' \to V'$ is one-to-one and onto, but now V' is an open ball (Exercise 12).

The Inverse Function Theorem has an interpretation as a statement about the solvability of systems of equations. Given two functions $\psi: \mathbb{R}^2 \to \mathbb{R}$ and $\phi: \mathbb{R}^2 \to \mathbb{R}$ and two numbers a and b, consider the system of equations

$$\begin{aligned} \psi(x, y) &= a \\ \phi(x, y) &= b. \end{aligned} \tag{16.9}$$

We can ask whether there are any solutions of this system of equations and, if there is a solution, whether there is only one solution. If we define the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (\psi(x, y), \phi(x, y))$ for (x, y) in $\mathbb{R}^2$, these two questions about the existence and the uniqueness of the solutions of system (16.9) can be rephrased as questions about the image of the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ and whether it has the property of being one-to-one. The following example shows how the Inverse Function Theorem provides information about systems of equations.

Example 16.6 Consider the system of equations

$$\begin{aligned} e^{x-y} + x^2y + x(y-1)^4 &= 2 \\ 1 + x^2 + x^4 + (xy)^5 &= 4. \end{aligned} \tag{16.10}$$

Observe that the point $(x, y) = (1, 1)$ is a solution of this system. The mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = (e^{x-y} + x^2y + x(y-1)^4, 1 + x^2 + x^4 + (xy)^5)$ for (x, y) in $\mathbb{R}^2$ is precisely the mapping considered in Example 16.3. From the analysis in that example, we conclude that there is a positive number r and a neighborhood U of the point $(1, 1)$ such that for any numbers a and b with $(a-2)^2 + (b-4)^2 < r^2$, the system of equations

$$\begin{aligned} e^{x-y} + x^2y + x(y-1)^4 &= a \\ 1 + x^2 + x^4 + (xy)^5 &= b, \qquad (x, y) \text{ in } U, \end{aligned}$$

has exactly one solution. ■

EXERCISES FOR SECTION 16.1

1. Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 - 3x + 1 \qquad \text{for } x \text{ in } \mathbb{R}.$$

At what points x in $\mathbb{R}$ does the Inverse Function Theorem apply?

2. Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 + x + \cos x \qquad \text{for } x \text{ in } \mathbb{R}.$$

At what points x in $\mathbb{R}$ does the Inverse Function Theorem apply? Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is one-to-one and onto.

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3. Suppose that the function $f:\mathbb{R}\to\mathbb{R}$ is differentiable and is one-to-one and onto. Suppose that $f(1)=0$, $f(0)=1$, $f'(0)=-4$, and $f'(1)=-10$. Find $(f^{-1})'(1)$ and $(f^{-1})'(0)$.
4. a. Give an example of a continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ that is one-to-one but not onto.
   b. Provide an example of a continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ that is onto but not one-to-one.
5. Suppose that the continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$ has the property that there is some positive number c such that
$$f'(x)\ge c \qquad \text{for every } x \text{ in } \mathbb{R}.$$
Show that the function $f:\mathbb{R}\to\mathbb{R}$ is both one-to-one and onto.
6. Define $f(x)=x^3$ for x in $\mathbb{R}$. Show that the function $f:\mathbb{R}\to\mathbb{R}$ is one-to-one and onto and that its inverse $f^{-1}:\mathbb{R}\to\mathbb{R}$ is continuous. At what points is the inverse differentiable?
7. Let $\mathcal{O}$ and V be open subsets of $\mathbb{R}$ and suppose that the differentiable function $f:\mathcal{O}\to V$ is one-to-one and onto. Suppose that x_0 is a point in $\mathcal{O}$ at which $f'(x_0)=0$. Show that the inverse function $f^{-1}:V\to\mathbb{R}$ cannot be differentiable at the point $f(x_0)$. [*Hint:* Argue by contradiction and use the Chain Rule to differentiate both sides of the following identity: $f^{-1}(f(x))=x$ for x in $\mathcal{O}$.]
8. For each of the following mappings $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$, apply the Inverse Function Theorem at the point $(x_0,\ y_0)=(0,0)$ and calculate the partial derivatives of the components of the inverse mapping at the point $(u_0,\ v_0)=\mathbf{F}(0,0)$:
   a. $\mathbf{F}(x,\ y)=(x+x^2+e^{x^2y^2},\ -x+y+\sin(xy))$ for $(x,\ y)$ in $\mathbb{R}^2$
   b. $\mathbf{F}(x,\ y)=(e^{x+y},\ e^{x-y})$ for $(x,\ y)$ in $\mathbb{R}^2$
9. Define the function $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$ by
$$\mathbf{F}(x,\ y)=(e^x\cos y,\ e^x\sin y) \qquad \text{for } (x,\ y) \text{ in } \mathbb{R}^2.$$
   a. Show that the Inverse Function Theorem is applicable at every point $(x_0,\ y_0)$ in the plane $\mathbb{R}^2$.
   b. Show that the function $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$ is not one-to-one.
   c. Does (b) contradict (a)?
10. Define the mapping $\mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2$ by
$$\mathbf{F}(r,\ \theta)=(r\cos\theta,\ r\sin\theta) \qquad \text{for } (r,\ \theta) \text{ in } \mathbb{R}^2.$$
   a. At what points $(r_0,\ \theta_0)$ in $\mathbb{R}^2$ can we apply the Inverse Function Theorem to this mapping?
   b. Find some explicit formula for the local inverse about the point $(r,\ \theta)=(1,\ \pi/2)$. (*Hint:* The local inverse corresponds to the assignment of polar coordinates.)
11. For a pair of real numbers a and b, consider the system of nonlinear equations
$$\begin{aligned} x+x^2\cos y+xye^{x^3y^2} &= a \\ y+x^5+y^3-x^2\cos(xy) &= b. \end{aligned}$$

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Use the Inverse Function Theorem to show that there is some positive number r such that if $a^2 + b^2 < r^2$, then this system of equations has at least one solution.

12. We observed that the conclusion of the Inverse Function Theorem could be refined in that the neighborhood V could be chosen to be an open ball. Use a similar argument to show that it is possible to choose the neighborhood U to be an open ball. Show that it is generally not possible to choose both U and V simultaneously to be open balls. [*Hint:* Consider the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = (x, 2y)$ for (x, y) in $\mathbb{R}^2$.]

13. Let the continuously differentiable mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ be represented in component functions by $\mathbf{F}(x, y) = (\psi(x, y), \varphi(x, y))$ for (x, y) in $\mathbb{R}^2$. Suppose that the point (x_0, y_0) in $\mathbb{R}^2$ has the property that

$$\psi(x, y) \geq \psi(x_0, y_0) \qquad \text{for all } (x, y) \text{ in } \mathbb{R}^2.$$

a. Explain analytically why the hypotheses of the Inverse Function Theorem cannot hold at (x_0, y_0).

b. Explain geometrically why the conclusion of the Inverse Function Theorem cannot hold at (x_0, y_0).

14. Suppose that the function $\psi: \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and define the mapping $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{F}(x, y) = (\psi(x, y), -\psi(x, y)) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

a. Explain analytically why the hypotheses of the Inverse Function Theorem fail at each point (x_0, y_0) in $\mathbb{R}^2$.

b. Explain geometrically why the conclusion of the Inverse Function Theorem must fail at each point (x_0, y_0) in $\mathbb{R}^2$.

16.2 STABILITY OF NONLINEAR MAPPINGS

In this section we study the concept of *stability* for nonlinear mappings. To motivate this, we first restate the following characterization of invertibility for an $n \times n$ matrix (Theorem 15.21).

Theorem 16.7 For an $n \times n$ matrix $\mathbf{A}$, the following two assertions are equivalent:

i. The matrix $\mathbf{A}$ is invertible.

ii. There is a positive number c such that

$$\|\mathbf{A}\mathbf{h}\| \geq c\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n.$$

As discussed in Section 15.1, for a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ associated with the $n \times n$ matrix $\mathbf{A}$, the assertion that the mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is invertible is equivalent to the assertion that the matrix $\mathbf{A}$ is an invertible matrix. Thus, if we set $\mathbf{h} = \mathbf{u} - \mathbf{v}$ and observe that, because of linearity, $\mathbf{T}(\mathbf{u} - \mathbf{v}) = \mathbf{T}(\mathbf{u}) - \mathbf{T}(\mathbf{v})$, it follows from the above theorem that a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if there is a positive

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number c such that

$$\|\mathbf{T}(\mathbf{u}) - \mathbf{T}(\mathbf{v})\| \geq c\|\mathbf{u} - \mathbf{v}\| \qquad \text{for all points } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathbb{R}^n.$$

This leads us to make the following definition for general mappings.

Definition Let $\mathcal{O}$ be a subset of $\mathbb{R}^n$. Then a mapping $\mathbf{F}:\mathcal{O} \to \mathbb{R}^n$ is called *stable* provided that there is some positive number c, called a *stability constant* for the mapping, such that

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})\| \geq c\|\mathbf{u} - \mathbf{v}\| \qquad \text{for all points } \mathbf{u} \text{ and } \mathbf{v} \text{ in } \mathcal{O}. \tag{16.11}$$

The main result of this section is that if a continuously differentiable mapping has an invertible derivative matrix at a point, then there is a neighborhood of that point on which the mapping is stable and on which at each point the derivative matrix is invertible. To prove this, it is useful first to establish the following result about perturbations of invertible matrices.

Lemma 16.8 Let $\mathbf{A}$ be an invertible $n \times n$ matrix and let c be a positive number such that

$$\|\mathbf{A}\mathbf{h}\| \geq c\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n.$$

If $\mathbf{B}$ is an $n \times n$ matrix such that $\|\mathbf{B} - \mathbf{A}\| \leq c/2$, then

$$\|\mathbf{B}\mathbf{h}\| \geq \frac{c}{2}\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n. \tag{16.12}$$

In particular, if for each pair of indices i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$,

$$|a_{ij} - b_{ij}| \leq \frac{c}{2n},$$

then (16.12) holds.

Proof

Let $\mathbf{h}$ be a point in $\mathbb{R}^n$. Since

$$\mathbf{A}\mathbf{h} = \mathbf{B}\mathbf{h} + [\mathbf{A} - \mathbf{B}]\mathbf{h},$$

from the Triangle Inequality it follows that

$$\|\mathbf{A}\mathbf{h}\| \leq \|\mathbf{B}\mathbf{h}\| + \|[\mathbf{A} - \mathbf{B}]\mathbf{h}\|,$$

and therefore, by the Generalized Cauchy–Schwarz Inequality,

$$\|\mathbf{A}\mathbf{h}\| \leq \|\mathbf{B}\mathbf{h}\| + \|\mathbf{A} - \mathbf{B}\| \cdot \|\mathbf{h}\|.$$

But, by assumption,

$$c\|\mathbf{h}\| \leq \|\mathbf{A}\mathbf{h}\| \qquad \text{and} \qquad \|\mathbf{A} - \mathbf{B}\| < c/2.$$

Thus,

$$\|\mathbf{B}h\| \geq c/2.$$

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Finally, if the absolute value of each entry of $\mathbf{B} - \mathbf{A}$ is less than $c/2n$, then by the very definition of the norm of a matrix,

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{\sum_{1 \le i, j \le n} (a_{ij} - b_{ij})^2} \le \frac{c}{2},$$

since $\sum_{1 \le i, j \le n} (a_{ij} - b_{ij})^2$ is the sum of n^2 terms each of which is less than $c^2/4n^2$. ■

Theorem 16.9 The Nonlinear Stability Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^n$ is continuously differentiable. Suppose that x_* is a point in $\mathcal{O}$ at which the derivative matrix

$$\mathbf{DF}(\mathrm{x}_*) \text{ is invertible.}$$

Then there is a neighborhood U of x_* such that

i. The mapping $\mathbf{F} : U \to \mathbb{R}^n$ is stable.
ii. The derivative matrix $\mathbf{DF}(\mathbf{x})$ is invertible at each point $\mathbf{x}$ in U.

Proof
Since the matrix $\mathbf{DF}(\mathrm{x}_*)$ is invertible, by Theorem 16.7 we can choose a positive number c such that

$$\|\mathbf{DF}(\mathrm{x}_*)\mathbf{h}\| \ge c\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n. \tag{16.13}$$

Because the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^n$ is continuously differentiable, we can select a positive number r such that the open ball $U = \mathcal{B}_r(\mathrm{x}_*)$ is contained in $\mathcal{O}$ and has the property that for each point $\mathbf{z}$ in U and each pair of indices i and j with $1 \le i \le n$ and $1 \le j \le n$,

$$\left| \frac{\partial F_i}{\partial x_j}(\mathbf{z}) - \frac{\partial F_i}{\partial x_j}(\mathrm{x}_*) \right| < \frac{c}{2n}. \tag{16.14}$$

The estimates (16.13) and (16.14), together with Lemma 16.8, imply that if $\mathbf{B}$ is any $n \times n$ matrix that, for a pair of indices i and j with $1 \le i \le n$ and $1 \le j \le n$, have

$$b_{ij} = \frac{\partial \mathbf{F}_i}{\partial x_j}(\mathbf{z}_{ij}) \qquad \text{for some point } \mathbf{z}_{ij} \text{ in } U, \tag{16.15}$$

then

$$\|\mathbf{Bh}\| \ge \frac{c}{2}\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n.$$

To show that $\mathbf{F} : U \to \mathbb{R}^n$ is stable, let $\mathbf{u}$ and $\mathbf{v}$ be distinct points in U. According to the Mean Value Theorem for Mappings, there are n points $\mathbf{z}_1, \ldots, \mathbf{z}_n$ lying on the segment between the points $\mathbf{u}$ and $\mathbf{v}$ such that if $\mathbf{B}$ is the $n \times n$ matrix whose ith row is $\nabla \mathbf{F}_i(\mathbf{z}_i)$ for each index i with $1 \le i \le n$, then

$$\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v}) = \mathbf{B}(\mathbf{u} - \mathbf{v}).$$

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But $\mathbf{B}$ is a matrix of the type just described by (16.15), in which $\mathbf{z}_{ij} = \mathbf{z}_i$. Thus,

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})\| = \|\mathbf{B}(\mathbf{u} - \mathbf{v})\| \geq \frac{c}{2}\|\mathbf{u} - \mathbf{v}\|.$$

This proves that the mapping $\mathbf{F}: U \to \mathbb{R}^n$ is stable.

To verify that $\mathbf{DF}(\mathbf{x})$ is invertible for each $\mathbf{x}$ in U, just observe that for $\mathbf{x}$ in U, $\mathbf{DF}(\mathbf{x})$ is a matrix of the type just described by (16.15), in which $\mathbf{z}_{ij} = \mathbf{x}$ and therefore

$$\|\mathbf{DF}(\mathbf{x})\mathbf{h}\| \geq \frac{c}{2}\|\mathbf{h}\| \qquad \text{for all points } \mathbf{h} \text{ in } \mathbb{R}^n.$$

Thus, by Theorem 16.7, the derivative matrix $\mathbf{DF}(\mathbf{x})$ is invertible. ∎

EXERCISES FOR SECTION 16.2

1. Let n be an odd positive integer and define $f(x) = x^n$ for each x in $\mathbb{R}$.
   a. Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is one-to-one and onto.
   b. Use the Difference of Powers Formula to show that if $n > 1$, then the function $f: \mathbb{R} \to \mathbb{R}$ is not stable.

2. Let I be an open interval and suppose that the function $f: I \to \mathbb{R}$ is differentiable. Prove that $f: I \to \mathbb{R}$ is stable, with a stability constant c, if and only if

$$|f'(x)| \geq c \qquad \text{for all } x \text{ in } I.$$

3. Define

$$\mathbf{G}(x, y) = (x^2, y) \qquad \text{for}(x, y) \text{ in } \mathbb{R}^2.$$

Show that there is no neighborhood U of the point $(0, 0)$ such that the mapping $\mathbf{G}: \mathbb{R}^2 \to \mathbb{R}^2$ is stable.

4. Is the sum of stable mappings also a stable mapping?

5. Suppose that the function $f: [0, \infty) \to [0, \infty)$ is continuous and stable and $f(0) = 0$. Prove that $f: [0, \infty) \to [0, \infty)$ is one-to-one and onto.

6. Find a constant $\gamma > 0$ so that the matrix

$$\begin{bmatrix} 1 & a & 0 \\ b & 1 & c \\ c & 0 & 1 \end{bmatrix}$$

is invertible if $|a| < \gamma$, $|b| < \gamma$, and $|c| < \gamma$.

7. (A Global Inverse Function Theorem) Suppose that the function $f: \mathbb{R} \to \mathbb{R}$ is continuous and stable. Prove that $f: \mathbb{R} \to \mathbb{R}$ is one-to-one and onto. [*Hint:* Show that both $f([0, \infty))$ and $f((-\infty, 0])$ are unbounded, with one set unbounded above and the other unbounded below.]

8. Let U be an open subset of $\mathbb{R}^n$ and suppose that the continuously differentiable mapping $\mathbf{F}: U \to \mathbb{R}^n$ is stable. Prove that at each point $\mathbf{x}$ in U, the derivative matrix $\mathbf{DF}(\mathbf{x})$ is invertible. (*Hint:* Use the First-Order Approximation Theorem.)

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16.3 A MINIMIZATION PRINCIPLE AND THE GENERAL INVERSE FUNCTION THEOREM

In order to prove the Inverse Function Theorem in the Plane and its extension to general Euclidean spaces, we must confront the question of how it is possible to show that a particular point lies in the image of a mapping. For real-valued functions of a single variable, the Intermediate Value Theorem is very useful since in order to show that a point lies in the image of a continuous function defined on an interval, it is necessary only to show that there are functional values both greater than and less than the point. The Intermediate Value Theorem does not easily generalize to mappings that have their images in $\mathbb{R}^m$ with $m > 1$. Hence we pursue a different strategy. In order to show that a particular equation has a solution, we introduce an auxiliary function with the property that the *minimizers* of the auxiliary function are *solutions* of the given equation.

Proposition 16.10 A Minimization Principle Let U be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: U \to \mathbb{R}^n$ is continuously differentiable. Moreover, suppose that at each point $\mathbf{x}$ in U the derivative matrix $\mathbf{DF}(\mathbf{x})$ is invertible. Let $\mathbf{y}$ be a point in $\mathbb{R}^n$. Define the real-valued function[2] $\mathrm{E}: U \to \mathbb{R}$ by

$$\mathrm{E}(\mathbf{x}) = \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\|^2 \qquad \text{for } \mathbf{x} \text{ in } U.$$

Then the following two assertions are equivalent:

i. The point $\mathbf{y}$ lies in the image of the mapping $\mathbf{F}: U \to \mathbb{R}^n$.
ii. The function $\mathrm{E}: U \to \mathbb{R}$ attains a minimum value.

Proof
Since $\mathrm{E}(\mathbf{x}) \geq 0$ for each point $\mathbf{x}$ in U, if the point $\mathbf{y}$ lies in the image of the mapping $\mathbf{F}: U \to \mathbb{R}^n$, then the function $\mathrm{E}: U \to \mathbb{R}$ attains the value 0, and 0 is certainly the minimum value. Thus, assertion (i) implies assertion (ii).

To prove the converse, suppose that the function $\mathrm{E}: U \to \mathbb{R}$ attains a minimum value. Choose a point $\mathbf{x}$ in U that is a minimizer for the function $\mathrm{E}: U \to \mathbb{R}$. Then at the point $\mathbf{x}$,

$$\nabla \mathrm{E}(\mathbf{x}) = \mathbf{0}.$$

But observe that

$$0 = \frac{\partial \mathrm{E}}{\partial x_i}(\mathbf{x}) = \sum_{j=1}^{n} 2(\mathrm{F}_j(\mathbf{x}) - y_j)\frac{\partial \mathrm{F}_j}{\partial x_i}(\mathbf{x}) \qquad \text{for each index } i \text{ with } 1 \leq i \leq n, \tag{16.16}$$

so the gradient of E can be rewritten in terms of the transpose of $\mathbf{DF}(\mathbf{x})$ as

$$\nabla \mathrm{E}(\mathbf{x}) = 2[\mathbf{DF}(\mathbf{x})]^{\mathbf{T}}(\mathbf{F}(\mathbf{x}) - \mathbf{y}).$$

[2] The symbol E is used to connote *error* since the number $\mathrm{E}(\mathbf{x}) = \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\|^2$ is a measure of how far point $\mathbf{x}$ is from being a solution of the equation $\mathbf{F}(\mathbf{x}) = \mathbf{y}$.

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Thus,

$$2[\mathbf{DF}(\mathbf{x})]^{\mathbf{T}}(\mathbf{F}(\mathbf{x}) - \mathbf{y}) = \mathbf{0}. \tag{16.17}$$

Since by assumption the matrix $\mathbf{DF}(\mathbf{x})$ is invertible, by Theorem 15.23, its transpose matrix $\mathbf{DF}(\mathbf{x})^{\mathbf{T}}$ also is invertible.

But for any invertible $n \times n$ matrix $\mathbf{A}$ and point $\mathbf{u}$ in $\mathbb{R}^n$,

$$\mathbf{Au} = \mathbf{0} \qquad \text{if and only if } \mathbf{u} = \mathbf{0}.$$

Thus, (16.17) implies that $\mathbf{F}(\mathbf{x}) - \mathbf{y} = \mathbf{0}$, so that

$$\mathbf{F}(\mathbf{x}) = \mathbf{y}. \qquad \blacksquare$$

In order to use the above Minimization Principle it is necessary to establish that certain continuous functions defined on open subsets of $\mathbb{R}^n$ have minimum values. Recall that Corollary 11.23 asserts that every continuous real-valued function defined on a closed bounded subset of $\mathbb{R}^n$ attains a minimum value. For a point $\mathbf{x}_0$ in $\mathbb{R}^n$ and $r > 0$, the *closed ball* about $\mathbf{x}_0$ of radius r, denoted by $\mathcal{CB}_r(\mathbf{x}_0)$, is defined by

$$\mathcal{CB}_r(\mathbf{x}_0) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{x}, \mathbf{x}_0) \leq r\}.$$

The closed unit ball $\mathcal{CB}_{r_0}(\mathbf{x}_0)$ is a closed bounded subset of $\mathbb{R}^n$. Suppose that $f : \mathcal{CB}_{r_0}(\mathbf{x}_0) \to \mathbb{R}$ is continuous and that

$$f(\mathbf{x}_0) < f(\mathbf{x}) \qquad \text{if dist}(\mathbf{x}, \mathbf{x}_0) = r. \tag{16.18}$$

Corollary 11.23 implies that the function $f : \mathcal{CB}_{r_0}(\mathbf{x}_0) \to \mathbb{R}$ attains a minimum value on $\mathcal{CB}_{r_0}(\mathbf{x}_0)$. Assumption (16.18) implies that the minimum does not occur at a point at which $\text{dist}(\mathbf{x}, \mathbf{x}_0) = r$. Thus, the restricted function $f : \mathcal{B}_{r_0}(\mathbf{x}_0) \to \mathbb{R}$, which is a continuous function on an open set, attains a minimum value. This is the strategy we use to apply the Minimization Principle.

Lemma 16.11 The Open-Image Lemma Let U be an open subset of $\mathbb{R}^n$ and suppose that the continuously differentiable mapping $\mathbf{F} : U \to \mathbb{R}^n$ is stable and has an invertible derivative matrix at each point. Then its image $\mathbf{F}(U)$ is also open.

Proof

Let $\mathbf{y}_0$ be a point in $\mathbf{F}(U)$. We must show that $\mathbf{y}_0$ is an interior point of $\mathbf{F}(U)$; that is, we must find some positive number r such that the open ball $\mathcal{B}_r(\mathbf{y}_0)$ is contained in $\mathbf{F}(U)$. To do so, let $\mathbf{x}_0$ be the point in U at which $\mathbf{F}(\mathbf{x}_0) = \mathbf{y}_0$. Since U is open, we can choose a positive number r_0 such that the closed ball $\mathcal{CB}_{r_0}(\mathbf{x}_0)$ is contained in U. As observed above, $\mathcal{CB}_{r_0}(\mathbf{x}_0)$ is closed and bounded. Define $S = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \text{dist}(\mathbf{x}, \mathbf{x}_0) = r_0\}$.

By assumption, the mapping $\mathbf{F} : U \to \mathbb{R}^n$ is stable; we can therefore select a positive number c such that

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})\| \geq c\|\mathbf{u} - \mathbf{v}\| \qquad \text{for all points } \mathbf{u} \text{ and } \mathbf{v} \text{ in } U.$$

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In particular,

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{y}_0\| = \|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)\| \geq c \operatorname{dist}(\mathbf{x}, \mathbf{x}_0) = cr_0 \qquad \text{for } \mathbf{x} \text{ in } S. \quad (16.19)$$

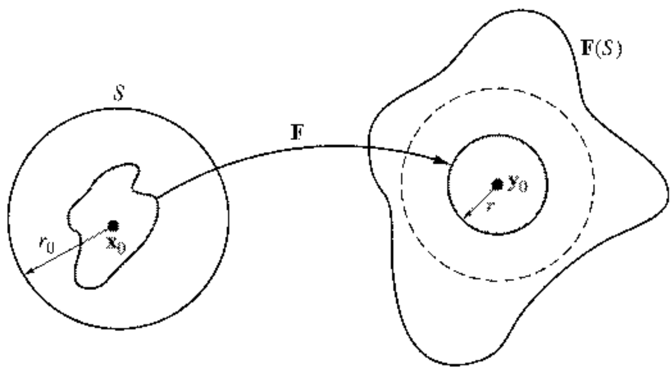

FIGURE 16.3 Constructing an open ball in the image.

Define $r = cr_0/2$. We claim that the open ball $\mathcal{B}_r(\mathbf{y}_0)$ is contained in $\mathbf{F}(U)$, and to prove this we use the Minimization Principle. Choose a point $\mathbf{y}$ in $\mathcal{B}_r(\mathbf{y}_0)$.

Define the auxiliary function $\mathrm{E}: \mathcal{CB}_{r_0}(\mathbf{x}_0) \to \mathbb{R}$ by

$$\mathrm{E}(\mathbf{x}) = \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\|^2 \qquad \text{for } \mathbf{x} \text{ in } \mathcal{CB}_{r_0}(\mathbf{x}_0).$$

From the estimate (16.19) and the Triangle Inequality, if $\operatorname{dist}(\mathbf{x}, \mathbf{x}_0) = r$,

$$cr_0 \leq \|\mathbf{F}(\mathbf{x}) - \mathbf{y}_0\| \leq \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\| + \|\mathbf{y} - \mathbf{y}_0\| < \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\| + cr_0/2,$$

and so

$$cr_0/2 < \|\mathbf{F}(\mathbf{x}) - \mathbf{y}\|.$$

Thus,

$$\mathrm{E}(\mathbf{x}_0) < \mathrm{E}(\mathbf{x}) \qquad \text{if } \operatorname{dist}(\mathbf{x}, \mathbf{x}_0) = r.$$

Following the argument detailed immediately preceding the lemma, we conclude that E attains a minimum value on the open ball $\mathcal{B}_{r_0}(\mathbf{x}_0)$. The Minimization Principle implies that y is in the image $\mathbf{F}(\mathcal{B}_{r_0}(\mathbf{x}_0))$. Thus, the open ball of radius r about y is contained in $\mathbf{F}(U)$. ∎

Theorem 16.12 The Inverse Function Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^n$ is continuously differentiable. Let x_* be a point in $\mathcal{O}$ at which the derivative matrix

$$\mathbf{DF}(\mathrm{x}_*) \text{ is invertible.} \qquad (16.20)$$

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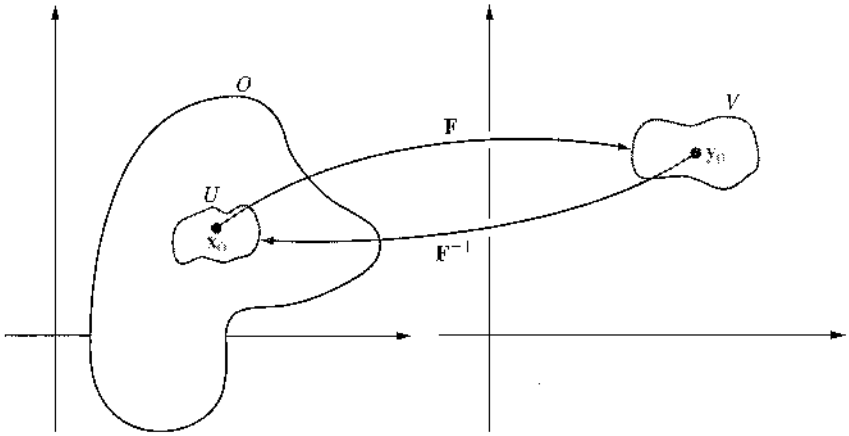

FIGURE 16.4 F maps a neighborhood onto a neighborhood.

Then there is a neighborhood U of the point $\mathbf{x}_*$ and a neighborhood V of its image $\mathbf{F}(\mathbf{x}_*)$ such that the mapping

$$\mathbf{F}: U \to V \text{ is one-to-one and onto.} \tag{16.21}$$

Moreover, the inverse mapping $\mathbf{F}^{-1}: V \to U$ is also continuously differentiable, and for a point $\mathbf{y}$ in V, if $\mathbf{x}$ is the point in U at which $\mathbf{F}(\mathbf{x}) = \mathbf{y}$, then

$$\mathbf{DF}^{-1}(\mathbf{y}) = [\mathbf{DF}(\mathbf{x})]^{-1}. \tag{16.22}$$

Proof

Because of the Nonlinear Stability Theorem, we can choose a neighborhood U of $\mathbf{x}_*$ and a positive number c such that

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{v})\| \geq c\|\mathbf{u} - \mathbf{v}\| \qquad \text{for all points } \mathbf{u} \text{ and } \mathbf{v} \text{ in } U, \tag{16.23}$$

and

$$\mathbf{DF}(\mathbf{x}) \text{ is invertible for all points } \mathbf{x} \text{ in } U. \tag{16.24}$$

We can invoke the Open-Image Lemma to conclude that the image $\mathbf{F}(U)$, which we denote by V, is open. Thus, U is a neighborhood of $\mathbf{x}_*$ and V is a neighborhood of its image $\mathbf{F}(\mathbf{x}_*) = \mathbf{y}_*$, the mapping $\mathbf{F}: U \to V$ is one-to-one and onto, and $\mathbf{DF}(\mathbf{x})$ is invertible for all points $\mathbf{x}$ in U. It remains only to prove that the inverse mapping $\mathbf{F}^{-1}: V \to U$ is also continuously differentiable and that formula (16.22) holds.

First, we show that the inverse mapping is continuous; then we establish formula (16.22). Once this has been done, the continuous differentiability of the inverse mapping follows from the formula relating the entries of the inverse of an $n \times n$ matrix to the entries of the matrix (Exercise 10).

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Let $\mathbf{y}$ be a point in V and let $\mathbf{x}$ be the point in U at which $\mathbf{F}(\mathbf{x}) = \mathbf{y}$. For a point $\mathbf{y} + \mathbf{k}$ in V, it is convenient to define $\mathbf{h} = \mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - \mathbf{F}^{-1}(\mathbf{y})$, so that

$$\mathbf{F}(\mathbf{x}) = \mathbf{y} \qquad \text{and} \qquad \mathbf{F}(\mathbf{x} + \mathbf{h}) = \mathbf{y} + \mathbf{k}. \tag{16.25}$$

Observe that from the stability estimate (16.23) we obtain the following estimate:

$$\|\mathbf{k}\| = \|\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x})\| \geq c\|\mathbf{h}\|. \tag{16.26}$$

Since $\mathbf{h} = \mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - \mathbf{F}^{-1}(\mathbf{y})$, this estimate can be rewritten as

$$\|\mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - \mathbf{F}^{-1}(\mathbf{y})\| \leq \frac{1}{c}\|\mathbf{k}\|,$$

from which the continuity of $\mathbf{F}^{-1} : V \to U$ at point $\mathbf{y}$ follows.

We now establish formula (16.22). To do so, define

$$\mathbf{B} = (\mathbf{DF}(\mathbf{x}))^{-1}.$$

According to Theorem 15.32, in order to prove (16.22) it is sufficient to show that

$$\lim_{\mathbf{k} \to \mathbf{0}} \frac{\|\mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - [\mathbf{F}^{-1}(\mathbf{y}) + \mathbf{B}\mathbf{k}]\|}{\|\mathbf{k}\|} = 0. \tag{16.27}$$

Note that (16.26) provides a useful lower bound for the denominator on the left-hand side of (16.27). To find a useful upper bound for the numerator, observe that

$$\begin{aligned}
\mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - [\mathbf{F}^{-1}(\mathbf{y}) + \mathbf{B}\mathbf{k}] &= \mathbf{h} - \mathbf{B}\mathbf{k} \\
&= \mathbf{B}[\mathbf{B}^{-1}\mathbf{h} - \mathbf{k}] \\
&= \mathbf{B}[\mathbf{DF}(\mathbf{x})\mathbf{h} - \mathbf{F}(\mathbf{x} + \mathbf{h}) + \mathbf{F}(\mathbf{x})].
\end{aligned}$$

Hence, using the Generalized Cauchy–Schwarz Inequality, it follows that

$$\|\mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - [\mathbf{F}^{-1}(\mathbf{y}) + \mathbf{B}\mathbf{k}]\| \leq \|\mathbf{B}\| \cdot \|\mathbf{F}(\mathbf{x} + \mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}]\|. \tag{16.28}$$

From the estimates (16.26) and (16.28), we have the following estimate for the quotient on the left-hand side of (16.27):

$$\frac{\|\mathbf{F}^{-1}(\mathbf{y} + \mathbf{k}) - [\mathbf{F}^{-1}(\mathbf{y}) + \mathbf{B}\mathbf{k}]\|}{\|\mathbf{k}\|} \leq \frac{\|\mathbf{B}\|}{c}\left\{\frac{\|\mathbf{F}(\mathbf{x} + \mathbf{h}) - [\mathbf{F}(\mathbf{x}) + \mathbf{DF}(\mathbf{x})\mathbf{h}]\|}{\|\mathbf{h}\|}\right\}. \tag{16.29}$$

But the estimate (16.26) implies that $\mathbf{h} \to \mathbf{0}$ as $\mathbf{k} \to \mathbf{0}$, so from the First-Order Approximation Theorem, the right-hand side of the estimate (16.29) converges to 0 as $\mathbf{k} \to \mathbf{0}$ and hence so does the left-hand side of (16.29). This proves (16.27). ■

EXERCISES FOR SECTION 16.3

1. For each of the following mappings $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, determine the points (x_0, y_0, z_0) in $\mathbb{R}^3$ at which the Inverse Function Theorem applies:

a. $\mathbf{F}(x, y, z) = (e^x \cos y,\ e^x \sin y,\ z^2)$ for (x, y, z) in $\mathbb{R}^3$

b. $\mathbf{F}(x, y, z) = (yz, xz, xy)$ for (x, y, z) in $\mathbb{R}^3$

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2. For a point $(\rho,\ \theta,\ \phi)$ in $\mathbb{R}^3$, define

$$\mathbf{F}(\rho,\ \theta,\ \phi) = (\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi).$$

At what points $(\rho_0,\ \theta_0,\ \phi_0)$ in $\mathbb{R}^3$ does the Inverse Function Theorem apply to the mapping $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$?

3. Suppose that the functions $\psi: \mathbb{R}^3 \to \mathbb{R}$ and $\phi: \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable. Define, for $(x,\ y,\ z)$ in $\mathbb{R}^3$,

$$\mathbf{F}(x,\ y,\ z) = (\psi(x,\ y,\ z),\ \varphi(x,\ y,\ z),\ (\psi(x,\ y,\ z))^2 + (\varphi(x,\ y,\ z))^2).$$

a. Explain analytically why there is no point $(x_0,\ y_0,\ z_0)$ in $\mathbb{R}^3$ at which the assumptions of the Inverse Function Theorem hold for the mapping $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$.

b. Explain geometrically why there is no point $(x_0,\ y_0,\ z_0)$ in $\mathbb{R}^3$ at which the conclusion of the Inverse Function Theorem holds for the mapping $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$.

4. In the case where $n = 3$, write out explicitly, in terms of the first-order partial derivatives of the component functions of the mapping and its inverse, the meaning of the matrix formula (16.22).

5. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{c}$ and $\mathbf{x}_*$ be points in $\mathbb{R}^n$. Define the affine mapping $\mathbf{G}: \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathbf{G}(\mathbf{x}) = \mathbf{c} + \mathbf{A}(\mathbf{x} - \mathbf{x}_*) \qquad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n.$$

Show that the mapping $\mathbf{G}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto if and only if the matrix $\mathbf{A}$ is invertible.

6. Define $f(x) = x^2$ for x in $\mathbb{R}$.

a. Show that if $\mathcal{O}$ is any open subset of $\mathbb{R}$ that does not contain 0, then $f(\mathcal{O})$ is open.

b. Show that if $\mathcal{O}$ is an open subset of $\mathbb{R}$ that contains 0, then $f(\mathcal{O})$ is not open.

7. Give an example of a continuously differentiable mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$ with the property that there is no open subset U of $\mathbb{R}^n$ for which $\mathbf{F}(U)$ is open in $\mathbb{R}^n$.

8. Let I be an open interval of real numbers and suppose that the function $f: I \to \mathbb{R}$ is continuous. Let c be a real number. Fix a number x_0 in the interval I and define the auxiliary function $H: \mathbb{R} \to \mathbb{R}$ by

$$H(x) = cx - \int_{x_0}^{x} f(s)\, ds \qquad \text{for } x \text{ in } I.$$

For a point x in I, show that $f(x) = c$ if $H'(x) = 0$. Conclude that c is in the image of $f: I \to \mathbb{R}$ provided that the function $H: I \to \mathbb{R}$ has a local extreme point.

9. Provide a proof of Theorem 16.1 in which, instead of the Intermediate Value Theorem, the Minimization Principle is used to show that J is contained in $f(I)$.

10. Use Cramer's Rule and the continuity of the inverse mapping $\mathbf{F}^{-1}: V \to U$ to show that formula (16.22) implies that the inverse mapping $\mathbf{F}^{-1}: V \to U$ is continuously differentiable.

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11. (A Global Inverse Function Theorem) Suppose that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and stable. By verifying each of the following parts, prove that the mapping $\mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n$ is invertible.

a. Show that at each point $\mathbf{x}$ in $\mathbb{R}^n$ the derivative matrix $\mathbf{DF}(\mathbf{x})$ is invertible.

b. Show that the image $\mathbf{F}(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$. (*Hint:* Apply the Inverse Function Theorem.)

c. Show that the image $\mathbf{F}(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$. (*Hint:* Use stability and the definition of closedness.)

d. Show that the image $\mathbf{F}(\mathbb{R}^n) = \mathbb{R}^n$. [*Hint:* The set $\mathbf{F}(\mathbb{R}^n)$ is both open and closed in $\mathbb{R}^n$, and $\mathbb{R}^n$ is connected.]

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CHAPTER 17

THE IMPLICIT FUNCTION THEOREM AND ITS APPLICATIONS

17.1 A SCALAR EQUATION IN TWO UNKNOWNS: DINI'S THEOREM

For $\mathcal{O}$ an open subset of the plane $\mathbb{R}^2$ and a continuously differentiable function $f : \mathcal{O} \to \mathbb{R}$, the set of solutions of the equation

$$f(x, y) = 0, \qquad (x, y) \text{ in } \mathcal{O} \tag{17.1}$$

is in general a very complicated subset of the plane. In particular, the set of solutions is not the graph of a function prescribing y as a function of x. However, if the point (x_0, y_0) is a solution of this equation and $\partial f/\partial y(x_0, y_0) \neq 0$, then there is a neighborhood of the point (x_0, y_0) having the property that the solutions of the above equation that are in this neighborhood make up the graph of a continuously differentiable function $g : I \to \mathbb{R}$, where I is an open interval about x_0. Moreover, the derivative of the implicitly defined function $g : I \to \mathbb{R}$ can be computed in terms of the partial derivatives of the function $f : \mathcal{O} \to \mathbb{R}$. This assertion is called Dini's Theorem. It has an extension, called the General Implicit Function Theorem, that provides a similar local description of the set of solutions of an equation of the form

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} \text{ in } \mathcal{O},$$

where $\mathcal{O}$ is an open subset of Euclidean space $\mathbb{R}^{n+k}$ and the mapping $\mathbf{F} : \mathcal{O} \to \mathbb{R}^k$ is continuously differentiable.

In this first section, we provide a proof of Dini's Theorem and consider a number of examples. In Section 17.2, we will prove the General Implicit Function Theorem. The last two sections consist of applications, first to the geometry of paths and surfaces in $\mathbb{R}^3$ and then to the consideration of constrained extrema problems.

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Example 17.1 The set of solutions of the equation

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \qquad (x, y) \text{ in } \mathbb{R}^2 \tag{17.2}$$

consists of the points lying on an ellipse centered at the origin. First, consider the solution $(x_0, y_0) = (0, 4)$, which is the upper vertex of the ellipse. Then for a number r between 0 and 3, define I to be the open interval $(-r, r)$ and define the function $g : I \to \mathbb{R}$ by $g(x) = 4\sqrt{1 - x^2/9}$ for x in I. Then there is a neighborhood of the solution $(0, 4)$ having the property that the set of solutions of equation (17.2) in this neighborhood consists of points of the form $(x, g(x))$ for x in I. Now consider the second component of the solution $(0, 4)$. Observe that it is not possible to find a neighborhood J of the number 4, a function $h : J \to \mathbb{R}$, and a neighborhood of the solution $(0, 4)$ in which the set of solutions of equation (17.2) consists of points of the form $(h(y), y)$ for y in J. At each of the other vertices of the ellipse, it is possible to find a neighborhood of the vertex in which the set of solutions of equation (17.2) has a similar description. On the other hand, at a solution (x_0, y_0) of equation (17.2) that is not a vertex of the ellipse, in a neighborhood of (x_0, y_0), the set of solutions of equation (17.2) determines both x as a function of y and y as a function of x. We leave the calculation of the specific functions as an exercise.

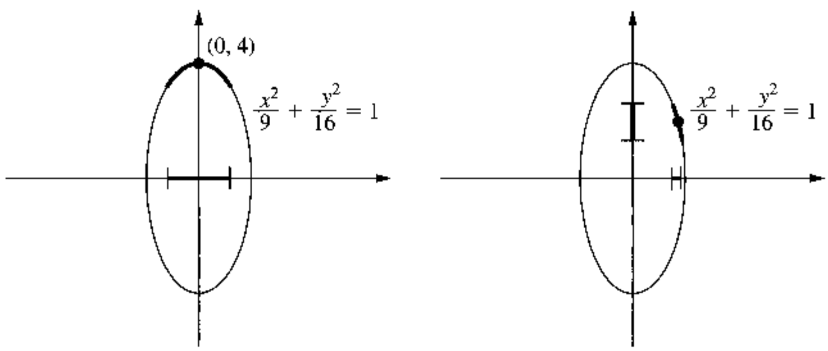

FIGURE 17.1 Solutions of the equation $f(x, y) = 0$ near the solution (x_0, y_0). ∎

Example 17.2 The set of solutions of the equation

$$y^2 - x^2 = 0, \qquad (x, y) \text{ in } \mathbb{R}^2, \tag{17.3}$$

consists of the points in the plane that lie on the line $y = x$ or on the line $y = -x$. At each solution (x_0, y_0) of equation (17.3) not equal to $(0, 0)$, there is a neighborhood of (x_0, y_0) in which the set of solutions of equation (17.3) determines both x as a function of y and y as a function of x. The origin $(0, 0)$ is a solution of equation (17.3), but there is no neighborhood of $(0, 0)$ in which the set of solutions coincides with the graph of a function expressing one of the components of the point (x, y) as a function of the other component. ∎

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The equations in the above two examples are so simple that we could explicitly determine all the solutions of each of these nonlinear equations. In general, this is certainly not possible. For this reason, the following theorem is important.

Theorem 17.3 Dini's Theorem Let $\mathcal{O}$ be an open subset of the plane $\mathbb{R}^2$ and suppose that the function $f:\mathcal{O}\to\mathbb{R}$ is continuously differentiable. Let $(x_0,\ y_0)$ be a point in $\mathcal{O}$ at which $f(x_0,\ y_0)=0$ and

$$\frac{\partial f}{\partial y}(x_0,\ y_0)\neq 0. \tag{17.4}$$

Then there is a positive number r and a continuously differentiable function $g: I\to\mathbb{R}$, where I is the open interval $(x_0-r,\ x_0+r)$, such that

$$f(x,\ g(x))=0 \qquad \text{for all } x \text{ in } I \tag{i}$$

and

$$\text{whenever } |x-x_0|<r, \qquad |y-y_0|<r, \ \text{ and } \ f(x,\ y)=0, \qquad \text{then } \ y=g(x). \tag{ii}$$

Moreover,

$$\frac{\partial f}{\partial x}(x,\ g(x))+\frac{\partial f}{\partial y}(x,\ g(x))g'(x)=0 \qquad \text{for all } x \text{ in } I. \tag{iii}$$

Proof

We assume that $\partial f/\partial y(x_0,\ y_0)>0$. Since $\mathcal{O}$ is open and the function $\partial f/\partial y:\mathcal{O}\to\mathbb{R}$ is continuous and positive at the point $(x_0,\ y_0)$, we can choose positive numbers a and c such that the closed square $R=[x_0-a,\ x_0+a]\times[y_0-a,\ y_0+a]$ is contained in $\mathcal{O}$ and

$$\frac{\partial f}{\partial y}(x,\ y)\geq c \qquad \text{for all points } (x,\ y) \text{ in } R. \tag{17.5}$$

It follows from the Mean Value Theorem for scalar functions of a single real variable that

$$f(x,\ y_1)<f(x,\ y_2) \qquad \text{if } |x-x_0|\leq a \text{ and } y_0-a\leq y_1<y_2\leq y_0+a. \tag{17.6}$$

In particular, since $f(x_0,\ y_0)=0$, it follows that $f(x_0,\ y_0-a)<0<f(x_0,y_0+a)$. Moreover, the function $f:\mathcal{O}\to\mathbb{R}$ is continuous since it is continuously differentiable. Thus, we can choose a positive number r less than a such that, if we let $I=(x_0-r,x_0+r)$,

$$f(x,\ y_0-a)<0<f(x,\ y_0+a) \qquad \text{for all } x \text{ in } I.$$

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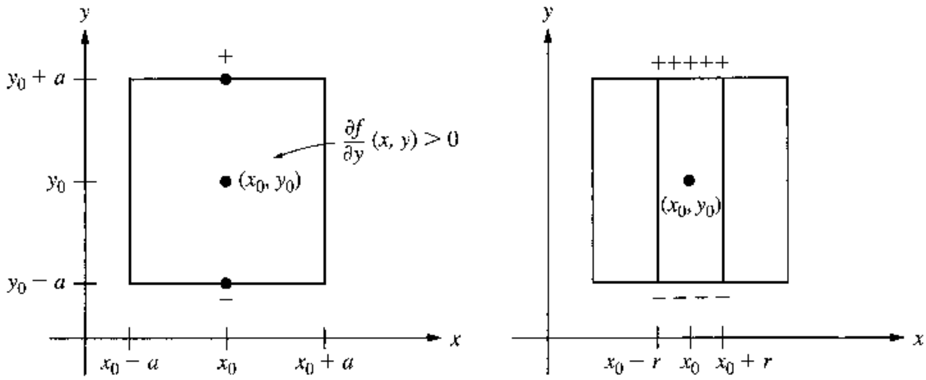

FIGURE 17.2 $f(x, y_0 - a) < 0 < f(x, y_0 + a)$ for all x in I.

Let x be a point in I. Since $f(x, y_0 - a) < 0$ and $f(x, y_0 + a) > 0$, according to the Intermediate Value Theorem, there is some point y between $y_0 - a$ and $y_0 + a$ at which $f(x, y) = 0$, and (17.6) implies that there is only one such point. Define $g(x)$ to be this point. This clearly defines a function $g : I \to \mathbb{R}$ having properties (i) and (ii).

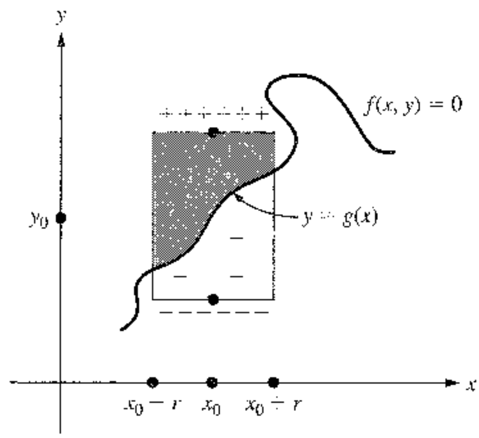

FIGURE 17.3 Changes in sign of $f(x, y)$.

We now show that $g : I \to \mathbb{R}$ is continuously differentiable and that the differentiation formula (iii) holds at the point x_0. Indeed, let $x_0 + h$ be a point in I. Then, by definition, $f(x_0 + h, g(x_0 + h)) = 0$ and $f(x_0, g(x_0)) = 0$. In particular,

$$0 = f(x_0 + h, g(x_0 + h)) - f(x_0, g(x_0)).$$

According to the Mean Value Theorem for scalar functions of two real variables, there is some point on the segment between the points $(x_0, g(x_0))$ and $(x_0 + h, g(x_0 + h))$, which we label $\mathbf{p}(h)$, at which

$$f(x_0+h, g(x_0+h)) - f(x_0, g(x_0)) = \frac{\partial f}{\partial x}(\mathbf{p}(h))h + \frac{\partial f}{\partial y}(\mathbf{p}(h))[g(x_0+h) - g(x_0)].$$

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But the left-hand side is 0, and hence,

$$g(x_0+h)-g(x_0) = -\left[\frac{\partial f/\partial x(\mathbf{p}(h))}{\partial f/\partial y(\mathbf{p}(h))}\right] h. \tag{17.7}$$

Since the function $\partial f/\partial x : \mathcal{O} \to \mathbb{R}$ is continuous and the closed square R is a sequentially compact subset of the plane, by the Extreme Value Theorem we can choose a positive number M such that

$$\left|\frac{\partial f}{\partial x}(x,\ y)\right| \leq M \qquad \text{for all points } (x,\ y) \text{ in } R.$$

Using this inequality, together with inequality (17.5), it follows from formula (17.7) that

$$|g(x_0+h)-g(x_0)| \leq \frac{M}{c}|h| \qquad \text{if } x_0+h \text{ is in } I.$$

Hence the function $g : I \to \mathbb{R}$ is continuous at the point x_0. Since the point $\mathbf{p}(h)$ lies on the segment between the points $(x_0,\ g(x_0))$ and $(x_0+h,\ g(x_0+h))$, we conclude that

$$\lim_{h\to 0} \mathbf{p}(h) = (x_0,\ y_0).$$

If we now divide (17.7) by h and use the continuity of the first-order partial derivatives of $f : \mathcal{O} \to \mathbb{R}$ at the point $(x_0,\ y_0)$, it follows from (17.7) that

$$\lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} = -\frac{\partial f/\partial x(x_0,\ y_0)}{\partial f/\partial y(x_0,\ y_0)},$$

which means that g is differentiable at x_0 and formula (iii) holds at x_0. But any other point x in the interval I satisfies the same assumptions as does the point x_0, and hence (iii) holds at all points in I. ■

Example 17.4 Consider the equation

$$\cos(x+y) + e^{y+x^2} + 3x - 2 - x^3y^3 = 0, \qquad (x,\ y) \text{ in } \mathbb{R}^2. \tag{17.8}$$

Define

$$f(x,\ y) = \cos(x+y) + e^{y+x^2} + 3x - 2 - x^3y^3 \qquad \text{for } (x,\ y) \text{ in } \mathbb{R}^2.$$

Then $(x,\ y)$ is a solution of equation (17.8) if and only if $f(x,\ y) = 0$. Observe that $(0, 0)$ is a solution of (17.8) and that

$$\frac{\partial f}{\partial x}(0,\ 0) = 3 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(0,\ 0) = 1.$$

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Dini's Theorem implies that there is a positive number r and a continuously differentiable function $g : I \to \mathbb{R}$, where I is the open interval $(-r, r)$, such that

$$\cos(x + g(x)) + e^{g(x)+x^2} + 3x - 2 - x^3(g(x))^3 = 0 \qquad \text{for all } x \text{ in } I.$$

Moreover, if (x, y) is a solution of equation (17.8) with $|x| < r$ and $|y| < r$, then $y = g(x)$. Finally, $g'(0)$ is determined by the formula

$$\frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial y}(0, 0)g'(0) = 0,$$

so $g'(0) = -3$. ∎

The assumption in Dini's Theorem that $\partial f/\partial y(x_0, y_0) \neq 0$ can be replaced by the assumption that $\partial f/\partial x(x_0, y_0) \neq 0$, and the conclusion remains the same except that the roles of x and y are interchanged. Thus, if $\mathcal{O}$ is an open subset of the plane $\mathbb{R}^2$, if the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable, and if at the point (x_0, y_0) in $\mathcal{O}$ we have $f(x_0, y_0) = 0$ and

$$\nabla f(x_0, y_0) \neq (0, 0),$$

then there is a neighborhood of the point (x_0, y_0) in which the solutions of equation (17.1) make up the graph of a continuously differentiable function prescribing at least one of the components of a solution (x, y) as a function of the other component.

Example 17.5 Consider the equation

$$e^{x-2+(y-1)^2} - 1 = 0, \qquad (x, y) \text{ in } \mathbb{R}^2. \tag{17.9}$$

Define

$$f(x, y) = e^{x-2+(y-1)^2} - 1 \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2.$$

Then the point (x, y) is a solution of equation (17.9) if and only if $f(x, y) = 0$. Observe that the point $(2, 1)$ is a solution of equation (17.9) and that

$$\frac{\partial f}{\partial x}(2, 1) = 1 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(2, 1) = 0.$$

Dini's Theorem, with the roles of the variables x and y interchanged, implies that there is a positive number r and a continuously differentiable function $h : J \to \mathbb{R}$, where J is the open interval $(1 - r, 1 + r)$, such that

$$e^{h(y)-2+(y-1)^2} - 1 = 0 \qquad \text{for all } y \text{ in } J.$$

Moreover, if (x, y) is a solution of equation (17.9) with $|x - 2| < r$ and $|y - 1| < r$, then $x = h(y)$. Finally, $h'(1)$ is determined by the formula

$$\frac{\partial f}{\partial x}(2, 1)h'(1) + \frac{\partial f}{\partial y}(2, 1) = 0,$$

so $h'(1) = 0$. ∎

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Of course, Dini's Theorem can be used to analyze solutions of equations of the form

$$\phi(x, y) = \eta(x, y), \qquad (x, y) \text{ in } \mathcal{O}. \tag{17.10}$$

Simply define

$$f(x, y) = \phi(x, y) - \eta(x, y) \qquad \text{for } (x, y) \text{ in } \mathcal{O},$$

so that the solutions of equation (17.10) are precisely the solutions of equation (17.1). In particular, for a real number c we can analyze solutions of the equation

$$f(x, y) = c, \qquad (x, y) \text{ in } \mathcal{O}. \tag{17.11}$$

For a given number c, the set of solutions of (17.11) is often called a *level curve* of the function $f : \mathcal{O} \to \mathbb{R}$. Because of Dini's Theorem, in order to justify the term *curve,* it seems reasonable to suppose that

$$\nabla f(x, y) \neq \mathbf{0} \text{ at each point } (x, y) \text{ in } \mathcal{O} \text{ at which } f(x, y) = c,$$

so that the set of solutions of equation (17.11) does indeed consist of the union of curves.

We conclude this section with a comment about implicit differentiation and higher-order derivatives. In the last assertion of Dini's Theorem, about the differentiability of the implicitly defined function, once we know that the function $g : I \to \mathbb{R}$ is differentiable and that

$$f(x, g(x)) = 0 \qquad \text{for all } x \text{ in } I, \tag{17.12}$$

the formula (iii) for the derivative $g' : I \to \mathbb{R}$ follows by differentiating each side of (17.12). This technique is known as *implicit differentiation.* Moreover, we can also find higher-order derivatives of the function $g : I \to \mathbb{R}$. For instance, if the function $f : \mathcal{O} \to \mathbb{R}$ in the statement of Dini's Theorem has continuous second-order partial derivatives, then from formula (iii) for the derivative of the implicitly defined function $g : I \to \mathbb{R}$, it follows that $g' : I \to \mathbb{R}$ itself is differentiable. We leave it as an exercise to differentiate each side of the formula

$$\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))g'(x) = 0, \qquad \text{for all } x \text{ in } I,$$

to obtain the following formula for the second derivative of the implicitly defined function g:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, g(x)) + 2\frac{\partial^2 f}{\partial x \partial y}(x, g(x))g'(x) + \frac{\partial^2 f}{\partial y^2}(x, g(x))[g'(x)]^2 \\ + \frac{\partial f}{\partial y}(x, g(x))g''(x) = 0. \end{aligned} \tag{17.13}$$

EXERCISES FOR SECTION 17.1

1. Consider the equation

$$\frac{x^2}{8} + \frac{y^2}{18} = 1, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

a. Graph the set of solutions and show that Dini's Theorem applies at the solution (2, 3).

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b. Explicitly define the function $g : I \to \mathbb{R}$ that has the property that in a neighborhood of the solution $(2, 3)$, all the solutions are of the form $(x, g(x))$ for x in I and check that formula (iii) holds for the derivative $g' : I \to \mathbb{R}$.

c. Explicitly define the function $h : J \to \mathbb{R}$ that has the property that in a neighborhood of the solution $(2, 3)$, all the solutions are of the form $(h(y), y)$ for y in J.

2. Consider the equation

$$f(x, y) = (x^2 + y^2 - 2)(x^2 - y^2) = 0, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

a. Compute partial derivatives to show that $\nabla f(x, y) = \mathbf{0}$ and hence that the assumptions of Dini's Theorem do not hold at each of the following solutions: $(0, 0)$, $(1, 1)$, $(1, -1)$, $(-1, -1)$, $(-1, 1)$.

b. By graphing the set of solutions of this equation, show that the conclusions of Dini's Theorem do not hold at each of the solutions listed in (a).

3. Consider the equation

$$x^2 + y^2 = 0, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

a. Show that the assumptions of Dini's Theorem do not hold at the solution $(0, 0)$.

b. Explain, by graphing the set of solutions of this equation, why the conclusion of Dini's Theorem does not hold at the solution $(0, 0)$.

4. Consider the equation

$$e^{2x-y} + \cos(x^2 + xy) - 2 - 2y = 0, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

Does the set of solutions of this equation in a neighborhood of the solution $(0, 0)$ implicitly define one of the components of the point (x, y) as a function of the other component? If so, compute the derivative of this function (these functions?) at the point 0.

5. The point $(0, 0)$ is a solution of the equation

$$\ln(x^2 + y^2 + 1) = x, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

Find the derivative at the point 0 of the function(s) of a single variable defined by the set of solutions of this equation in a neighborhood of $(0, 0)$.

6. Find the explicit formula for all the solutions of the following equation considered in Example 17.5:

$$e^{x-2+(y-1)^2} - 1 = 0, \qquad (x, y) \text{ in } \mathbb{R}^2.$$

7. Let $\mathcal{O}$ be an open subset of the plane and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. At the point (x_0, y_0) in $\mathcal{O}$, suppose that $f(x_0, y_0) = 0$ and that $\nabla f(x_0, y_0) \neq (0, 0)$. Show that the vector $\nabla f(x_0, y_0)$ is orthogonal to the tangent line at (x_0, y_0) of the implicitly defined function.

8. Let $\mathcal{O}$ be an open subset of the plane and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. At the point (x_0, y_0) in $\mathcal{O}$, suppose that $f(x_0, y_0) = 0$

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and that

$$\frac{\partial f}{\partial x}(x_0, y_0) \neq 0, \qquad \frac{\partial f}{\partial y}(x_0, y_0) \neq 0.$$

Show that the two functions implicitly defined by Dini's Theorem, when their domains are properly chosen, are inverses of each other.

9. Suppose that the continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ can be factored as

$$f(x, y) = (x^2 + y^2)h(x, y) \qquad \text{for } (x, y) \text{ in } \mathbb{R}^2,$$

where the function $h : \mathbb{R}^2 \to \mathbb{R}$ is also continuously differentiable. Show that Dini's Theorem cannot be directly applied to analyze the solutions of the equation $f(x, y) = 0$ in a neighborhood of the solution $(0, 0)$. If $h(0, 0) = 0$ and $\nabla h(0, 0) \neq (0, 0)$, use Dini's Theorem to analyze the solutions of $f(x, y) = 0$.

10. Suppose that the function $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that at the point x_0 in $\mathbb{R}$, $\phi'(x_0) \neq 0$. Set $y_0 = \phi(x_0)$ and define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = y - \phi(x)$ for (x, y) in $\mathbb{R}^2$. Apply Dini's Theorem to the function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point (x_0, y_0) and compare the result with the conclusion of the Inverse Function Theorem applied to the function $\phi : \mathbb{R} \to \mathbb{R}$ at the point x_0.

11. Suppose that the function $h : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a positive number c such that $h'(t) \geq c$ for all points t in $\mathbb{R}$. Prove that there is exactly one number t at which $h(t) = 0$.

12. (A Global Implicit Function Theorem) Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable and that there is a positive number c such that

$$\frac{\partial f}{\partial y}(x, y) \geq c \qquad \text{for every } (x, y) \text{ in } \mathbb{R}^2.$$

Prove that there is a continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$ with

$$f(x, g(x)) = 0 \qquad \text{for every } x \text{ in } \mathbb{R}$$

and that if $f(x, y) = 0$, then $y = g(x)$. (*Hint:* Use Exercise 11.)

13. Suppose that the function $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable and that (x_0, y_0, z_0) is a point in $\mathbb{R}^3$ at which $f(x_0, y_0, z_0) = 0$ and $\partial f/\partial z(x_0, y_0, z_0) > 0$. Follow the proof of Dini's Theorem to show that there is a positive number r and a function $g : \mathcal{B}_r(x_0, y_0) \to \mathbb{R}$ such that

$$f(x, y, g(x, y)) = 0 \qquad \text{for all } (x, y) \text{ in } \mathcal{B}_r(x_0, y_0).$$

14. In addition to the assumptions of Dini's Theorem, assume also that the function $f : \mathcal{O} \to \mathbb{R}$ has continuous second-order partial derivatives.

a. Verify formula (17.13).

b. Moreover, suppose that

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x_0, y_0)\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0.$$

Prove that the graph of $g : I \to \mathbb{R}$ lies below the line $y = y_0$ if I is chosen sufficiently small. [*Hint:* Use formula (17.13) to determine the sign of $g''(x_0)$.]

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17.2 THE GENERAL IMPLICIT FUNCTION THEOREM

Let k and n be positive integers, let $\mathcal{O}$ be an open subset of Euclidean space $\mathbb{R}^{n+k}$, and let the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^k$ be continuously differentiable. Consider the equation

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} \text{ in } \mathcal{O}. \tag{17.14}$$

In the case where $n = 1$ and $k = 1$, we already considered this equation in Section 17.1, where we proved Dini's Theorem. The object of this section is to prove the General Implicit Function Theorem, which extends Dini's Theorem to more general equations of the form (17.14). In order to emphasize the analogy between the general case and the case where $n = 1$ and $k = 1$, it is useful to introduce the following notation: For a point $\mathbf{u}$ in $\mathbb{R}^{n+k}$, we separate the first n components of $\mathbf{u}$ from the last k components and label them as follows:

$$\mathbf{u} = (\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_k).$$

Then equation (17.14) can be rewritten as

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}, \qquad (\mathbf{x}, \mathbf{y}) \text{ in } \mathcal{O}. \tag{17.15}$$

If the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^k$ is written in terms of its component functions, $\mathbf{F} = (\mathrm{F}_1, \ldots, \mathrm{F}_k)$, this equation in turn can be written as the following system of k nonlinear scalar equations in $n + k$ scalar unknowns:

$$\begin{aligned}
\mathrm{F}_1(x_1, \ldots, x_n, y_1, \ldots, y_k) &= 0 \\
&\vdots \\
\mathrm{F}_i(x_1, \ldots, x_n, y_1, \ldots, y_k) &= 0 \\
&\vdots \\
\mathrm{F}_k(x_1, \ldots, x_n, y_1, \ldots, y_k) &= 0
\end{aligned} \tag{17.16}$$

for $(x_1, \ldots, x_n, y_1, \ldots, y_k)$ in $\mathcal{O}$.

This system is "underdetermined" in the sense that there are fewer equations than there are variables. It is highly unlikely that we can explicitly find all the solutions of such a complicated system of nonlinear equations. Thus, we seek the type of information already provided by Dini's Theorem for a single scalar equation with two scalar unknowns.

Just as we have defined the partial derivatives for a function, we now define the *partial derivative matrices* for a mapping. For a point $(\mathbf{x}, \mathbf{y})$ in $\mathcal{O}$, fix $\mathbf{x}$ and consider the mapping $\mathbf{y} \mapsto \mathbf{F}(\mathbf{x}, \mathbf{y})$, which is a mapping from an open subset of $\mathbb{R}^k$ into $\mathbb{R}^k$; we denote the $k \times k$ derivative matrix of this mapping by $\mathbf{D_y F}(\mathbf{x}, \mathbf{y})$. Similarly, if $\mathbf{y}$ is fixed, consider the mapping $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x}, \mathbf{y})$, which is a mapping from an open subset of $\mathbb{R}^n$ into $\mathbb{R}^k$; we denote the $k \times n$ derivative matrix of this mapping by $\mathbf{D_x F}(\mathbf{x}, \mathbf{y})$. Displayed in

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terms of their entries, these matrices are

$$\mathbf{D_y F}(\mathbf{x},\ \mathbf{y}) = \begin{bmatrix} \partial F_1/\partial y_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_1/\partial y_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_1/\partial y_k(\mathbf{x},\ \mathbf{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial F_i/\partial y_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_i/\partial y_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_i/\partial y_k(\mathbf{x},\ \mathbf{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial F_k/\partial y_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_k/\partial y_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_k/\partial y_k(\mathbf{x},\ \mathbf{y}) \end{bmatrix}$$

and

$$\mathbf{D_x F}(\mathbf{x},\ \mathbf{y}) = \begin{bmatrix} \partial F_1/\partial x_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_1/\partial x_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_1/\partial x_n(\mathbf{x},\ \mathbf{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial F_i/\partial x_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_i/\partial x_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_i/\partial x_n(\mathbf{x},\ \mathbf{y}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \partial F_k/\partial x_1(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_k/\partial x_j(\mathbf{x},\ \mathbf{y}) & \cdots & \partial F_k/\partial x_n(\mathbf{x},\ \mathbf{y}) \end{bmatrix}.$$

The following theorem, describing the solutions of equation (17.14), is a direct generalization of Dini's Theorem.

Theorem 17.6 The General Implicit Function Theorem Let n and k be positive integers, let $\mathcal{O}$ be an open subset of $\mathbb{R}^{n+k}$, and suppose that the mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^k$ is continuously differentiable. At the point $(\mathbf{x}_0,\ \mathbf{y}_0)$ in $\mathcal{O}$, suppose that $\mathbf{F}(\mathbf{x}_0,\ \mathbf{y}_0) = \mathbf{0}$ and that the $k \times k$ partial derivative matrix

$$\mathbf{D_y F}(\mathbf{x}_0,\ \mathbf{y}_0) \text{ is invertible.} \tag{17.17}$$

Then there is a positive number r and a continuously differentiable mapping $\mathbf{G}: \mathcal{B} \to \mathbb{R}^k$, where $\mathcal{B} = \mathcal{B}_r(\mathbf{x}_0)$, such that

$$\mathbf{F}(\mathbf{x},\ \mathbf{G}(\mathbf{x})) = \mathbf{0} \qquad \text{for all points } \mathbf{x} \text{ in } \mathcal{B}, \tag{i}$$

and

$$\text{whenever } \|\mathbf{x}-\mathbf{x}_0\| < r,\ \|\mathbf{y}-\mathbf{y}_0\| < r, \qquad \text{and} \qquad \mathbf{F}(\mathbf{x},\mathbf{y}) = \mathbf{0}, \qquad \text{then } \mathbf{y} = \mathbf{G}(\mathbf{x}). \tag{ii}$$

Moreover,

$$\mathbf{D_x F}(\mathbf{x},\ \mathbf{G}(\mathbf{x})) + \mathbf{D_y F}(\mathbf{x},\ \mathbf{G}(\mathbf{x})) \cdot \mathbf{DG}(\mathbf{x}) = \mathbf{0} \qquad \text{for all points } \mathbf{x} \text{ in } \mathcal{B}. \tag{iii}$$

Proof

We define an auxiliary mapping

$$\mathbf{H}: \mathcal{O} \to \mathbb{R}^{n+k}$$

by

$$\mathbf{H}(\mathbf{x},\ \mathbf{y}) = (\mathbf{x},\ \mathbf{F}(\mathbf{x},\ \mathbf{y})) \qquad \text{for } (\mathbf{x},\ \mathbf{y}) \text{ in } \mathcal{O}.$$

Observe that

$$\mathbf{F}(\mathbf{x},\ \mathbf{y}) = \mathbf{0} \qquad \text{if and only if } \mathbf{H}(\mathbf{x},\ \mathbf{y}) = (\mathbf{x},\ \mathbf{0}). \tag{17.18}$$

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Now the mapping $\mathbf{H}: \mathcal{O} \to \mathbb{R}^{n+k}$ is a continuously differentiable mapping between Euclidean spaces of the same dimension, so the Inverse Function Theorem can be applied in order to analyze its image, and therefore, because of the correspondence to (17.18), to analyze the points $(\mathbf{x}, \mathbf{y})$ at which $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

The derivative matrix $\mathbf{DH}(\mathbf{x}_0, \mathbf{y}_0)$ can be partitioned as

$$\mathbf{DH}(\mathbf{x}_0, \mathbf{y}_0) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{D}_{\mathbf{x}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) & \mathbf{D}_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) \end{bmatrix}, \tag{17.19}$$

where $\mathbf{I}_n$ denotes the $n \times n$ identity matrix and the upper right-hand matrix is the $n \times k$ matrix all of whose entries are 0. The assumption that $\mathbf{D}_{\mathbf{y}}\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0)$ is an invertible $k \times k$ matrix implies that the derivative matrix $\mathbf{DH}(\mathbf{x}_0, \mathbf{y}_0)$ is an invertible $(n+k) \times (n+k)$ matrix (Exercise 7). Hence we can apply the Inverse Function Theorem to the mapping $\mathbf{H}: \mathcal{O} \to \mathbb{R}^{n+k}$ at the point $(\mathbf{x}_0, \mathbf{y}_0)$ in order to conclude that there is a neighborhood U of $(\mathbf{x}_0, \mathbf{y}_0)$ in $\mathbb{R}^{n+k}$ and a neighborhood V of its image $\mathbf{H}(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{x}_0, \mathbf{0})$ in $\mathbb{R}^{n+k}$ such that $\mathbf{H}: U \to V$ is one-to-one and onto and that the inverse mapping $\mathbf{H}^{-1}: V \to U$ is also continuously differentiable.

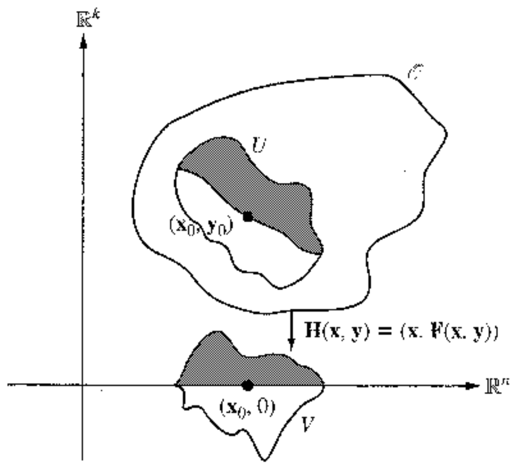

FIGURE 17.4 Projecting the $U \subseteq \mathbb{R}^{n+k}$ onto $\mathbb{R}^n$.

Write the inverse mapping as

$$\mathbf{H}^{-1}(\mathbf{x}, \mathbf{y}) = (\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{N}(\mathbf{x}, \mathbf{y})) \qquad \text{for } (\mathbf{x}, \mathbf{y}) \text{ in } V.$$

The definition of inverse mapping and the definition of $\mathbf{H}$ yields the following identity for all points $(\mathbf{x}, \mathbf{y})$ in V:

$$(\mathbf{x}, \mathbf{y}) = \left(\mathbf{H} \circ \mathbf{H}^{-1}\right)(\mathbf{x}, \mathbf{y}) = (\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{F}(\mathbf{M}(\mathbf{x}, \mathbf{y}), \mathbf{N}(\mathbf{x}, \mathbf{y}))). \tag{17.20}$$

But equating the first components in this identity, we see that

$$\mathbf{M}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \qquad \text{for all } (\mathbf{x}, \mathbf{y}) \text{ in } V,$$

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which, when we equate the second components of the same identity, gives

$$\mathbf{F}(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{y})) = \mathbf{y} \qquad \text{for all } (\mathbf{x}, \mathbf{y}) \text{ in } V. \tag{17.21}$$

Since the point $(\mathbf{x}_0, \mathbf{y}_0)$ belongs to V, an open subset of $\mathbb{R}^{n+k}$, we can choose $r > 0$ such that $(\mathbf{x}, \mathbf{0})$ belongs to V if the point $\mathbf{x}$ in $\mathbb{R}^n$ belongs to $\mathcal{B} = \mathcal{B}_r(\mathbf{x}_0)$, the open ball in $\mathbb{R}^n$ of radius r about $\mathbf{x}_0$. Define the mapping $\mathbf{G}: \mathcal{B} \to \mathbb{R}^k$ by

$$\mathbf{G}(\mathbf{x}) = \mathbf{N}(\mathbf{x}, \mathbf{0}) \qquad \text{for all } \mathbf{x} \text{ in } \mathcal{B}.$$

Then the mapping $\mathbf{G}: \mathcal{B} \to \mathbb{R}^k$ is continuously differentiable, and from 17.21 it follows that

$$\mathbf{F}(\mathbf{x}, \mathbf{G}(\mathbf{x})) = \mathbf{0} \qquad \text{for all } \mathbf{x} \text{ in } \mathcal{B}. \tag{17.22}$$

Thus, property (i) holds.

To verify property (ii), we use the fact that $\mathbf{H}: U \to V$ is one-to-one. Indeed, if the point $(\mathbf{x}, \mathbf{y})$ belongs to U and $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, then, since the point $(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{0})) = \mathbf{H}^{-1}(\mathbf{x}, \mathbf{0})$ also belongs to U and

$$\mathbf{H}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0}) = \mathbf{H}(\mathbf{x}, \mathbf{N}(\mathbf{x}, \mathbf{0})),$$

it follows that

$$\mathbf{y} = \mathbf{N}(\mathbf{x}, \mathbf{0}).$$

Thus, property (ii) holds.

Finally, we verify formula (iii) for the derivative matrix. Indeed, if we represent the mapping $\mathbf{F}$ in component functions as $\mathbf{F} = (\mathrm{F}_1, \ldots, \mathrm{F}_k)$, then equation (17.22) can be written in components as

$$\mathrm{F}_i(\mathbf{x}, \mathbf{G}(\mathbf{x})) = 0 \qquad \text{for all points } \mathbf{x} \text{ in } \mathcal{B} \text{ and all indices } i \text{ with } 1 \leq i \leq k.$$

Using the Chain Rule to differentiate the preceding system of equations with respect to the component x_j, we obtain

$$\frac{\partial \mathrm{F}_i}{\partial x_j}(\mathbf{x}, \mathbf{G}(\mathbf{x})) + \sum_{\ell=1}^{k} \frac{\partial \mathrm{F}_i}{\partial y_\ell}(\mathbf{x}, \mathbf{G}(\mathbf{x})) \frac{\partial G_\ell}{\partial x_j}(\mathbf{x}) = 0 \tag{17.23}$$

for all points $\mathbf{x}$ in $\mathcal{B}$ and pairs of indices i and j with $1 \leq i \leq k$ and $1 \leq j \leq n$. However, the matrix identity (iii) is just a rewriting in matrix notation of the above system (17.23). ∎

In the statement of the Implicit Function Theorem, we singled out the first n components and the last k components of a point $\mathbf{u}$ in $\mathcal{O}$. But, in fact, this separation of components was rather arbitrary. A $k \times (n+k)$ matrix is said to have *maximal rank* if it has a $k \times k$ submatrix that is invertible. The Implicit Function Theorem holds for a continuously differentiable mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^k$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^{n+k}$, at a point $\mathbf{u}_0$ in $\mathcal{O}$ at which $\mathbf{F}(\mathbf{u}_0) = \mathbf{0}$, provided that the derivative matrix $\mathbf{DF}(\mathbf{u}_0)$ has maximal rank. When this is so, we select a $k \times k$ submatrix of $\mathbf{DF}(\mathbf{u}_0)$ that is invertible

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and has column indices $j_1, \ldots, j_k$. Then the components $u_{j_1}, \ldots, u_{j_k}$ of the solutions $\mathbf{u}$ of the equation

$$\mathbf{F}(\mathbf{u}) = \mathbf{0}, \qquad \mathbf{u} \text{ in } \mathcal{O}$$

that lie in a neighborhood of the point $\mathbf{u}_0$ can be expressed as continuously differentiable functions of the remaining n components.

Example 17.7 Consider the system of equations

$$\begin{cases} \ln(1 + x^2 + t^2) + st + e^{s+z} - 1 = 0 \\ s^3 e^{\cos(x^2+z^2)} + s + 2z + (x + s + z)^4 = 0, \qquad (s, x, t, z) \text{ in } \mathbb{R}^4. \end{cases} \tag{17.24}$$

Observe that the point $(0, 0, 0, 0)$ is a solution of this system of equations. For a point (s, x, t, z) in $\mathbb{R}^4$, define

$$\mathbf{F}(s, x, t, z) = (\ln(1 + x^2 + t^2) + st + e^{s-z} - 1,\ s^3 e^{\cos(x^2+z^2)} \\ + s + 2z + (x + s + z)^4).$$

Then we can readily check that the mapping $\mathbf{F}: \mathbb{R}^4 \to \mathbb{R}^2$ is continuously differentiable and that its derivative matrix at the point $\mathbf{0} = (0, 0, 0, 0)$ is

$$\mathbf{DF}(\mathbf{0}) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Thus, the 2×2 matrix

$$\begin{bmatrix} \partial F_1/\partial s(0, 0, 0, 0) & \partial F_1/\partial z(0, 0, 0, 0) \\ \partial F_2/\partial s(0, 0, 0, 0) & \partial F_2/\partial z(0, 0, 0, 0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

is invertible. We apply the Implicit Function Theorem to choose a positive number r and continuously differentiable functions $g: \mathcal{B} \to \mathbb{R}$ and $h: \mathcal{B} \to \mathbb{R}$, where $\mathcal{B} = \mathcal{B}_r(0, 0)$, such that if $x^2 + t^2 < r^2$, then $(g(x, t), x, t, h(x, t))$ is a solution of the system of equations (17.24). Moreover, if the point (s, x, t, z) in $\mathbb{R}^4$ is a solution of the system (17.24) and if $s^2 + z^2 < r^2$ and $x^2 + t^2 < r^2$, then $s = g(x, t)$ and $z = h(x, t)$. ■

We will consider further examples of the Implicit Function Theorem in Section 17.3.

EXERCISES FOR SECTION 17.2

For Exercises 1 through 6, use the Implicit Function Theorem to analyze the solutions of the given systems of equations near the solution $\mathbf{0}$.

1. $$\begin{cases} (x^2 + y^2 + z^2)^3 - x + z = 0 \\ \cos(x^2 + y^4) + e^z - 2 = 0, \qquad (x, y, z) \text{ in } \mathbb{R}^3. \end{cases}$$

2. $$\begin{cases} a^3 + a^2 b + \sin(a + b + c) = 0 \\ \ln(1 + a^2) + 2a + (bc)^4 = 0, \qquad (a, b, c) \text{ in } \mathbb{R}^3. \end{cases}$$

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3. $\begin{cases} (uv)^4 + (u+s)^3 + t = 0 \\ \sin(uv) + e^{v+t^2} - 1 = 0, \qquad (u,\ v,\ s,\ t) \text{ in } \mathbb{R}^4. \end{cases}$

4. $\begin{cases} x + 2y + x^2 + (yz)^2 + t^3 = 0 \\ -x + z + \sin(y^2 + z^2 + t^3) = 0, \qquad (x,\ y,\ z,\ t) \text{ in } \mathbb{R}^4. \end{cases}$

5. $e^{x^2} + y^2 + z - 4xy^3 - 1 = 0, \qquad (x,\ y,\ z) \text{ in } \mathbb{R}^3.$

6. $e^{xy} + x^2 + 2y - 1 = 0, \qquad (x,\ y) \text{ in } \mathbb{R}^2.$

7. In the proof of the Implicit Function Theorem, it was asserted that the invertibility of the $k \times k$ matrix $\mathbf{D_y F}(\mathbf{x}_0,\ \mathbf{y}_0)$ implies the invertibility of the $(n+k) \times (n+k)$ matrix $\mathbf{DH}(\mathbf{x}_0,\ \mathbf{y}_0)$. Verify this assertion.

8. Consider the linear system of equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0, \qquad (x,\ y,\ z) \text{ in } \mathbb{R}^3. \end{cases}$$

Define η to be the vector $(a_{11},\ a_{12},\ a_{13})$ and β to be the vector $(a_{21},\ a_{22},\ a_{23})$.

a. Show that if $\eta \times \beta \neq \mathbf{0}$, then the above system of equations defines two of the variables as a function of the remaining variable.

b. Interpret (a) in the light of the geometry of lines and planes in $\mathbb{R}^3$.

9. Graph the solutions of the equation

$$y^3 - x^2 = 0, \qquad (x,\ y) \text{ in } \mathbb{R}^2.$$

Does the Implicit Function Theorem apply at the point $(0,\ 0)$? Does this equation define one of the components of a solution $(x,\ y)$ as a function of the other component?

17.3 EQUATIONS OF SURFACES AND PATHS IN $\mathbb{R}^3$

In this section we obtain, in the cases of one scalar equation in three scalar unknowns and of two scalar equations in three scalar unknowns, more detailed geometric information about the graphs of the mappings implicitly defined by the solutions of such systems of equations.

Proposition 17.8 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Assume that $(x_0,\ y_0,\ z_0)$ is a point in $\mathcal{O}$ at which

$$f(x_0,\ y_0,\ z_0) = 0 \qquad \text{and} \qquad \nabla f(x_0,\ y_0,\ z_0) \neq \mathbf{0}. \tag{17.25}$$

Then at the point $(x_0,\ y_0,\ z_0)$ the vector $\nabla f(x_0,\ y_0,\ z_0)$ is normal to the surface consisting of the solutions, in a neighborhood of the point $(x_0,\ y_0,\ z_0)$, of the equation

$$f(x,\ y,\ z) = 0, \qquad (x,\ y,\ z) \text{ in } \mathcal{O}.$$

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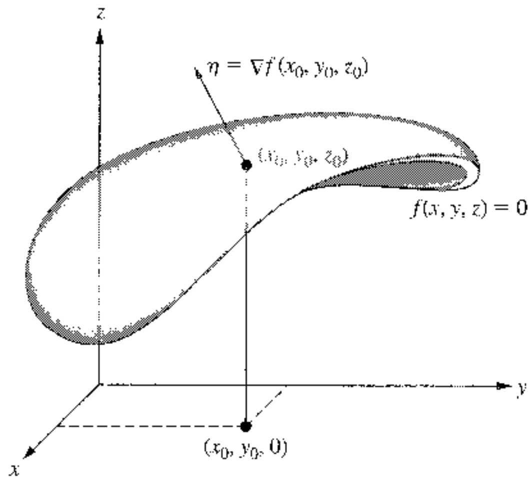

FIGURE 17.5 $\nabla f(x_0, y_0, z_0)$ is normal to the surface S at the point (x_0, y_0, z_0).

Proof

We can assume that it is the third component of $\nabla f(x_0, y_0, z_0)$ that is nonzero. The Implicit Function Theorem implies that there is a positive number r and a continuously differentiable function $g : \mathcal{B} \to \mathbb{R}$, where $\mathcal{B} = \mathcal{B}_r(x_0, y_0)$, such that

$$f(x, y, g(x, y)) = 0 \qquad \text{for all } (x, y) \text{ in } \mathcal{B} \tag{17.26}$$

and moreover that, in a neighborhood of the point (x_0, y_0, z_0), all the solutions belong to the graph of $g : \mathcal{B} \to \mathbb{R}$.

In Section 14.1, we showed that at the point (x_0, y_0, z_0) the vector

$$\eta = \left(-\frac{\partial g}{\partial x}(x_0, y_0), -\frac{\partial g}{\partial y}(x_0, y_0), 1\right)$$

is normal to the surface defined by the graph of the function $g : \mathcal{B} \to \mathbb{R}$. On the other hand, if we differentiate (17.26), first with respect to x and then with respect to y, we get the following two equations:

$$\frac{\partial f}{\partial x}(x_0, y_0, g(x_0, y_0)) + \frac{\partial f}{\partial z}(x_0, y_0, g(x_0, y_0))\frac{\partial g}{\partial x}(x_0, y_0) = 0$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0, g(x_0, y_0)) + \frac{\partial f}{\partial z}(x_0, y_0, g(x_0, y_0))\frac{\partial g}{\partial y}(x_0, y_0) = 0,$$

to which we can adjoin the identity

$$\frac{\partial f}{\partial z}(x_0, y_0, g(x_0, y_0)) + \frac{\partial f}{\partial z}(x_0, y_0, g(x_0, y_0))(-1) = 0.$$

The last three equations can be written in vector form as

$$\nabla f(x_0, y_0, z_0) = \alpha\eta,$$

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where $\alpha = \partial f/\partial z(x_0, y_0, z_0)$. Since $\alpha \neq 0$, this means that at the point (x_0, y_0, z_0), the vector $\nabla f(x_0, y_0, z_0)$ is normal to the surface consisting of the solutions of the equation $f(x, y, z) = 0$ in a neighborhood of the point (x_0, y_0, z_0). ∎

Example 17.9 The set of solutions of the equation

$$x^2 + y^2 + z^2 - 1 = 0, \qquad (x, y, z) \text{ in } \mathbb{R}^3,$$

is the sphere of radius 1 centered at the origin. Define $f(x, y, z) = x^2 + y^2 + z^2 - 1$ for (x, y, z) in $\mathbb{R}^3$ and observe that for a point (x_0, y_0, z_0) on this sphere,

$$\nabla f(x_0, y_0, z_0) = (2x_0, 2y_0, 2z_0).$$

Proposition 17.8 implies that the vector $(2x_0, 2y_0, 2z_0)$ is normal to the sphere at the point (x_0, y_0, z_0). This confirms what is already geometrically clear.

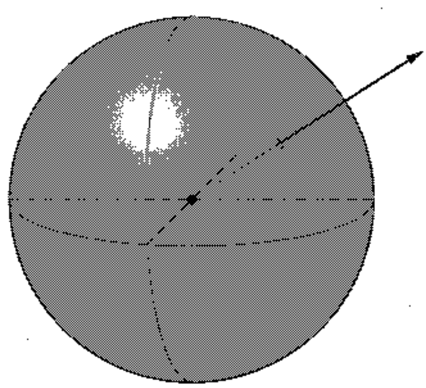

FIGURE 17.6 The normal to the unit sphere at a point. ∎

Let us now examine the set of solutions of a system of two equations in three unknowns. Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^3$ and that the functions $g : \mathcal{O} \to \mathbb{R}$ and $h : \mathcal{O} \to \mathbb{R}$ are continuously differentiable. Consider the system of equations

$$\begin{cases} g(x, y, z) = 0 \\ h(x, y, z) = 0, \qquad (x, y, z) \text{ in } \mathcal{O}. \end{cases} \tag{17.27}$$

Suppose that the point (x_0, y_0, z_0) is a solution of this system and also assume that

$$\nabla g(x_0, y_0, z_0) \neq \mathbf{0} \qquad \text{and} \qquad \nabla h(x_0, y_0, z_0) \neq \mathbf{0}.$$

Define

$$S_1 = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid g(x, y, z) = 0\}$$

and

$$S_2 = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid h(x, y, z) = 0\},$$

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so that the set of solutions of the system of equations (17.27) consists of the intersection of the sets S_1 and S_2. Because of Proposition 17.8, in a neighborhood of the point (x_0, y_0, z_0), S_1 is a surface having $\nabla g(x_0, y_0, z_0)$ as a normal at (x_0, y_0, z_0); S_2 is a surface having $\nabla h(x_0, y_0, z_0)$ as a normal at (x_0, y_0, z_0). *If these two normals are not parallel, these surfaces should intersect in a path having* $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ *as a tangent vector at the point* (x_0, y_0, z_0). These normals are nonparallel precisely when

$$\nabla h(x_0, y_0, z_0) \times \nabla g(x_0, y_0, z_0) \neq \mathbf{0}.$$

The following proposition justifies the above geometric argument.

Proposition 17.10 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and suppose that the functions $g : \mathcal{O} \to \mathbb{R}$ and $h : \mathcal{O} \to \mathbb{R}$ are continuously differentiable. Let (x_0, y_0, z_0) be a point in $\mathcal{O}$ at which

$$g(x_0, y_0, z_0) = h(x_0, y_0, z_0) = 0$$

and

$$\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0) \neq \mathbf{0}. \tag{17.28}$$

Then there is a neighborhood of the point (x_0, y_0, z_0) in which the set of solutions of the system

$$\begin{cases} g(x, y, z) = 0 \\ h(x, y, z) = 0, \qquad (x, y, z) \text{ in } \mathcal{O}, \end{cases} \tag{17.29}$$

consists of a path that at the point (x_0, y_0, z_0) has the vector $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ as a tangent vector.

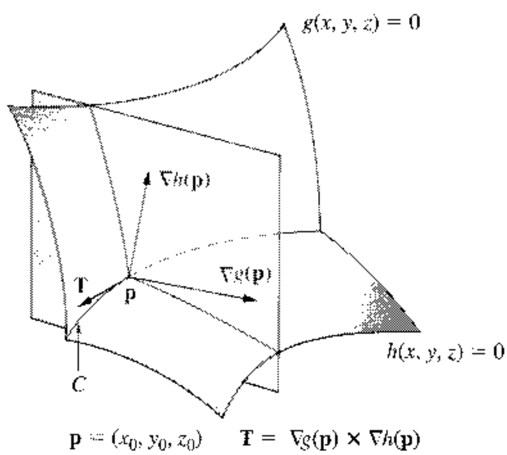

FIGURE 17.7 **T** is tangent to the curve.

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Proof

Since the vector $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ is nonzero, at least one of its components is nonzero. We suppose that the first component is nonzero; that is,

$$\frac{\partial g}{\partial y}(x_0, y_0, z_0)\frac{\partial h}{\partial z}(x_0, y_0, z_0) - \frac{\partial g}{\partial z}(x_0, y_0, z_0)\frac{\partial h}{\partial y}(x_0, y_0, z_0) \neq 0. \qquad (17.30)$$

Observe that this means that the 2×2 matrix

$$\begin{bmatrix} \partial g/\partial y(x_0, y_0, z_0) & \partial g/\partial z(x_0, y_0, z_0) \\ \partial h/\partial y(x_0, y_0, z_0) & \partial h/\partial z(x_0, y_0, z_0) \end{bmatrix}$$

is invertible. Thus, we can apply the Implicit Function Theorem at the point (x_0, y_0, z_0) to the continuously differentiable mapping $\mathbf{F}: \mathcal{O} \to \mathbb{R}^2$ defined by

$$\mathbf{F}(x, y, z) = (g(x, y, z), h(x, y, z)) \qquad \text{for } (x, y, z) \text{ in } \mathcal{O}.$$

It follows that there is an open interval I containing the point x_0 and two continuously differentiable functions

$$\alpha : I \to \mathbb{R} \qquad \text{and} \qquad \beta : I \to \mathbb{R}$$

such that

$$\mathbf{F}(x, \alpha(x), \beta(x)) = \mathbf{0} \qquad \text{for all } x \text{ in } I,$$

and in a neighborhood of the point (x_0, y_0, z_0), all the solutions of the system are of the form $(x, \alpha(x), \beta(x))$ for x in I. Consequently, near (x_0, y_0, z_0), the set of solutions of the system (17.29) coincides with the image of the parametrized path $\gamma : I \to \mathbb{R}^3$ defined by

$$\gamma(t) = (t, \alpha(t), \beta(t)) \qquad \text{for } t \text{ in } I.$$

The tangent vector to this path at $t = x_0$ is

$$\mathbf{T} = \gamma'(x_0) = (1, \alpha'(x_0), \beta'(x_0)).$$

Now since $g(t, \alpha(t), \beta(t)) = 0$ and $h(t, \alpha(t), \beta(t)) = 0$ for all t in I, we can differentiate each of these identities to obtain

$$\langle \nabla g(x_0, y_0, z_0), \mathbf{T} \rangle = 0$$

and

$$\langle \nabla h(x_0, y_0, z_0), \mathbf{T} \rangle = 0.$$

Hence $\mathbf{T}$ is orthogonal to both $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$, so

$$\mathbf{T} = \alpha(\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0))$$

for some $\alpha \neq 0$. Thus, $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ is tangent to the path of solutions of (17.29) at (x_0, y_0, z_0). ■

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Example 17.11 The set of solutions of the system of equations

$$\begin{cases} x - y = 0 \\ x^2 + y^2 + z^2 - 2 = 0, \qquad (x,\ y,\ z) \text{ in } \mathbb{R}^3, \end{cases}$$

consists of the intersection of a sphere and a plane. The point $(1, 1, 0)$ is a solution. By the preceding proposition and a brief computation of partial derivatives, we conclude that the vector $(0, 0, 4)$ is tangent at the point $(1,\ 1,\ 0)$ to the path of solutions of this system. ■

EXERCISES FOR SECTION 17.3

1. Consider the system of equations

$$\begin{cases} (x-1)^2 + (y-1)^2 + z^2 - 2 = 0 \\ (x+1)^2 + (y-2)^2 + z^2 - 5 = 0, \qquad (x,\ y,\ z) \text{ in } \mathbb{R}^3. \end{cases}$$

   a. Apply Proposition 17.10 at the point $(0, 0, 0)$ to find the tangent line to the path defined by the solutions of this system near $(0, 0, 0)$.
   b. Solve this system of equations near $(0, 0, 0)$ and explicitly check the tangent to the path of solutions.

2. Explicitly find the circle that is the set of solutions of the system in Example 17.11 and directly find the tangent to this circle at the point $(1,\ 1,\ 0)$.

3. Consider the system of equations

$$\begin{cases} (x-1)^2 + y^2 + z^2 - 1 = 0 \\ (x-1/4)^2 + y^2 + z^2 - 1/16 = 0, \qquad (x,\ y,\ z) \text{ in } \mathbb{R}^3. \end{cases}$$

   a. Show that assumption (17.28) of Proposition 17.10 is not satisfied at the point $(0, 0, 0)$.
   b. By graphing each of the surfaces defined by the individual equations in the system, explain why there is exactly one solution of this system.

4. Construct examples demonstrating that when $1 \le i \le 3$ and the ith component of the vector $\nabla g(x_0,\ y_0,\ z_0) \times \nabla h(x_0,\ y_0,\ z_0)$ is zero, then it may be that the path of solutions of (17.29) passing through $(x_0,\ y_0,\ z_0)$ cannot be parametrized by the ith component.

5. Suppose that the functions $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^3 \to \mathbb{R}$, and $h : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable and let $(x_0,\ y_0,\ z_0)$ be a point in $\mathbb{R}^3$ at which

$$f(x_0,\ y_0,\ z_0) = g(x_0,\ y_0,\ z_0) = h(x_0,\ y_0,\ z_0) = 0$$

   and

$$\langle \nabla f(x_0,\ y_0,\ z_0),\ \nabla g(x_0,\ y_0,\ z_0) \times \nabla h(x_0,\ y_0,\ z_0)\rangle \neq 0.$$

   By considering the set of solutions of this system as consisting of the intersection of a surface with a path, explain why that in a neighborhood of the point $(x_0,\ y_0,\ z_0)$

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the system of equations

$$\begin{aligned} f(x, y, z) &= 0 \\ g(x, y, z) &= 0 \\ h(x, y, z) &= 0, \qquad (x, y, z) \text{ in } \mathbb{R}^3, \end{aligned}$$

has exactly one solution. Also explain this by using the Inverse Function Theorem.

17.4 CONSTRAINED EXTREMA PROBLEMS AND LAGRANGE MULTIPLIERS

A point in the domain of a real-valued function is called an *extremum,* or an *extreme point,* for the function if the function attains either a maximum or a minimum value at that point. If $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives, then for the point $\mathbf{x}$ in $\mathcal{O}$ to be an extreme point for the function $f : \mathcal{O} \to \mathbb{R}$, it is necessary that

$$\nabla f(\mathbf{x}) = \mathbf{0}. \tag{17.31}$$

The Second-Derivative Test, established in Section 14.3, prescribes sufficient conditions for such a point $\mathbf{x}$ to be an extreme point for the function $f : \mathcal{O} \to \mathbb{R}$. There is no assertion regarding the *existence* of extrema for a real-valued function defined on an open subset of $\mathbb{R}^n$. However, in Section 11.2, we proved that a continuous function defined on a closed bounded subset of $\mathbb{R}^n$ attains a maximum and a minimum value.

Now suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, that the function $f : \mathcal{O} \to \mathbb{R}$ has first-order partial derivatives, and that K is a subset of $\mathcal{O}$. Let $\mathbf{x}$ be a point in K that is an extreme point for the restricted function $f : K \to \mathbb{R}$. Such an extreme point $\mathbf{x}$ is called a *constrained extremum* for the function $f : \mathcal{O} \to \mathbb{R}$ because the values of the function $f : \mathcal{O} \to \mathbb{R}$ with which we can compare $f(\mathbf{x})$ are constrained to functional values of points in K; the set K is called the *constraint set*. We note that if K is closed and bounded then constrained extrema do, in fact, exist.

It is not true that at a constrained extremum of a function all the partial derivatives must be zero since, while the set $\mathcal{O}$ is an open subset of $\mathbb{R}^n$, the constraint set K need not be open in $\mathbb{R}^n$. The conditions that are necessarily satisfied by the partial derivatives at a constrained extremum depend on the nature of the constraint set. We devote this section to describing what these conditions are. Let us begin with two examples.

Example 17.12 Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = z$ for all (x, y, z) in $\mathbb{R}^3$. Choose as the constraint set the unit sphere $K = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Then clearly the point $(0, 0, 1)$ is a maximizer for $f : K \to \mathbb{R}$. However, we have

$$\frac{\partial f}{\partial x}(0, 0, 1) = 0, \qquad \frac{\partial f}{\partial y}(0, 0, 1) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial z}(0, 0, 1) = 1, \tag{17.32}$$

so (17.31) is not satisfied at this constrained extremum since $\partial f / \partial z(0, 0, 1) \neq 0$.

■

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Example 17.13 Define the function $f(x, y, z) = (x - 4y + 3z) + z^2$ for (x, y, z) in $\mathbb{R}^3$. Choose as the constraint set the line $K = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x = y = z\}$. Then clearly the point $(0, 0, 0)$ is a minimizer for the function $f : K \to \mathbb{R}$. We have

$$\frac{\partial f}{\partial x}(0, 0, 0) = 1, \qquad \frac{\partial f}{\partial y}(0, 0, 0) = -4, \qquad \text{and} \qquad \frac{\partial f}{\partial z}(0, 0, 0) = 3, \tag{17.33}$$

so (17.31) is not satisfied at this constrained extremum since in fact none of the components of $\nabla f(0, 0, 0)$ are zero. ∎

In the first example, the constraint set is a surface in $\mathbb{R}^3$; in the second example, the constraint set is a path in $\mathbb{R}^3$. We now turn to describing the conditions that must be satisfied by the partial derivatives of a function at a constrained extremum. First we consider the case where the constraint set K is a surface in $\mathbb{R}^3$. Then we consider the case where the constraint set K is a path in $\mathbb{R}^3$. With these two cases understood, we will then be in a position to appreciate the significance of a general theorem on constrained extrema.

Theorem 17.14 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and suppose that the functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are continuously differentiable. Define

$$S = \{(x, y, z) \text{ in } \mathcal{O} \mid g(x, y, z) = 0\}.$$

Suppose that the point (x_0, y_0, z_0) in S is an extreme point for the function

$$f : S \to \mathbb{R}$$

and that

$$\nabla g(x_0, y_0, z_0) \neq \mathbf{0}. \tag{17.34}$$

Then there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0). \tag{17.35}$$

Proof

Assumption (17.34) means that at least one of the components of the derivative vector $\nabla g(x_0, y_0, z_0)$ is nonzero. We suppose that it is the third component; that is,

$$\frac{\partial g}{\partial z}(x_0, y_0, z_0) \neq 0.$$

According to the Implicit Function Theorem, there is a neighborhood $\mathcal{N}$ of the point (x_0, y_0) in $\mathbb{R}^2$ and a continuously differentiable function $\phi : \mathcal{N} \to \mathbb{R}$ such that $\phi(x_0, y_0) = z_0$ and

$$g(x, y, \phi(x, y)) = 0 \qquad \text{for all } (x, y) \text{ in } \mathcal{N}. \tag{17.36}$$

Thus, the graph of the function $\phi : \mathcal{N} \to \mathbb{R}$ lies in the constraint set S.

Define an auxiliary function $\psi : \mathcal{N} \to \mathbb{R}$ by

$$\psi(x, y) = f(x, y, \phi(x, y)) \qquad \text{for all } (x, y) \text{ in } \mathcal{N}. \tag{17.37}$$

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Since the graph of the function $\phi : \mathcal{N} \to \mathbb{R}$ lies in the constraint set S, it follows that the point (x_0, y_0) is an extreme point of the function $\psi : \mathcal{N} \to \mathbb{R}$. Since $\mathcal{N}$ is an open subset of $\mathbb{R}^2$, the point (x_0, y_0) is an *unconstrained extremum* of the function of two variables $\psi : \mathcal{N} \to \mathbb{R}$. Thus,

$$\begin{aligned} \frac{\partial \psi}{\partial x}(x_0, y_0) &= 0 \\ \frac{\partial \psi}{\partial y}(x_0, y_0) &= 0. \end{aligned} \tag{17.38}$$

Using the Chain Rule to express the partial derivatives of the function $\psi : \mathcal{N} \to \mathbb{R}$ in terms of the partial derivatives of the function $f : \mathcal{O} \to \mathbb{R}$, we rewrite the equations (17.38) as

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, y_0, z_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\frac{\partial \phi}{\partial x}(x_0, y_0) &= 0 \\ \frac{\partial f}{\partial y}(x_0, y_0, z_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\frac{\partial \phi}{\partial y}(x_0, y_0) &= 0. \end{aligned} \tag{17.39}$$

On the other hand, by differentiating the identity (17.36), first with respect to x and then with respect to y, we obtain

$$\begin{aligned} \frac{\partial g}{\partial x}(x_0, y_0, z_0) + \frac{\partial g}{\partial z}(x_0, y_0, z_0)\frac{\partial \phi}{\partial x}(x_0, y_0) &= 0 \\ \frac{\partial g}{\partial y}(x_0, y_0, z_0) + \frac{\partial g}{\partial z}(x_0, y_0, z_0)\frac{\partial \phi}{\partial y}(x_0, y_0) &= 0. \end{aligned} \tag{17.40}$$

Now define

$$\lambda = \frac{\partial f/\partial z(x_0, y_0, z_0)}{\partial g/\partial z(x_0, y_0, z_0)}. \tag{17.41}$$

From the first equations in (17.39) and (17.40), respectively, we obtain

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \lambda \frac{\partial g}{\partial x}(x_0, y_0, z_0); \tag{17.42}$$

from the second equations in (17.39) and (17.40), respectively, we obtain

$$\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \lambda \frac{\partial g}{\partial y}(x_0, y_0, z_0). \tag{17.43}$$

Finally, equations (17.41), (17.42), and (17.43) can be written in vector form as

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0). \qquad \blacksquare$$

Returning to the extremum problem in Example 17.11, if for a point (x, y, z) in $\mathbb{R}^3$ we define

$$f(x, y, z) = z \qquad \text{and} \qquad g(x, y, z) = x^2 + y^2 + z^2 - 1$$

at the constrained extremum $(0, 0, 1)$,

$$\nabla g(0, 0, 1) = (0, 0, 2) \qquad \text{and} \qquad \nabla f(0, 0, 1) = (0, 0, 1),$$

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so with $\lambda = 1/2$,

$$\nabla f(0, 0, 1) = \lambda \nabla g(0, 0, 1).$$

Example 17.15 Suppose we wish to find the minimum value of $x + y + 2z$ on the set

$$K = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

Define

$$f(x, y, z) = x + y + 2z \qquad \text{for } (x, y, z) \text{ in } \mathbb{R}^3$$

and observe that since K is sequentially compact, there is a minimizer (x_0, y_0, z_0) for the function $f : K \to \mathbb{R}$. This minimizer cannot lie in the interior of K, since if it did, we would have

$$\nabla f(x_0, y_0, z_0) = (0, 0, 0).$$

But $\nabla f(x_0, y_0, z_0) = (1, 1, 2)$. Thus, the point (x_0, y_0, z_0) lies on the boundary of K; that is, $x_0^2 + y_0^2 + z_0^2 = 1$. If we define $S = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0\}$, the point (x_0, y_0, z_0) is a minimizer of the function $f : S \to \mathbb{R}$. We can apply Theorem 17.14 to assert that there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0);$$

that is, $(1, 1, 2) = \lambda(2x_0, 2y_0, 2z_0)$. Thus, $2x_0 = 2y_0 = z_0$, and since $x_0^2 + y_0^2 + z_0^2 = 1$, the minimum occurs at the point $-(1/\sqrt{6}, 1/\sqrt{6}, \sqrt{2}/\sqrt{3})$. ∎

Theorem 17.14 provides necessary conditions for a point to be a constrained extremum of a function of three variables that is constrained to a *surface*. The next theorem gives sufficient conditions in the case where the constraint set is a *path*.

Theorem 17.16 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^3$ and suppose that the functions $f : \mathcal{O} \to \mathbb{R}$, $g : \mathcal{O} \to \mathbb{R}$, and $h : \mathcal{O} \to \mathbb{R}$ are continuously differentiable. Define

$$C = \{(x, y, z) \text{ in } \mathcal{O} \mid g(x, y, z) = h(x, y, z) = 0\}.$$

Suppose that the point (x_0, y_0, z_0) in C is an extreme point for the function

$$f : C \to \mathbb{R}$$

and that

$$\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0) \neq \mathbf{0}. \tag{17.44}$$

Then there are numbers λ and μ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0). \tag{17.45}$$

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Proof

Since the vector $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ is nonzero, at least one of its components is nonzero. We suppose that it is the first component; that is,

$$\frac{\partial g}{\partial y}(x_0, y_0)\frac{\partial h}{\partial z}(x_0, y_0) - \frac{\partial g}{\partial z}(x_0, y_0)\frac{\partial h}{\partial y}(x_0, y_0) \neq 0.$$

The Implicit Function Theorem, in the form proved as Proposition 17.10, asserts that there is a neighborhood I of the point x_0 in $\mathbb{R}$, and continuously differentiable functions $\alpha : I \to \mathbb{R}$ and $\beta : I \to \mathbb{R}$, such that $\alpha(x_0) = y_0$, $\beta(x_0) = z_0$, and

$$\begin{cases} g(x, \alpha(x), \beta(x)) = 0 \\ h(x, \alpha(x), \beta(x)) = 0 \qquad \text{for all } x \text{ in } I. \end{cases} \tag{17.46}$$

This means that if we define the parametrized path $\gamma : I \to \mathbb{R}$ by

$$\gamma(t) = (t, \alpha(t), \beta(t)) \qquad \text{for } t \text{ in } I,$$

then the image of this parametrized path lies in the constraint set C. Hence if we define the auxiliary function $\psi : I \to \mathbb{R}$ by

$$\psi(t) = (f \circ \gamma)(t) = f(t, \alpha(t), \beta(t)) \qquad \text{for } t \text{ in } I,$$

then this function $\psi : I \to \mathbb{R}$ attains an extreme value at the point x_0. However, I is an open subset of $\mathbb{R}$, so the point x_0 is an *unconstrained extremum* of the function of a single variable $\psi : I \to \mathbb{R}$. Thus,

$$\psi'(x_0) = 0,$$

which, because of the Chain Rule, means that

$$\psi'(x_0) = \langle \nabla f(x_0, y_0, z_0), \gamma'(x_0)\rangle = 0. \tag{17.47}$$

According to Proposition 17.10, the vector $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$ is a nonzero multiple of the tangent vector $\gamma'(x_0)$. Hence (17.47) implies that

$$\langle \nabla f(x_0, y_0, z_0), \nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)\rangle = 0; \tag{17.48}$$

that is, the vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the vector cross-product $\nabla g(x_0, y_0, z_0) \times \nabla h(x_0, y_0, z_0)$. But the only vectors that are perpendicular to this nonzero cross-product are vectors that are linear combinations of the vectors $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. This means that (17.45) holds. ∎

Observe that the constrained extremum problem in Example 17.13 is of the form described by the preceding theorem. Indeed, for a point (x, y, z) in $\mathbb{R}^3$, define

$$f(x, y, z) = x - 4y + 3z + z^2, \qquad g(x, y, z) = x - y, \qquad \text{and} \qquad h(x, y, z) = y - z.$$

Then the constraint set is given by

$$K = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid x = y = z\} = \{(x, y, z) \text{ in } \mathbb{R}^3 \mid g(x, y, z) = h(x, y, z) = 0\}.$$

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At the point $(0,\ 0,\ 0)$, which is a minimizer for $f : K \to \mathbb{R}$, we have

$$\nabla f(0,\ 0,\ 0) = (1,\ -4,\ 3), \qquad \nabla g(0,\ 0,\ 0) = (1,\ -1,\ 0), \qquad \text{and}$$
$$\nabla h(0,\ 0,\ 0) = (0,\ 1,\ -1),$$

so, setting $\lambda = 1$ and $\mu = -3$, we have

$$\nabla f(0,\ 0,\ 0) = \lambda \nabla g(0,\ 0,\ 0) + \mu \nabla h(0,\ 0,\ 0).$$

We arrived at our formulations of Theorems 17.14 and 17.16 aided by geometric reasoning about paths and surfaces in $\mathbb{R}^3$. The following is the general constrained extremum result, which includes these two results as particular cases.

Theorem 17.17 The General Lagrange Multiplier Theorem Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the function $f : \mathcal{O} \to \mathbb{R}$ is continuously differentiable. Let k be a positive integer less than n and suppose that the mapping $\mathbf{G} : \mathcal{O} \to \mathbb{R}^k$ is also continuously differentiable. Define

$$S = \{\mathbf{x} \text{ in } \mathcal{O} \mid \mathbf{G}(\mathbf{x}) = \mathbf{0}\}.$$

Suppose that the point $\mathbf{u}$ in S is an extreme point for the function

$$f : S \to \mathbb{R}$$

and that the $k \times n$ matrix

$$\mathbf{DG}(\mathbf{u}) \text{ has maximal rank.} \tag{17.49}$$

Then there are k numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\nabla f(\mathbf{u}) = \sum_{i=1}^{k} \lambda_i \nabla \mathbf{G}_i(\mathbf{u}). \tag{17.50}$$

Proof

Let $n = m + k$ and write points in $\mathbb{R}^n$ as $(\mathbf{x},\ \mathbf{y})$, where $\mathbf{x}$ is in $\mathbb{R}^m$ and $\mathbf{y}$ is in $\mathbb{R}^k$. Since by assumption the $k \times n$ matrix $\mathbf{DG}(\mathbf{u})$ has rank k, we can, by relabeling components if necessary, suppose that at the point $\mathbf{u} = (\mathbf{x}_0,\ \mathbf{y}_0)$ the $k \times k$ matrix

$$\mathbf{D_y G}(\mathbf{x}_0,\ \mathbf{y}_0) \text{ is invertible.} \tag{17.51}$$

According to the Implicit Function Theorem, there is a neighborhood $\mathcal{N}$ of the point $\mathbf{x}_0$ in $\mathbb{R}^m$ and a mapping $\psi : \mathcal{N} \to \mathbb{R}^k$ such that $\mathbf{y}_0 = \psi(\mathbf{x}_0)$ and

$$\mathbf{G}(\mathbf{x},\ \psi(\mathbf{x})) = \mathbf{0} \qquad \text{for } \mathbf{x} \text{ in } \mathcal{N}. \tag{17.52}$$

Thus, the graph of the mapping $\psi : \mathcal{N} \to \mathbb{R}^k$ lies in the constraint set S.

Define an auxiliary function $\eta : \mathcal{N} \to \mathbb{R}$ by

$$\eta(\mathbf{x}) = f(\mathbf{x},\ \psi(\mathbf{x})) \qquad \text{for } \mathbf{x} \text{ in } \mathcal{N}.$$

Observe that since $\mathcal{N}$ is an open subset of $\mathbb{R}^m$, the point $\mathbf{x}_0$ is an *unconstrained extremum* of the function $\eta : \mathcal{N} \to \mathbb{R}$. Thus,

$$\nabla \eta(\mathbf{x}_0) = \mathbf{0}. \tag{17.53}$$

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Hence, from the definition of the function $\eta : \mathcal{N} \to \mathbb{R}$ and the Chain Rule, we have

$$\nabla_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{y}_0) + \nabla_{\mathbf{y}} f(\mathbf{x}_0, \mathbf{y}_0)\mathbf{D}\psi(\mathbf{x}_0) = \mathbf{0}. \tag{17.54}$$

On the other hand, differentiating the identity (17.52), it follows that

$$\mathbf{D}_{\mathbf{x}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0) + \mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)\mathbf{D}\psi(\mathbf{x}_0) = \mathbf{0}. \tag{17.55}$$

But the $k \times k$ matrix $\mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)$ is invertible, so (17.55) yields

$$-[\mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)]^{-1}\mathbf{D}_{\mathbf{x}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{D}\psi(\mathbf{x}_0),$$

which, when substituted in (17.54), yields

$$\nabla_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{y}_0) = \nabla_{\mathbf{y}} f(\mathbf{x}_0, \mathbf{y}_0)[\mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)]^{-1}\mathbf{D}_{\mathbf{x}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0). \tag{17.56}$$

To verify (17.50), first observe that for the $1 \times k$ row matrix $[\lambda_1, \ldots, \lambda_k]$, (17.50) is equivalent to the matrix identity

$$\nabla f(\mathbf{x}_0, \mathbf{y}_0) = [\lambda_1, \ldots, \lambda_k]\mathbf{D}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0),$$

which in turn can be written in components as

$$\begin{aligned} \nabla_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{y}_0) &= [\lambda_1, \ldots, \lambda_k]\mathbf{D}_{\mathbf{x}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0) \\ \nabla_{\mathbf{y}} f(\mathbf{x}_0, \mathbf{y}_0) &= [\lambda_1, \ldots, \lambda_k]\mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0). \end{aligned} \tag{17.57}$$

Define

$$[\lambda_1, \ldots, \lambda_k] = \nabla_{\mathbf{y}} f(\mathbf{x}_0, \mathbf{y}_0)[\mathbf{D}_{\mathbf{y}}\mathbf{G}(\mathbf{x}_0, \mathbf{y}_0)]^{-1}. \tag{17.58}$$

This definition ensures that the second equation in the system (17.57) is satisfied. On the other hand, equation (17.56) is the assertion that this choice of $[\lambda_1, \ldots, \lambda_k]$ also satisfies the first equation in the system (17.57). ■

The λ_i's in formula (17.50) are often referred to as *Lagrange multipliers*.

We have the following immediate corollary of the General Lagrange Multiplier Theorem.

Corollary 17.18 Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and suppose that the functions $f : \mathcal{O} \to \mathbb{R}$ and $g : \mathcal{O} \to \mathbb{R}$ are continuously differentiable. Define

$$S = \{\mathbf{x} \text{ in } \mathcal{O} \mid g(\mathbf{x}) = 0\}.$$

Suppose that the point $\mathbf{u}$ in S is an extreme point for the function $f : S \to \mathbb{R}$ and that

$$\nabla g(\mathbf{u}) \neq \mathbf{0}.$$

Then there is a number λ such that

$$\nabla f(\mathbf{u}) = \lambda \nabla g(\mathbf{u}). \tag{17.59}$$

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The above corollary has the following interesting application to the question about the existence of eigenvalues of matrices. Recall that for an $n \times n$ matrix $\mathbf{A}$ of real numbers, the real number λ is called an *eigenvalue* of $\mathbf{A}$ provided that there is some nonzero point $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

In general, there may be no such real eigenvalues (Exercise 11). Recall that a matrix $\mathbf{A}$ is called *symmetric* provided that $\mathbf{A} = \mathbf{A}^{\mathbf{T}}$. For symmetric matrices, we have the following result.

Proposition 17.19 Every symmetric matrix has a real eigenvalue.

Proof

Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Define the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle - 1 \qquad \text{and} \qquad f(\mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle \qquad \text{for } \mathbf{x} \text{ in } \mathbb{R}^n$$

and then define

$$S = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid g(\mathbf{x}) = 0\}.$$

The set S consists of all points in $\mathbb{R}^n$ of norm 1, so it is closed and bounded. Therefore, since the function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, it follows from the Extreme Value Theorem that the function $f : S \to \mathbb{R}$ assumes a minimum value at a point $\mathbf{x}$ in S. From the preceding corollary, we conclude that there is a number λ such that

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}). \tag{17.60}$$

However, we can explicitly compute $\nabla f(\mathbf{x})$ and $\nabla g(\mathbf{x})$. Indeed, for an index i with $1 \le i \le n$, we have

$$\begin{aligned}
\frac{\partial f}{\partial x_i}(\mathbf{x}) &= \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t} \\
&= \lim_{t \to 0} \frac{\langle \mathbf{A}(\mathbf{x} + t\mathbf{e}_i),\ \mathbf{x} + t\mathbf{e}_i \rangle - \langle \mathbf{A}x, \mathbf{x} \rangle}{t} \\
&= \lim_{t \to 0} \frac{t\langle \mathbf{A}x,\ \mathbf{e}_i \rangle + t\langle \mathbf{A}e_i,\ \mathbf{x} \rangle + t^2\langle \mathbf{A}e_i,\ \mathbf{e}_i \rangle}{t} \\
&= \langle \mathbf{A}x,\ \mathbf{e}_i \rangle + \langle \mathbf{x},\ \mathbf{A}e_i \rangle \\
&= \langle \mathbf{A}x,\ \mathbf{e}_i \rangle + \langle \mathbf{A}^{\mathbf{T}}\mathbf{x},\ \mathbf{e}_i \rangle.
\end{aligned}$$

But $\mathbf{A}$ is a symmetric matrix, meaning that $\mathbf{A} = \mathbf{A}^{\mathbf{T}}$, so

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = 2\langle \mathbf{A}x,\ \mathbf{e}_i \rangle.$$

It follows that

$$\nabla f(\mathbf{x}) = 2\mathbf{A}x.$$

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When $\mathbf{A} = \mathbf{I}_n$, the above formula reads

$$\nabla g(\mathbf{x}) = 2\mathbf{x}.$$

Substituting $\nabla f(\mathbf{x}) = 2\mathbf{A}x$ and $\nabla g(\mathbf{x}) = 2\mathbf{x}$ in formula (17.60), we see that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{and} \qquad \|\mathbf{x}\| = 1. \qquad \blacksquare$$

EXERCISES FOR SECTION 17.4

1. Find the minimum of $\{x + y \mid x^2 + y^2 = 1\}$.
2. Find the maximum of $\{x^2 + y^2 \mid y^2 + x^2 + z^2 = 6\}$ by inspection and by using Lagrange multipliers.
3. Find the maximum of $\{x + y + z \mid |x| + |y| + |z| \leq 1\}$ by inspection.
4. Find the maximum of $\{x^2 + y^2 + z^2 \mid 2x^2 + y^2 + 3z^2 \leq 1\}$.
5. Verify the details of the applications of the Chain Rule in the proof of the General Lagrange Multiplier Theorem.
6. For numbers a, b, and c, find the minimum of

$$\{ax + by + cz \mid x^2 + y^2 + z^2 \leq 1\}.$$

Give a geometric interpretation of the answer by viewing $ax + by + cz$ as the inner product of (x, y, z) and (a, b, c).

7. Find the point on the plane $ax + by + cz + d = 0$ that is closest to the point $(0, 0, 0)$.
8. For positive numbers a, b, and c, find a point on the ellipsoid

$$S = \left\{ (x, y, z) \text{ in } \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

that is closest to the point $(0, 0, 0)$.

9. Show, by giving an example, that Corollary 17.18 is false if we drop the assumption that $\nabla g(\mathbf{u})$ is nonzero.
10. Use Dini's Theorem to provide a proof of Corollary 17.18 in the case that $n = 2$.
11. Show that the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has no real eigenvalues.

12. Let $\mathbf{A}$ be an $n \times n$ symmetric matrix and define λ to be the maximum of

$$\{\langle \mathbf{A}\mathbf{x}, \mathbf{x}\rangle \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}.$$

Follow the proof of Proposition 17.19 to show that λ is an eigenvalue of the matrix $\mathbf{A}$.

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13. Let p and q be numbers with $p > 1$ and $q > 1$.
a. Show that

$$\frac{x^p}{p} + \frac{y^q}{q} \geq \frac{1}{p} + \frac{1}{q}$$

for all (x, y) in $\mathbb{R}^2$ such that $x > 0$, $y > 0$, and $xy = 1$.
b. Use (a) to verify the following inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

if $a \geq 0$, $b \geq 0$, $p > 1$, $q > 1$, and $1/p + 1/q = 1$.

14. a. Show that

$$x + y + z \geq 3$$

for all (x, y, z) in $\mathbb{R}^3$ such that $x > 0$, $y > 0$, $z > 0$, and $xyz = 1$.
b. Use (a) to verify the following Geometric Mean–Arithmetic Mean Inequality. If a_1, a_2, and a_3 are positive numbers, then

$$(a_1 a_2 a_3)^{1/3} \leq \frac{a_1 + a_2 + a_3}{3}.$$

c. Generalize the above inequality from $n = 3$ to general positive integers n.

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CHAPTER 18

INTEGRATING FUNCTIONS OF SEVERAL VARIABLES

In Chapters 6 and 7, we considered integration for functions $f : I \to \mathbb{R}$ defined on a closed bounded interval I of real numbers. We will devote this and the following two chapters to the study of integration for functions of several variables. In Section 18.1, we will consider integration for bounded functions $f : \mathbf{I} \to \mathbb{R}$ defined on a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$. Many of the results of Chapter 6 carry over with very little change: In fact, here we omit a number of proofs since they are entirely similar to corresponding results we obtained for functions of a single variable. In Section 18.2 we will consider integration for continuous functions. We prove that a continuous function defined on a generalized rectangle is integrable. We also introduce the concept of a set of Jordan content 0 and show that a bounded function defined on a generalized rectangle and continuous except for a set of Jordan content 0 is integrable; this result is new even in the case of functions of a single variable. In the final Section 18.3 we will introduce the concept of a Jordan domain in $\mathbb{R}^n$ and define the integral for certain functions defined on Jordan domains including those that are continuous.

18.1 INTEGRATION OF FUNCTIONS ON GENERALIZED RECTANGLES

Recall that if $I = [a, b]$ is a closed bounded interval of real numbers, m is a positive integer, and $P = \{x_0, \dots, x_m\}$ are $m + 1$ real numbers such that

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_m = b,$$

then P is called a *partition* of $[a, b]$, and the intervals $[x_{i-1}, x_i]$, for i an index between 1 and m, are called *intervals* in the partition P. We define the *length* of the interval[1] $I = [a, b]$ to be $b - a$.

[1] Unless explicitly stated otherwise, it is assumed that an interval $I = [a, b]$ is nondegenerate, that is, $a < b$.

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Let n be a positive integer and for each index i between 1 and n let $I_i = [a_i, b_i]$ be a closed bounded interval of real numbers. Recall that the Cartesian product of these intervals,

$$\mathbf{I} = I_1 \times \cdots \times I_i \times \cdots \times I_n = \{\mathbf{x} = (x_1, \ldots, x_n) \text{ in } \mathbb{R}^n \mid x_i \text{ in } I_i \text{ for } 1 \leq i \leq n\},$$

is called a *generalized rectangle*. It is convenient to refer to the interval I_i as being the ith *edge* of $\mathbf{I}$. We define the *volume* of $\mathbf{I}$, denoted by vol $\mathbf{I}$, to be the product of the lengths of the n edges; that is,

$$\text{vol } \mathbf{I} \equiv \prod_{i=1}^{n} [b_i - a_i].$$

In the case where $n = 1$, the volume is simply the length; in the case where $n = 2$, the volume is called the *area*.

Definition Given a generalized rectangle $\mathbf{I} = I_1 \times \cdots \times I_i \times \cdots \times I_n$, for each index i between 1 and n, let P_i be a partition of the ith edge I_i. The collection of generalized rectangles of the form

$$\mathbf{J} = J_1 \times \cdots \times J_i \times \cdots \times J_n,$$

where each J_i is an interval in the partition P_i, is called a partition of $\mathbf{I}$ and is denoted by

$$\mathbf{P} \equiv (P_1, \ldots, P_n).$$

Consider the rectangle $[a, b] \times [c, d]$ in the plane $\mathbb{R}^2$. Let $P_1 = \{x_0, \ldots, x_m\}$ and $P_2 = \{y_0, \ldots, y_\ell\}$ be partitions of $[a, b]$ and $[c, d]$, respectively, and define $\mathbf{P} = (P_1, P_2)$. Then

$$\begin{aligned}
\sum_{\mathbf{J} \text{ in } \mathbf{P}} \text{vol } \mathbf{J} &= \sum_{j=1}^{\ell} \sum_{i=1}^{m} [x_i - x_{i-1}][y_j - y_{j-1}] \\
&= \sum_{j=1}^{\ell} \left\{ \sum_{i=1}^{m} [x_i - x_{i-1}] \right\} [y_j - y_{j-1}] \\
&= \sum_{j=1}^{\ell} \{[b - a]\}[y_j - y_{j-1}] \\
&= [b - a] \sum_{j=1}^{\ell} [y_j - y_{j-1}] \\
&= [b - a][d - c] = \text{vol } \mathbf{I}.
\end{aligned}$$

An induction argument shows that the above formula also holds in general: For each natural number n, if $\mathbf{P}$ is a partition of the generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$, then

$$\text{vol } \mathbf{I} = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \text{vol } \mathbf{J}. \tag{18.1}$$

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Upper and Lower Darboux Sums

Now suppose that $f : \mathbf{I} \to \mathbb{R}$ is a bounded function whose domain $\mathbf{I}$ is a generalized rectangle and let $\mathbf{P}$ be a partition of $\mathbf{I}$. For $\mathbf{J}$ a generalized rectangle in the partition $\mathbf{P}$, we define

$$m(f, \mathbf{J}) \equiv \inf\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} \qquad \text{and} \qquad M(f, \mathbf{J}) \equiv \sup\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\}.$$

We then define the *lower Darboux sum* for the function $f : \mathbf{I} \to \mathbb{R}$ with respect to the partition $\mathbf{P}$, denoted by $L(f, \mathbf{P})$, by the formula

$$L(f, \mathbf{P}) \equiv \sum_{\mathbf{J} \text{ in } \mathbf{P}} m(f, \mathbf{J}) \operatorname{vol} \mathbf{J}$$

and we define the *upper Darboux sum* for the function $f : \mathbf{I} \to \mathbb{R}$ with respect to the partition $\mathbf{P}$, denoted by $U(f, \mathbf{P})$, by the formula

$$U(f, \mathbf{P}) \equiv \sum_{\mathbf{J} \text{ in } \mathbf{P}} M(f, \mathbf{J}) \operatorname{vol} \mathbf{J}.$$

Lemma 18.1 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Suppose that the two numbers m and M have the property that

$$m \le f(\mathbf{x}) \le M \qquad \text{for all points } \mathbf{x} \text{ in } \mathbf{I}.$$

Then for any partition $\mathbf{P}$ of $\mathbf{I}$,

$$m \operatorname{vol} \mathbf{I} \le L(f, \mathbf{P}) \le U(f, \mathbf{P}) \le M \operatorname{vol} \mathbf{I}. \tag{18.2}$$

Proof

Let $\mathbf{P}$ be a partition of $\mathbf{I}$. For a generalized rectangle $\mathbf{J}$ in $\mathbf{I}$, it is clear that

$$m \le \inf\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} = m(f, \mathbf{J}) \le M(f, \mathbf{J}) = \sup\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} \le M,$$

so

$$m \operatorname{vol} \mathbf{J} \le m(f, \mathbf{J}) \operatorname{vol} \mathbf{J} \le M(f, \mathbf{J}) \operatorname{vol} \mathbf{J} \le M \operatorname{vol} \mathbf{J}.$$

Summing over all the generalized rectangles $\mathbf{J}$ in the partition $\mathbf{P}$ and using the sum of volumes formula (18.1), we conclude that the inequality (18.2) holds. ∎

Given a partition $\mathbf{P} = (P_1, \ldots, P_n)$ of a generalized rectangle $\mathbf{I}$, another partition $\mathbf{P}^* = (P_1^*, \ldots, P_n^*)$ of $\mathbf{I}$ is said to be a *refinement* of $\mathbf{P}$ provided that for each index i between 1 and n, P_i^* is a refinement of P_i. Observe that if $\mathbf{P}^*$ is a refinement of $\mathbf{P}$, then (i) each generalized rectangle $\mathbf{J}$ in $\mathbf{P}^*$ is contained in exactly one generalized rectangle in $\mathbf{P}$, and (ii) given a generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the collection of generalized rectangles in $\mathbf{P}^*$ contained in $\mathbf{J}$ induce a partition of $\mathbf{J}$ that we denote by $\mathbf{P}^*(\mathbf{J})$. The following distribution formulas for the lower and upper Darboux sums follow from these two properties:

$$L(f, \mathbf{P}^*) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} L(f, \mathbf{P}^*(\mathbf{J})) \qquad \text{and} \qquad U(f, \mathbf{P}^*) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} U(f, \mathbf{P}^*(\mathbf{J})). \tag{18.3}$$

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Lemma 18.2 The Refinement Lemma Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Let $\mathbf{P}$ be a partition of $\mathbf{I}$ and let $\mathbf{P}^*$ be a refinement of $\mathbf{P}$. Then

$$L(f, \mathbf{P}) \leq L(f, \mathbf{P}^*) \leq U(f, \mathbf{P}^*) \leq U(f, \mathbf{P}). \tag{18.4}$$

Proof

Let $\mathbf{J}$ be a generalized rectangle in $\mathbf{P}$ and denote by $\mathbf{P}^*(\mathbf{J})$ the partition of $\mathbf{J}$ induced by $\mathbf{P}^*$. From Lemma 18.1, with $\mathbf{J}$ playing the role of $\mathbf{I}$, it follows that

$$m(f, \mathbf{J}) \text{ vol } \mathbf{J} \leq L(f, \mathbf{P}^*(\mathbf{J})) \leq U(f, \mathbf{P}^*(\mathbf{J})) \leq M(f, \mathbf{J}) \text{ vol } \mathbf{J}.$$

If we sum these inequalities over all generalized rectangles $\mathbf{J}$ in $\mathbf{P}$ and use the distribution formulas (18.3), we arrive at the inequality (18.4). ■

For two partitions P and P' of a closed bounded interval of real numbers I, by taking the partition consisting of all points that are partition points in at least one of the two partitions, we obtain a partition that is a *common refinement* of the two given partitions, meaning that it is a refinement of both P and P'. Similarly, suppose that $\mathbf{P}$ and $\mathbf{P}'$ are two partitions of a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$ represented as $\mathbf{P} = (P_1, \ldots, P_n)$ and $\mathbf{P}' = (P_1', \ldots, P_n')$. For each index i between 1 and n, choose P_i'' to be a common refinement of P_i and P_i' and define $\mathbf{P}'' = (P_1'', \ldots, P_n'')$. Then $\mathbf{P}''$ is a partition of $\mathbf{I}$ that is a common refinement of the partitions $\mathbf{P}$ and $\mathbf{P}'$. The existence of common refinements is what is necessary to establish the following proposition.

Proposition 18.3 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. For any two partitions $\mathbf{P}_1$ and $\mathbf{P}_2$ of $\mathbf{I}$,

$$L(f, \mathbf{P}_1) \leq U(f, \mathbf{P}_2).$$

Proof

Choose $\mathbf{P}$ to be a common refinement of the two partitions $\mathbf{P}_1$ and $\mathbf{P}_2$. By the Refinement Lemma,

$$L(f, \mathbf{P}_1) \leq L(f, \mathbf{P}) \leq U(f, \mathbf{P}) \leq U(f, \mathbf{P}_2).$$ ■

Upper and Lower Integrals

Definition Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. We define the *lower integral* of f on $\mathbf{I}$, denoted by $\underline{\int}_{\mathbf{I}} f$, by

$$\underline{\int}_{\mathbf{I}} f \equiv \sup\{L(f, \mathbf{P}) \mid \mathbf{P} \text{ a partition of the generalized rectangle } \mathbf{I}\}. \tag{18.5}$$

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We define the *upper integral* of f on $[a, b]$, denoted by $\overline{\int_{\mathbf{I}}} f$, by

$$\overline{\int_{\mathbf{I}}} f \equiv \inf\{U(f, \mathbf{P}) \mid \mathbf{P} \text{ a partition of the generalized rectangle } \mathbf{I}\}. \tag{18.6}$$

Lemma 18.4 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Then

$$\underline{\int_{\mathbf{I}}} f \leq \overline{\int_{\mathbf{I}}} f. \tag{18.7}$$

Proof

Let $\mathbf{P}$ be a partition of $\mathbf{I}$. Lemma 18.3 asserts that $U(f, \mathbf{P})$ is an upper bound for the collection of all lower Darboux sums for f. Therefore, by the definition of supremum,

$$\underline{\int_{\mathbf{I}}} f \leq U(f, \mathbf{P}).$$

But this inequality asserts that $\underline{\int_{\mathbf{I}}} f$ is a lower bound for the collection of upper Darboux sums for f. Thus, by the definition of infimum,

$$\underline{\int_{\mathbf{I}}} f \leq \overline{\int_{\mathbf{I}}} f.$$ ■

Example 18.5 Let $\mathbf{I}$ be a generalized rectangle, let c be a real number, and define $f : \mathbf{I} \to \mathbb{R}$ to be the constant function that assumes the value c at every point. Then $f : \mathbf{I} \to \mathbb{R}$ is integrable and

$$\int_{\mathbf{I}} f = c \text{ vol } \mathbf{I}.$$

This follows directly from the definition. Indeed, if $\mathbf{P}$ is any partition of $\mathbf{I}$, then for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, we have $m(f, \mathbf{J}) = c = M(f, \mathbf{J})$, so that by the sum of volumes formula (18.1),

$$L(f, \mathbf{P}) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} c \text{ vol } \mathbf{J} = c \text{ vol } \mathbf{I} = \sum_{\mathbf{J} \text{ in } \mathbf{P}} c \text{ vol } \mathbf{J} = U(f, \mathbf{P}).$$

Thus, the collection of lower Darboux sums consists of the single number c vol $\mathbf{I}$, as does the collection of upper Darboux sums. By the definitions of the lower and upper integrals,

$$\underline{\int_{\mathbf{I}}} f = c \text{ vol } \mathbf{I} \quad \text{and} \quad \overline{\int_{\mathbf{I}}} f = c \text{ vol } \mathbf{I}.$$ ■

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Example 18.6 Dirichlet's Function For a generalized rectangle $\mathbf{I}$, define $f : \mathbf{I} \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ has a rational component} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that each generalized rectangle contains points with a rational component and points that have no rational component; this is a consequence of the density of the rational and irrational numbers in $\mathbb{R}$. Hence, if $\mathbf{P}$ is any partition of $\mathbf{I}$, by the sum of volumes formula (18.1),

$$L(f, \mathbf{P}) = 0 \qquad \text{and} \qquad U(f, \mathbf{P}) = \text{vol } \mathbf{I}.$$

Thus, the collection of lower Darboux sums consists of the single number 0. Consequently, by the definition of supremum,

$$\underline{\int_{\mathbf{I}}} f = 0.$$

On the other hand, the collection of upper Darboux sums consists of the single number vol $\mathbf{I}$, and hence, by the definition of infimum,

$$\overline{\int_{\mathbf{I}}} f = \text{vol } \mathbf{I}.$$

■

Definition Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. Then we say that $f : \mathbf{I} \to \mathbb{R}$ is *integrable*, or f is integrable on $\mathbf{I}$, provided that

$$\underline{\int_{\mathbf{I}}} f = \overline{\int_{\mathbf{I}}} f.$$

When this is so, the integral of the function $f : \mathbf{I} \to \mathbb{R}$, denoted by $\int_{\mathbf{I}} f$, is defined by

$$\int_{\mathbf{I}} f \equiv \underline{\int_{\mathbf{I}}} f = \overline{\int_{\mathbf{I}}} f.$$

We showed in Example 18.5 that a function $f : \mathbf{I} \to \mathbb{R}$ that has constant value c is integrable and that its integral equals c vol $\mathbf{I}$. We also have seen in Example 18.6 that Dirichlet's function is not integrable.

The Archimedes–Riemann Theorem

The above definitions of integrability and of the integral are direct extensions of the concepts defined for functions of a single variable in Section 6.1. The Archimedes–Riemann Theorem also extends to functions of several variables.

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Definition Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle. For each natural number k, let $\mathbf{P}_k$ be a partition of $\mathbf{I}$. The sequence of partitions $\{\mathbf{P}_k\}$ is said to be an *Archimedean sequence of partitions* for the function $f : \mathbf{I} \to \mathbb{R}$ provided that

$$\lim_{k\to\infty} [U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)] = 0.$$

The arguments used in Chapter 6 to prove Lemma 6.7 and the Archimedes–Riemann Theorem directly extend to the case of bounded functions defined on a generalized rectangle. We leave it to the reader to reread these proofs and observe that they do indeed directly extend to provide a proof of the following theorem.

Theorem 18.7 The Archimedes–Riemann Theorem Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Then f is integrable on $\mathbf{I}$ if and only if there is an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$. Moreover, for any such Archimedean sequence of partitions $\{\mathbf{P}_k\}$,

$$\lim_{k\to\infty} L(f, \mathbf{P}_k) = \int_{\mathbf{I}} f \quad \text{and} \quad \lim_{k\to\infty} U(f, \mathbf{P}_k) = \int_{\mathbf{I}} f. \tag{18.8}$$

Example 18.8 For the rectangle $\mathbf{I} = [0, 1] \times [0, 1]$ in the plane $\mathbb{R}^2$, define

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \text{ is in } \mathbf{I} \text{ and } y > x \\ 0 & \text{if } (x, y) \text{ is in } \mathbf{I} \text{ and } y \leq x. \end{cases}$$

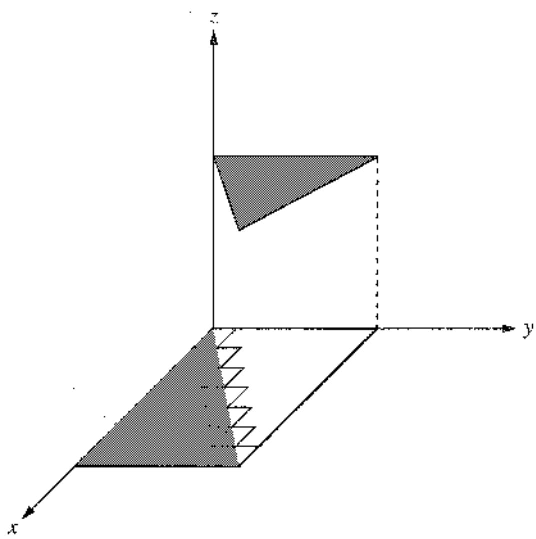

FIGURE 18.1 Construction of an Archimedean sequence of partitions

We use the Archimedes–Riemann Theorem to show that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. For a natural number k, let P_k be the partition of the interval $[0, 1]$

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into k intervals of equal length $1/k$ and define the partition of $\mathbf{P}_k$ of $\mathbf{I}$ to be (P_k, P_k). Observe that the only terms in the sum $U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)$ that can possibly be nonzero are those arising from the rectangles in the partition $\mathbf{P}_k$ that intersect the diagonal. Each of these contributions is $1/k^2$, and there are fewer than $2k$ of such rectangles. Hence

$$U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) < 2/k.$$

Thus,

$$\lim_{k\to\infty} [U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)] = 0,$$

and therefore, by the Archimedes–Riemann Theorem, $f : \mathbf{I} \to \mathbb{R}$ is integrable. ■

Example 18.9 For the rectangle $\mathbf{I} = [0, 1] \times [0, 1]$ in the plane $\mathbb{R}^2$, define $f : \mathbf{I} \to \mathbb{R}$ by

$$f(x, y) = x^2y^2 \qquad \text{for } (x, y) \text{ in } \mathbf{I}.$$

We use the Archimedes–Riemann Theorem to show that $f : \mathbf{I} \to \mathbb{R}$ is integrable and that $\int_{\mathbf{I}} f = 1/9$. Indeed, for k a natural number, let P_k be the partition of $[0, 1]$ into k intervals of equal length $1/k$ and define $\mathbf{P}_k = (P_k, P_k)$. Each rectangle $\mathbf{J}$ in $\mathbf{P}_k$ has area $1/k^2$. Moreover, we see that for two indices i and j between 1 and k, if

$$\mathbf{J} = \left[\frac{i-1}{k}, \frac{i}{k}\right] \times \left[\frac{j-1}{k}, \frac{j}{k}\right],$$

then

$$m(f, \mathbf{J}) = \frac{(i-1)^2(j-i)^2}{k^4} \qquad \text{and} \qquad M(f, \mathbf{J}) = \frac{i^2j^2}{k^4}.$$

Thus,

$$\begin{aligned}
U(f, \mathbf{P}_k) &= \sum_{\mathbf{J} \text{ in } \mathbf{P}_k} M(f, \mathbf{J}) \operatorname{vol} \mathbf{J} \\
&= \sum_{1\le i,j\le k} \frac{i^2j^2}{k^6} \\
&= \frac{1}{k^6} \sum_{i=1}^{k} i^2 \left[\sum_{j=1}^{k} j^2\right] \\
&= \frac{1}{k^6} \sum_{i=1}^{k} i^2 \left[\frac{k(k+1)(2k+1)}{6}\right] \\
&= \frac{1}{k^6} \cdot \frac{k(k+1)(2k+1)}{6} \sum_{i=1}^{k} i^2 \\
&= \frac{1}{k^6} \cdot \left[\frac{[k(k+1)(2k+1)]}{6}\right]^2.
\end{aligned}$$

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A similar addition shows that

$$L(f, \mathbf{P}_k) = \frac{1}{k^6} \cdot \left[\frac{[(k-1)(k)(2k-1)]}{6}\right]^2.$$

Thus,

$$\lim_{k\to\infty} U(f, \mathbf{P}_k) = 1/9 \qquad \text{and} \qquad \lim_{k\to\infty} L(f, \mathbf{P}_k) = 1/9.$$

Therefore, $\{\mathbf{P}_k\}$ is an Archimedean sequence for $f : \mathbf{I} \to \mathbb{R}$. It follows from the Archimedes–Riemann Theorem that $f : \mathbf{I} \to \mathbb{R}$ is integrable and

$$\int_{\mathbf{I}} f = \lim_{k\to\infty} U(f, \mathbf{P}_k) = 1/9. \qquad \blacksquare$$

The following theorem presents a criterion for establishing that there is an Archimedean sequence for a function $f : \mathbf{I} \to \mathbb{R}$.

Theorem 18.10 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Then the following two assertions are equivalent:

i. There is an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$.
ii. For each $\epsilon > 0$ there is a partition $\mathbf{P}$ of $\mathbf{I}$ such that

$$U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon.$$

Proof

First we suppose that (i) holds. Let $\{\mathbf{P}_k\}$ be an Archimedean sequence of partitions for $f : \mathbf{I} \to \mathbb{R}$. To verify criterion (ii) we let ϵ be any positive number. By the definition of convergent sequence we can choose an index k such that $U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) < \epsilon$. Thus, setting $\mathbf{P} = \mathbf{P}_k$, we have $U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon$. Thus, criterion (ii) holds.

Now suppose that criterion (ii) holds. Let k be a natural number. Then, setting $\epsilon = 1/k$, according to (ii) there is a partition $\mathbf{P}$ such that $U(f, \mathbf{P}) - L(f, \mathbf{P}) < 1/k$. Choose such a partition and label it $\mathbf{P}_k$. This defines a sequence of partitions $\{\mathbf{P}_k\}$ of the generalized interval $\mathbf{I}$ that is Archimedean since

$$0 \le \lim_{k\to\infty} [U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)] \le \lim_{k\to\infty} 1/k = 0. \qquad \blacksquare$$

Additivity, Monotonicity, and Linearity of Integration

In Chapter 6 we proved Theorem 6.12, which asserts that if a function $f : [a, b] \to \mathbb{R}$ is integrable and c is a point in the open interval (a, b), then the restrictions of f to the intervals $[a, c]$ and $[c, b]$ also are integrable and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

We need the following extension of this result and also now need its converse.

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Theorem 18.11 Additivity over Partitions Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on the generalized rectangle $\mathbf{I}$. Let $\mathbf{P}$ be a partition of $\mathbf{I}$. Then the function $f : \mathbf{I} \to \mathbb{R}$ is integrable if and only if for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the restriction of f to $\mathbf{J}$, $f : \mathbf{J} \to \mathbb{R}$, is integrable: In this case,

$$\int_{\mathbf{I}} f = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \int_{\mathbf{J}} f. \tag{18.9}$$

Proof

First we suppose that for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the function $f : \mathbf{J} \to \mathbb{R}$ is integrable. We will use the Archimedes–Riemann Theorem to show that $f : \mathbf{I} \to \mathbb{R}$ is integrable. Suppose that there are m generalized rectangles in $\mathbf{P}$. Let k be a natural number. Using the Archimedes–Riemann Theorem and criterion (ii) of Theorem 18.10, we see that for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$ we can select a partition $\mathbf{P}_k(\mathbf{J})$ of $\mathbf{J}$ such that

$$U(f, \mathbf{P}_k(\mathbf{J})) - L(f, \mathbf{P}_k(\mathbf{J})) < \frac{1}{km}.$$

Choose $\mathbf{P}_k$ to be a partition of $\mathbf{I}$ that contains all the generalized rectangles in any one of the $\mathbf{P}_k(\mathbf{J})$'s. By the distribution formula (18.3) and the Refinement Lemma,

$$\begin{aligned} U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) &\leq \sum_{\mathbf{J} \text{ in } \mathbf{P}} U(f, \mathbf{P}_k(\mathbf{J})) - L(f, \mathbf{P}_k(\mathbf{J})) \\ &< m\frac{1}{km} = \frac{1}{k}. \end{aligned}$$

Thus,

$$\lim_{k \to \infty} [U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k] = 0,$$

and therefore, by the Archimedes–Riemann Theorem, the function f is integrable on $\mathbf{I}$.

To prove the converse, suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. By the Archimedes–Riemann Theorem we can choose an Archimedean sequence $\{\mathbf{P}_k\}$ of partitions for $f : \mathbf{I} \to \mathbb{R}$. Using the Refinement Lemma and possibly replacing each $\mathbf{P}_k$ by a common refinement of $\mathbf{P}_k$ and $\mathbf{P}$, we can suppose that each $\mathbf{P}_k$ is a refinement of $\mathbf{P}$. For each natural number k, observe that if $\mathbf{P}_k(\mathbf{J})$ is the partition that $\mathbf{P}_k$ induces on the generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, then

$$U(f, \mathbf{P}_k(\mathbf{J})) - L(f, \mathbf{P}_k(\mathbf{J})) \leq U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k).$$

Thus, for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the sequence of partitions $\{\mathbf{P}_k(\mathbf{J})\}$ is an Archimedean sequence for $f : \mathbf{J} \to \mathbb{R}$ and therefore, by the Archimedes–Riemann Theorem, the function f is integrable on $\mathbf{J}$.

It remains to verify formula (18.9). However, we have the following distribution formula for the Darboux sums:

$$L(f, \mathbf{P}_k) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} L(f, \mathbf{P}_k(\mathbf{J})).$$

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According to the Archimedes–Riemann Theorem, the sequence of lower Darboux sums associated with an Archimedean sequence of partitions converges to the value of the integral. Thus,

$$\int_{\mathbf{I}} f = \lim_{k\to\infty} L(f, \mathbf{P}_k) = \lim_{k\to\infty} \sum_{\mathbf{J} \text{ in } \mathbf{P}} L(f, \mathbf{P}_k(\mathbf{J})) = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \int_{\mathbf{J}} f. \quad \blacksquare$$

In Section 6.3 we used the Refinement Lemma and the Archimedes–Riemann Theorem to prove the monotonicity and linearity properties for integrals of functions of a single variable. We leave it to the reader to reread these proofs and see that the proofs of the single variable theorems (Theorems 6.13 and 6.15) directly extend to provide proofs of the following two theorems.

Theorem 18.12 Monotonicity of the Integral Suppose that the functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are integrable, where $\mathbf{I}$ is a generalized rectangle in $\mathbb{R}^n$, and also suppose that

$$f(\mathbf{x}) \leq g(\mathbf{x}) \qquad \text{for all points } \mathbf{x} \text{ in } \mathbf{I}.$$

Then

$$\int_{\mathbf{I}} f \leq \int_{\mathbf{I}} g.$$

Theorem 18.13 Linearity of the Integral Suppose that the functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are integrable, where $\mathbf{I}$ is a generalized rectangle in $\mathbb{R}^n$. Then for any two numbers α and β, the function $\alpha f + \beta g : \mathbf{I} \to \mathbb{R}$ also is integrable and

$$\int_{\mathbf{I}} [\alpha f + \beta g] = \alpha \int_{\mathbf{I}} f + \beta \int_{\mathbf{I}} g. \tag{18.10}$$

The Darboux Sum Convergence Criterion

In Chapter 7 we introduced the concept of gap for a partition and proved that for an integrable function $f : I \to \mathbb{R}$ defined on a closed bounded interval I, a sequence of partitions $\{P_k\}$ of I is an Archimedean sequence of partitions for $f : I \to \mathbb{R}$ provided that

$$\lim_{k\to\infty} \text{gap } P_k = 0.$$

This result extends to functions of several variables. To state the extension we first need a few definitions.

Definition For a bounded subset D of $\mathbb{R}^n$, the diameter of D, denoted by diam D, is defined by

$$\text{diam } D \equiv \sup\{\text{dist}(\mathbf{u}, \mathbf{v}) \mid \mathbf{u} \text{ and } \mathbf{v} \text{ in } D\}.$$

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Observe that for a closed bounded interval of real numbers $I = [a, b]$, the diameter of I equals its length. Hence, from the definition of the distance between points in $\mathbb{R}^n$, it directly follows that if the generalized rectangle $\mathbf{I}$ is the Cartesian product

$$\mathbf{I} = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

then

$$\operatorname{diam} \mathbf{I} = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2};$$

that is, the diameter of $\mathbf{I}$ is the distance between the points $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$.

Definition For a partition $\mathbf{P}$ of a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$, we define the gap of $\mathbf{P}$, denoted by gap $\mathbf{P}$, to be the largest of the diameters of the generalized rectangles in $\mathbf{P}$.

It is not difficult to check that for a partition $\mathbf{P}$ of a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$, if $\mathbf{P} = (P_1, \ldots, P_n)$, then

$$\text{gap } \mathbf{P} = \sqrt{[\text{gap } P_1]^2 + \cdots + [\text{gap } P_n]^2}.$$

The proof of the following theorem proceeds exactly the same as the proof of the single-variable case (Theorem 7.12). Therefore, we leave the proof to the reader.

Theorem 18.14 The Darboux Sum Convergence Criterion Let $\mathbf{I}$ be a generalized rectangle and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. Let $\{\mathbf{P}_k\}$ be a sequence of partitions of $\mathbf{I}$. If

$$\lim_{k \to \infty} \text{gap } \mathbf{P}_k = 0,$$

then $\{\mathbf{P}_k\}$ is an Archimedean sequence for $f : \mathbf{I} \to \mathbb{R}$ and therefore,

$$\lim_{k \to \infty} L(f, \mathbf{P}_k) = \int_{\mathbf{I}} f \qquad \text{and} \qquad \lim_{k \to \infty} U(f, \mathbf{P}_k) = \int_{\mathbf{I}} f.$$

EXERCISES FOR SECTION 18.1

1. For $\mathbf{I}$ a generalized rectangle in $\mathbb{R}^n$ and a number $\delta > 0$, show that there is a partition $\mathbf{P}$ of $\mathbf{I}$ such that gap $\mathbf{P} < \delta$.
2. Let $\mathbf{I}$ be a generalized rectangle and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. Let $\{\mathbf{P}_k\}$ be an Archimedean sequence of partitions of $\mathbf{I}$. Is it necessarily the case that
$$\lim_{k \to \infty} \text{gap } \mathbf{P}_k = 0?$$
3. In Example 18.8, find the exact values of $U(f, \mathbf{P}_k)$ and $L(f, \mathbf{P}_k)$.
4. Use the Archimedes–Riemann Theorem to evaluate the value of the integral of the function in Example 18.8.
5. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. Assume that $f(\mathbf{x}) \geq 0$ if $\mathbf{x}$ is a point in $\mathbf{I}$ with a rational component. Prove that $\int_{\mathbf{I}} f \geq 0$.

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6. Show that each generalized rectangle in $\mathbb{R}^n$ contains a point with a rational component and a point without any rational component.

7. For the generalized rectangle $\mathbf{I} = [0, 1] \times [0, 1]$ in the plane $\mathbb{R}^2$, define

$$f(x, y) = \begin{cases} 5 & \text{if } (x, y) \text{ is in } \mathbf{I} \text{ and } x > 1/2 \\ 1 & \text{if } (x, y) \text{ is in } \mathbf{I} \text{ and } x \le 1/2. \end{cases}$$

Use the Archimedes–Riemann Theorem to show that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable.

8. For a partition $\mathbf{P} = (P_1, \ldots, P_n)$ of a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$, verify the formula

$$\text{gap } \mathbf{P} = \sqrt{[\text{gap } P_1]^2 + \cdots + [\text{gap } P_1]^2}.$$

9. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^2$ and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ assumes the value 0 except at a single point $\mathbf{x}$ in $\mathbf{I}$. Show that $f : \mathbf{I} \to \mathbb{R}$ is integrable. Then show that $\int_{\mathbf{I}} f = 0$. Is the same result true for a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$?

10. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^2$ and suppose that the bounded function $f : \mathbf{I} \to \mathbb{R}$ has the value 0 on the interior of $\mathbf{I}$. Show that $f : \mathbf{I} \to \mathbb{R}$ is integrable and that $\int_{\mathbf{I}} f = 0$. Is the same result true for a generalized rectangle $\mathbf{I}$ in $\mathbb{R}^n$?

11. For the rectangle $\mathbf{I} = [0, 1] \times [0, 1]$ in the plane $\mathbb{R}^2$, define the function $f : \mathbf{I} \to \mathbb{R}$ by

$$f(x, y) = xy \qquad \text{for } (x, y) \text{ in } \mathbf{I}.$$

Use the Archimedes–Riemann Theorem to evaluate $\int_{\mathbf{I}} f$.

12. For the rectangle $\mathbf{I} = [0, 1] \times [-1, 0]$ in the plane $\mathbb{R}^2$, define the function $f : \mathbf{I} \to \mathbb{R}$ by

$$f(x, y) = x^2 y \qquad \text{for } (x, y) \text{ in } \mathbf{I}.$$

Use the Archimedes–Riemann Theorem to evaluate $\int_{\mathbf{I}} f$.

13. For the rectangle $\mathbf{I} = [0, 2] \times [0, 1]$ in the plane $\mathbb{R}^2$, define the function $f : \mathbf{I} \to \mathbb{R}$ by

$$f(x, y) = x + 2y \qquad \text{for } (x, y) \text{ in } \mathbf{I}.$$

Use the Archimedes–Riemann Theorem to evaluate $\int_{\mathbf{I}} f$.

14. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable. Let the number M have the property that $|f(\mathbf{x})| \le M$ for all $\mathbf{x}$ in $\mathbf{I}$. Prove that

$$\left| \int_{\mathbf{I}} f \right| \le M \cdot \text{vol } \mathbf{I}.$$

18.2 CONTINUITY AND INTEGRABILITY

Our first goal in this section is to show that a continuous function on a generalized rectangle is integrable. We then prove a more general result regarding the extent to which a bounded function on a generalized rectangle can fail to be continuous and still be integrable.

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The proof we provided of Theorem 6.18, which states that a continuous function on a closed bounded interval is integrable, relied on two fundamental results: A continuous function on a closed bounded interval (i) attains maximum and minimum functional values and (ii) is uniformly continuous. But Corollaries 11.23 and 11.26, respectively, extend these results to continuous functions on a generalized rectangle in $\mathbb{R}^n$. Thus, we have the following generalizations of Lemma 6.17 and Theorem 6.18 to continuous functions defined on a generalized rectangle.

Lemma 18.15 Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function on the generalized rectangle $\mathbf{I}$. Let $\mathbf{P}$ be a partition of $\mathbf{I}$. Then there is a generalized rectangle in the partition $\mathbf{P}$ that contains two points $\mathbf{u}$ and $\mathbf{v}$ for which the following estimate holds:

$$0 \le U(f, \mathbf{P}) - L(f, \mathbf{P}) \le [f(\mathbf{u}) - f(\mathbf{v})] \operatorname{vol} \mathbf{I}. \tag{18.11}$$

Proof

Let $\mathbf{J}$ be a generalized rectangle in the partition $\mathbf{P}$. Since $f : \mathbf{I} \to \mathbb{R}$ is continuous, according to Corollary 11.23, f assumes a maximum value and a minimum value on $\mathbf{J}$; that is, there are points $\mathbf{u}(\mathbf{J})$ and $\mathbf{v}(\mathbf{J})$ in $\mathbf{J}$ such that

$$f(\mathbf{v}(\mathbf{J})) = m(f, \mathbf{J}) \equiv \inf\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\}$$

and

$$f(\mathbf{u}(\mathbf{J})) = M(f, \mathbf{J}) \equiv \sup\{f(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\}.$$

Since there are only finitely many generalized rectangles in $\mathbf{P}$, we can choose a generalized rectangle $\mathbf{J}_*$ in $\mathbf{P}$ such that

$$M(f, \mathbf{J}_*) - m(f, \mathbf{J}_*) = \max_{\mathbf{J} \text{ in } \mathbf{I}} [M(f, \mathbf{J}) - m(f, \mathbf{J})]$$

and define

$$\mathbf{u} \equiv \mathbf{u}(\mathbf{J}_*) \qquad \text{and} \qquad \mathbf{v} \equiv \mathbf{v}(\mathbf{J}_*).$$

Then

$$M(f, \mathbf{J}) - m(f, \mathbf{J}) \le M(f, \mathbf{J}_*) - m(f, \mathbf{J}_*) = f(\mathbf{u}) - f(\mathbf{v}) \qquad \text{for all } \mathbf{J} \text{ in } \mathbf{P}.$$

Therefore,

$$\begin{aligned} U(f, \mathbf{P}) - L(f, \mathbf{P}) &= \sum_{\mathbf{J} \text{ in } \mathbf{P}} [M(\mathbf{J}) - m(\mathbf{J})] \operatorname{vol} \mathbf{J} \\ &\le \sum_{\mathbf{J} \text{ in } \mathbf{I}} [f(\mathbf{u}) - f(\mathbf{v})] \operatorname{vol} \mathbf{J} \\ &= [f(\mathbf{u}) - f(\mathbf{v})] \sum_{\mathbf{J} \text{ in } \mathbf{P}} \operatorname{vol} \mathbf{J} \\ &= [f(\mathbf{u}) - f(\mathbf{v})] \operatorname{vol} \mathbf{I}. \end{aligned}$$

■

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According to Corollary 11.26 a continuous function $f : \mathbf{I} \to \mathbb{R}$ on a generalized rectangle $\mathbf{I}$ is uniformly continuous; that is, for any two sequences $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ in $\mathbf{I}$,

$$\lim_{k\to\infty} [f(\mathbf{u}_k) - f(\mathbf{v}_k)] = 0 \qquad \text{if } \lim_{k\to\infty} \operatorname{dist}(\mathbf{u}_k, \mathbf{v}_k) = 0.$$

This is the property of a continuous function on a generalized rectangle that implies that it is integrable.

Theorem 18.16 Let $f : \mathbf{I} \to \mathbb{R}$ be a continuous function on a generalized rectangle $\mathbf{I}$. Then f is integrable on $\mathbf{I}$.

Proof

To prove the theorem, we use the Archimedes–Riemann Theorem. Let $\{\mathbf{P}_k\}$ be any sequence of partitions of $\mathbf{I}$ such that

$$\lim_{k\to\infty} \text{gap } \mathbf{P}_k = 0. \tag{18.12}$$

We show that the sequence $\{\mathbf{P}_k\}$ is an Archimedean sequence of partitions for f on $\mathbf{I}$. By the preceding lemma, for each index k, we can choose a generalized rectangle in the partition $\mathbf{P}_k$ that contains two points $\mathbf{u}_k$ and $\mathbf{v}_k$ for which the following estimate holds:

$$0 \le U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) \le [f(\mathbf{u}_k) - f(\mathbf{v}_k)] \text{ vol } \mathbf{I}. \tag{18.13}$$

Observe that since $\mathbf{u}_k$ and $\mathbf{v}_k$ belong to a common generalized rectangle in the partition $\mathbf{P}_k$,

$$\operatorname{dist}(\mathbf{u}_k, \mathbf{v}_k) \le \text{gap } \mathbf{P}_k.$$

From this estimate and (18.12), we conclude that $\{\mathbf{u}_k\}$ and $\{\mathbf{v}_k\}$ are sequences in the generalized rectangle $\mathbf{I}$ having the property that

$$\lim_{k\to\infty} \operatorname{dist}(\mathbf{u}_k, \mathbf{v}_k) = 0.$$

But by Corollary 11.26 a continuous function on a generalized rectangle is uniformly continuous. Thus,

$$\lim_{k\to\infty} [f(\mathbf{u}_k) - f(\mathbf{v}_k)] = 0.$$

This limit, together with the inequality (18.13), implies that

$$0 \le \lim_{k\to\infty} [U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k)] \le \lim_{k\to\infty} [f(\mathbf{u}_k) - f(\mathbf{v}_k)] \text{ vol } \mathbf{I} = 0.$$

Thus, the sequence $\{\mathbf{P}_k\}$ is an Archimedean sequence of partitions for f on $\mathbf{I}$. According to the Archimedes–Riemann Theorem, f is integrable on $\mathbf{I}$. ■

In order to extend the concept of integral to functions defined on sets that are much more general than generalized rectangles, the following extension of the preceding theorem is useful.

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Proposition 18.17 Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is bounded and that its restriction to the interior of $\mathbf{I}$ is continuous. Then the function $f : \mathbf{I} \to \mathbb{R}$ is integrable.

Proof

Once more we use the Archimedes–Riemann Theorem. Let k be a natural number. Choose $\mathbf{I}_k$ to be a generalized rectangle that is contained in the interior of $\mathbf{I}$ and has the property that (Exercise 5)

$$\text{vol}\,\mathbf{I} - \text{vol}\,\mathbf{I}_k < 1/k. \tag{18.14}$$

According to the preceding theorem, since f is continuous on the interior of $\mathbf{I}$, the restriction $f : \mathbf{I}_k \to \mathbb{R}$ is integrable. It follows from the Archimedes–Riemann Theorem and criterion (ii) in Theorem 18.10 that there is a partition $\mathbf{P}_k$ of $\mathbf{I}_k$ such that

$$U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) < 1/k. \tag{18.15}$$

Choose a partition $\mathbf{P}'_k$ of $\mathbf{I}$ such that each generalized rectangle of $\mathbf{P}_k$ is also a generalized rectangle of $\mathbf{P}'_k$ (Exercise 6).

We claim that $\{\mathbf{P}'_k\}$ is an Archimedean sequence of partitions for f on $\mathbf{I}$. Once this is shown, the integrability of f on $\mathbf{I}$ follows from the Archimedes–Riemann Theorem. Since $f : \mathbf{I} \to \mathbb{R}$ is bounded we can choose a number $M > 0$ such that

$$|f(\mathbf{x})| \le M \qquad \text{for all } \mathbf{x} \text{ in } \mathbf{I}.$$

Then for any generalized rectangle $\mathbf{J}$ in $\mathbf{P}'_k$,

$$M(f, \mathbf{J}) - m(f, \mathbf{J}) \le 2M\ \text{vol}\,\mathbf{J}. \tag{18.16}$$

Observe that

$$U(f, \mathbf{P}'_k) - L(f, \mathbf{P}'_k) = U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) + E_k, \tag{18.17}$$

where E_k is the sum of the terms $[M(f, \mathbf{J}) - m(f, \mathbf{J})]$ vol $\mathbf{J}$, where $\mathbf{J}$ is a generalized rectangle in the partition $\mathbf{P}'_k$ that is not in the partition $\mathbf{P}_k$. By the estimates (18.14) and (18.16),

$$E_k \le 2M \cdot [\text{vol}\,\mathbf{I} - \text{vol}\,\mathbf{I}_k] \le 2M/k.$$

This estimate, together with (18.17) and (18.15), yields the estimate

$$0 \le U(f, \mathbf{P}'_k) - L(f, \mathbf{P}'_k) = U(f, \mathbf{P}_k) - L(f, \mathbf{P}_k) + E_k < 1/k + 2M/k.$$

Thus,

$$\lim_{k\to\infty} [U(f, \mathbf{P}'_k) - L(f, \mathbf{P}'_k)] = 0;$$

that is, $\{\mathbf{P}'_k\}$ is an Archimedean sequence of partitions for f on $\mathbf{I}$. ∎

Sets of Jordan Content 0

In the following section we will extend the concept of an integral to functions of several variables whose domains are more general than generalized rectangles. In order to do

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this it is necessary to provide a description of the extent to which a bounded function defined on a generalized rectangle can fail to be continuous and yet still be integrable.

For a subset S of $\mathbb{R}^n$, a collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$ is said to *cover* S provided that the union of the sets in the collection contains the set S—that is,

$$S \subseteq \bigcup_{F \text{ in } \mathcal{F}} F.$$

Definition A bounded subset S of $\mathbb{R}^n$ is said to have *Jordan content* 0 provided that for each $\epsilon > 0$ there is a finite collection $\mathcal{F}$ of generalized rectangles in $\mathbb{R}^n$ that cover S, the sum of whose volumes is less than ϵ; that is, if $\mathcal{F} = \{\mathbf{I}_1, \ldots, \mathbf{I}_m\}$, then

$$S \subseteq \bigcup_{1 \le j \le m} \mathbf{I}_j \qquad \text{and} \qquad \sum_{j=1}^{m} \operatorname{vol} \mathbf{I}_j < \epsilon.$$

It is clear that if a set D has Jordan content 0, then each subset of D also has Jordan content 0. Moreover, the union of a finite number of sets, each of which has Jordan content 0, also has Jordan content 0. To verify this, for k a positive integer, let $\{S_i\}_{1 \le i \le k}$ be a collection of k subsets of $\mathbb{R}^n$ such that each S_i has Jordan content 0. We claim that the union $S = \bigcup_{1 \le i \le k} S_i$ also has Jordan content 0. Indeed, let $\epsilon > 0$. For each index i between 1 and k, since ϵ/k is a positive number, we can choose a finite number of generalized rectangles that cover S_i, the sum of whose volumes is less than ϵ/k. Taking the union of these k finite collections of generalized rectangles, we obtain a finite collection of generalized rectangles that covers S, the sum of whose volumes is less than $k[\epsilon/k] = \epsilon$.

Example 18.18 Define the segment S in the plane $\mathbb{R}^2$ by

$$S = \{(x, y) \mid 0 \le x \le 1, \ y = x\}.$$

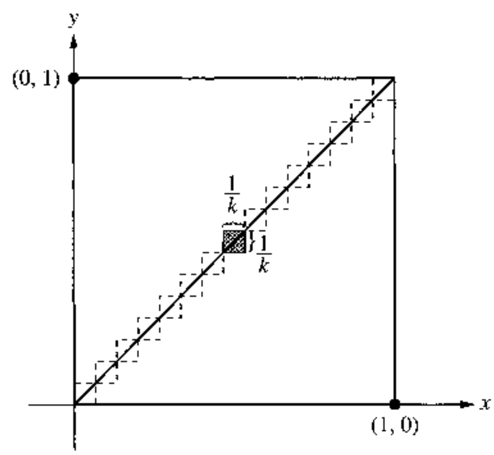

FIGURE 18.2 The segment S in $\mathbb{R}^2$ has Jordan content 0.

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Then S has Jordan content 0. To verify this, let $\epsilon > 0$. Then for each natural number k, let $\{\mathbf{I}_j\}_{1\le j\le k}$ be the collection of k generalized rectangles defined by

$$\mathbf{I}_j = \left[\frac{j-1}{k}, \frac{j}{k}\right] \times \left[\frac{j-1}{k}, \frac{j}{k}\right] \qquad \text{for } 1 \le j \le k.$$

Clearly, this collection of generalized rectangles covers the set S. Each of these rectangles has area $1/k^2$, so the sum of their areas is $1/k$. If k is chosen so that $1/k < \epsilon$, then S is covered by a finite collection of rectangles, the sum of whose areas is less than ϵ. ■

Example 18.19 Define the set S in $\mathbb{R}^3$ by

$$S = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, z = 4\}.$$

Then S also has Jordan content 0. Again, to verify this, let $\epsilon > 0$. Then the single generalized rectangle

$$\mathbf{I} = [0, 1] \times [0, 1] \times [4 - \epsilon/3, 4 + \epsilon/3]$$

contains S and has volume $2\epsilon/3 < \epsilon$. ■

Theorem 18.20 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$. If the set of discontinuities of $f : \mathbf{I} \to \mathbb{R}$ has Jordan content 0, then f is integrable on $\mathbf{I}$.

Proof

We use the Archimedes–Riemann Theorem and criterion (ii) of Theorem 18.10. Let $\epsilon > 0$. It is necessary to find a partition $\mathbf{P}$ of $\mathbf{I}$ such that

$$U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon.$$

Since the function $f : \mathbf{I} \to \mathbb{R}$ is bounded, we can choose a number $M > 0$ such that

$$|f(\mathbf{x})| \le M \qquad \text{for all } \mathbf{x} \text{ in } \mathbf{I}.$$

Denote by D the set of points in $\mathbf{I}$ at which the function $f : \mathbf{I} \to \mathbb{R}$ fails to be continuous. Since the number $\epsilon/4M$ is positive and the set D has Jordan content 0, we can choose a finite collection $\mathcal{F}$ of generalized rectangles that cover D, the sum of whose volumes is less than $\epsilon/4M$. We can also assume that each generalized rectangle in $\mathcal{F}$ is a subset of $\mathbf{I}$.

For each index i between 1 and n, define P_i to be a partition of the ith edge of $\mathbf{I}$ containing all the endpoints of the ith edges of the generalized rectangles in $\mathcal{F}$. Define $\mathbf{P} = (P_1, \ldots, P_n)$. The partition $\mathbf{P}$ has been constructed to have the property that each generalized rectangle in $\mathcal{F}$ is the union of rectangles in the partition $\mathbf{P}$. We divide the set of generalized rectangles in the partition $\mathbf{P}$ into those contained in one of the generalized rectangles in $\mathcal{F}$, which we list as $\mathbf{J}'_1, \ldots, \mathbf{J}'_\ell$, and those that do

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not have this property, which we list as $\mathbf{J}_1, \ldots, \mathbf{J}_m$. Since the sum of the volumes of the generalized rectangles in $\mathcal{F}$ is less than $\epsilon/4M$,

$$\sum_{i=1}^{\ell} \text{vol}\, \mathbf{J}'_i < \frac{\epsilon}{4M},$$

and hence, by the choice of M,

$$\sum_{i=1}^{\ell}[M(f, \mathbf{J}'_i) - m(f, \mathbf{J}'_i)] \text{ vol}\, \mathbf{J}' \leq \sum_{i=1}^{\ell} 2M \cdot \text{vol}\, \mathbf{J}' < \frac{\epsilon}{2}. \tag{18.18}$$

On the other hand, for each index i between 1 and m, because the collection of rectangles $\{\mathbf{J}'_1, \ldots, \mathbf{J}'_\ell\}$ covers D, the function $f : \mathbf{J}_i \to \mathbb{R}$ is continuous on the interior of $\mathbf{J}_i$. Thus, by Proposition 18.17, $f : \mathbf{J}_i \to \mathbb{R}$ is integrable, so by the Archimedes–Riemann Theorem and criterion (ii) of Theorem 18.10, we can choose a partition $\mathbf{P}_i$ of $\mathbf{J}_i$ such that

$$U(f, \mathbf{P}_i) - L(f, \mathbf{P}_i) < \frac{\epsilon}{2m}. \tag{18.19}$$

Choose $\mathbf{P}^*$ to be a refinement of the partition $\mathbf{P}$ that for each index i between 1 and m induces a partition of $\mathbf{J}_i$ that is a refinement of $\mathbf{P}_i$. Then, by the Refinement Lemma,

$$\begin{aligned} U(f, \mathbf{P}^*) - L(f, \mathbf{P}^*) &\leq \sum_{i=1}^{\ell}[M(f, \mathbf{J}'_i) - m(f, \mathbf{J}'_i)] \text{ vol}\, \mathbf{J}'_i + \sum_{i=1}^{m}[U(f, \mathbf{P}_i) - L(f, \mathbf{P}_i)] \\ &< \frac{\epsilon}{2} + m\left[\frac{\epsilon}{2m}\right] = \epsilon. \end{aligned}$$

■

The above theorem is an extension of Proposition 18.17 since (Exercise 9) the boundary of a generalized rectangle has Jordan content 0.

EXERCISES FOR SECTION 18.2

1. For positive numbers a and b, show that the ellipse

$$\{(x,\ y) \text{ in } \mathbb{R}^2 \mid |x| + |y| = 1\}$$

has Jordan content 0.

2. Show that the set of real numbers $\{1/n \mid n \text{ in } \mathbb{N}\}$ has Jordan content 0.

3. Show that the ellipsoid

$$\{(x,\ y,\ z) \text{ in } \mathbb{R}^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, y = z\}$$

has Jordan content 0.

4. For a subset S of $\mathbb{R}^n$, the *characteristic function* of S, denoted by $\chi : \mathbb{R}^n \to \mathbb{R}$, is defined by

$$\chi(\mathbf{x}) \equiv \begin{cases} 1 & \text{for } \mathbf{x} \text{ in } S \\ 0 & \text{for } \mathbf{x} \text{ not in } S. \end{cases}$$

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Show that the set of discontinuities of this characteristic function consists of the boundary of S.

5. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let $\epsilon > 0$. Show that there is a generalized rectangle $\mathbf{J}$ that is contained in the interior of $\mathbf{I}$ and has the property that $\text{vol}\,\mathbf{I} - \text{vol}\,\mathbf{J} < \epsilon$.
6. Let $\mathbf{J}$ and $\mathbf{I}$ be generalized rectangles in $\mathbb{R}^n$ such that $\mathbf{J}$ is contained in the interior of $\mathbf{J}$. Given a partition $\mathbf{P}$ of $\mathbf{J}$, show that there is a partition $\mathbf{P}'$ of $\mathbf{I}$ such that each generalized rectangle in $\mathbf{P}$ is also a generalized rectangle in $\mathbf{P}'$.
7. Let $A = \{x \text{ in } \mathbb{R} \mid 0 \le x \le 1\}$ and let $B = \{(x, y) \text{ in } \mathbb{R}^2 \mid 0 \le x \le 1,\ y = 0\}$. Show that A does not have Jordan content 0, whereas B has Jordan content 0. Is this consistent?
8. Let $\{\mathbf{u}_k\}$ be a convergent sequence in $\mathbb{R}^n$. Show that the set $\{\mathbf{u}_k \mid k \text{ in } \mathbb{N}\}$ has Jordan content 0.
9. Show that the boundary of a generalized rectangle has Jordan content 0.
10. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is continuous. Assume that $f(\mathbf{x}) \ge 0$ for all points $\mathbf{x}$ in $\mathbf{I}$. Prove that if $\int_{\mathbf{I}} f = 0$, then $f(\mathbf{x}) = 0$ for all points $\mathbf{x}$ in $\mathbf{I}$. Is continuity necessary for this to hold?

18.3 INTEGRATION OF FUNCTIONS ON JORDAN DOMAINS

In the study of integration of functions of a single variable, we considered only functions that have as their domains closed bounded intervals. For functions of several variables, generalized rectangles do not play a similar preeminent role: It is necessary to integrate functions of several variables that have quite general domains.

Integrating the Zero Extension of a Function

Definition For a bounded subset D of $\mathbb{R}^n$ and a bounded function $f : D \to \mathbb{R}$, if $\mathbf{I}$ is a generalized rectangle that contains D, we define the *zero extension* of $f : D \to \mathbb{R}$ to $\mathbf{I}$, denoted by $\hat{f} : \mathbf{I} \to \mathbb{R}$, to be the function defined by

$$\hat{f}(\mathbf{x}) \equiv \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \text{ is in } D \\ 0 & \text{if } \mathbf{x} \text{ is in } \mathbf{I} \backslash D. \end{cases}$$

Definition Let D be a bounded subset of $\mathbb{R}^n$ and let the function $f : D \to \mathbb{R}$ be bounded. Then $f : D \to \mathbb{R}$ is said to be *integrable* provided that there is a generalized rectangle $\mathbf{I}$ that contains D for which the zero extension $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable; in this case, we define

$$\int_D f \equiv \int_{\mathbf{I}} \hat{f}. \tag{18.20}$$

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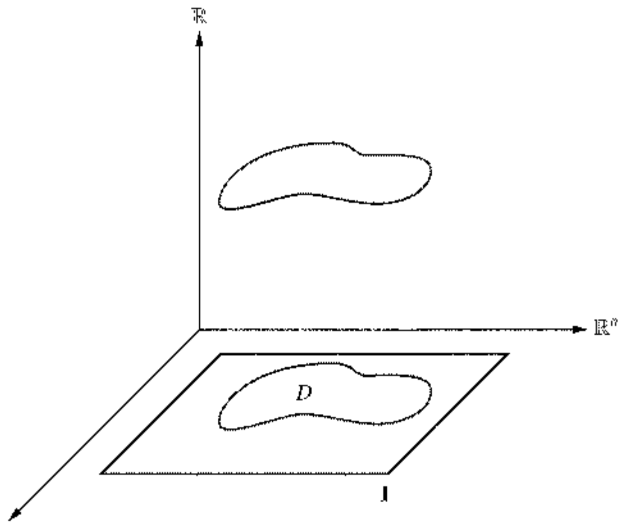

FIGURE 18.3 The zero extension of $f : D \to \mathbb{R}$ to $\mathbf{I}$.

It is necessary to show that the above definition is unambiguous, that is, that it is independent of the choice of generalized rectangle containing D. To do this, we first establish that this is so in the case where D itself is a generalized rectangle.

Lemma 18.21 Let $f : \mathbf{J} \to \mathbb{R}$ be an integrable function on the generalized rectangle $\mathbf{J}$ and let $\mathbf{I}$ be a generalized rectangle that contains $\mathbf{J}$. Then $\hat{f} : \mathbf{I} \to \mathbb{R}$, the zero extension of f to $\mathbf{I}$, is integrable and

$$\int_{\mathbf{I}} \hat{f} = \int_{\mathbf{J}} f.$$

Proof

For each index i between 1 and n, let the ith edge of $\mathbf{I}$ be $[a_i, b_i]$, let the ith edge of $\mathbf{J}$ be $[a_i', b_i']$, and consider the partition $P_i = \{a_i, a_i', b_i', b_i\}$ of the interval $[a_i, b_i]$. The partition $\mathbf{P} = (P_1, \ldots, P_n)$ of $\mathbf{I}$ has the property that it contains the generalized rectangle $\mathbf{J}$. Furthermore, for each generalized rectangle $\mathbf{J}'$ in $\mathbf{P}$ other than $\mathbf{J}$, the restriction $\hat{f} : \mathbf{J}' \to \mathbb{R}$ is zero on the interior of $\mathbf{J}'$. By Proposition 18.17, the function $\hat{f} : \mathbf{J}' \to \mathbb{R}$ is integrable; it is not difficult (Exercise 5) to see that $\int_{\mathbf{J}'} \hat{f} = 0$. It follows from the additivity of integrals over partitions (Theorem 18.11) that the function $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable if and only if its restriction $f : \mathbf{J} \to \mathbb{R}$ is integrable and that

$$\int_{\mathbf{I}} \hat{f} = \sum_{\mathbf{J}' \text{ in } \mathbf{P}} \int_{\mathbf{J}'} \hat{f} = \int_{\mathbf{J}} \hat{f} = \int_{\mathbf{J}} f. \qquad \blacksquare$$

To show that the above definition of the integral of f over a general bounded set D is properly defined, let $\mathbf{I}_1$ and $\mathbf{I}_2$ be two generalized rectangles that contain D. Define $\mathbf{I}$ to be the intersection of $\mathbf{I}_1$ and $\mathbf{I}_2$. By Lemma 18.21, the function $\hat{f} : \mathbf{I}_1 \to \mathbb{R}$ is integrable

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if and only if the function $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable, in which case

$$\int_{\mathbf{I}} \hat{f} = \int_{\mathbf{J}} \hat{f},$$

and the function $\hat{f} : \mathbf{I}_2 \to \mathbb{R}$ is integrable if and only if the function $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable, in which case

$$\int_{\mathbf{I}} \hat{f} = \int_{\mathbf{I}_2} \hat{f}.$$

Thus, $\hat{f} : \mathbf{I}_1 \to \mathbb{R}$ is integrable if and only if $\hat{f} : \mathbf{I}_2 \to \mathbb{R}$ is integrable, in which case

$$\int_{\mathbf{I}_1} \hat{f} = \int_{\mathbf{I}_2} \hat{f}.$$

It follows that the above definitions of integrability and of the integral are unambiguous.

Monotonicity and Linearity of Integration

Theorem 18.22 Monotonicity of the Integral For D a bounded subset of $\mathbb{R}^n$, suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are integrable and also that

$$f(\mathbf{x}) \leq g(\mathbf{x}) \qquad \text{for all points } \mathbf{x} \text{ in } D.$$

Then

$$\int_D f \leq \int_D g.$$

Proof

Choose $\mathbf{I}$ to be a generalized rectangle that contains D. Observe that

$$\hat{f}(\mathbf{x}) \leq \hat{g}(\mathbf{x}) \qquad \text{for all points } \mathbf{x} \text{ in } \mathbf{I}.$$

By the monotonicity property of integration for functions defined on generalized rectangles (Theorem 18.12),

$$\int_D f = \int_{\mathbf{I}} \hat{f} \leq \int_{\mathbf{I}} \hat{g} = \int_D g. \qquad \blacksquare$$

Theorem 18.23 Linearity of the Integral For D a bounded subset of $\mathbb{R}^n$, suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are integrable. Then for any two numbers α and β, the function $\alpha f + \beta g : D \to \mathbb{R}$ also is integrable and

$$\int_D [\alpha f + \beta g] = \alpha \int_D f + \beta \int_D g. \tag{18.21}$$

Proof

Let $\mathbf{I}$ be a generalized rectangle that contains D. Observe that the zero extension of $\alpha f + \beta g : D \to \mathbb{R}$ to $\mathbf{I}$ equals $\alpha\hat{f} + \beta\hat{g} : \mathbf{I} \to \mathbb{R}$. By the linearity property of

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integration of functions defined on rectangles (Theorem 18.13),

$$\int_D [\alpha f + \beta g] = \int_{\mathbf{I}} [\alpha \hat{f} + \beta \hat{g}] = \alpha \int_{\mathbf{I}} \hat{f} + \beta \int_{\mathbf{I}} \hat{g} = \alpha \int_D f + \beta \int_D g. \qquad \blacksquare$$

Jordan Domains

For a continuous function defined on a bounded subset D of $\mathbb{R}^n$, the set of discontinuities of any zero extension are contained in the boundary of D (Exercise 6). Thus, since according to Theorem 18.20 a bounded function on a generalized rectangle is integrable provided that its set of discontinuities has Jordan content 0, we are led to focus on the following class of bounded subsets of $\mathbb{R}^n$.

Definition A bounded subset D of $\mathbb{R}^n$ is said to be a *Jordan domain* provided that its boundary has Jordan content 0.

Theorem 18.24 Let D be a Jordan domain in $\mathbb{R}^n$ and let the function $f : D \to \mathbb{R}$ be bounded. If the set of discontinuities of $f : D \to \mathbb{R}$ has Jordan content 0, then $f : D \to \mathbb{R}$ is integrable.

Proof

Choose $\mathbf{I}$ to be a generalized rectangle in $\mathbb{R}^n$ that contains the domain D; it is necessary to show that the zero extension $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable. According to Theorem 18.20, to do so it suffices to show that the set of discontinuities of $\hat{f} : \mathbf{I} \to \mathbb{R}$ has Jordan content 0. But the zero extension is clearly continuous at each point $\mathbf{x}$ in the interior of D at which the function $f : D \to \mathbb{R}$ is continuous since there is a neighborhood of $\mathbf{x}$ on which $\hat{f} : \mathbf{I} \to \mathbb{R}$ agrees with the function $f : D \to \mathbb{R}$. On the other hand, for each point $\mathbf{x}$ in $\mathbf{I}$ that is in the exterior of D, there is a neighborhood of $\mathbf{x}$ on which $\hat{f}$ is identically equal to 0, and so $\hat{f} : \mathbf{I} \to \mathbb{R}$ is certainly continuous at $\mathbf{x}$. Thus, the set of discontinuities of $\hat{f} : \mathbf{I} \to \mathbb{R}$ is contained in the union of the boundary of D and the set of discontinuities of $f : D \to \mathbb{R}$ that are in the interior of D. By assumption, each of these two sets has Jordan content 0, and hence so does their union. $\blacksquare$

Graphs of Integrable Functions

For the above theorem to be useful, it is necessary to provide criteria for a set to have Jordan content 0. The following proposition and its corollary provide such criteria.

Proposition 18.25 Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^{n-1}$ and suppose that the function $g : \mathbf{I} \to \mathbb{R}$ is integrable. Then the subset of $\mathbb{R}^n$ consisting of the graph of $g : \mathbf{I} \to \mathbb{R}$,

$$\{(\mathbf{x}, g(\mathbf{x})) \text{ in } \mathbb{R}^n \mid \mathbf{x} \text{ in } \mathbf{I}\},$$

has Jordan content 0.

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Proof

Let $\epsilon > 0$. According to the Archimedes–Riemann Theorem and criterion (ii) of Theorem 18.10, there is a partition $\mathbf{P}$ of $\mathbf{I}$ having the property that $U(g, \mathbf{P}) - L(g, \mathbf{P}) < \epsilon$; that is,

$$\sum_{\mathbf{J} \text{ in } \mathbf{P}} [M(g, \mathbf{J}) - m(g, \mathbf{J})] \operatorname{vol} \mathbf{J} < \epsilon, \tag{18.22}$$

where for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$,

$$M(g, \mathbf{J}) = \sup\{g(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\} \qquad \text{and} \qquad m(g, \mathbf{J}) = \inf\{g(\mathbf{x}) \mid \mathbf{x} \text{ in } \mathbf{J}\}.$$

Observe that for each generalized rectangle $\mathbf{J}$ in $\mathbf{P}$, the Cartesian product generalized rectangle

$$\mathbf{J} \times [m(g, \mathbf{J}), M(g, \mathbf{J})]$$

is a generalized rectangle in $\mathbb{R}^n$ and that by the definition of Darboux sums, the graph of $g : \mathbf{I} \to \mathbb{R}$ is contained in the union of the products as $\mathbf{J}$ varies in $\mathbf{P}$. The inequality (18.22) means precisely that the sum of the volumes of these products is less than ϵ. Thus, the graph has Jordan content 0. ∎

Corollary 18.26 Let D be a Jordan domain in $\mathbb{R}^{n-1}$, let the function $f : D \to \mathbb{R}$ be bounded, and let the set of discontinuities of $f : D \to \mathbb{R}$ have Jordan content 0. Then the subset of $\mathbb{R}^n$ consisting of the graph of $f : D \to \mathbb{R}$,

$$\{(\mathbf{x}, g(\mathbf{x})) \text{ in } \mathbb{R}^n \mid \mathbf{x} \text{ in } D\},$$

has Jordan content 0.

Proof

Let $\mathbf{I}$ be a generalized rectangle that contains D. Since $f : D \to \mathbb{R}$ is integrable, the zero extension of $f : D \to \mathbb{R}$ to $\mathbf{I}$, $\hat{f} : \mathbf{I} \to \mathbb{R}$, also is integrable. By Proposition 18.25, the graph of $\hat{f} : \mathbf{I} \to \mathbb{R}$ has Jordan content 0. Since the graph of $f : D \to \mathbb{R}$ is a subset of the graph of $\hat{f} : \mathbf{I} \to \mathbb{R}$, the graph of $f : D \to \mathbb{R}$ also has Jordan content 0. ∎

Example 18.27 The following subsets of the plane $\mathbb{R}^2$ are Jordan domains:

$$\{(x, y) \mid x^2 + y^2 < 1\}, \qquad \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Indeed, each of these sets has as its boundary the unit circle $S = \{(x, y) \mid x^2 + y^2 = 1\}$, so it is necessary to show that S has Jordan content 0. But S is the union of two graphs of continuous functions defined on the interval $[-1, 1]$. The preceding corollary asserts that each of these graphs has Jordan content 0, and hence so does their union. ∎

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Example 18.28 For $c > 0$, define the cylinder C in $\mathbb{R}^3$ by

$$C = \{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq z \leq c\}.$$

We claim that C is a Jordan domain. Indeed, the boundary of C is the union of the top, $\{(x, y, z) \mid x^2+y^2 \leq 1, z = c\}$, the bottom, $\{(x, y, z) \mid x^2+y^2 \leq 1, z = 0\}$, and the lateral side

$$\{(x, y, z) \mid x^2 + y^2 = 1, 0 \leq z \leq c\}.$$

The top and the bottom each have Jordan content 0 since each is the graph of a constant function on a Jordan domain. The lateral side can be written as

$$\left\{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq z \leq c, y = \pm\sqrt{1 - x^2}\right\}.$$

Since this set is the union of two sets, each of which is the graph of a continuous function defined on a rectangle, the lateral side has Jordan content 0. Hence C is a Jordan domain. ■

The above three examples illustrate a general way to check that a subset D of $\mathbb{R}^n$ is a Jordan domain: It suffices to show that the boundary of D is the union of a finite number of graphs of integrable functions of $n - 1$ variables. In particular (Exercise 1), any open ball and any generalized rectangle in $\mathbb{R}^n$ is a Jordan domain.

Additivity of Integration

In preparation for a general additivity over domains result for the integral, we establish the following lemma.

Lemma 18.29 Let $f : \mathbf{I} \to \mathbb{R}$ be a bounded function on a generalized rectangle $\mathbf{I}$ with the property that there is a subset S of $\mathbf{I}$ having Jordan content 0 such that

$$f(\mathbf{x}) = 0 \qquad \text{for all points } \mathbf{x} \text{ in } \mathbf{I}\backslash S.$$

Then $f : \mathbf{I} \to \mathbb{R}$ is integrable and $\int_{\mathbf{I}} f = 0$.

Proof

It is not difficult to see that since the set S has Jordan content 0, the boundary of S also has Jordan content 0 (Exercise 12). Thus, S is a Jordan domain. Moreover, the function $f : \mathbf{I} \to \mathbb{R}$ is the zero extension of the function $f : S \to \mathbb{R}$. Since the function $f : \mathbf{I} \to \mathbb{R}$ is continuous at each point in $\mathbf{I}$ that is an exterior point of S, the set of discontinuities $f : \mathbf{I} \to \mathbb{R}$ has Jordan content 0. It follows from the proof of Theorem 18.20 that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable, so it is necessary only to prove that $\int_{\mathbf{I}} f = 0$. Establishing this is equivalent to showing that for each $\epsilon > 0$,

$$-\epsilon < \int_{\mathbf{I}} f < \epsilon. \tag{18.23}$$

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Let $\epsilon > 0$. Following the proof of Theorem 18.20, it is not difficult to find a partition $\mathbf{P}$ of $\mathbf{I}$ such that

$$-\epsilon < L(f, \mathbf{P}) \le \int_{\mathbf{I}} f \le U(f, \mathbf{P}) < \epsilon,$$

and therefore (18.23) holds. ■

Theorem 18.30 Let D_1 and D_2 be bounded subsets of $\mathbb{R}^n$. For $D = D_1 \cup D_2$, suppose that the function $f : D \to \mathbb{R}$ has the property that both $f : D_1 \to \mathbb{R}$ and $f : D_2 \to \mathbb{R}$ are integrable. If the intersection $D_1 \cap D_2$ has Jordan content 0, then the function $f : D \to \mathbb{R}$ also is integrable and

$$\int_D f = \int_{D_1} f + \int_{D_2} f. \tag{18.24}$$

Proof

Choose $\mathbf{I}$ to be a generalized rectangle that contains D. Define $\hat{f}_1 : \mathbf{I} \to \mathbb{R}$ to be the zero extension of $f : D_1 \to \mathbb{R}$ to $\mathbf{I}$, $\hat{f}_2 : \mathbf{I} \to \mathbb{R}$ to be the zero extension of $f : D_2 \to \mathbb{R}$ to $\mathbf{I}$, and $\hat{f} : \mathbf{I} \to \mathbb{R}$ to be the zero extension of $f : D \to \mathbb{R}$ to $\mathbf{I}$. It is necessary to show that $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable and that

$$\int_{\mathbf{I}} \hat{f} = \int_{\mathbf{I}} \hat{f}_1 + \int_{\mathbf{I}} \hat{f}_2. \tag{18.25}$$

To do so, define the auxiliary function $g : \mathbf{I} \to \mathbb{R}$ by

$$g(\mathbf{x}) = \hat{f}(\mathbf{x}) - [\hat{f}_1(\mathbf{x}) + \hat{f}_2(\mathbf{x})] \qquad \text{for all points } \mathbf{x} \text{ in } \mathbf{I}.$$

Observe that the only possible points $\mathbf{x}$ in $\mathbf{I}$ at which $g(\mathbf{x}) \neq 0$ are those in $D_1 \cap D_2$. By assumption, the set $D_1 \cap D_2$ has Jordan content 0, so it follows from Lemma 18.29 that the function $g : \mathbf{I} \to \mathbb{R}$ is integrable and that $\int_{\mathbf{I}} g = 0$. By the linearity of the integrals of functions defined on generalized rectangles (Theorem 18.13), since $\hat{f} = \hat{f}_1 + \hat{f}_2 + g$, the function $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable and

$$\int_{\mathbf{I}} \hat{f} = \int_{\mathbf{I}} [\hat{f}_1 + \hat{f}_2 + g] = \int_{\mathbf{I}} \hat{f}_1 + \int_{\mathbf{I}} \hat{f}_2 + \int_{\mathbf{I}} g = \int_{\mathbf{I}} \hat{f}_1 + \int_{\mathbf{I}} \hat{f}_2;$$

that is, (18.25) holds. ■

Volume of Sets

We have defined the *volume* of a generalized rectangle to be the product of the lengths of its edges. It is useful to define volume for more general types of sets.

Definition For a bounded subset D of $\mathbb{R}^n$, suppose that the function $f : D \to \mathbb{R}$ that is identically equal to 1 is integrable. Then the set D is said to have volume, and its volume, denoted by vol D, is defined by

$$\text{vol } D = \int_D f.$$

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Since a constant function is certainly continuous, it follows from Theorem 18.24 that every Jordan domain D in $\mathbb{R}^n$ has volume; moreover, since by definition its boundary ∂D has Jordan content 0, by Lemma 18.29, the set ∂D also has volume and vol $\partial D = 0$. Thus, by the additivity of integration formula (18.24), the set $D \cup \partial D$ also has volume and

$$\text{vol}(D \cup \partial D) = \text{vol } D + \text{vol } \partial D = \text{vol } D. \tag{18.26}$$

It is useful to record for future reference an additivity of volume result.

Corollary 18.31 Additivity of Volume Suppose that D_1 and D_2 are Jordan domains in $\mathbb{R}^n$ that are disjoint. Then $D = (D_1 \cup D_2)$ also is a Jordan domain, as is $D \cup \partial D$, and

$$\text{vol}(D \cup \partial D) = \text{vol } D = \text{vol } D_1 + \text{vol } D_2.$$

Proof

It is not difficult to see that $\partial D \subseteq \partial D_1 \cup \partial D_2$. By definition, ∂D_1 and ∂D_2 have Jordan content 0, and hence so does ∂D. Thus, D is a Jordan domain. Thus, by the additivity of integration formula (18.24) and formula (18.26),

$$\text{vol}(D \cup \partial D) = \text{vol } D = \text{vol } D_1 + \text{vol } D_2. \qquad \blacksquare$$

It is not the case that all bounded subsets of $\mathbb{R}^n$ have volume. For instance, consider the set S of all rational numbers in the interval $[0, 1]$. The zero extension to the interval $[0, 1]$ of the function that is identically equal to 1 on S is Dirichlet's function. Dirichlet's function is not integrable (Example 18.6), and therefore the set S does not have volume.

EXERCISES FOR SECTION 18.3

1. Show that any open ball and any generalized rectangle in $\mathbb{R}^n$ is a Jordan domain.
2. For a subset S of $\mathbb{R}^n$ contained in the generalized rectangle $\mathbf{I}$, define the function $f : \mathbf{I} \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \text{ in } S \\ 0 & \text{for } \mathbf{x} \text{ not in } S. \end{cases}$$

Show that $f : \mathbf{I} \to \mathbb{R}$ is integrable if S has Jordan content 0.

3. For two subsets D_1 and D_2 of $\mathbb{R}^n$, show that

$$\partial(D_1 \cup D_2) \subseteq \partial D_1 \cup \partial D_2.$$

Provide examples where there is equality and where there fails to be equality.

4. Prove that the union of two Jordan domains is a Jordan domain.
5. Suppose that $f : \mathbf{I} \to \mathbb{R}$ is a bounded function on a generalized rectangle that has the value 0 on the interior of $\mathbf{I}$. Show that $\int_{\mathbf{I}} f = 0$.
6. For a continuous function defined on a bounded subset D of $\mathbb{R}^n$, show that the set of discontinuities of any zero extension are contained in the boundary of D.

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7. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$. By observing that there is a bounded function $f : \mathbf{I} \to \mathbb{R}$ that is not integrable, use Lemma 18.29 to conclude that $\mathbf{I}$ does not have Jordan content 0.

8. Show that a subset S of $\mathbb{R}^n$ that has Jordan content 0 has an empty interior. (*Hint:* By Exercise 7, S cannot contain a generalized rectangle.)

9. For $\mathbf{I}$ a generalized rectangle in $\mathbb{R}^n$, let A be a subset of $\mathbf{I}$ of Jordan content 0 and suppose that the integrable functions $f : \mathbf{I} \to \mathbb{R}$ and $g : \mathbf{I} \to \mathbb{R}$ are such that

$$f(\mathbf{x}) = g(\mathbf{x}) \qquad \text{for } \mathbf{x} \text{ in } \mathbf{I} \backslash A.$$

Show that

$$\int_{\mathbf{I}} f = \int_{\mathbf{I}} g.$$

10. For $\mathbf{I}$ a generalized rectangle in $\mathbb{R}^n$, define $f : \mathbf{I} \to \mathbb{R}$ to be the function with constant value 1. Find a subset D of $\mathbf{I}$ such that the restriction $f : D \to \mathbb{R}$ is not integrable.

11. Let $\mathbf{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let the function $f : \mathbf{I} \to \mathbb{R}$ be integrable. Denote the interior of $\mathbf{I}$ by D. Show that the restriction $f : D \to \mathbb{R}$ is integrable and that

$$\int_{\mathbf{I}} f = \int_D f.$$

12. **a.** Let S and F be subsets of $\mathbb{R}^n$ such that $S \subseteq F$. If F is closed, show that $\partial S \subseteq F$.

b. Use part (a) and the fact that the union of a finite number of generalized rectangles is closed to show that if S has Jordan content 0, then ∂S also has Jordan content 0.

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CHAPTER 19

ITERATED INTEGRATION AND CHANGES OF VARIABLES

At this point, we have no general method to actually evaluate the integral of an integrable function of several variables. The first section of this chapter will be devoted to Fubini's Theorem, which reduces the evaluation of certain integrals to the problem of evaluating integrals of functions of a single variable, in which case it is often possible to use the First Fundamental Theorem of Calculus (Integrating Derivatives). In the second section, we will study a general change of variables result for evaluating integrals of functions of several variables. This is the extension of the change of variables result for functions of a single variable. It can often be used to reduce the evaluation of complicated integrals to ones that can be evaluated more easily, for instance, by using Fubini's Theorem.

19.1 FUBINI'S THEOREM

Theorem 19.1 Fubini's Theorem in the Plane Suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable, where $\mathbf{I} = [a, b] \times [c, d]$ is a rectangle in the plane $\mathbb{R}^2$. For each point x in $[a, b]$, define the function $F_x : [c, d] \to \mathbb{R}$ by $F_x(y) = f(x, y)$ for y in $[c, d]$, suppose that the function $F_x : [c, d] \to \mathbb{R}$ is integrable, and define

$$A(x) = \int_c^d f(x, y)\, dy.$$

Then the function $A : [a, b] \to \mathbb{R}$ is integrable, and

$$\int_{\mathbf{I}} f = \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx. \tag{19.1}$$

Proof

The crucial point of the proof is to verify the following inequality that is satisfied by the Darboux sums for the two functions $f : \mathbf{I} \to \mathbb{R}$ and $A : [a, b] \to \mathbb{R}$: for every

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partition $\mathbf{P} = (P_1, P_2)$ of $\mathbf{I}$,

$$L(f, \mathbf{P}) \leq L(A, P_1) \leq U(A, P_1) \leq U(f, \mathbf{P}). \tag{19.2}$$

Indeed, suppose that this inequality has been verified. Since f is integrable on $\mathbf{I}$, according to the Archimedes–Riemann Theorem, we can select an Archimedean sequence of partitions $\{\mathbf{P}_k\}$ for f on $\mathbf{I}$. For each index k, set $\mathbf{P}_k = (P_k^1, P_k^2)$. Then from the inequality (19.2) it follows that $\{P_k^1\}$ is an Archimedean sequence of partitions for the function A on $[a, b]$. Once more using the Archimedes–Riemann Theorem, we conclude that A is integrable on $[a, b]$. Since the Darboux sums associated with an Archimedean sequence of partitions converge to the value of the integral, it follows from the inequality (19.2) that formula (19.3) holds.

It remains to verify the inequality (19.2). Let $\mathbf{P} = (P_1, P_2)$ be a partition of $\mathbf{I}$, where $P_1 = \{x_0, \ldots, x_m\}$ and $P_2 = \{y_0, \ldots, y_\ell\}$ are partitions of $[a, b]$ and $[c, d]$, respectively. For a pair of indices i and j such that $1 \leq i \leq m$ and $1 \leq j \leq \ell$, define

$$M_{ij} = \sup\{f(x, y) \mid (x, y) \text{ in } [x_{i-1}, x_i] \times [y_{j-1}, y_j]\};$$

and

$$M_i = \sup\{A(x) \mid x \text{ in } [x_{i-1}, x_i]\}.$$

Then, by definition,

$$U(f, \mathbf{P}) = \sum_{i=1}^{m} \sum_{j=1}^{\ell} M_{ij}[x_i - x_{i-1}][y_j - y_{j-1}];$$

and

$$U(A, P_1) = \sum_{i=1}^{m} M_i[x_i - x_{i-1}].$$

Fix an index i between 1 and m and a point x in the interval $[x_{i-1}, x_i]$. Then for each index j between 1 and ℓ,

$$f(x, y) \leq M_{ij} \qquad \text{for all points } y \text{ in } [y_{j-1}, y_j].$$

The monotonicity property of the integral of functions defined on the interval $[y_{j-1}, y_j]$ implies that

$$\int_{y_{j-1}}^{y_j} f(x, y)\, dy \leq M_{ij}[y_j - y_{j-1}].$$

Summing this inequality for $j = 1, \ldots, \ell$ and using the additivity over intervals property of the integral, we obtain

$$\int_d^c f(x, y)\, dy \leq \sum_{j=1}^{\ell} M_{ij}[y_j - y_{j-1}].$$

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Since this inequality holds for each point x in $[x_{i-1}, x_i]$, it follows from the definition of m_i and M_i that

$$M_i \leq \sum_{j=1}^{\ell} M_{ij}[y_j - y_{j-1}].$$

Multiply this inequality by $x_i - x_{i-1}$ and sum the resulting m inequalities for $i = 1, \ldots, m$ to obtain

$$U(A, P_1) \leq U(f, \mathbf{P}).$$

A similar argument shows that

$$L(f, \mathbf{P}) \leq L(A, P_1). \qquad \blacksquare$$

Example 19.2 Define $f(x, y) = e^{xy}x$ for (x, y) in $\mathbf{I} = [1, 2] \times [0, 1]$. Since the function $f : \mathbf{I} \to \mathbb{R}$ is continuous, it follows from Fubini's Theorem that iterated integration is permissible. By the First Fundamental Theorem of Calculus (Integrating Derivatives),

$$\int_{\mathbf{I}} f = \int_1^2 \left[\int_0^1 e^{xy} x \, dy \right] dx = \int_1^2 [e^x - 1] \, dx = e^2 - e - 1. \qquad \blacksquare$$

Theorem 19.3 For continuous functions $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ with the property that $h(x) \leq g(x)$ for all points x in $[a, b]$, define

$$D = \{(x, y) \mid a \leq x \leq b, \ h(x) \leq y \leq g(x)\}.$$

Suppose that the function $f : D \to \mathbb{R}$ is continuous and bounded. Then

$$\int_D f = \int_a^b \left[\int_{h(x)}^{g(x)} f(x, y) \, dy \right] dx. \tag{19.3}$$

Proof

The set D is a Jordan domain since its boundary consists of the union of four graphs, each of which is the graph of a continuous function on a bounded interval. Choose an interval $[c, d]$ such that the rectangle $\mathbf{I} = [a, b] \times [c, d]$ contains D and let $\hat{f} : \mathbf{I} \to \mathbb{R}$ be the zero extension of $f : D \to \mathbb{R}$ to $\mathbf{I}$. Theorem 18.24 implies that the function $\hat{f} : \mathbf{I} \to \mathbb{R}$ is integrable. Using Lemma 18.21 in the case where $n = 1$, with $I = [c, d]$ and $J = [h(x), g(x)]$, we conclude that

$$A(x) \equiv \int_c^d \hat{f}(x, y) \, dy = \int_{h(x)}^{g(x)} f(x, y) \, dy \qquad \text{for all } x \text{ in } [a, b].$$

Formula (19.3) now follows from Fubini's Theorem in the plane. $\blacksquare$

Example 19.4 For $D = \{(x, y) \mid x^2 + y^2 \leq 1, y \geq 0\}$, define $f : D \to \mathbb{R}$ to be the constant function with value 1. Then

$$D = \left\{(x, y) \mid -1 \leq x \leq 1, \ 0 \leq y \leq \sqrt{1 - x^2}\right\},$$

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so by Theorem 19.3,

$$\int_D f = \int_{-1}^{1}\left[\int_0^{\sqrt{1-x^2}} dy\right] dx = \int_{-1}^{1}\left[\sqrt{1-x^2}\right] dx = \frac{\pi}{2}. \qquad \blacksquare$$

In the case where the function $f : \mathbf{I} \to \mathbb{R}$ fails to be continuous, some care is needed in verifying formula (19.1). As the next example shows, it is possible for the integral on the right-hand side of formula (19.1) to be properly defined and yet for the function $f : \mathbf{I} \to \mathbb{R}$ to fail to be integrable.

Example 19.5 Define the function $f : \mathbf{I} \to \mathbb{R}$, where $\mathbf{I} = [0, 1] \times [0, 1]$, by

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}$$

For each point x in $[0, 1]$, $\int_0^1 f(x, y)\,dy = 1$, so that

$$\int_0^1 \left[\int_0^1 f(x, y)\,dy\right] dx = 1.$$

But it is not difficult to see that the function $f : \mathbf{I} \to \mathbb{R}$ is not integrable (Exercise 8). $\blacksquare$

Formula (19.1) singles out the variables x and y in a particular order. In fact, there is a corresponding formula if the order of integration is reversed. For an integrable function $f : [a, b] \times [c, d] \to \mathbb{R}$, for each point y in $[c, d]$, define the function $F_y : [a, b] \to \mathbb{R}$ by $F_y(x) = f(x, y)$ for x in $[a, b]$. Suppose that the function $F_y : [a, b] \to \mathbb{R}$ is integrable and define

$$B(y) = \int_a^b f(x, y)\,dx.$$

Then the function $B : [c, d] \to \mathbb{R}$ is integrable, and

$$\int_{\mathbf{I}} f = \int_c^d B(y)\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx\right] dy. \tag{19.4}$$

The proof of this formula is precisely the same as the proof of Theorem 19.1, except that it requires replacing

$$\sum_{i=1}^{m}\sum_{j=1}^{\ell} M_{ij}[y_j - y_{j-1}][x_i - x_{i-1}]$$

with

$$\sum_{j=1}^{\ell}\sum_{i=1}^{m} M_{ij}[x_i - x_{i-1}][y_j - y_{j-1}].$$

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Corollary 19.6 Suppose that the function $f : \mathbf{I} \to \mathbb{R}$ is integrable, where $\mathbf{I} = [a, b] \times [c, d]$ is a rectangle in the plane $\mathbb{R}^2$. Then the following formula holds provided that each side is defined:

$$\int_a^b \left[\int_c^d f(x, y)\, dy \right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy. \tag{19.5}$$

In particular, this formula holds if the function $f : \mathbf{I} \to \mathbb{R}$ is continuous.

Proof

Theorem 19.1 asserts that

$$\int_a^b \left[\int_c^d f(x, y)\, dy \right] dx = \int_{\mathbf{I}} f$$

provided that the integral on the left is defined. As discussed above, a symmetric argument shows that

$$\int_c^d \left[\int_a^b f(x, y)\, dx \right] dy = \int_{\mathbf{I}} f$$

provided that the integral on the left is defined. Thus, each of the sides of (19.5) is equal to $\int_{\mathbf{I}} f$ provided that they are defined. ■

The iterated integration formula (19.1) extends to the integral of functions over generalized rectangles in Euclidean space $\mathbb{R}^m$ of dimension $m \geq 2$. Moreover, once the proper notation is introduced, the statement and proof of the general result are exactly the same as for $\mathbb{R}^2$. Indeed, write $m = n + k$, where n and k are natural numbers. As we have done before, we write a point $\mathbf{u}$ in $\mathbb{R}^{n+k}$ as

$$\mathbf{u} = (\mathbf{x}, \mathbf{y}), \qquad \text{where } \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } \mathbf{y} \text{ is in } \mathbb{R}^k.$$

Moreover, a generalized rectangle $\mathbf{I} = I_1 \times \cdots \times I_n \times I_{n+1} \times \cdots \times I_{n+k}$ in $\mathbb{R}^{n+k}$ can be represented as the Cartesian product

$$\mathbf{I} = \mathbf{I}_{\mathbf{x}} \times \mathbf{I}_{\mathbf{y}},$$

where

$$\mathbf{I}_{\mathbf{x}} = I_1 \times \cdots \times I_n \qquad \text{and} \qquad \mathbf{I}_{\mathbf{y}} = I_{n+1} \times \cdots \times I_{n+k}.$$

The proof given for Fubini's Theorem in the plane extends directly to provide a proof of the following extension.

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Theorem 19.7 Fubini's Theorem Suppose that the function $f:\mathbf{I}\to\mathbb{R}$ is integrable, where $\mathbf{I}=\mathbf{I}_\mathbf{x}\times\mathbf{I}_\mathbf{y}$ is a generalized rectangle in $\mathbb{R}^{n+k}$. For each point $\mathbf{x}$ in $\mathbf{I}_\mathbf{x}$, define the function $F_\mathbf{x}:\mathbf{I}_\mathbf{y}\to\mathbb{R}$ by

$$F_\mathbf{x}(\mathbf{y})=f(\mathbf{x},\,\mathbf{y})\qquad\text{for }\mathbf{y}\text{ in }\mathbf{I}_\mathbf{y};$$

suppose that the function $F_\mathbf{x}:\mathbf{I}_\mathbf{y}\to\mathbb{R}$ is integrable and define

$$A(\mathbf{x})=\int_{\mathbf{I}_\mathbf{y}} f(\mathbf{x},\,\mathbf{y})\,d\mathbf{y}.$$

Then the function $A:\mathbf{I}_\mathbf{x}\to\mathbb{R}$ is integrable, and

$$\int_\mathbf{I} f=\int_{\mathbf{I}_\mathbf{x}} A(\mathbf{x})\,d\mathbf{x}=\int_{\mathbf{I}_\mathbf{x}}\left[\int_{\mathbf{I}_\mathbf{y}} f(\mathbf{x},\,\mathbf{y})\,d\mathbf{y}\right]d\mathbf{x}.\tag{19.6}$$

The proof of the next theorem is exactly the same as the proof of Theorem 19.3.

Theorem 19.8 For a Jordan domain K in $\mathbb{R}^n$, let $h:K\to\mathbb{R}$ and $g:K\to\mathbb{R}$ be continuous bounded functions with the property that

$$h(\mathbf{x})\le g(\mathbf{x})\qquad\text{for all points }\mathbf{x}\text{ in }K.$$

Define

$$D=\{(\mathbf{x},\,\mathbf{y})\text{ in }\mathbb{R}^{n+1}\mid \mathbf{x}\text{ in }K,\ h(\mathbf{x})\le y\le g(\mathbf{x})\}.$$

Suppose that the function $f:D\to\mathbb{R}$ is continuous and bounded. Then

$$\int_D f=\int_K\left[\int_{h(\mathbf{x})}^{g(\mathbf{x})} f(\mathbf{x},\,y)\,dy\right]d\mathbf{x}.\tag{19.7}$$

In calculating the value of the integral of a function of a single variable, notation involving Leibnitz symbols is often useful. It is also useful for functions of several variables. For an integrable function $f:\mathbf{I}\to\mathbb{R}$ defined on a generalized rectangle $\mathbf{I}=[a_1,\,b_1]\times\cdots\times[a_n,\,b_n]$, the value of the integral is often denoted by

$$\int_\mathbf{I} f(\mathbf{x})\,d\mathbf{x}$$

or by

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} f(x_1,\ldots,x_n)\,dx_n\ldots dx_1.$$

For $n=2$ or $n=3$, we use the notation

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,\,y)\,dy\,dx\qquad\text{and}\qquad\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,\,y,\,z)\,dz\,dy\,dx.$$

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In the case of $\mathbb{R}^3$ and $\mathbf{I} = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, the above iterated integration formula (19.6) becomes

$$\int_{\mathbf{I}} f(x, y, z)\, dx\, dy\, dz = \int_{a_3}^{b_3} \left[\iint_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y, z)\, dx\, dy \right] dz. \tag{19.8}$$

Moreover, if we use the two-variable iterated integration formula in the above inner integral, we obtain

$$\int_{\mathbf{I}} f(x, y, z)\, dx\, dy\, dz = \int_{a_3}^{b_3} \left[\int_{a_2}^{b_2} \left\{ \int_{a_1}^{b_1} f(x, y, z)\, dx \right\} dy \right] dz. \tag{19.9}$$

We emphasize that for the above formulas to be valid, it is necessary to make sure that the integrals on both sides are properly defined.

EXERCISES FOR SECTION 19.1

1. Evaluate

$$\iint_{[0,1]\times[0,1]} \sin^2 x \sin^2 y\, dx\, dy.$$

2. For the following three functions, evaluate $\iint_{\mathbf{I}} f$, where $\mathbf{I} = [0, 1] \times [0, 1]$:

a. $f(x, y) = \begin{cases} 1 - x - y & \text{if } x + y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

b. $f(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

c. $f(x, y) = \begin{cases} x + y & \text{if } x^2 \leq y \leq 2x^2 \\ 0 & \text{otherwise.} \end{cases}$

3. Show that

$$\int_0^3 \left[\int_1^{\sqrt{4-y}} (x + y)\, dx \right] dy = \int_1^2 \left[\int_0^{4-x^2} (x + y)\, dy \right] dx = \frac{241}{60}.$$

4. For a continuous function $f : [a, b] \times [a, b] \to \mathbb{R}$, prove Dirichlet's formula:

$$\int_a^b \left[\int_a^x f(x, y)\, dy \right] dx = \int_a^b \left[\int_y^b f(x, y)\, dx \right] dy.$$

5. Suppose that the function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that for each $x \geq 0$,

$$\int_0^x \left[\int_0^t \phi(s)\, ds \right] dt = \int_0^x (x - s)\phi(s)\, ds.$$

6. Suppose that the function $f : [a, b] \to \mathbb{R}$ is continuous. Prove that

$$2 \int_a^b \left[f(x) \int_x^b f(y)\, dy \right] dx = \left[\int_a^b f(x)\, dx \right]^2.$$

7. Follow the proof of Theorem 19.3 and thereby provide a proof of Theorem 19.8.

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8. Show that the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined in Example 19.5 is not integrable.
9. For a Jordan domain K in $\mathbb{R}^n$, let $h : K \to \mathbb{R}$ and $g : K \to \mathbb{R}$ be continuous bounded functions with the property that

$$h(\mathbf{x}) \leq g(\mathbf{x}) \qquad \text{for all points } \mathbf{x} \text{ in } K.$$

Define

$$D = \{(\mathbf{x}, y) \mid \mathbf{x} \text{ in } K,\ h(\mathbf{x}) \leq y \leq g(\mathbf{x})\}.$$

Prove that the set D also is a Jordan domain.
10. Follow the proof of Theorem 19.1 and thereby provide a proof of formula (19.4).

19.2 THE CHANGE OF VARIABLES THEOREM: STATEMENTS AND EXAMPLES

The Change of Variables Theorem, a substitution method for evaluating integrals, for the integral of a function of a single variable was considered in Section 7.2. In this section, we extend this theorem to functions of several variables. It is convenient first to state the general Change of Variables Theorem. Then we consider some important special cases, including those of a domain in the plane $\mathbb{R}^2$ described by polar coordinates and a domain in $\mathbb{R}^3$ described by spherical coordinates. We postpone the proof of the Change of Variables Theorem until the following section.

In Section 7.2 we proved a change of variables result for evaluating integrals that implies the following: Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}$ that contains the closed bounded interval $I = [a, b]$ and let $\psi : \mathcal{O} \to \mathbb{R}$ be a continuously differentiable function such that $\psi'(x) \neq 0$ for all x in $\mathcal{O}$. Then for any continuous function $f : \psi(I) \to \mathbb{R}$,

$$\int_{\psi(a)}^{\psi(b)} f(x)\, dx = \int_a^b f(\psi(u))\psi'(u)\, du.$$

In the case where $\psi'(u) > 0$ for all u in I, $\psi(I) = [\psi(a), \psi(b)]$; in the case where $\psi'(u) < 0$ for all u in I, $\psi(I) = [\psi(b), \psi(a)]$. Therefore, the preceding formula can be rewritten in the following equivalent form:

$$\int_{\psi(I)} f(x)\, dx = \int_I f(\psi(u))|\psi'(u)|\, du. \tag{19.10}$$

It is in this form that the change of variables formula will be extended to functions of several variables.

Definition Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. A continuously differentiable mapping $\Psi : \mathcal{O} \to \mathbb{R}^n$ is called a *smooth change of variables* provided that the following two properties hold:

i. The mapping $\Psi : \mathcal{O} \to \mathbb{R}^n$ is one-to-one.
ii. For each point $\mathbf{x}$ in $\mathcal{O}$, the derivative matrix $\mathbf{D}\Psi(\mathbf{x})$ is invertible.

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Theorem 19.9 The Change of Variables Theorem Suppose that the mapping $\Psi : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$. Let D be an open Jordan domain such that $K = D \cup \partial D$ is contained in $\mathcal{O}$. Then $\Psi(K)$ is a Jordan domain with the property that for any continuous function $f : \Psi(K) \to \mathbb{R}$, the following integral transformation formula holds:

$$\int_{\Psi(K)} f(\mathbf{x})\, d\mathbf{x} = \int_K f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u}. \tag{19.11}$$

Polar Coordinates

For each point $\mathbf{u} = (x, y) \neq (0, 0)$ in the plane $\mathbb{R}^2$, if we define $r = \sqrt{x^2 + y^2}$, then there is a unique number θ in the interval $[0, 2\pi)$ such that

$$(x, y) = (r\cos\theta, r\sin\theta).$$

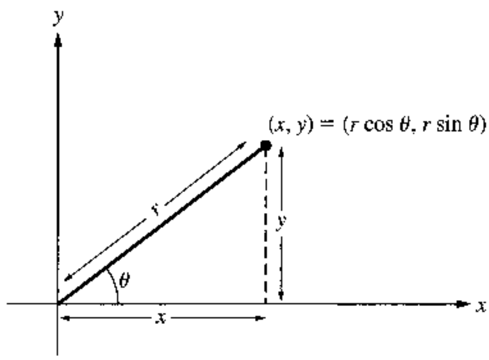

FIGURE 19.1 Polar coordinates.

The pair of numbers (r, θ) is called a choice of *polar coordinates* for the point $\mathbf{u}$.

Define $\mathcal{O}$ to be the subset of $\mathbb{R}^2$ consisting of points (r, θ) with $r > 0$ and $0 < \theta < 2\pi$ and then define the mapping $\Psi : \mathcal{O} \to \mathbb{R}^2$ by

$$\Psi(r, \theta) = (r\cos\theta, r\sin\theta) \qquad \text{for } (r, \theta) \text{ in } \mathcal{O}.$$

It is clear that the mapping $\Psi : \mathcal{O} \to \mathbb{R}^2$ is both continuously differentiable and one-to-one. Also, at each point (r, θ) in $\mathcal{O}$,

$$\det \mathbf{D}\Psi(r, \theta) = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r \neq 0,$$

so that the derivative matrix $\mathbf{D}\Psi(r, \theta)$ is invertible. Thus, the mapping $\Psi : \mathcal{O} \to \mathbb{R}^2$ is a smooth change of variables. For $0 < r_1 < r_2$ and $0 < \theta_1 < \theta_2 < 2\pi$, define $K = [r_1, r_2] \times [\theta_1, \theta_2]$. Suppose that the function $f : \Psi(K) \to \mathbb{R}$ is continuous. It

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follows directly from formula (19.11) and Fubini's Theorem that

$$\begin{aligned}\int_{\Psi(K)} f(x, y)\,dx\,dy &= \int_K [f(r\cos\theta, r\sin\theta)r]\,dr\,d\theta \\ &= \int_{\theta_1}^{\theta_2}\left[\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)r\,dr\right]d\theta.\end{aligned} \tag{19.12}$$

We note that there is no neighborhood of the origin on which polar coordinates provide a smooth change of variables. Nevertheless, by using an approximation argument, the change of variables formula (19.12) for polar coordinates can be extended to Jordan domains K that intersect the boundary of $\mathcal{O}$. For instance, for the above domain K, suppose that we allow the possibility that $r_1 = 0$, $\theta_1 = 0$, or $\theta_2 = 2\pi$. For $0 < \epsilon < \min\{r_2, [\theta_2 - \theta_1]/2\}$, define

$$K_\epsilon = \{(r, \theta) \mid \epsilon \le r \le r_2, \theta_1 + \epsilon \le \theta \le \theta_2 - \epsilon\}.$$

Thus, formula (19.12) holds when applied to K_ϵ. Now choose a number M such that $|f(x, y)| \le M$ for all (x, y) in $\Psi(K)$. Then, by the addition over domains formula for the integrals and formula (19.12) when applied to K_ϵ, we have

$$\begin{aligned}&\left|\int_{\Psi(K)} f(x, y)\,dx\,dy - \int_K [f(r\cos\theta, r\sin\theta)r]\,dr\,d\theta\right| \\ &\quad= \left|\int_{\Psi(K\backslash K_\epsilon)} f(x, y)\,dx\,dy - \int_{K\backslash K_\epsilon} [f(r\cos\theta, r\sin\theta)r]\,dr\,d\theta\right| \\ &\quad\le M \operatorname{vol} \Psi(K\backslash K_\epsilon) + Mr_2 \operatorname{vol}(K\backslash K_\epsilon).\end{aligned}$$

Since

$$\lim_{\epsilon\to 0} \operatorname{vol} \Psi(K\backslash K_\epsilon) = \lim_{\epsilon\to 0} \operatorname{vol}(K\backslash K_\epsilon) = 0, \tag{19.13}$$

we see that formula (19.12) also holds in this limiting case.

Example 19.10 Define $D = \{(x, y) \mid x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$. We use formula (19.12) to evaluate

$$\int_D e^{x^2+y^2}\,dx\,dy.$$

Indeed, $D = \{(r\cos\theta, r\sin\theta) \mid 0 \le r \le 1, 0 \le \theta \le \pi/2\}$, so that by formula (19.12),

$$\int_D e^{x^2+y^2}\,dx\,dy = \int_0^{\pi/2}\left[\int_0^1 e^{r^2} r\,dr\right]d\theta = \int_0^{\pi/2}\left[\frac{e-1}{2}\right]d\theta = \frac{\pi}{4}(e-1). \quad \blacksquare$$

Spherical Coordinates

We now turn to a useful change of variables formula in three dimensions. For each point $\mathbf{u} = (x, y, z)$ in $\mathbb{R}^3$ that does not lie on the z-axis, we define $\rho = \sqrt{x^2 + y^2 + z^2}$.

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It is not difficult to see that there are unique numbers θ in the interval $[0, 2\pi)$ and ϕ in the interval $(0, \pi)$ such that

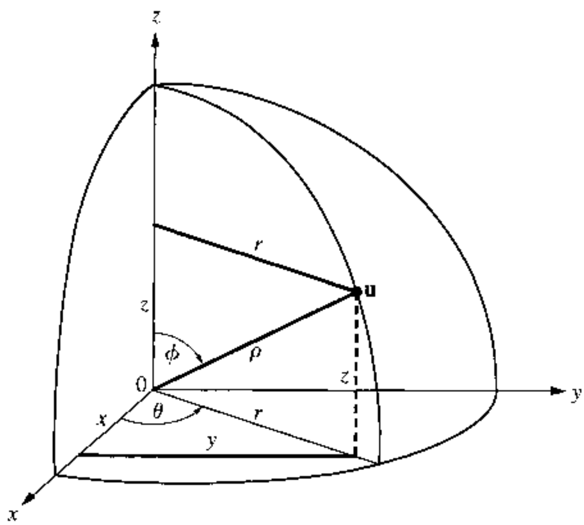
$$\mathbf{u} = (x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

The triple of numbers (ρ, ϕ, θ) is called a choice of *spherical coordinates* for the point $\mathbf{u}$. Define $\mathcal{O}$ to be the open subset of $\mathbb{R}^3$ consisting of points (ρ, ϕ, θ) with $\rho > 0$, $0 < \phi < \pi$, and $0 < \theta < 2\pi$ and then define $\Psi : \mathcal{O} \to \mathbb{R}^3$ by

$$\Psi(\rho, \phi, \theta) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \qquad \text{for } (\rho, \phi, \theta) \text{ in } \mathcal{O}.$$

It is clear that the mapping $\Psi : \mathcal{O} \to \mathbb{R}^3$ is both continuously differentiable and one-to-one. Also, at each point (ρ, ϕ, θ) in $\mathcal{O}$, the derivative matrix is given by

$$\mathbf{D}\Psi(\rho, \phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}.$$



$\mathbf{u} = (x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$

FIGURE 19.2 Spherical coordinates.

A brief computation yields

$$\det \mathbf{D}\Psi(\rho, \phi, \theta) = \rho^2 \sin\phi \neq 0.$$

Thus, the derivative matrix $\mathbf{D}\Psi(\rho, \phi, \theta)$ is invertible, so $\Psi : \mathcal{O} \to \mathbb{R}^3$ is a smooth change of variables. For $0 < \rho_1 < \rho_2$, $0 < \phi_1 < \phi_2 < \pi$, and $0 < \theta_1 < \theta_2 < 2\pi$, define

$$K = [\rho_1, \rho_2] \times [\phi_1, \phi_2] \times [\theta_1, \theta_2].$$

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Suppose that the function $f : \Psi(K) \to \mathbb{R}$ is continuous. Then by the integral transformation formula (19.11) and Fubini's Theorem,

$$\begin{aligned}
&\int_{\Psi(K)} f(x, y, z)\, dx\, dy\, dz \\
&\quad = \int_K [f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\rho^2 \sin\phi]\, d\rho\, d\phi\, d\theta \\
&\quad = \int_{\theta_1}^{\theta_2} \left[\int_{\phi_1}^{\phi_2} \left\{ \int_{\rho_1}^{\rho_2} f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\rho^2 \sin\phi\, d\rho \right\} d\phi \right] d\theta.
\end{aligned} \tag{19.14}$$

Spherical coordinates do not define a smooth change of variables on any domain that intersects the boundary of $\mathcal{O}$. However, by an approximation argument similar to that used for polar coordinates, formula (19.14) can be extended to allow domains K that do intersect the boundary of $\mathcal{O}$.

Example 19.11 For $a > 0$, we find the volume of the ball[1] in $\mathbb{R}^3$ of radius a,

$$B_a = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq a^2\}.$$

Indeed, by formula (19.14),

$$\begin{aligned}
\operatorname{vol} B_a &= \int_{B_a} 1\, dx\, dy\, dz \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \left\{ \int_0^a \rho^2 \sin\phi\, d\rho \right\} d\phi \right] d\theta \\
&= [4/3]\pi a^3.
\end{aligned}$$

■

Example 19.12 For positive numbers a and b, consider the ellipse

$$D = \{(x, y) \mid x^2/a^2 + y^2/b^2 \leq 1\}.$$

Define $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Psi(u, v) = (au, bv)$ for all (u, v) in $\mathbb{R}^2$. Then $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear mapping, so it is a smooth change of variables. By the change of variables formula for Ψ and also for polar coordinates, we see that for any continuous function $f : D \to \mathbb{R}$,

$$\begin{aligned}
\int_D f(x, y)\, dx\, dy &= ab \int_{u^2+v^2 \leq 1} f(au, bv)\, du\, dv \\
&= ab \int_0^{2\pi} \left[\int_0^1 f(ar\cos\theta,\ br\sin\theta) r\, dr \right] d\theta.
\end{aligned}$$

■

[1] Archimedes discovered the formula for the volume of a ball. He was so proud of this accomplishment that he had the formula inscribed on his tomb.

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EXERCISES FOR SECTION 19.2

1. Evaluate

$$\iint_{x^2+y^2\le 1} x^2y^2\,dx\,dy.$$

2. Evaluate

$$\iiint_V |xyz|\,dx\,dy\,dz,$$

where

$$V=\left\{(x,\ y,\ z) \text{ in } \mathbb{R}^3 \;\middle|\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}.$$

3. Find the volume of the Jordan domain in $\mathbb{R}^3$ bounded by the xy plane and the paraboloid $z=2-x^2-y^2$.

4. Show that the volume of the Jordan domain in $\mathbb{R}^3$ bounded by the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ is equal to $16a^3/3$.

5. For $r>0$ and $h>0$, show that the volume of the cone $\{(x,\ y,\ z) \text{ in } \mathbb{R}^3 \mid x^2+y^2\le r^2,\ 0\le z\le h/r^2(r^2-x^2-y^2)\}$ is equal to $[1/3]\pi r^2 h$.

6. Suppose that $\Psi:\mathcal{O}\to\mathbb{R}^n$ is a smooth change of variables on an open subset $\mathcal{O}$ of $\mathbb{R}^n$. Use the Inverse Function Theorem to show that the inverse mapping $\Psi^{-1}:\Psi(\mathcal{O})\to\mathbb{R}^n$ is a smooth change of variables on the open subset $\Psi(\mathcal{O})$ of $\mathbb{R}^n$.

7. (Hyperbolic Coordinates) Define $\mathcal{U}=\{(x,\ y)\mid x>0,\ y>0\}$ and define $\Phi:\mathcal{U}\to\mathbb{R}^2$ by $\Phi(x,\ y)=(x^2-y^2,\ xy)$ for $(x,\ y)$ in $\mathcal{U}$. Show that $\Phi:\mathcal{U}\to\mathbb{R}^2$ is a smooth change of variables. For a point $(x,\ y)$ in $\mathcal{U}$, the pair of numbers $(x^2-y^2,\ xy)=\Phi(x,\ y)$ are called *hyperbolic coordinates* for $(x,\ y)$.

8. Define $D=\{(x,\ y)\mid x>0,\ y>0,\ 1\le x^2-y^2\le 9,\ 2\le xy\le 4\}$. For a continuous function $f:D\to\mathbb{R}$, use the hyperbolic coordinates from Exercise 7 to show that

$$\int_D [x^2+y^2]\,dx\,dy=8.$$

9. Verify the limits (19.13) by finding a positive number C such that for each $\epsilon>0$,

$$|\text{vol}\ \Psi(K\backslash K_\epsilon)+\text{vol}(K\backslash K_\epsilon)|\le C\epsilon.$$

19.3 PROOF OF THE CHANGE OF VARIABLES THEOREM

We now turn to the proof of the Change of Variables Theorem. In order to prove this theorem, it is necessary to precisely compare the volume of $\Psi(\mathbf{J})$ and $\mathrm{d}\Psi(\mathbf{x})(\mathbf{J})$,[2] where $\mathbf{J}$ is a generalized rectangle of small diameter in $\mathcal{O}$ that contains the point $\mathbf{x}$. The following

[2] Recall that $\mathrm{d}\Psi(\mathbf{x}):\mathbb{R}^n\to\mathbb{R}^n$ is the linear mapping, called the *differential* of $\Psi:\mathcal{O}\to\mathbb{R}^n$ at the point $\mathbf{x}$, associated with the derivative matrix $\mathbf{D}\Psi(\mathbf{x})$ by the formula $\mathrm{d}\Psi(\mathbf{x})(\mathbf{h})=\mathbf{D}\Psi(\mathbf{x})\mathbf{h}$.

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result is the precise comparison of volume result on which the proof of the Change of Variables Theorem depends.

Theorem 19.13 The Volume Comparison Theorem Let $\Psi : \mathcal{O} \to \mathbb{R}^n$ be a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$. Let K be a closed bounded subset of $\mathcal{O}$ and let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $\mathbf{J}$ is any generalized rectangle of diameter less than δ and $\mathbf{x}$ is a point in $K \cap \mathbf{J}$, then $\mathbf{J}$ is contained in $\mathcal{O}$ and

$$\text{vol } \Psi(\mathbf{J}) = |\det D\Psi(\mathbf{x})| \text{ vol } \mathbf{J} + \mathbf{E} \text{ vol } \mathbf{J}, \qquad \text{where } |\mathbf{E}| < \epsilon. \tag{19.15}$$

It turns out that the proof of this theorem is rather technical, and it depends on properties of determinants that we will not prove in this book. Thus, we prefer to describe the two fundamental ideas that underlie the theorem and then provide completely elementary proofs of this Volume Comparison Theorem for polar coordinates and for spherical coordinates.

i. For an invertible linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ represented by the $n \times n$ matrix $\mathbf{A}$ and a generalized rectangle $\mathbf{J}$ in $\mathbb{R}^n$, the volume of $\mathbf{T}(\mathbf{J})$ is given by

$$\text{vol } \mathbf{T}(\mathbf{J}) = |\det \mathbf{A}| \text{ vol } \mathbf{J}.$$

In $\mathbb{R}^3$ this formula follows from the discussion in Appendix B, on linear algebra, of the relationships among volumes, determinants, and cross-products. The general case follows from the product property of determinants and from a result that permits general linear mappings to be written as the composition of linear mappings of a very elementary kind. From the above volume transformation formula for linear mappings, it follows that for a point $\mathbf{x}$ in $\mathcal{O}$ and a generalized rectangle $\mathbf{J}$,

$$\text{vol } d\Psi(\mathbf{x})(\mathbf{J}) = |\det \mathbf{D}\Psi(\mathbf{x})| \text{ vol } \mathbf{J}. \tag{19.16}$$

ii. Recall that in Chapter 13 we proved the First-Order Approximation Theorem, from which it follows that if $\Psi : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables, then for each point $\mathbf{x}$ in $\mathcal{O}$,

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\|\Psi(\mathbf{x} + \mathbf{h}) - [\Psi(\mathbf{x}) + d\Psi(\mathbf{x})(\mathbf{h})]\|}{\|\mathbf{h}\|} = 0.$$

The First-Order Approximation Theorem is a precise assertion of the way in which in a neighborhood of a point $\mathbf{x}$ in $\mathcal{O}$, the mapping Ψ is approximated by the differential of Ψ at the point $\mathbf{x}$. Simply on the basis of the continuity of the mapping Ψ, it is not difficult to see that vol $\Psi(\mathbf{J})$ is small if diam $\mathbf{J}$ is small, so the difference

$$\text{vol } \Psi(\mathbf{J}) - |\det \mathbf{D}\Psi(\mathbf{x})| \text{ vol } \mathbf{J}$$

also is small if diam $\mathbf{J}$ is small. The First-Order Approximation Theorem and formula (19.16) provide a much stronger result: Even if we divide the above difference by

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vol **J**, the result,

$$\frac{\text{vol } \Psi(\mathbf{J})}{\text{vol } \mathbf{J}} - |\det \mathbf{D}\Psi(\mathbf{x})|, \tag{19.17}$$

remains small provided that **J** is a generalized rectangle of small diameter that contains the point **x**. The Volume Comparison Theorem is a precise assertion of what is true in this respect.

For the cases of polar coordinates and spherical coordinates, we now explicitly calculate the volume of the image of a generalized rectangle under the induced change of coordinates.

Proposition 19.14 Area Change under Polar Coordinates For $0 \leq r_1 < r_2$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$, let $\mathbf{J} = [r_1, r_2] \times [\theta_1, \theta_2]$ and define $\Psi(r, \theta) = (r\cos\theta, r\sin\theta)$ for (r, θ) in $\mathbf{J}$. Then

$$\text{area } \Psi(\mathbf{J}) = \frac{1}{2}\left[r_2^2 - r_1^2\right][\theta_2 - \theta_1] = \frac{[r_1 + r_2]}{2}(\text{area } \mathbf{J}). \tag{19.18}$$

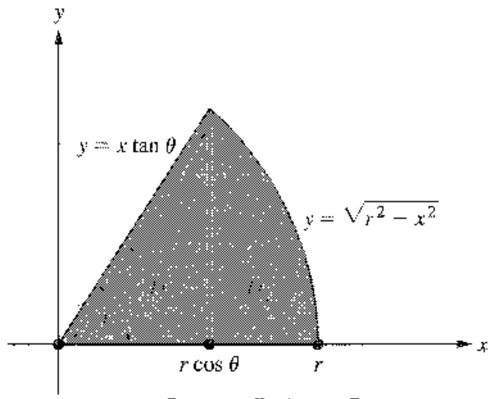

FIGURE 19.3 Area change under polar coordinates.

Proof

By the additivity of volume property asserted in Corollary 18.30, it suffices to prove formula (19.18) in the case where $r_1 = 0, \theta_1 = 0$, and $0 \leq \theta_2 \leq \pi/2$. Define $r = r_2$ and $\theta = \theta_2$; it is necessary to show that

$$\text{area } D = \frac{1}{2}r^2\theta.$$

Observe that D is the union of the two sets $D_1 = \{(x, y) \mid 0 \leq x \leq r\cos\theta,\ 0 \leq y \leq \tan\theta\, x\}$ and $D_2 = \{(x, y) \mid r\cos\theta \leq x \leq r,\ 0 \leq y \leq \sqrt{r^2 - x^2}\}$.

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Thus, by the additivity property of volume and Fubini's Theorem in the Plane (Theorem 19.3), we have

$$\begin{aligned}\text{area } D &= \text{area } D_1 + \text{area } D_2 \\ &= \int_0^{r\cos\theta} \tan\theta\, x\, dx + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx \\ &= \frac{r^2}{2} \sin\theta\cos\theta + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx.\end{aligned}$$

But by the change of variables $x = r\cos t$, $0 \le t \le \theta$, it follows from the single-variable change of variables formula that

$$\begin{aligned}\int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx = r^2 \int_0^{\theta} \sin^2 t\, dt = r^2 \int_0^{\theta} \left[\frac{1-\cos 2t}{2}\right] dt \\ = r^2 \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]\Bigg|_{t=0}^{t=\theta}.\end{aligned}$$

Thus,

$$\text{area } D = \frac{r^2}{2}\sin\theta\cos\theta + r^2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right] = \frac{1}{2}r^2\theta. \qquad \blacksquare$$

From this area transformation formula for polar coordinates, we immediately obtain the Volume Comparison Theorem for the case of polar coordinates. Indeed, for $\mathbf{J}$ as above and any point (r, θ) in $\mathbf{J}$, it follows from formula (19.18) that

$$\text{area } \Psi(\mathbf{J}) = r \text{ area } \mathbf{J} + \mathbf{E} \text{ area } \mathbf{J}, \qquad \text{where } \mathbf{E} = \frac{r_1 + r_2}{2} - r.$$

Clearly, $|\mathbf{E}| < \operatorname{diam} \mathbf{J}$, so we can let $\delta = \epsilon$ in the statement of the Volume Comparison Theorem for polar coordinates.

Proposition 19.15 Volume Change under Spherical Coordinates For $0 \le \rho_1 < \rho_2$, $0 \le \phi_1 < \phi_2 \le \pi$, and $0 \le \theta_1 < \theta_2 \le 2\pi$, let

$$\mathbf{J} = [\rho_1, \rho_2] \times [\phi_1, \phi_2] \times [\theta_1, \theta_2]$$

and define

$$\Psi(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi) \qquad \text{for } (\rho, \phi, \theta) \text{ in } \mathbf{J}.$$

Then

$$\text{vol } \Psi(\mathbf{J}) = \frac{1}{3}\left[\rho_2^3 - \rho_1^3\right][\cos\phi_1 - \cos\phi_2][\theta_2 - \theta_1]. \tag{19.19}$$

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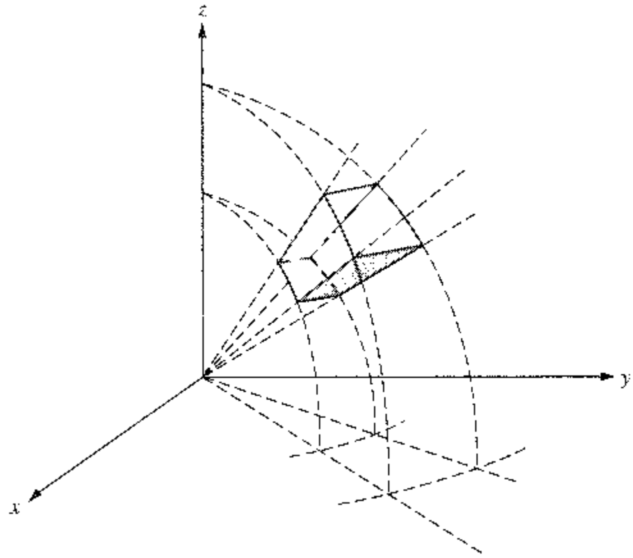

FIGURE 19.4 Volume change under spherical coordinates.

Proof

The additivity of volume property asserted in Corollary 18.30 implies that it is sufficient to establish formula (19.19) in the case where $\rho_1 = 0$, $0 = \phi_1 < \phi_2 \le \pi/2$, and $\theta_1 = 0$. It is necessary to show that

$$\text{vol}\,\Psi(\mathbf{J}) = \frac{1}{3}\rho_2^3[1 - \cos\phi_2]\theta_2.$$

But observe that $\Psi(\mathbf{J}) = \{(x, y, z) \mid (x, y) \text{ in } D,\ g(x, y) \le z \le h(x, y)\}$, where $D = \{(x, y) = (r\cos\theta, r\sin\theta) \mid 0 \le \theta \le \theta_2,\ 0 \le r \le \rho_2 \sin\phi_2\}$, and that for $(x, y) = (r\cos\theta, r\sin\theta)$ in D,

$$h(x, y) = \sqrt{\rho_2^2 - r^2} \qquad \text{and} \qquad g(x, y) = \frac{r}{\tan\phi_2}.$$

Using the iterated integration formula (19.7) and the change of variables formula (from Cartesian to polar coordinates) in the plane $\mathbb{R}^2$, we have

$$\begin{aligned}
\text{vol}\,\Psi(\mathbf{J}) &= \int_D [h(x, y) - g(x, y)]\,dx\,dy \\
&= \int_0^{\theta_2}\int_0^{\rho_2\sin\phi_2}\left[\sqrt{\rho_2^2 - r^2} - \frac{r}{\tan\phi_2}\right] r\,dr\,d\theta \\
&= \int_0^{\theta_2}\left\{\int_0^{\rho_2\sin\phi_2}\left[\sqrt{\rho_2^2 - r^2} - \frac{r}{\tan\phi_2}\right] r\,dr\right\} d\theta.
\end{aligned}$$

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However, by the First Fundamental Theorem of Calculus (Integrating Derivatives),

$$\int_0^{\rho_2 \sin\phi_2}\left[\sqrt{\rho_2^2 - r^2} - \frac{r}{\tan\phi_2}\right] r\,dr = \left\{-\frac{1}{3}\left(\rho_2^2 - r^2\right)^{3/2} - \frac{r^3}{3}\frac{1}{\tan\phi_2}\right\}\Bigg|_{r=0}^{r=\rho_2\sin\phi_2}$$
$$= \frac{\rho_2^3}{3}\left\{-\cos^3\phi_2 - \frac{\sin^3\phi_2}{\tan\phi_2} + 1\right\}$$
$$= \frac{\rho_2^3}{3}[1 - \cos\phi_2].$$

Thus,

$$\operatorname{vol}\Psi(\mathbf{J}) = \int_0^{\theta_2}\left\{\int_0^{\rho_2\sin\phi_2}\left[\sqrt{\rho_2^2 - r^2} - \frac{r}{\tan\phi_2}\right] r\,dr\right\} d\theta = \frac{\rho_2^3}{3}[1 - \cos\phi_2]\theta_2.$$

■

Formula (19.19) implies the Volume Comparison Theorem for the case of spherical coordinates. Indeed, using the Mean Value Theorem twice, we can select points ρ' in the interval (ρ_1, ρ_2) and ϕ' in the interval (ϕ_1, ϕ_2) such that

$$\frac{1}{3}[\rho_2^3 - \rho_1^3][\cos\phi_1 - \cos\phi_2][\theta_2 - \theta_1] = [\rho']^2[\sin\phi'][\rho_2 - \rho_1][\phi_2 - \phi_1][\theta_2 - \theta_1].$$

Thus, for a point (ρ, ϕ, θ) in $\mathbf{J}$, we have

$$\operatorname{vol}\ \Psi(\mathbf{J}) = [\rho^2\sin\phi]\operatorname{vol}\mathbf{J} + \mathbf{E}\operatorname{vol}\mathbf{J},$$

where

$$\mathbf{E} = \{[\rho']^2\sin\phi' - \rho^2\sin\phi\}.$$

Since $\det \mathbf{D}\Psi(\rho, \phi, \theta) = \rho^2\sin\phi$, the uniform continuity on bounded sets of the function $\det\mathbf{D}\Psi : \mathbb{R}^3 \to \mathbb{R}$ shows that for each closed bounded subset K of $\mathbb{R}^3$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\mathbf{E} < \epsilon$ if $\operatorname{diam}\mathbf{J} < \delta$ and $\mathbf{J}$ contains a point of K.

We have now established the Volume Comparison Theorem for polar coordinates and spherical coordinates. We will not prove the general theorem. Instead, we now prove the Change of Variables Theorem under the assumption that the general Volume Comparison Theorem is true.

Proof of the Change of Variables Theorem (Theorem 19.9)

Since K is closed and bounded (Exercise 4) and the functions $\det\mathbf{D}\Psi : K \to \mathbb{R}$ and $f\circ\Psi : K \to \mathbb{R}$ are continuous, it follows from the Extreme Value Theorem (Theorem 11.22) that there is a number $M > 0$ such that

$$|f(\Psi(\mathbf{x}))| \le M \qquad \text{and} \qquad |\det\mathbf{D}\Psi(\mathbf{x})| \le M \qquad \text{for all points } \mathbf{x} \text{ in } K. \quad (19.20)$$

Fix $\mathbf{I}$ to be a generalized rectangle that contains K. Let $\epsilon > 0$.

The Inverse Function Theorem implies that $\Psi(D)$ is an open subset of $\mathbb{R}^n$ and that the boundary of $\Psi(D)$ equals $\Psi(\partial D)$ (Exercise 9). Since by assumption D is a Jordan domain, the set ∂D has Jordan content 0 (Exercise 6). Using the Volume Comparison

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Theorem, it follows that $\Psi(\partial D)$ also has Jordan content 0. Thus, $\Psi(K)$ is a Jordan domain, as is $\Psi(\mathbf{J})$ for any generalized rectangle $\mathbf{J}$ contained in $\mathcal{O}$.

According to Theorem 11.25, a continuous function on a closed bounded set is uniformly continuous. Hence, the function $(f \circ \Psi) \cdot |\det \mathbf{D}\Psi| : K \to \mathbb{R}$ is uniformly continuous. Thus, using the ϵ-δ criterion for uniform continuity described in Theorem 11.27, we can select $\delta_1 > 0$ such that for any two points $\mathbf{u}$ and $\mathbf{v}$ in K,

$$|f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})| - f(\Psi(\mathbf{v}))|\det \mathbf{D}\Psi(\mathbf{v})|| < \epsilon \qquad \text{if } \|\mathbf{u} - \mathbf{v}\| < \delta_1. \quad (19.21)$$

By the Volume Comparison Theorem, we can select $\delta_2 > 0$ such that if $\mathbf{J}$ is any generalized rectangle of diameter less than δ_2 and $\mathbf{x}$ is a point in $K \cap \mathbf{J}$, then $\mathbf{J}$ is contained in $\mathcal{O}$ and

$$\text{vol } \Psi(\mathbf{J}) = |\det \mathbf{D}\Psi(\mathbf{x})| \text{ vol } \mathbf{J} + \mathbf{E} \text{ vol } \mathbf{J}, \qquad \text{where } |\mathbf{E}| < \epsilon. \quad (19.22)$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Since ∂D has Jordan content 0, arguing as we have in the proof of Theorem 18.20, we can select a partition $\mathbf{P}$ of $\mathbf{I}$ such that

$$\text{gap } \mathbf{P} < \delta \qquad \text{and} \qquad \sum_{\mathbf{J} \text{ in } \mathbf{P},\, \mathbf{J} \cap \partial D \neq \phi} \text{vol } \mathbf{J} < \epsilon. \quad (19.23)$$

By the additivity over domains properties of the integral, which we established in Theorems 18.11 and 18.30,

$$\begin{aligned}
&\int_{\Psi(K)} f(\mathbf{x})\, d\mathbf{x} - \int_K f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u} \\
&\quad = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \int_{\Psi(K \cap \mathbf{J})} f(\mathbf{x})\, d\mathbf{x} - \sum_{\mathbf{J} \text{ in } \mathbf{P}} \int_{K \cap \mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u} \qquad (19.24) \\
&\quad = \sum_{\mathbf{J} \text{ in } \mathbf{P}} \left[\int_{\Psi(K \cap \mathbf{J})} f(\mathbf{x})\, d\mathbf{x} - \int_{K \cap \mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u}\right];
\end{aligned}$$

we estimate separately the contribution to the right-hand sum from generalized rectangles that intersect ∂D and from those contained in D.

For a generalized rectangle $\mathbf{J}$ in $\mathbf{P}$ of diameter less than δ, it follows from (19.20) and (19.22) that

$$\begin{aligned}
&\left|\int_{\Psi(K \cap \mathbf{J})} f(\mathbf{x})\, d\mathbf{x} - \int_{K \cap \mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u}\right| \\
&\quad \leq \left|\int_{\Psi(K \cap \mathbf{J})} f(\mathbf{x})\, d\mathbf{x}\right| + \left|\int_{K \cap \mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\, d\mathbf{u}\right| \\
&\quad \leq M \text{ vol } \Psi(K \cap \mathbf{J}) + M^2 \text{ vol } (K \cap \mathbf{J}) \\
&\quad \leq M \text{ vol } \Psi(\mathbf{J}) + M^2 \text{ vol } (\mathbf{J}) \\
&\quad \leq M(M + \epsilon) \text{ vol } \mathbf{J} + M^2 \text{ vol } \mathbf{J} \\
&\quad = \{M(M + \epsilon) + M^2\} \text{ vol } \mathbf{J}.
\end{aligned}$$

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Therefore, because the sum of the volumes of the generalized rectangles in $\mathbf{P}$ that intersect ∂D has a volume less than ϵ, we obtain the following estimate:

$$\begin{aligned}\Bigg| \sum_{\mathbf{J}\cap\partial D\neq\phi} \int_{\Psi(K\cap\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} \\ - \sum_{\mathbf{J}\cap\partial D\neq\phi} \int_{K\cap\mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u}\Bigg| \leq \{M(M+\epsilon)+M^2\}\epsilon\end{aligned} \tag{19.25}$$

It remains to estimate the contribution to the right-hand side of (19.24) from the sum of the terms of the form

$$\int_{\Psi(\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} - \int_{\mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u},$$

where $\mathbf{J}$ is a generalized rectangle in $\mathbf{P}$ that is contained in D. Let $\mathbf{J}$ be such a generalized rectangle. It follows from the Mean Value Property of Integrals (Exercise 7) that there is a point $\mathbf{u}_0$ in $\mathbf{J}$ at which

$$\int_{\Psi(\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} = f(\Psi(\mathbf{u}_0)) \text{ vol } \Psi(\mathbf{J}),$$

and hence, by (19.22),

$$\int_{\Psi(\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} = f(\Psi(\mathbf{u}_0))|\det \mathbf{D}\Psi(\mathbf{u}_0)| \text{ vol } \mathbf{J} + f(\Psi(\mathbf{u}_0))\mathbf{E} \text{ vol } \mathbf{J}, \qquad \text{where } |\mathbf{E}| < \epsilon.$$

Again using the Mean Value Property of Integrals (Exercise 7), there is a point $\mathbf{v}_0$ in $\mathbf{J}$ at which

$$\int_{\mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u} = f(\Psi(\mathbf{v}_0))|\det \mathbf{D}\Psi(\mathbf{v}_0)| \text{ vol } \mathbf{J}.$$

Since $\|\mathbf{u}_0 - \mathbf{v}_0\| \leq \operatorname{diam} \mathbf{J} < \delta$, from the uniform continuity assertion (19.21), we have

$$\big|f(\Psi(\mathbf{u}_0))|\det \mathbf{D}\Psi(\mathbf{u}_0)| - f(\Psi(\mathbf{v}_0))|\det \mathbf{D}\Psi(\mathbf{v}_0)|\big| < \epsilon.$$

Thus,

$$\begin{aligned}&\left|\int_{\Psi(\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} - \int_{\mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u}\right| \\ &\quad = \big|f(\Psi(\mathbf{u}_0))|\det \mathbf{D}\Psi(\mathbf{u}_0)| - f(\Psi(\mathbf{v}_0))|\det \mathbf{D}\Psi(\mathbf{v}_0)| + f(\Psi(\mathbf{u}_0))\mathbf{E}\big| \text{ vol } \mathbf{J} \\ &\quad \leq \{\epsilon + M\epsilon\} \text{ vol } \mathbf{J}.\end{aligned}$$

We sum this estimate over all the generalized rectangles $\mathbf{J}$ in $\mathbf{P}$ that are contained in D to obtain

$$\left|\sum_{\mathbf{J} \text{ in } \mathbf{P},\, \mathbf{J}\subseteq D} \left\{\int_{\Psi(\mathbf{J})} f(\mathbf{x})\,d\mathbf{x} - \int_{\mathbf{J}} f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u}\right\}\right| < \{\epsilon + M\epsilon\} \text{ vol } \mathbf{I}.$$

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Combining this with the estimate (19.25), we see that

$$\begin{aligned}\left|\int_{\Psi(K)} f(\mathbf{x})\,dx - \int_K f(\Psi(\mathbf{u}))|\det \mathbf{D}\Psi(\mathbf{u})|\,d\mathbf{u}\right| \\ < \{M(M+\epsilon)+M^2\}\epsilon + \{\epsilon + M\epsilon\} \text{ vol } \mathbf{I} \\ = \epsilon\{M(M+\epsilon)+M^2+(1+M) \text{ vol } \mathbf{I}\}.\end{aligned}$$

Since the numbers M and vol $\mathbf{I}$ are fixed and the preceding estimate holds for all positive numbers ϵ, we conclude that the integral transformation formula (19.11) holds.

EXERCISES FOR SECTION 19.3

1. Suppose that the smooth change of variables $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ has the special form $\Psi(x, y) = (x, g(x, y))$ for all (x, y) in $\mathbb{R}^2$, where the function g is strictly increasing with respect to its second variable. For $K = [a, b] \times [0, 1]$, explicitly find $\Psi(K)$ and thus provide an independent proof of the change of variables formula in this particular case.
2. Show that the linear volume transformation formula (19.16) is a special case of formula (19.11).
3. Show that the volume transformation formulas (19.18) and (19.19) are special cases of formula (19.11).
4. Let K be a bounded subset of $\mathbb{R}^n$. Show that $K \cup \partial K$ is sequentially compact. (*Hint:* Show that the complement of $K \cup \partial K$ in $\mathbb{R}^n$ is the exterior of K.)
5. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ that contains the closed bounded set K. Show that there is a positive number δ such that if $\mathbf{J}$ is a generalized rectangle that has diameter less than δ and contains a point of K, then $\mathbf{J}$ is contained in $\mathcal{O}$. (*Hint:* Argue by contradiction.)
6. Suppose that $\Psi : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$. Show that if A is a closed bounded subset of $\mathcal{O}$ with Jordan content 0, then its image $\Psi(A)$ also has Jordan content 0. (*Hint:* Use the Volume Comparison Theorem.)
7. (Mean Value Property for Integrals) Suppose that $G : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$, $h : G(\mathcal{O}) \to \mathbb{R}$ is continuous, and $\mathbf{J}$ is a generalized rectangle contained in $\mathcal{O}$.
   a. Use the Extreme Value Theorem to choose points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{J}$ at which the function $f \circ G : \mathbf{J} \to \mathbb{R}$ assumes a minimum and a maximum value, respectively, and then use the monotonicity property of the integral to show that
   $$h(G(\mathbf{u})) \le \frac{1}{\operatorname{vol} G(\mathbf{J})} \int_{G(\mathbf{J})} h(\mathbf{x})\,d\mathbf{x} \le h(G(\mathbf{v})).$$
   b. Apply the Intermediate Value Theorem to the function $\alpha : [0, 1] \to \mathbb{R}$ defined by $\alpha(t) = h(G(t\mathbf{u} + (1-t)\mathbf{v}))$, $0 \le t \le 1$, and part (a) to find a point $\mathbf{w}$ on the segment between $\mathbf{u}$ and $\mathbf{v}$ at which
   $$\int_{G(\mathbf{J})} h(\mathbf{x})\,d\mathbf{x} = h(G(\mathbf{w})) \text{ vol } G(\mathbf{J}).$$

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c. In the case where $G(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathcal{O}$, obtain the following consequence of part (c):

$$\int_{\mathbf{J}} h(\mathbf{x})\, d\mathbf{x} = h(\mathbf{w}) \text{ vol } \mathbf{J}.$$

8. Use the mean value property of integrals to show that the Volume Comparison Theorem can be derived as a consequence of formula (19.11).

9. Suppose that $\Psi : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$. Let D be an open subset of $\mathbb{R}^n$ such that $D \cup \partial D$ is contained in $\mathcal{O}$. Use the Inverse Function Theorem to show that $\Psi(\partial D) = \partial \Psi(D)$—that is, the boundary of the image is the image of the boundary.

10. Suppose that $\Psi : \mathcal{O} \to \mathbb{R}^2$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$ and that K is a closed bounded subset of $\mathcal{O}$. Prove that there is a positive number c such that for any two points $\mathbf{u}$ and $\mathbf{v}$ in K,

$$\|\Psi(\mathbf{u}) - \Psi(\mathbf{v})\| \leq c\|\mathbf{u} - \mathbf{v}\|.$$

(*Hint:* Use the Mean Value Theorem.)

11. Suppose that $\Psi : \mathcal{O} \to \mathbb{R}^n$ is a smooth change of variables on the open subset $\mathcal{O}$ of $\mathbb{R}^n$ and that K is a closed bounded subset of $\mathcal{O}$. Prove that there is a positive number c such that for any two points $\mathbf{u}$ and $\mathbf{v}$ in K,

$$\|\Psi(\mathbf{u}) - \Psi(\mathbf{v})\| \geq c\|\mathbf{u} - \mathbf{v}\|.$$

(*Hint:* Argue by contradiction using the Nonlinear Stability Theorem.)

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CHAPTER 20

LINE AND SURFACE INTEGRALS

20.1 ARCLENGTH AND LINE INTEGRALS

Smooth Paths and Piecewise Smooth Paths

Recall that in Section 11.3 we called a continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ a *parametrized path* whose parameter space is the interval $[a, b]$, and we called the image of this mapping a *path*. We emphasize that a path is a *subset* of $\mathbb{R}^n$, whereas a parametrized path is a *mapping*, and that a path is the image of different parametrized paths.

Definition A parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is called *smooth* provided that the restriction $\gamma : (a, b) \to \mathbb{R}^n$ has a continuous bounded derivative such that $\gamma'(t) \neq 0$ for all parameter values t in (a, b).

Definition A parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is called *piecewise smooth* provided that there is a partition $P = \{x_0, \ldots, x_m\}$ of the parameter space $[a, b]$ such that for each index i between 1 and m, the restriction $\gamma : [x_{i-1}, x_i] \to \mathbb{R}^n$ is a smooth parametrized path.

The image of a smooth parametrized path is called a *smooth path*, and the image of a piecewise smooth parametrized path is called a *piecewise smooth path*.

Example 20.1 Recall that for any two distinct points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$, the mapping $\gamma : [0, 1] \to \mathbb{R}^n$ defined by

$$\gamma(t) = t\mathbf{q} + (1 - t)\mathbf{p} \qquad \text{for } 0 \leq t \leq 1$$

is called the parametrized segment from the point $\mathbf{p}$ to the point $\mathbf{q}$. This parametrized segment is clearly a smooth parametrized path. ■

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Example 20.2 Let Γ be the ellipse in the plane $\mathbb{R}^2$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > 0$ and $b > 0$. Then Γ is a smooth path. To verify this, we must find a smooth parametrized path that has this ellipse as its image. But define

$$\gamma(\theta) = (a\cos\theta,\ b\sin\theta) \qquad \text{for } 0 \le \theta \le 2\pi.$$

This is clearly a smooth parametrized path that has Γ as its image. ∎

Example 20.3 The mapping $\gamma : [0,\ 4\pi] \to \mathbb{R}^3$ defined by

$$\gamma(t) = (\cos t,\ \sin t,\ t) \qquad \text{for } 0 \le t \le 4\pi$$

is a smooth parametrized path. The image of this smooth parametrized path is called a *helix*.

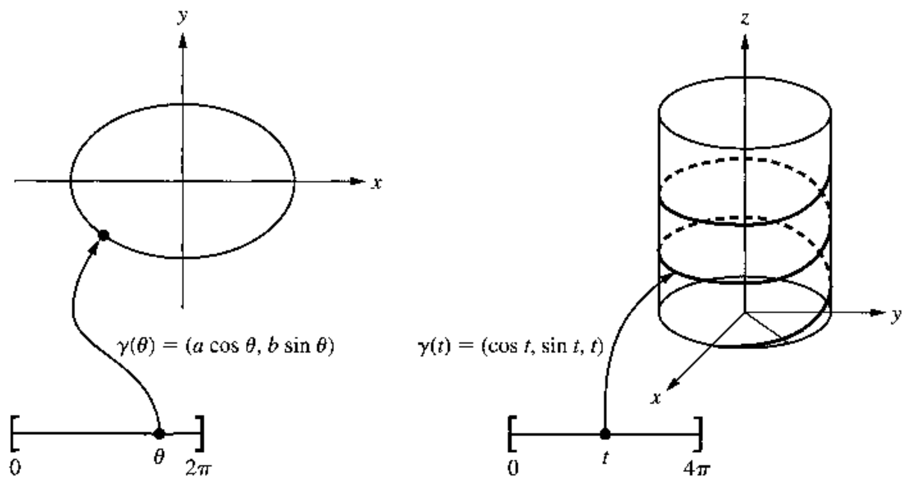

FIGURE 20.1 Parametrizing an ellipse and a helix. ∎

Example 20.4 Let the continuous functions $g : [a,\ b] \to \mathbb{R}$ and $h : [a,\ b] \to \mathbb{R}$ have the property that $g(x) < h(x)$ for all x in $(a,\ b)$ and have restrictions $g : (a,\ b) \to \mathbb{R}$ and $h : (a,\ b) \to \mathbb{R}$ that have continuous bounded derivatives. Define

$$\Omega = \{(x,\ y) \mid a < x < b,\ g(x) < y < h(x)\}.$$

Then the boundary of Ω is a piecewise smooth parametrized path. To see this, observe that if $g(a) < h(a)$ and $g(b) < h(b)$, then the boundary of Ω is the image

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of the mapping $\gamma : [0, 1] \to \mathbb{R}^2$ defined by

$$\gamma(t) = \begin{cases} (a + 4t(b-a),\ g(a + 4t(b-a))) & \text{if } 0 \le t \le 1/4 \\ (b,\ g(b) + 4(t - 1/4)(h(b) - g(b))) & \text{if } 1/4 \le t \le 1/2 \\ (b - 4(t - 1/2)(b-a),\ h(b - 4(t-1/2)(b-a))) & \text{if } 1/2 \le t \le 3/4 \\ (a,\ h(a) + 4(t - 3/4)(g(a) - h(a))) & \text{if } 3/4 \le t \le 1. \end{cases}$$

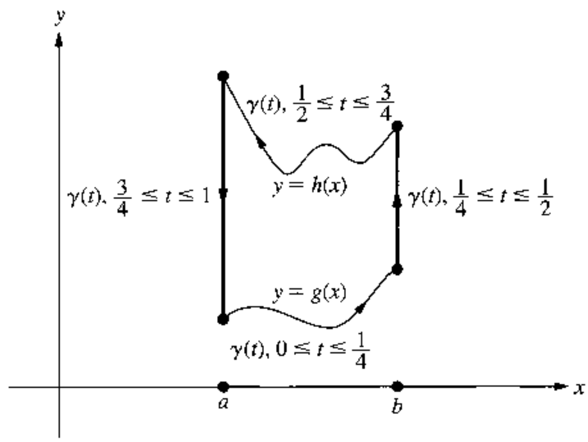

FIGURE 20.2 An explicit piecewise smooth parametrizing of the boundary of Ω. ■

Arclength of a Path

We have defined the length of the interval of real numbers $[a, b]$ to be $b - a$. Moreover, we define the length of the segment joining the points $\mathbf{p}$ and $\mathbf{q}$ in Euclidean space $\mathbb{R}^n$ to be the Euclidean distance between these points,

$$\operatorname{dist}(\mathbf{p}, \mathbf{q}) \equiv \|\mathbf{p} - \mathbf{q}\|.$$

We wish to extend this definition to the concept of *arclength* of a parametrized path. To do so, it is necessary to restrict the class of parametrized paths. The following definition of arclength is motivated by the geometric concept of polygonal approximation.

Definition A parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is said to have *arclength* (or to be *rectifiable*) provided that there is a number ℓ with the following property: For each positive number ϵ there is a positive number δ such that

$$\left| \left[\sum_{i=1}^{m} \|\gamma(x_i) - \gamma(x_{i-1})\| \right] - \ell \right| < \epsilon$$

for each partition $P = \{x_0, \ldots, x_m\}$ of $[a, b]$ with gap $P < \delta$. We call ℓ the arclength of the parametrized path $\gamma : [a, b] \to \mathbb{R}^n$.

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For a parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ and a partition $P = \{x_0, \ldots, x_m\}$ of its parameter space $[a, b]$, it is convenient to call the sum

$$\sum_{i=1}^{m} \|\gamma(x_i) - \gamma(x_{i-1})\|$$

the *polygonal approximation* of the arclength of $\gamma : [a, b] \to \mathbb{R}^n$ based on the partition P.

For a rectifiable parametrized path $\gamma : [a, b] \to \mathbb{R}^n$, there is only one number ℓ that has the defining polygonal approximation property (Exercise 7).

We have defined the arclength of a parametrized path; it certainly seems reasonable that there should be a sense in which the arclength is a property possessed by the image of this parametrized path and does not depend on the manner of parametrization. To understand the sense in which this is so, it is necessary to consider the relationship between different parametrizations of a given path.

Definition Two parametrized paths $\gamma : [a, b] \to \mathbb{R}^n$ and $\alpha : [c, d] \to \mathbb{R}^n$ are said to be *equivalent* provided that there is a strictly increasing parametrized path $u : [c, d] \to \mathbb{R}$ with image $[a, b]$ such that

$$\alpha = \gamma \circ u : [c, d] \to \mathbb{R}.$$

Proposition 20.5 Suppose that the parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable. Then every parametrized path equivalent to $\gamma : [a, b] \to \mathbb{R}^n$ is also rectifiable and has the same arclength as $\gamma : [a, b] \to \mathbb{R}^n$.

Proof

Let $u : [c, d] \to \mathbb{R}$ be a continuous, strictly increasing function with image $[a, b]$. Then the composition $\alpha \equiv \gamma \circ u : [c, d] \to \mathbb{R}^n$ is equivalent to $\gamma : [a, b] \to \mathbb{R}^n$. We must show that $\alpha : [c, d] \to \mathbb{R}^n$ is rectifiable and has arclength equal to the arclength of $\gamma : [a, b] \to \mathbb{R}^n$, which we denote by ℓ.

Let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $P = \{x_0, \ldots, x_m\}$ is a partition of the interval $[a, b]$ such that gap $P < \delta$, then

$$\left| \sum_{i=1}^{m} \|\gamma(x_i) - \gamma(x_{i-1})\| - \ell \right| < \epsilon. \tag{20.1}$$

According to Theorem 3.17, since the function $u : [c, d] \to \mathbb{R}$ is continuous, it is uniformly continuous. Thus, using the ϵ-δ criterion for uniform continuity asserted in Theorem 3.20, there is a $\delta' > 0$ such that $|u(t_1) - u(t_2)| < \delta$ for any two points t_1 and t_2 in $[c, d]$ such that $|t_1 - t_2| < \delta'$. Since the function $u : [c, d] \to \mathbb{R}$ is strictly increasing, it is clear that a partition P' of $[c, d]$ is mapped by $u : [c, d] \to \mathbb{R}$ onto a partition P of $[a, b]$ and that the polygonal approximation for the arclength of $\alpha : [c, d] \to \mathbb{R}^n$ based on P' is equal to the polygonal approximation for the arclength of $\gamma : [a, b] \to \mathbb{R}^n$ based on P. Thus, by the choice of δ and δ', if gap $P' < \delta'$, then gap $P < \delta$, and so the polygonal approximation of the arclength of $\alpha : [c, d] \to \mathbb{R}^n$ based on P' differs from ℓ by at most ϵ. ∎

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Arclength of a Piecewise Smooth Path

We show that a piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable and that its arclength equals $\int_a^b \|\gamma'\|$. The first step toward proving this is to establish the following approximation lemma.

Lemma 20.6 A smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ has the following approximation property: For each subinterval $[a', b']$ of $[a, b]$, with $a < a' < b' < b$, and each positive number ϵ, there is a positive number δ such that if $[c, d]$ is a subinterval of $[a', b']$ of length less than δ and t is a parameter value in the open interval (c, d), then

$$\left\| \frac{\gamma(d) - \gamma(c)}{d - c} - \gamma'(t) \right\| < \epsilon. \tag{20.2}$$

Proof

We express $\gamma : [a, b] \to \mathbb{R}^n$ in component functions as follows:

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \qquad \text{for } a \le t \le b.$$

By the very definition of a smooth parametrized path, the mapping $\gamma' : [a', b'] \to \mathbb{R}^n$ is continuous. Since a continuous function defined on a closed bounded interval is uniformly continuous, for each index i between 1 and n, the function $\gamma_i' : [a', b'] \to \mathbb{R}$ is uniformly continuous.

Let $\epsilon > 0$. Define $\epsilon' \equiv \epsilon/\sqrt{n}$. For each index i between 1 and n, we can select $\delta_i > 0$ such that if t_1 and t_2 are parameter values in the interval $[a', b']$ with $|t_2 - t_1| < \delta_i$, then $|\gamma_i'(t_2) - \gamma_i'(t_1)| < \epsilon'$. Define $\delta = \min_{1 \le i \le n} \delta_i$.

Suppose that $[c, d]$ is a subinterval of $[a', b']$ of length less than δ. Then for each index i, we can apply the Mean Value Theorem for scalar functions to choose a point η_i in the open interval (c, d) such that

$$\gamma_i(d) - \gamma_i(c) = \gamma_i'(\eta_i)(d - c).$$

Let t be a parameter value in the open interval (c, d). Then for each index i with $1 \le i \le n$, $|\eta_i - t| < \delta \le \delta_i$, so by the choice of δ_i,

$$\begin{aligned} &|[\gamma_i(d) - \gamma_i(c)] - \gamma_i'(t)(d - c)| \\ &\quad = |\gamma_i'(\eta_i) - \gamma_i'(t)|(d - c) < \epsilon'(d - c) = \left(\epsilon/\sqrt{n}\right)(d - c). \end{aligned}$$

Consequently,

$$\begin{aligned} &\|[\gamma(d) - \gamma(c)] - \gamma'(t)(d - c)\| \\ &\quad = \sqrt{\sum_{i=1}^{n} [|[\gamma_i(d) - \gamma_i(c)] - \gamma_i'(t)(d - c)|]^2} < \epsilon(d - c). \end{aligned}$$

Dividing this inequality by $d - c$, we obtain (20.2). ■

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Proposition 20.7 A smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable, and its arclength ℓ is given by

$$\ell = \int_a^b \|\gamma'\|. \tag{20.3}$$

Proof
Define the function $h : (a, b) \to \mathbb{R}$ by $h(t) = \|\gamma'(t)\|$ for $a < t < b$. Then, by definition, h is continuous and bounded on the open interval (a, b), and hence it is integrable on the closed interval $[a, b]$. For each index i between 1 and n, we can choose $M_i > 0$ such that $|\gamma_i'(t)| \leq M_i$ for all t in (a, b). Define $M = \sqrt{M_1^2 + \cdots + M_n^2}$. Using the Mean Value Theorem for each of the component functions of the map γ and arguing as in the proof of Lemma 20.6, it follows that for any subinterval $[c, d]$ of $[a, b]$,

$$\left| \frac{\|\gamma(d) - \gamma(c)\|}{d - c} - \|\gamma'(t)\| \right| < 2M \qquad \text{for all } t \text{ in } [c, d].$$

Thus, by the monotonicity property of the integral,

$$\left| \|\gamma(d) - \gamma(c)\| - \int_c^d \|\gamma'\| \right| < 2M(d - c) \quad \text{for every subinterval } [c, d] \text{ of } [a, b]. \tag{20.4}$$

Let $\epsilon > 0$. Choose a subinterval $[a', b']$ of $[a, b]$, with $a < a' < b' < b$, such that if we define $\eta = \max\{a' - a, b - b'\}$, then $16\eta M < \epsilon$. By Lemma 20.6, setting $\epsilon' = \epsilon/2(b' - a')$, we can choose $\delta' > 0$ such that if $[c, d]$ is a subinterval of $[a', b']$ of length less than δ' and t is a parameter value in the open interval (c, d), then

$$\left\| \frac{\gamma(d) - \gamma(c)}{d - c} - \gamma'(t) \right\| < \epsilon',$$

which, by the Triangle Inequality, implies that

$$\left| \frac{\|\gamma(d) - \gamma(c)\|}{d - c} - \|\gamma'(t)\| \right| < \epsilon' \qquad \text{for all } t \text{ in } (c, d).$$

Hence, by the monotonicity property of the integral,

$$\left| \|\gamma(d) - \gamma(c)\| - \int_c^d \|\gamma'\| \right| < \epsilon'(d - c) \qquad \text{if } [c, d] \subseteq [a', b'] \text{ and } d - c < \delta'. \tag{20.5}$$

Define $\delta = \min\{\delta', \eta\}$. Let $P = \{x_0, \ldots, x_m\}$ be a partition of $[a, b]$ such that gap $P < \delta$. By the Triangle Inequality,

$$\left| \sum_{i=1}^m \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_a^b \|\gamma'\| \right| \leq \sum_{i=1}^m \left| \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_{x_{i-1}}^{x_i} \|\gamma'\| \right|. \tag{20.6}$$

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By the estimate (20.5), for each index i with $1 \leq i \leq m$,

$$\left| \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_{x_{i-1}}^{x_i} \|\gamma'\| \right| < \epsilon'(x_i - x_{i-1}) \qquad \text{if } [x_{i-1}, x_i] \subseteq [a', b'].$$

Thus, the contribution to the sum on the right-hand side of (20.6) from intervals $[x_{i-1}, x_i]$ contained in $[a', b']$ is less than $\epsilon'(b' - a') = \epsilon/2$. On the other hand, for each interval $[x_{i-1}, x_i]$ not contained in $[a', b']$, the estimate (20.4) implies that

$$\left| \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_{x_{i-1}}^{x_i} \|\gamma'\| \right| < 2M(x_i - x_{i-1}),$$

and since the sum of the lengths of such intervals is at most 4η, the contribution to the sum on the right-hand side of (20.6) from intervals $[x_{i-1}, x_i]$ not contained in $[a', b']$ is at most $8\eta M < \epsilon/2$. From these two estimates it follows that

$$\left| \sum_{i=1}^{m} \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_a^b \|\gamma'\| \right| < \epsilon. \qquad \blacksquare$$

Theorem 20.8 A piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable, and its arclength ℓ is given by

$$\ell = \int_a^b \|\gamma'\|. \tag{20.7}$$

Proof

Let $\epsilon > 0$. It is necessary to find $\delta > 0$ such that if $P = \{x_0, \ldots, x_m\}$ is a partition of $[a, b]$ with gap $P < \delta$, then

$$\left| \sum_{i=1}^{m} \|\gamma(x_i) - \gamma(x_{i-1})\| - \int_a^b \|\gamma'\| \right| < \epsilon.$$

Choose a partition $\{z_0, \ldots, z_k\}$ of the interval $[a, b]$ such that the restriction of γ to each subinterval of this partition is a smooth parametrized path. Since the mapping $\gamma : [a, b] \to \mathbb{R}^n$ is uniformly continuous, we can choose a positive number δ' such that $\|\gamma(s) - \gamma(t)\| < \epsilon/6k$ for any two parameter values s and t in $[a, b]$ such that $|s - t| < \delta'$.

The crucial point in the proof is the following observation: If P is any partition of $[a, b]$ with gap $P < \delta'$ and P' is the refinement of P obtained by inserting all the z_i's, then the difference between the polygonal approximation of the arclength of $\gamma : [a, b] \to \mathbb{R}^n$ based on P and that based on P' is at most $\epsilon/2$. To see this, observe that the difference between these approximations consists of fewer than $3k$ terms of the form $\|\gamma(t) - \gamma(s)\|$, where s and t are parameter values such that $|s - t| < \delta'$. Since each of these terms is at most $\epsilon/6k$, the polygonal approximations do indeed differ by at most $\epsilon/2$.

For each index i between 1 and k, by Proposition 20.7, the parametrized path $\gamma : [z_{i-1}, z_i] \to \mathbb{R}^n$ is rectifiable, so we can choose $\delta_i > 0$ such that if P_i is any partition of $[z_{i-1}, z_i]$ such that gap $P_i < \delta_i$, then the polygonal approximation of the arclength of $\gamma : [z_{i-1}, z_i] \to \mathbb{R}^n$ differs from the integral of $\|\gamma'\|$ on $[z_{i-1}, z_i]$ by

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at most $\epsilon/2k$. Define $\delta = \min\{\delta', \delta_1, \ldots, \delta_k\}$. Let P be any partition of $[a, b]$ with gap $P < \delta$. By the choice of the δ_i's, the polygonal approximation of the arclength of $\gamma : [a, b] \to \mathbb{R}^n$ based on P' differs from the integral of $\|\gamma'\|$ on $[a, b]$ by at most $\epsilon/2$. By the estimate in the preceding paragraph, it follows that the polygonal approximation of the arclength of $\gamma : [a, b] \to \mathbb{R}^n$ based on P differs from the integral of $\|\gamma'\|$ on $[a, b]$ by at most ϵ. Thus, (20.7) holds. ■

Example 20.9 For the helix defined by the parametrization

$$\gamma(t) = (\cos t,\ \sin t,\ t) \qquad \text{for } 0 \leq t \leq 4\pi,$$

observe that

$$\gamma'(t) = (-\sin t,\ \cos t,\ 1) \qquad \text{for } 0 \leq t \leq 4\pi.$$

By Theorem 20.8, this parametrized path is rectifiable, and its arclength is

$$\int_0^{4\pi} \|\gamma'(t)\|\, dt = \int_0^{4\pi} \sqrt{2}\, dt = 4\sqrt{2}\pi.$$ ■

Suppose that the parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable. Then for P a partition of $[a, b]$ and P' a refinement of P, the polygonal approximation of the arclength based on P' is larger than the polygonal approximation based on P; this is an immediate consequence of the Triangle Inequality. From this it is not difficult to show that the arclength of a rectifiable parametrized path is the supremum of all polygonal approximations of the arclength (Exercise 8).

A Nonrectifiable Parametrized Path

In proving that a smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is rectifiable, we used the fact that for such a parametrized path the derivative $\gamma' : (a, b) \to \mathbb{R}^n$ is *bounded*. In fact, as the following example shows, it is not true that a parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ that has only the property that $\gamma : (a, b) \to \mathbb{R}^n$ is continuously differentiable is necessarily rectifiable: If the derivative is not bounded, the path may fail to be rectifiable.

Example 20.10 First, define

$$f(x) = \begin{cases} x \sin(\pi/2 + \pi/x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0, \end{cases}$$

and then define $\gamma(t) = (t, f(t))$ for $0 \leq t \leq 1$. Then it is clear that $\gamma : [0, 1] \to \mathbb{R}^2$ is a parametrized path and that $\gamma : (0, 1) \to \mathbb{R}^2$ is continuously differentiable. However, the path $\gamma : [0, 1] \to \mathbb{R}^2$ is not rectifiable. Indeed, observe that

$$f\left(\frac{1}{k}\right) = \begin{cases} 1/k & \text{if } k \text{ is an even natural number} \\ -1/k & \text{if } k \text{ is an odd natural number,} \end{cases}$$

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so that

$$\left\|\gamma\left(\frac{1}{k}\right) - \gamma\left(\frac{1}{(k+1)}\right)\right\| \geq \frac{1}{k} \qquad \text{for each natural number } k.$$

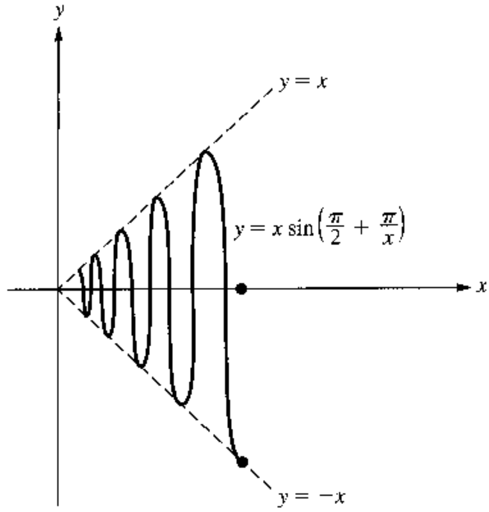

FIGURE 20.3 A nonrectifiable continuously differentiable parametrized path.

It follows that for each natural number n, if we define the partition $P = \{0, 1/(n+1), 1/n, \ldots, 1/2, 1\}$, then the polygonal approximation based on P is greater than $\sum_{k=1}^{n} 1/k$. Since the sequence of partial sums of the Harmonic Series $\sum_{k=1}^{\infty} 1/k$ is unbounded, we conclude that the polygonal approximations can be arbitrarily large, so the parametrized path $\gamma : [0, 1] \to \mathbb{R}^2$ is not rectifiable. ■

Parametrization by Arclength

Among the equivalent parametrizations of a piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$, there is a particular parametrization, called *parametrization by arclength,* that is often convenient. It is defined as follows: For each parameter value t in $[a, b]$, define

$$u(t) \equiv \int_a^t \|\gamma'\|.$$

Then, since $\gamma'(t) \neq \mathbf{0}$, except possibly at a finite number of points in (a, b), the continuous function $u : [a, b] \to \mathbb{R}$ is strictly increasing and $u([a, b]) = [0, \ell]$, where ℓ is the arclength of $\gamma : [a, b] \to \mathbb{R}^n$. The inverse function $u^{-1} : [0, \ell] \to [a, b]$ is also strictly increasing and continuous. The parametrized path $\alpha : [0, \ell] \to \mathbb{R}^n$ defined by the composition $\alpha = \gamma \circ u^{-1}$ is the parametrization by arclength for the parametrized

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path $\gamma:[a, b] \to \mathbb{R}^n$. It has the property that for each parameter value s in $[0, \ell]$, the arclength of the parametrized path $\alpha:[0, s] \to \mathbb{R}^n$ equals the parameter value s. The Second Fundamental Theorem of Calculus (Differentiating Integrals) implies that except for possibly finitely many points, $u'(t) = \|\gamma'(t)\|$, so that using the Chain Rule, it follows that the parametrization by arclength $\alpha:[0, \ell] \to \mathbb{R}^n$ has the property that at each parameter value s in $[0, \ell]$ at which it is differentiable, $\|\alpha'(s)\| = 1$.

Example 20.11 Consider the helix defined by $\gamma(t) = (\cos t, \sin t, t)$ for $0 \le t \le 4\pi$. For each t in $[0, 4\pi]$,

$$u(t) = \int_0^t \|\gamma'\| = \int_0^t \sqrt{2}\,dt = \sqrt{2}t.$$

Thus, the parametrization by arclength of the helix, $\alpha:[0, 4\sqrt{2}\pi] \to \mathbb{R}^3$, is defined by

$$\alpha(s) = \left(\cos\left(s/\sqrt{2}\right), \sin\left(s/\sqrt{2}\right), s/\sqrt{2}\right) \qquad \text{for } 0 \le s \le 4\sqrt{2}\pi. \qquad \blacksquare$$

Integration along a Path: Line Integrals

We now consider the concept of line integral. Given a piecewise smooth parametrized path $\gamma:[a, b] \to \mathbb{R}^n$ having as its image the path $\mathbf{\Gamma}$ and a continuous function $f:\mathbf{\Gamma} \to \mathbb{R}$, we define the *line integral* of $f:\mathbf{\Gamma} \to \mathbb{R}$ on the parametrized path $\gamma:[a, b] \to \mathbb{R}^n$ by the formula

$$\int_{\gamma:[a,b]\to\mathbb{R}^n} f \equiv \int_a^b f(\gamma(t))\|\gamma'(t)\|\,dt.$$

Our first result is that replacing one parametrization of a path by an equivalent parametrization leaves the above line integral unchanged.

Theorem 20.12 Suppose that $\gamma:[a, b] \to \mathbb{R}^n$ and $\alpha:[c, d] \to \mathbb{R}^n$ are two piecewise smooth parametrizations of the same path $\mathbf{\Gamma}$ that are equivalent. Then for any continuous function $f:\mathbf{\Gamma} \to \mathbb{R}$,

$$\int_{\gamma:[a,b]\to\mathbb{R}^n} f = \int_{\alpha:[c,d]\to\mathbb{R}^n} f.$$

Proof

Using the addition over partitions property of the integral, it suffices to consider the case where the paths are smooth. By the definition of equivalence of parametrizations, there is a strictly increasing path $u:[c, d] \to \mathbb{R}$ such that $u([c, d]) = [a, b]$ and $\alpha = \gamma \circ u:[c, d] \to \mathbb{R}$. For each parameter value t in (c, d) there is an index i such that $\gamma_i'(u(t)) \neq 0$, so that since $\gamma_i \circ u = \alpha_i:(c, d) \to \mathbb{R}$, from the Inverse Function Theorem and the Chain Rule it follows that $u:(c, d) \to R$ is differentiable at t. Thus, $\alpha'(t) = \gamma'(u(t))u'(t)$, and so, in particular, $u'(t) \neq 0$ for all parameter values t in (c, d). Since $u:[c, d] \to \mathbb{R}$ is strictly increasing and $u'(t) \neq 0$ for

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all t in (c, d), $u'(t) > 0$ for all t in (c, d). By the Change of Variables Theorem and the Chain Rule,

$$\begin{aligned}\int_{\gamma:[a,b]\to\mathbb{R}^n} f &= \int_a^b f(\gamma(t))\|\gamma'(t)\|\,dt \\ &= \int_{u(c)}^{u(d)} f(\gamma(t))\|\gamma'(t)\|\,dt \\ &= \int_c^d f(\gamma(u(t)))\|\gamma'(u(t))\|u'(t)\,dt \\ &= \int_c^d f(\alpha(t))\|\alpha'(t)\|\,dt \\ &= \int_{\alpha:[c,d]\to\mathbb{R}^n} f.\end{aligned}$$

■

For a piecewise smooth parametrization $\gamma:[a, b]\to\mathbb{R}^n$ of a path $\mathbf{\Gamma}$ and a continuous function $f:\mathbf{\Gamma}\to\mathbb{R}$, if the arclength of this path is ℓ and $\alpha:[0, \ell]\to\mathbb{R}^n$ is the equivalent parametrization by arclength, then since $\|\alpha(s)\| = 1$ except at possibly finitely many points, by Theorem 20.12,

$$\int_{\gamma:[a,b]\to\mathbb{R}^n} f = \int_0^\ell f(\alpha(s))\,ds.$$

For this reason, it is common to denote the line integral by

$$\int_{\mathbf{\Gamma}} f(s)\,ds$$

and explicitly specify a choice of smooth parametrization of the path $\mathbf{\Gamma}$.

There are other types of line integrals associated with continuous functions defined on paths. For a parametrized path $\gamma:[a, b]\to\mathbb{R}^n$ having image $\mathbf{\Gamma}$ and a continuous function $f:\mathbf{\Gamma}\to\mathbb{R}$, for an index i between 1 and n with the property that $\gamma_i:(a, b)\to\mathbb{R}^n$ has a continuous bounded derivative, we define

$$\int_{\mathbf{\Gamma}} f(\mathbf{x})\,dx_i = \int_{\gamma:[a,b]\to\mathbb{R}^n} f(\mathbf{x})\,dx_i = \int_a^b f(\gamma(t))\gamma_i'(t)\,dt.$$

Arguing as we did in the proof of Theorem 20.12, it follows that changing from one parametrization to another by composing with a strictly increasing smooth function does not change the value of the integral. Naturally, in $\mathbb{R}^2$ and $\mathbb{R}^3$, when points are denoted by (x, y) or (x, y, z), we use a more familiar notation for these integrals: In $\mathbb{R}^3$ we use

$$\int_{\mathbf{\Gamma}} f(x, y, z)\,dx, \qquad \int_{\mathbf{\Gamma}} f(x, y, z)\,dy, \qquad \text{and} \qquad \int_{\mathbf{\Gamma}} f(x, y, z)\,dz$$

and specify a choice of parametrization of the path $\mathbf{\Gamma}$.

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Example 20.13 Consider the piecewise smooth parametrization $\gamma : [0, 1] \to \mathbb{R}^2$ of the path $\mathbf{\Gamma}$ that is the boundary of Ω given in Example 20.4. For this parametrization, it follows from the very definition of these line integrals and the change of variables formula for integrals of functions of a single variable that

$$\begin{aligned}\int_{\mathbf{\Gamma}} y\,dx &= \int_{\mathbf{\Gamma}_1} y\,dx + \int_{\mathbf{\Gamma}_3} y\,dx \\ &= \int_0^{1/4} g(a + 4t(b-a))4(b-a)\,dt \\ &\quad + \int_{1/2}^{3/4} h(b - 4(t-1/2)(b-a))(-4(b-a))\,dt \\ &= \int_a^b [g(t) - h(t)]\,dt.\end{aligned}$$

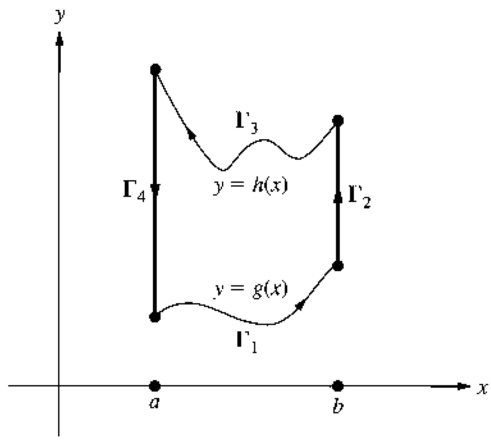

FIGURE 20.4 $\int_{\mathbf{\Gamma}} y\,dx = \int_{\mathbf{\Gamma}_1} y\,dx + \int_{\mathbf{\Gamma}_3} y\,dx.$ ■

EXERCISES FOR SECTION 20.1

1. For two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$, use formula (20.3) to check that the arclength of the parametrized segment from $\mathbf{p}$ to $\mathbf{q}$ is $\|\mathbf{p} - \mathbf{q}\|$.
2. Find the arclength of the parametrized path $\gamma : [0, 2\pi] \to \mathbb{R}^3$ defined by
$$\gamma(t) = (3\cos t,\ 3\sin t,\ 2t) \qquad \text{for } 0 \le t \le 2\pi.$$
3. For the parametrized path $\gamma : [0, 4] \to \mathbb{R}^3$ defined by
$$\gamma(t) = (t,\ t^2,\ 1) \qquad \text{for } 0 \le t \le 4,$$
find a smoothly equivalent parametrization by arclength.
4. For $0 < a < b$, find an integral that equals the arclength of the ellipse $\{(x, y) \mid x^2/a^2 + y^2/b^2 = 1\}$. (It is necessary to use numerical approximation methods to actually evaluate this integral.)

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5. Evaluate

$$\int_{\Gamma} x \sin y \, dx + y \cos x \, dy$$

where Γ is the image of the parametrized segment $\gamma : [0, 1] \to \mathbb{R}^2$ defined by

$$\gamma(t) = (t, mt) \qquad \text{for } 0 \le t \le 1.$$

6. Evaluate

$$\int_{\Gamma} \frac{x}{x^2 + y^2} \, dx + \frac{y}{x^2 + y^2} \, dy$$

where Γ is the image of the parametrized path $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\gamma(t) = (e^t \cos t, \, e^t \sin t) \qquad \text{for } 0 \le t \le 2\pi.$$

7. Show that the definition of arclength is unambiguous in that if a parametrized path is rectifiable, then there is only one number ℓ that has the polynomial approximation property in the definition.

8. Show that the arclength of a rectifiable parametrized path is the supremum of all polygonal approximations of the arclength.

9. For a rectifiable parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ having image Γ, define $\alpha : [a, b] \to \mathbb{R}^n$ by $\alpha(t) = \gamma(a + b - t)$ for $a \le t \le b$.

a. Show that $\alpha : [a, b] \to \mathbb{R}^n$ also is a parametrization of the path Γ that is rectifiable and has the same arclength as $\gamma : [a, b] \to \mathbb{R}^n$.

b. For a continuous function $f : \Gamma \to \mathbb{R}$ show that

$$\int_{\gamma:[a, b] \to \mathbb{R}^n} f = \int_{\alpha:[a, b] \to \mathbb{R}^n} f.$$

c. If $\gamma : [a, b] \to \mathbb{R}^n$ is smooth, $f : \Gamma \to \mathbb{R}$ is continuous, and $1 \le i \le n$, show that

$$\int_{\gamma:[a, b] \to \mathbb{R}^n} f \, dx_i = -\int_{\alpha:[a, b] \to \mathbb{R}^n} f \, dx_i.$$

10. Suppose that $\gamma : [a, b] \to \mathbb{R}^n$ and $\alpha : [c, d] \to \mathbb{R}^n$ are parametrized paths, each of which is one-to-one and such that $\gamma(a) = \alpha(c)$. Show that these parametrized paths are equivalent if and only if they have the same image. [*Hint:* For each parameter value t in $[c, d]$, define $u(t) = s$ to be the unique parameter value s in $[a, b]$ such that $\alpha(t) = \gamma(s)$.]

11. For a piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ and a continuous function $f : \Gamma \to \mathbb{R}$, where Γ is the image of $\gamma : [a, b] \to \mathbb{R}^n$, prove that

$$\left| \int_{\Gamma} f(s) \, ds \right| \le M\ell,$$

where M is such that $|f(\mathbf{p})| \le M$ for all points $\mathbf{p}$ on Γ and ℓ is the arclength of $\gamma : [a, b] \to \mathbb{R}^n$.

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20.2 SURFACE AREA AND SURFACE INTEGRALS

In the preceding section, we defined parametrized paths, paths, and line integrals; in this section, we define the corresponding notions of parametrized surfaces, surfaces, and surface integrals.

Regions in the Plane and Parametrized Surfaces

We call a subset $\mathcal{R}$ of the plane $\mathbb{R}^2$ a *region* provided that it is open and is a Jordan domain—that is, it is open and bounded and its boundary has Jordan content 0. By Theorem 18.17, if $\mathcal{R}$ is a region in $\mathbb{R}^2$ and the function $g:\mathcal{R} \to \mathbb{R}$ is continuous and bounded, then its integral $\int_{\mathcal{R}} g(x, y)\,dx\,dy$ is properly defined.

Definition Let $\mathcal{R}$ be a region in $\mathbb{R}^2$. Then a continuously differentiable mapping $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ is called a *parametrized surface* with parameter space $\mathcal{R}$ provided that the following three properties hold:

i. The component functions of the mapping $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ have bounded first-order partial derivatives.
ii. The mapping $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ is one-to-one.
iii. For each point (u, v) in $\mathcal{R}$,

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \neq \mathbf{0}.$$

Definition A subset $\mathcal{S}$ of $\mathbb{R}^3$ is called a *surface* provided that it is the image of a parametrized surface.

For a parametrized surface $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ and a point (u_0, v_0) in the parameter space $\mathcal{R}$, since $\mathcal{R}$ is open, if the positive number r is sufficiently small, then the parametrized path $\gamma:(-r, r) \to \mathbb{R}^3$ defined by

$$\gamma(t) = \mathbf{r}(u_0 + t, v_0) \qquad \text{for } t \text{ in } (-r, r)$$

defines a smooth path in the surface $\mathcal{S}$ that has a tangent vector at the point $\mathbf{r}(u_0, v_0)$ given by $\partial\mathbf{r}/\partial u(u_0, v_0)$. Similarly, holding the u variable constant, we have a parametrized path whose image lies in the surface $\mathcal{S}$ and passes through the point $\mathbf{r}(u_0, v_0)$, at which point it has a tangent vector $\partial\mathbf{r}/\partial v(u_0, v_0)$. Thus, because of assumption (iii), if we define the vector $\boldsymbol{\eta}$ by

$$\boldsymbol{\eta} = \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0),$$

we conclude that the vector $\boldsymbol{\eta}$ is nonzero and, by the properties of the cross-product, it is orthogonal to the tangent vectors $\partial\mathbf{r}/\partial u(u_0, v_0)$ and $\partial\mathbf{r}/\partial v(u_0, v_0)$. For this reason, we call $\boldsymbol{\eta}$, or any nonzero scalar multiple of $\boldsymbol{\eta}$, a normal to the surface $\mathcal{S}$ at the point $\mathbf{r}(u_0, v_0)$. Thus, a surface has a normal at each point that, for a given parametrization, varies continuously with the parameters (u, v).

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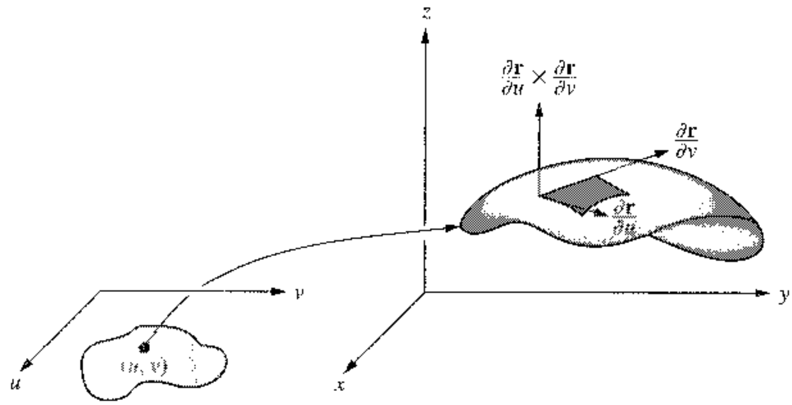

FIGURE 20.5 A parametrized surface with a normal vector.

Example 20.14 Fix a positive number a less than 1 and consider the set

$$\mathcal{S} = \{(x,\ y,\ z) \text{ in } \mathbb{R}^3 \mid x > 0,\ y > 0,\ z > 0,\ x^2+y^2+z^2 = 1,\ 0 < x^2+y^2 < a^2\}.$$

Then the set $\mathcal{S}$ is a surface. To verify this, we must find a parametrized surface with an image equal to $\mathcal{S}$. Define $\mathcal{R} = \{(x,\ y) \text{ in } \mathbb{R}^2 \mid x > 0,\ y > 0,\ 0 < x^2+y^2 < a^2\}$ and define $\mathbf{r}: \mathcal{R} \to \mathbb{R}^3$ by

$$\mathbf{r}(x,\ y) = \left(x,\ y,\ \sqrt{1-x^2-y^2}\right) \qquad \text{for } (x,\ y) \text{ in } \mathcal{R}.$$

Then the set $\mathcal{R}$ is certainly a region, the mapping $\mathbf{r}: \mathcal{R} \to \mathbb{R}^3$ is continuously differentiable, and a computation of partial derivatives shows that the partial derivatives of the components of this mapping are bounded: For example, the estimate for the partial derivatives of the third component with respect to x is

$$\left|\frac{\partial r_3}{\partial x}(x,\ y)\right| = \left|\frac{-x}{\sqrt{1-x^2-y^2}}\right| \le \frac{1}{\sqrt{1-a^2}} \qquad \text{for all } (x,\ y) \text{ in } \mathcal{R}.$$

Furthermore,

$$\frac{\partial \mathbf{r}}{\partial x}(x,\ y) \times \frac{\partial \mathbf{r}}{\partial y}(x,\ y) = \left(\frac{x}{\sqrt{1-x^2-y^2}},\ \frac{y}{\sqrt{1-x^2-y^2}},\ 1\right) \ne \mathbf{0}.$$

Finally, it is clear that this mapping is one-to-one and has an image equal to $\mathcal{S}$. ∎

Example 20.15 In Example 20.14, if we allow $a = 1$, then the mapping $\mathbf{r}: \mathcal{R} \to \mathbb{R}^3$ defined above is not a parametrized surface since the partial derivatives of the third component are no longer bounded. Nevertheless, the image of this mapping, $\mathcal{S}$, is a surface. In order to verify this it is necessary to find a better parametrization

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of $\mathcal{S}$. Define $\mathcal{R}' = \{(u, v) \text{ in } \mathbb{R}^2 \mid 0 < u < \pi/2,\ 0 < v < \pi/2\}$ and then define $\mathbf{r}' : \mathcal{R}' \to \mathbb{R}^3$ by

$$\mathbf{r}'(u, v) = (\sin u \cos v,\ \sin u \sin v,\ \cos u) \qquad \text{for } (u, v) \text{ in } \mathcal{R}'.$$

The parameter space $\mathcal{R}'$ is a region in the plane. It is clear that this mapping is continuously differentiable and that its components have bounded partial derivatives. Moreover, geometrically interpreting the significance of the components (these are spherical coordinates), we see that the image of this map is $\mathcal{S}$ and that the map is one-to-one. Finally, a brief computation, which we leave as an exercise, shows that for each point (u, v) in $\mathcal{R}$,

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) = (\sin^2 u \cos v, \sin^2 u \sin v,\ \sin u\ \cos u) \neq \mathbf{0}. \qquad \blacksquare$$

Projectionally Parametrized Surfaces

The simplest type of parametrized surface is one that has as its image the graph of a function defined on a region in a coordinate plane. Such a parametrized surface is called a *projectionally parametrized surface*. For example, if $\mathcal{R}$ is a region in the plane $\mathbb{R}^2$ and a continuously differentiable function $g : \mathcal{R} \to \mathbb{R}$ with bounded partial derivatives is given, then each of the following three mappings describes a projectionally parametrized surface: the map $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$ defined by

$$\mathbf{r}(x, y) = (x, y, g(x, y)) \qquad \text{for } (x, y) \text{ in } \mathcal{R},$$

the map

$$\mathbf{r}(x, z) = (x, g(x, z), z) \qquad \text{for } (x, z) \text{ in } \mathcal{R},$$

and the map

$$\mathbf{r}(y, z) = (g(y, z), y, z) \qquad \text{for } (y, z) \text{ in } \mathcal{R}.$$

Observe that in the first case a short computation shows that a normal $\boldsymbol{\eta}$ to this first surface at the point $(x_0, y_0, g(x_0, y_0))$ is given by

$$\boldsymbol{\eta} = \frac{\partial \mathbf{r}}{\partial x}(x_0, y_0) \times \frac{\partial \mathbf{r}}{\partial y}(x_0, y_0) = \left(-\frac{\partial g}{\partial x}(x_0, y_0), -\frac{\partial g}{\partial y}(x_0, y_0), 1\right).$$

This shows that the definition of normal to a surface that is a graph of a function of two variables, described in Section 14.1, is consistent with the general definition of normal we are considering here.

Surface Area

We now wish to define surface area. To do so, we first describe some properties of the scalar product and the cross-product of vectors in $\mathbb{R}^3$. Recall that for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle$ and the cross-product $\mathbf{u} \times \mathbf{v}$ are defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle \equiv u_1 v_1 + u_2 v_2 + u_3 v_3$$

and

$$\mathbf{u} \times \mathbf{v} \equiv (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$$

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The scalar product and the cross-product provide a very useful way to describe geometric concepts in an analytic form. We make the following two observations regarding the scalar product and cross-product[1]

i. Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^3$ and let ℓ be the line through the origin that is parallel to $\mathbf{u}$; that is, ℓ consists of points of the form $t\mathbf{u}$ for t in $\mathbb{R}$. Then for any point $\mathbf{q}$ in $\mathbb{R}^3$,

$$\text{the point on } \ell \text{ that is closest to } \mathbf{q} \text{ is } \lambda\mathbf{u}, \text{ where } \lambda = \frac{\langle \mathbf{q},\ \mathbf{u}\rangle}{\langle \mathbf{u},\ \mathbf{u}\rangle}.$$

(The proof of this is a computation: Show that $\|\mathbf{q} - \lambda\mathbf{u}\|^2 \le \|\mathbf{q} - t\mathbf{u}\|^2$ for all t in $\mathbb{R}$.)

ii. Let the vectors $\mathbf{u}$ and $\mathbf{v}$ be linearly independent. For a point $\mathbf{p}$ in $\mathbb{R}^3$, we define the *parallelogram* based at $\mathbf{p}$ and bounded by the vectors $\mathbf{u}$ and $\mathbf{v}$ to be the set

$$S = \{\mathbf{p} + t\mathbf{u} + s\mathbf{v} \mid 0 < t < 1,\ 0 < s < 1\}.$$

The *area* of this parallelogram is defined to be the length of the vector $\mathbf{v}$ times the distance from the point $\mathbf{p} + \mathbf{u}$ to the line through the point $\mathbf{p}$ that is parallel to the vector $\mathbf{v}$. Since the distance between points is invariant under translation, the area is independent of the choice of base point $\mathbf{p}$. Suppose $\mathbf{p} = \mathbf{0}$; then, by (i),

$$\text{area } S = \|\mathbf{v}\| \cdot \|\mathbf{u} - \lambda\mathbf{v}\|, \qquad \text{where } \lambda = \frac{\langle \mathbf{u},\ \mathbf{v}\rangle}{\langle \mathbf{v},\ \mathbf{v}\rangle}.$$

The cross-product provides a simple formula for this:

$$\text{area } S = \|\mathbf{u} \times \mathbf{v}\|. \tag{20.8}$$

The verification of this formula is a straightforward computation. Indeed,

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2 - \langle \mathbf{u},\ \mathbf{v}\rangle^2 \\
&= \|\mathbf{v}\|^2 \left\{ \|\mathbf{u}\|^2 - \frac{\langle \mathbf{u},\ \mathbf{v}\rangle^2}{\|\mathbf{v}\|^2} \right\} \\
&= \|\mathbf{v}\|^2 \langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{u}\rangle \\
&= \|\mathbf{v}\|^2 \langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{u} - \lambda\mathbf{v}\rangle \qquad (\text{since } \langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{v}\rangle = 0) \\
&= \|\mathbf{v}\|^2 \cdot \|\mathbf{u} - \lambda\mathbf{v}\|^2.
\end{aligned}$$

Proposition 20.16 Let $\mathbf{J}$ be an open rectangle in the plane and define the parametrized surface $\mathbf{r} : \mathbf{J} \to \mathbb{R}^3$ by

$$\mathbf{r}(x,\ y) = (x,\ y,\ ax + by + c) \qquad \text{for } (x,\ y) \text{ in } \mathbf{J}.$$

Then the area of the surface $S = \mathbf{r}(\mathbf{J})$ is given by

$$\text{area } S = \sqrt{1 + a^2 + b^2} \cdot \text{area } \mathbf{J}. \tag{20.9}$$

[1] In Appendix B, there is a full description of the geometric properties of the scalar product and the cross-product of two vectors in $\mathbb{R}^3$.

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Proof

Suppose that $\mathbf{J} = (x_1, x_2) \times (y_1, y_2)$. Then $\mathcal{S}$ is the parallelogram based at the point $\mathbf{p} = (x_1, y_1, ax_1+by_1+c)$ and bounded by the vectors $\mathbf{u} = (0, y_2-y_1, b(y_2-y_1))$ and $\mathbf{v} = (x_2 - x_1, 0, a(x_2 - x_1))$. By formula (20.8),

$$\begin{aligned}\text{area } \mathcal{S} &= \|\mathbf{u} \times \mathbf{v}\| \\ &= \|(a(x_2 - x_1)(y_2 - y_1), b(x_2 - x_1)(y_2 - y_1), -(x_2 - x_1)(y_2 - y_1))\| \\ &= \sqrt{1 + a^2 + b^2} \cdot \text{area } \mathbf{J}.\end{aligned}$$

■

Formula (20.9) motivates the general definition of surface area. We begin with the definition of area for a projectionally parametrized surface. Let $\mathcal{R}$ be an open rectangle in the plane $\mathbb{R}^2$ and let $g : \mathcal{R} \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Then the mapping $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$ defined by

$$\mathbf{r}(x, y) = (x, y, g(x, y)) \qquad \text{for } (x, y) \text{ in } \mathcal{R}$$

is a parametrized surface. Suppose that $\mathbf{P}$ is a partition of the parameter space $\mathcal{R}$ and that $\mathbf{J}$ is a rectangle in $\mathbf{P}$. Select a point $\mathbf{p} = (x_0, y_0)$ in the interior of $\mathbf{J}$ and let $\mathbf{T}(\mathbf{p})$ be the tangent plane to the surface $\mathcal{S}$ at the point $\mathbf{r}(\mathbf{p})$. As we showed in Section 14.1, the tangent plane consists of points (x, y, z) such that

$$z = g(\mathbf{p}) + \frac{\partial g}{\partial x}(\mathbf{p})(x - x_0) + \frac{\partial g}{\partial y}(\mathbf{p})(y - y_0).$$

According to formula (20.9), the area $\mathcal{S}(\mathbf{J})$ of the part of the tangent plane $\mathbf{T}(\mathbf{p})$ that lies above the generalized rectangle $\mathbf{J}$ is given by

$$\text{area } \mathcal{S}(\mathbf{J}) = \sqrt{1 + \left(\frac{\partial g}{\partial x}(\mathbf{p})\right)^2 + \left(\frac{\partial g}{\partial y}(\mathbf{p})\right)^2} \text{ area } \mathbf{J}.$$

If we sum these areas over the partition $\mathbf{P}$, we obtain

$$\sum_{\mathbf{J} \text{ in } \mathbf{P}} \sqrt{1 + \left(\frac{\partial g}{\partial x}(\mathbf{p})\right)^2 + \left(\frac{\partial g}{\partial x}(\mathbf{p})\right)^2} \text{ area } \mathbf{J}. \tag{20.10}$$

If we now take a sequence of partitions of $\mathcal{R}$, the gaps of which converge to 0, then the sums (20.10), being Riemann sums, converge to

$$\int_{\mathcal{R}} \sqrt{1 + \left(\frac{\partial g}{\partial x}(x, y)\right)^2 + \left(\frac{\partial g}{\partial y}(x, y)\right)^2}\, dx\, dy.$$

This motivates the following definition of *surface area* for a projectionally parametrized surface.

Definition Suppose that $\mathcal{R}$ is a region in the plane $\mathbb{R}^2$ and that the continuously differentiable function $g : \mathcal{R} \to \mathbb{R}$ has bounded first-order partial derivatives. We define the area of the surface

$$\mathcal{S} = \{(x, y, g(x, y)) \text{ in } \mathbb{R}^3 \mid (x, y) \text{ in } \mathcal{R}\}$$

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by the formula

$$\text{area}\,\mathcal{S} \equiv \int_{\mathcal{R}} \sqrt{1+\left(\frac{\partial g}{\partial x}(x,\,y)\right)^2+\left(\frac{\partial g}{\partial y}(x,\,y)\right)^2}\,dx\,dy. \tag{20.11}$$

The general definition of surface area and surface integral is as follows:

Definition Suppose that $\mathcal{R}$ is a region in the plane $\mathbb{R}^2$ and let $\mathcal{S}$ be the surface that is the image of the parametrized surface $\mathbf{r}:\mathcal{R}\to\mathbb{R}^3$. We define the area of $\mathcal{S}$ by

$$\text{area}\,\mathcal{S} \equiv \int_{\mathcal{R}} \left\|\frac{\partial \mathbf{r}}{\partial u}(u,\,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,\,v)\right\|\,du\,dv. \tag{20.12}$$

Furthermore, for a continuous bounded function $f:\mathcal{S}\to\mathbb{R}$, the surface integral of the function $f:\mathcal{S}\to\mathbb{R}$ over the surface $\mathcal{S}$, which is denoted by $\int_{\mathcal{S}} f\,d\sigma$, is defined by the formula

$$\int_{\mathcal{S}} f\,d\sigma \equiv \int_{\mathcal{R}} f\left(\mathbf{r}(u,\,v)\right)\left\|\frac{\partial \mathbf{r}}{\partial u}(u,\,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,\,v)\right\|\,du\,dv. \tag{20.13}$$

Example 20.17 Consider the set

$$\mathcal{S} = \{(x,\,y,\,z) \text{ in } \mathbb{R}^3 \mid x>0,\ y>0,\ z>0,\ x^2+y^2+z^2=1.\}$$

We showed in Example 20.15 that $\mathcal{S}$ is a surface parametrized by $\mathbf{r}:\mathcal{R}\to\mathbb{R}^3$, where $\mathcal{R}=\{(u,\,v) \text{ in } \mathbb{R}^2 \mid 0<u<\pi/2,\ 0<v<\pi/2\}$, defined by

$$\mathbf{r}(u,\,v) = (\sin u\cos v,\ \sin u\sin v,\ \cos u) \qquad \text{for } (u,\,v) \text{ in } \mathcal{R}.$$

A short computation of partial derivatives and of the cross-product shows that for each $(u,\,v)$ in $\mathcal{R}$,

$$\left\|\frac{\partial \mathbf{r}}{\partial u}(u,\,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,\,v)\right\| = \sin u.$$

Thus, by the definition of area and Fubini's Theorem,

$$\begin{aligned}
\text{area } \mathcal{S} &= \int_{\mathcal{R}} \left\|\frac{\partial \mathbf{r}}{\partial u}(u,\,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,\,v)\right\|\,du\,dv \\
&= \int_{\mathcal{R}} \sin u\,du\,dv \\
&= \int_0^{\pi/2}\left[\int_0^{\pi/2} \sin u\,du\right]dv \\
&= \int_0^{\pi/2} 1\,dv = \frac{\pi}{2}.
\end{aligned}$$

From this we conclude that the surface area of the sphere about the origin of radius 1 is 4π. ∎

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As we have already seen, a surface always has different parametrizations. It is certainly desirable that the definition of surface integral, and in particular that of surface area, be independent of the choice of parametrization of the surface. For instance, the two definitions we have given of surface area for a projectionally parametrized surface should coincide. We now establish the independence of parametrization of the surface integral.

Theorem 20.18 Let $\mathcal{R}$ and $\mathcal{R}'$ be regions in the plane $\mathbb{R}^2$. Let $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ and $\mathbf{r}':\mathcal{R}' \to \mathbb{R}^3$ be parametrized surfaces with the same image $\mathcal{S}$. Suppose that the function $f:\mathcal{S} \to \mathbb{R}$ is continuous and bounded. Then

$$\begin{aligned}\int_{\mathcal{R}} f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\| du\,dv \\ = \int_{\mathcal{R}'} f(\mathbf{r}'(u', v')) \left\| \frac{\partial \mathbf{r}'}{\partial u'}(u', v') \times \frac{\partial \mathbf{r}'}{\partial u'}(u', v') \right\| du'dv'.\end{aligned} \tag{20.14}$$

Proof

By the definition of parametrized surface, each of the mappings $\mathbf{r}:\mathcal{R} \to \mathbb{R}^3$ and $\mathbf{r}':\mathcal{R}' \to \mathbb{R}^3$ is one-to-one and by assumption each has an image equal to $\mathcal{S}$. For a point (u, v) in $\mathcal{R}$, define $\mathbf{g}(u, v) = (u', v')$ to be the unique point in $\mathcal{R}'$ at which

$$\mathbf{r}'(u', v') = \mathbf{r}(u, v). \tag{20.15}$$

This defines a mapping $\mathbf{g}:\mathcal{R} \to \mathbb{R}^2$ that is one-to-one and has an image equal to $\mathcal{R}'$. If we write out the parametrized surfaces and the mapping $\mathbf{g}:\mathcal{R} \to \mathbb{R}^2$ in terms of their component functions, it is clear that formula (20.15) is equivalent to the following system of identities:

$$\begin{aligned} r_1'(g_1(u, v), g_2(u, v)) &= r_1(u, v), \\ r_2'(g_1(u, v), g_2(u, v)) &= r_2(u, v), \\ r_3'(g_1(u, v), g_2(u, v)) &= r_3(u, v). \end{aligned} \tag{20.16}$$

We claim that the mapping $\mathbf{g}:\mathcal{R} \to \mathbb{R}^2$ is a smooth change of variables, meaning that it is continuously differentiable and has an invertible derivative matrix at each point. Let (u_0, v_0) be a point in $\mathcal{R}$. By assumption,

$$\frac{\partial \mathbf{r}'}{\partial u}(\mathbf{g}(u_0, v_0)) \times \frac{\partial \mathbf{r}'}{\partial v}(\mathbf{g}(u_0, v_0)) \neq \mathbf{0},$$

and we can suppose that it is the last component of this cross-product that is nonzero. Thus, we can apply the Inverse Function in the Plane to the mapping $(r_1', r_2'):\mathcal{R} \to \mathbb{R}^2$ at the point $\mathbf{g}(u_0, v_0)$ to conclude that there is a neighborhood of $\mathbf{g}(u_0, v_0)$ on which the mapping $(r_1', r_2'):\mathcal{R} \to \mathbb{R}^2$ has a continuously differentiable inverse. From the first two equations of the system (20.16), it follows that there is a neighborhood $\mathcal{N}$

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of (u_0, v_0) on which

$$(g_1, g_2) = (r_1', r_2')^{-1} \circ (r_1, r_2).$$

Since the composition of continuously differentiable mappings is also continuously differentiable, it follows that $\mathbf{g}: \mathcal{R} \to \mathbb{R}^2$ is continuously differentiable. It remains to verify that at each point in $\mathcal{R}$ the derivative matrix of $\mathbf{g}: \mathcal{R} \to \mathbb{R}^2$ is invertible. This follows from the following identity: For each point (u, v) in $\mathcal{R}$,

$$\left\| \frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\| = \left\| \frac{\partial \mathbf{r}'}{\partial u'}(\mathbf{g}(u, v)) \times \frac{\partial \mathbf{r}'}{\partial v'}(\mathbf{g}(u, v)) \right\| \cdot \left| \det \mathbf{Dg}(u, v) \right|. \tag{20.17}$$

This identity is a consequence of the Chain Rule. Indeed, the system of identities (20.16) means that $\mathbf{r}: \mathcal{R} \to \mathbb{R}^3$ is the composition $\mathbf{r}' \circ \mathbf{g}: \mathcal{R} \to \mathbb{R}^3$. By the Chain Rule,

$$\mathbf{Dr}(u, v) = \mathbf{Dr}'(\mathbf{g}(u, v)) \cdot \mathbf{Dg}(u, v) \qquad \text{for all } (u, v) \text{ in } \mathcal{R}.$$

This is an equality between two 3×2 matrices. Equating all the 2×2 submatrices in this identity and using the product property of 2×2 determinants (Exercise 14), it follows that for each index $i = 1, 2, 3$ and each point (u, v) in $\mathcal{R}$,

$$\left\langle \frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v), \mathbf{e}_i \right\rangle = \left\langle \frac{\partial \mathbf{r}'}{\partial u'}(u', v') \times \frac{\partial \mathbf{r}'}{\partial v'}(u', v'), \mathbf{e}_i \right\rangle \cdot \det \mathbf{Dg}(u, v);$$

that is,

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) = \left(\frac{\partial \mathbf{r}'}{\partial u'}(u', v') \times \frac{\partial \mathbf{r}'}{\partial v'}(u', v') \right) \cdot \det \mathbf{Dg}(u, v). \tag{20.18}$$

We take the norm of each side to obtain the identity (20.17).

Finally, now that we have established that the mapping $\mathbf{g}: \mathcal{R} \to \mathbb{R}^2$ is a smooth change of variables, we can apply the change of variables formula (19.11) of Section 19.2 to evaluate the integral on the right-hand side of (20.14). Using formula (20.17) and the change of variables formula, we have

$$\begin{aligned}
&\int_{\mathcal{R}'} f(\mathbf{r}'(u', v')) \left\| \frac{\partial \mathbf{r}'}{\partial u'}(u', v') \times \frac{\partial \mathbf{r}'}{\partial v'}(u', v') \right\| du'\, dv' \\
&= \int_{\mathcal{R}} f(\mathbf{r}'(\mathbf{g}(u, v)) \left\| \frac{\partial \mathbf{r}'}{\partial u'}(\mathbf{g}(u, v)) \times \frac{\partial \mathbf{r}'}{\partial v'}(\mathbf{g}(u, v)) \right\| \cdot |\det \mathbf{Dg}(u, v)|\, du\, dv \\
&= \int_{\mathcal{R}} f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\| du\, dv.
\end{aligned}$$

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EXERCISES FOR SECTION 20.2

1. Let a, b, c, and d be numbers with $c \neq 0$ and consider the plane

$$ax + by + cz + d = 0.$$

Parametrize the plane by $\mathbf{r}(u, v) = (u, v, -(au + bv + d)/c)$ and use the formula

$$\eta = \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$$

to find a normal to the plane at each point (x, y, z).

2. Find a unit normal vector at each point on the surface

$$\mathcal{S} = \{(x, y, z) \mid z = x^2 + y^2, |x| + |y| < 4\}.$$

3. For $a > 0$ and $b > 0$, show that the cylindrical set

$$\mathcal{S} = \{(x, y, z) \mid 0 < x < a, z > 0, y^2 + z^2 = b^2\}$$

is a surface and find its surface area.

4. For $r > 0$ and $h > 0$, show that the conical set

$$\mathcal{S} = \{(x, y, z) \mid x > 0, y > 0, z > 0, z^2 = h^2(r^2 - x^2 - y^2)\}$$

is a surface and find its surface area.

5. Find the surface area of the plane

$$\mathcal{S} = \{(x, y, z) \mid x > 0, y > 0, z > 0, 2x + y + z = 16\}.$$

6. Show that

$$\iint_{\mathcal{S}} (x^2 + y^2)\, d\sigma = \frac{9\pi}{4},$$

where

$$\mathcal{S} = \{(x, y, z) \mid x > 0, y > 0, 3 > z > 0, z^2 = 3(x^2 + y^2)\}.$$

7. Compute

$$\iint_{\mathcal{S}} (x^2y^2 + y^2z^2 + z^2x^2)\, d\sigma,$$

where $\mathcal{S}$ is the portion of the cone $\{(x, y, z) \mid x^2 + y^2 = z^2, z > 0\}$ cut off by the cylinder $\{(x, y, z) \mid x^2 + y^2 - 2x = 0\}$.

8. For $0 < a < b$, a surface of the form

$$\{(u \cos v, u \sin v, g(u)) \mid a < u < b, 0 < v < 2\pi\},$$

where the function $g : (a, b) \to \mathbb{R}$ has a bounded continuous derivative, is called a *surface of revolution*. Show that the area of a surface of revolution is given by the formula

$$2\pi \int_a^b u\sqrt{1 + (g'(u))^2}\, du.$$

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9. Suppose that $\mathcal{R}$ is a convex region in the plane and that the function $g : \mathcal{R} \to \mathbb{R}$ has continuous bounded partial derivatives. Show that the surface $\mathcal{S} = \{(x, y, g(x, y)) \mid (x, y) \text{ in } \mathcal{R}\}$ has area equal to that of $\mathcal{R}$ if and only if the function $g : \mathcal{R} \to \mathbb{R}$ is constant.

10. For any two vectors **A** and **B** in $\mathbb{R}^3$, show that

$$\|\mathbf{A} \times \mathbf{B}\|^2 = \|\mathbf{A}\|^2 \cdot \|\mathbf{B}\|^2 - \langle \mathbf{A}, \mathbf{B}\rangle^2.$$

(*Hint:* Follow the verification of formula (20.8).)

11. Use formula (20.18) to show that the definition of normal to a surface is independent of the choice of parametrization.

12. For a parametrized surface $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$, at each point (u, v) in $\mathcal{R}$, define the real-valued functions $E : \mathcal{R} \to \mathbb{R}$, $F : \mathcal{R} \to \mathbb{R}$, and $G : \mathcal{R} \to \mathbb{R}$ by

$$E(u, v) = \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle, \quad F(u, v) = \left\langle \frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle, \quad \text{and} \quad G(u, v) = \left\langle \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle.$$

Use the identity in Exercise 10 to rewrite the formula for the surface area of the surface $\mathcal{S}$ parametrized by $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$ as

$$\text{area } \mathcal{S} = \int_{\mathcal{R}} \sqrt{EG - F^2}\, du\, dv.$$

13. Let **A**, **B**, **C**, and **D** be vectors in $\mathbb{R}^3$. Prove Lagrange's Identity:

$$\langle \mathbf{A} \times \mathbf{B}, \mathbf{C} \times \mathbf{D}\rangle = \langle \mathbf{A}, \mathbf{C}\rangle\langle \mathbf{B}, \mathbf{D}\rangle - \langle \mathbf{A}, \mathbf{D}\rangle\langle \mathbf{B}, \mathbf{C}\rangle.$$

(*Hint:* Prove this identity by showing that it is true if all four vectors are standard basis vectors and then use the linearity of the cross-product and of the scalar product to obtain the general case.)

14. For two 2×2 matrices **A** and **B**, by explicit calculation, show that

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}.$$

15. Let **A** and **C** be 3×2 matrices and let **B** be a 2×2 matrix such that $\mathbf{AB} = \mathbf{C}$. Prove the identity

$$\|\mathbf{A}_1 \times \mathbf{A}_2\| \cdot |\det \mathbf{B}| = \|\mathbf{C}_1 \times \mathbf{C}_2\|$$

where $\mathbf{A}_i$ and $\mathbf{C}_i$ are the ith columns of **A** and **C**.

16. For a parametrized surface $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$ and a parameter value (u_0, v_0) in $\mathcal{R}$, show that there is a neighborhood $\mathcal{N}$ of (u_0, v_0) such that $\mathbf{r}(\mathcal{N})$ is the image of a projectionally parametrized surface. (*Hint:* Use the Inverse Function Theorem as in the proof of Theorem 20.18.)

17. Suppose that the function $h : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Let **p** be a point in $\mathbb{R}^3$ at which $\nabla h(\mathbf{p}) \neq \mathbf{0}$ and define $c = h(\mathbf{p})$. Use the Implicit Function Theorem to show that there is a neighborhood $\mathcal{N}$ of **p** such that $\mathcal{S} = \{(x, y, z) \text{ in } \mathcal{N} \mid h(x, y, z) = c\}$ is a surface.

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18. For a parametrized surface $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$ and a continuous bounded function $f : S \to \mathbb{R}$, where S is the image of $\mathbf{r} : \mathcal{R} \to \mathbb{R}^3$, prove that

$$\left| \int_S f \, d\sigma \right| \leq MA,$$

where M is such that $|f(\mathbf{p})| \leq M$ for all points $\mathbf{p}$ on S and A is the surface area of S.

20.3 THE INTEGRAL FORMULAS OF GREEN AND STOKES

For a closed bounded interval $[a, b]$ in $\mathbb{R}$ and a continuous function $f : [a, b] \to \mathbb{R}$ that has a continuous bounded derivative on the open interval (a, b), the First Fundamental Theorem of Calculus (Integrating Derivatives) asserts that

$$\int_a^b f'(t) \, dt = f(b) - f(a). \tag{20.19}$$

There are significant generalizations of this formula for functions of several variables. In this section, we extend this formula to the case where the open interval (a, b) is replaced by certain regions $\mathcal{R}$ in the plane $\mathbb{R}^2$ (Green's Formula) and to the case where (a, b) is replaced by certain surfaces $\mathcal{S}$ in $\mathbb{R}^3$ (Stokes's Formula).

A parametrized path $\gamma : [a, b] \to \mathbb{R}^n$ is said to be *simple* provided that its restriction to the interval $[a, b)$ is one-to-one; it is said to be *closed* provided that $\gamma(a) = \gamma(b)$. We are concerned here with open bounded subsets Ω of the plane $\mathbb{R}^2$ with the property that the boundary of Ω, $\partial\Omega$, is the image, $\mathbf{\Gamma}$, of a simple closed parametrized path.

Example 20.19 Suppose that the functions $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ are continuous and that $g(x) < h(x)$ for all x in (a, b). Define

$$\Omega = \{(x, y) \mid a < x < b, \; g(x) < y < h(x)\} \qquad \text{and} \qquad \gamma = \partial\Omega.$$

This set was considered in Example 20.8. In the case where $g(a) < h(a)$ and $g(b) < h(b)$, the parametrization of γ given in Example 20.4 is a parametrization by a simple closed parametrized path. In the case of equality at an end, we omit the path corresponding to the end and again obtain a simple closed parametrization of the boundary. For the obvious geometric reason, this parametrization, or a simple closed parametrization equivalent to it, is called a *counterclockwise parametrization* of the boundary of Ω, $\mathbf{\Gamma}$. ■

Example 20.20 Suppose that the functions $g : [c, d] \to \mathbb{R}$ and $h : [c, d] \to \mathbb{R}$ are continuous and that $g(y) < h(y)$ for all y in (c, d). Define

$$\Omega = \{(x, y) \mid g(y) < x < h(y), \; c < y < d\} \qquad \text{and} \qquad \gamma = \partial\Omega.$$

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Again Ω is a bounded open subset of $\mathbb{R}^2$ whose boundary can be parametrized by a simple closed parametrized path. Indeed, in the case where $g(c) < h(c)$ and $g(d) < h(d)$, the following defines such a parametrization:

$$\gamma(t) = \begin{cases} (g(d + 4t(c-d)),\ d + 4t(c-d)) & \text{if } 0 \le t \le 1/4 \\ (g(c) + 4(t - 1/4)(h(c) - g(c)),\ c) & \text{if } 1/4 \le t \le 1/2 \\ (h(c - 4(t - 1/2)(c-d)),\ c - 4(t-1/2)(c-d)) & \text{if } 1/2 \le t \le 3/4 \\ (h(d) + 4(t - 3/4)(g(d) - h(d)),\ d) & \text{if } 3/4 \le t \le 1. \end{cases}$$

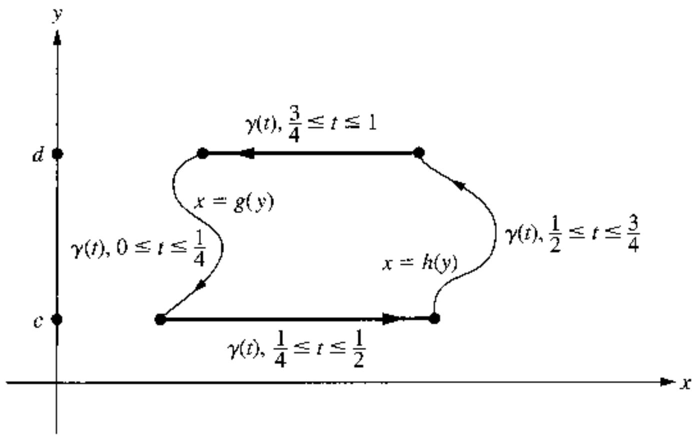

FIGURE 20.6 An explicit parametrization of $\partial\Omega$.

Again for the obvious geometric reason, this parametrization, or a simple closed parametrization equivalent to it, is called a counterclockwise parametrization of the boundary of Ω, $\mathbf{\Gamma}$. ∎

Proposition 20.21 Suppose that the functions $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ are continuous and that $g(x) < h(x)$ for all x in (a, b). Define

$$\Omega = \{(x, y) \mid a < x < b,\ g(x) < y < h(x)\} \qquad \text{and} \qquad \mathbf{\Gamma} = \partial\Omega.$$

Let the function $N : \Omega \cup \mathbf{\Gamma} \to \mathbb{R}$ be continuous and such that $\partial N/\partial y : \Omega \to \mathbb{R}$ exists and is both continuous and bounded. Then

$$\iint_\Omega \frac{\partial N}{\partial y}(x, y)\, dx\, dy = -\int_\Gamma N(x, y)\, dx, \tag{20.20}$$

where the right-hand integral is computed with respect to a counterclockwise parametrization of the path γ.

Proof

The heart of the proof consists in using Fubini's Theorem so that we can apply the First Fundamental Theorem of Calculus (Integrating Derivatives). First, it is

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convenient to express $\mathbf{\Gamma}$ as

$$\mathbf{\Gamma} = \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_2 \cup \mathbf{\Gamma}_3 \cup \mathbf{\Gamma}_4,$$

where $\mathbf{\Gamma}_1 = \{(x, g(x)) \mid a \le x \le b\}$, $\mathbf{\Gamma}_2 = \{(b, y) \mid g(b) \le y \le h(b)\}$, $\mathbf{\Gamma}_3 = \{(x, h(x)) \mid a \le x \le b\}$, and $\mathbf{\Gamma}_4 = \{(a, y) \mid g(a) \le y \le h(a)\}$. Using Fubini's Theorem, the First Fundamental Theorem of Calculus, and the very definition of line integral, we have

$$\begin{aligned}
\iint_\Omega \left[\frac{\partial N}{\partial y}(x, y)\right] dx\, dy &= \int_a^b \left\{\int_{g(x)}^{h(x)} \frac{\partial N}{\partial y}(x, y)\, dy\right\} dx \\
&= \int_a^b \left\{N(x, h(x)) - N(x, g(x))\right\} dx \\
&= \int_a^b N(x, h(x))\, dx - \int_a^b N(x, g(x))\, dx \\
&= -\left\{\int_a^b N(x, g(x))\, dx - \int_a^b N(x, h(x))\, dx\right\} \\
&= -\left\{\int_{\mathbf{\Gamma}_1} N\, dx + 0 + \int_{\mathbf{\Gamma}_3} N\, dx + 0\right\} \\
&= -\left\{\int_{\mathbf{\Gamma}_1} N\, dx + \int_{\mathbf{\Gamma}_2} N\, dx + \int_{\mathbf{\Gamma}_3} N\, dx + \int_{\mathbf{\Gamma}_4} N\, dx\right\} \\
&= -\int_{\mathbf{\Gamma}} N\, dx. \qquad \blacksquare
\end{aligned}$$

Proposition 20.22 Suppose that the functions $g : [c, d] \to \mathbb{R}$ and $h : [c, d] \to \mathbb{R}$ are continuous and that $g(y) < h(y)$ for all y in (c, d). Define

$$\Omega = \{(x, y) \mid g(y) < x < h(y),\ c < y < d\} \qquad \text{and} \qquad \mathbf{\Gamma} = \partial\Omega.$$

Let the function $M : \Omega \cup \mathbf{\Gamma} \to \mathbb{R}$ be continuous and such that $\partial M/\partial x : \Omega \to \mathbb{R}$ exists and is both continuous and bounded. Then

$$\iint_\Omega \frac{\partial M}{\partial x}(x, y)\, dx\, dy = \int_{\mathbf{\Gamma}} M(x, y)\, dy, \tag{20.21}$$

where the right-hand integral is computed with respect to a counterclockwise parametrization of the path $\mathbf{\Gamma}$.

Proof

As in the proof of Proposition 20.21, we use Fubini's Theorem in order to apply the First Fundamental Theorem of Calculus (Integrating Derivatives). Express $\mathbf{\Gamma}$ as $\mathbf{\Gamma} = \mathbf{\Gamma}_1 \cup \mathbf{\Gamma}_2 \cup \mathbf{\Gamma}_3 \cup \mathbf{\Gamma}_4$, where $\mathbf{\Gamma}_1 = \{(g(y), y) \mid c \le y \le d\}$, $\mathbf{\Gamma}_2 = \{(x, c) \mid g(c) \le x \le h(c)\}$, $\mathbf{\Gamma}_3 = \{(h(y), y) \mid c \le y \le d\}$, and $\mathbf{\Gamma}_4 = \{(x, d) \mid g(d) \le x \le h(d)\}$. Using Fubini's Theorem, the First Fundamental Theorem of Calculus,

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and the very definition of a line integral, we have

$$\begin{aligned}
\iint_\Omega \left[\frac{\partial M}{\partial x}(x, y)\right] dx\,dy &= \int_c^d \left\{\int_{g(y)}^{h(y)} \frac{\partial M}{\partial x}(x, y)\,dx\right\} dy \\
&= \int_c^d \left\{M(h(y), y) - M(g(y), y)\right\} dy \\
&= \int_c^d M(h(y), y)\,dy - \int_c^d M(g(y), y)\,dy \\
&= \int_{\Gamma_3} M\,dy + 0 + \int_{\Gamma_1} M\,dy + 0 \\
&= \int_{\Gamma_1} M\,dy + \int_{\Gamma_2} M\,dy + \int_{\Gamma_3} M\,dy + \int_{\Gamma_4} M\,dy \\
&= \int_{\Gamma} M\,dy.
\end{aligned}$$

■

Definition A region Ω in the plane $\mathbb{R}^2$ is called a *Green's domain* provided that its boundary $\boldsymbol{\Gamma} = \partial\Omega$ is the image of a simple closed piecewise smooth parametrized path $\gamma : I \to \mathbb{R}^2$ for which the following property holds: Suppose that the two functions $M : \Omega \cup \boldsymbol{\Gamma} \to \mathbb{R}$ and $N : \Omega \cup \boldsymbol{\Gamma} \to \mathbb{R}$ are continuous and that their restrictions $M : \Omega \to \mathbb{R}$ and $N : \Omega \to \mathbb{R}$ are continuously differentiable and have bounded partial derivatives. Then the following formula holds.

Green's Formula

$$\iint_\Omega \left[\frac{\partial M}{\partial x}(x, y) - \frac{\partial N}{\partial y}(x, y)\right] dx\,dy = \int_{\Gamma} [M(x, y)\,dy + N(x, y)\,dx], \tag{20.22}$$

where the right-hand integral is computed with respect to $\gamma : I \to \mathbb{R}^2$. We call such a parametrization a *Green's parametrization*.

By first taking $M(x, y)$ identically equal to 0 and then taking $N(x, y)$ identically equal to 0, we see that for a domain to be a Green's domain it is necessary only to verify separately the formulas

$$\iint_\Omega \frac{\partial N}{\partial y}(x, y)\,dx\,dy = -\int_{\Gamma} N(x, y)\,dx$$

and

$$\iint_\Omega \frac{\partial M}{\partial x}(x, y)\,dx\,dy = \int_{\Gamma} M(x, y)\,dy.$$

Proposition 20.21 and Proposition 20.22 provide a means for showing that domains, with appropriate parametrizations of their boundaries, are Green's domains.

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Example 20.23 Define $\Omega = \{(x, y) \mid x^2 + y^2 < 1\}$ so that its boundary $\mathbf{\Gamma}$ is the circle of radius 1 centered at the origin. It is clear that Ω is a domain to which both Proposition 20.21 and Proposition 20.22 apply, so it is a Green's domain for the counterclockwise parametrization $\gamma : [0, 2\pi] \to \mathbb{R}^2$ defined by $\gamma(\theta) = (\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$. Thus, for two continuous functions $M : \Omega \cup \mathbf{\Gamma} \to \mathbb{R}$ and $N : \Omega \cup \mathbf{\Gamma} \to \mathbb{R}$ whose restrictions $M : \Omega \to \mathbb{R}$ and $N : \Omega \to \mathbb{R}$ are continuously differentiable and have bounded partial derivatives, we have

$$\iint_{x^2+y^2\leq 1} \left[\frac{\partial M}{\partial x}(x, y) - \frac{\partial N}{\partial y}(x, y)\right] dx\, dy$$
$$= \int_0^{2\pi} [M(\cos\theta, \sin\theta)\cos\theta - N(\cos\theta, \sin\theta)\sin\theta]\, d\theta.$$
■

Example 20.24 Define $\Omega = \{(x, y) = (r\cos\theta, r\sin\theta) \mid 0 < \theta < 3\pi/2, 0 < r < 1\}$. Then Ω is a Green's domain with respect to the counterclockwise parametrization, although it is not a set to which either Proposition 20.21 or Proposition 20.22 directly applies. However, setting

$$\Omega_+ = \{(x, y) \text{ in } \Omega \mid x > 0\} \qquad \text{and} \qquad \Omega_- = \{(x, y) \text{ in } \Omega \mid x < 0\},$$

we see that both Ω_+ and Ω_- are Green's domains since both Proposition 20.21 and Proposition 20.22 apply to each of these sets. By adding Green's Formula for each of these sets, we obtain Green's Formula for Ω because, with respect to the counterclockwise parametrization of each, the contributions of the line integrals along the common boundary cancel.

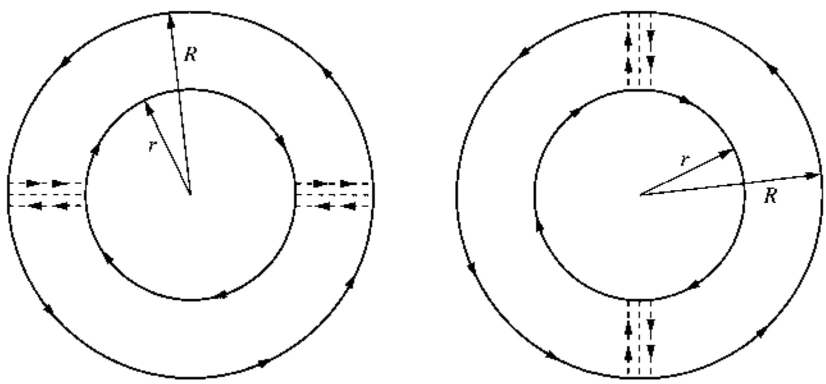

FIGURE 20.7 A cut annulus is a Green's domain.
■

Example 20.25 For $0 < r < R$, consider the annulus $\Omega = \{(x, y) \mid r^2 < x^2 + y^2 < R^2\}$. The boundary of this annulus is the union of two circles. Suppose that the two functions $M : \Omega \cup \partial\Omega \to \mathbb{R}$ and $N : \Omega \cup \partial\Omega \to \mathbb{R}$ are continuous and that their restrictions $M : \Omega \to \mathbb{R}$ and $N : \Omega \to \mathbb{R}$ are continuously differentiable and have

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bounded partial derivatives. Then

$$\iint_{r^2<x^2+y^2<R^2}\left[\frac{\partial M}{\partial x}(x,\ y)-\frac{\partial N}{\partial y}(x,\ y)\right]dx\,dy$$
$$=\int_{x^2+y^2=R}[M(x,\ y)\,dy+N(x,\ y)\,dx] \tag{20.23}$$
$$-\int_{x^2+y^2=r}[M(x,\ y)\,dy+N(x,\ y)\,dx],$$

where both integrals are computed with respect to a counterclockwise parametrization. To verify this, it suffices to verify the formula first when $M(x,\ y)$ is identically 0 and then when $N(x,\ y)$ is identically 0. In the case where $M(x,\ y)$ is identically 0, use the x-axis to divide the annulus into two domains Ω_+ and Ω_-, to each of which Proposition 20.21 applies. Apply Proposition 20.21 to each domain and sum the resulting integral equalities. Because of the choice of counterclockwise parametrization of each line integral, the contributions to the line integrals over the boundaries lying on the x-axis cancel out and we obtain the above formula in the case where $M(x,\ y)$ is identically 0. A similar computation, but now bisecting the annulus with the y-axis and using Proposition 20.22, establishes the above formula in the case where $N(x,\ y)$ is identically 0. ■

For a Green's domain Ω with boundary $\mathbf{\Gamma}$, taking

$$M(x,\ y)\equiv x \qquad \text{and} \qquad N(x,\ y)\equiv -y,$$

we obtain the following formula for the area of Ω:

$$\text{area}\,\Omega=\frac{1}{2}\int_{\mathbf{\Gamma}}[x\,dy-y\,dx], \tag{20.24}$$

where the line integral is computed with respect to the Green's parametrization of $\mathbf{\Gamma}$.

Example 20.26 The elliptical region $\Omega=\{(x,\ y)\mid x^2/a^2+y^2/b^2\le 1\}$ is a Green's domain since it is an open bounded subset of $\mathbb{R}^2$ to which both Proposition 20.21 and Proposition 20.22 apply. The above area formula gives

$$\text{area}\,\Omega=\frac{1}{2}\int_0^{2\pi}[a\cos\theta(b\cos\theta)-b\sin\theta(-a\sin\theta)]\,d\theta$$
$$=\frac{ab}{2}\int_0^{2\pi}d\theta=\pi ab. \qquad ■$$

The examples of Green's domains, with associated parametrizations of the boundary, exhibited above have all been built out of the types of domains described in Propositions 20.21 and 20.22. There is, in fact, a very general condition for an open bounded subset of $\mathbb{R}^2$ and an associated parametrization of its boundary to be a Green's domain: Let Ω be an open bounded subset of $\mathbb{R}^2$ whose boundary is the image of a simple

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closed piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^2$ with the property that

$$\int_{\Gamma:[a,b]\to\mathbb{R}^2} [x\,dy - y\,dx] > 0. \tag{20.25}$$

Then Ω, with this parametrization, is a Green's domain. The intuitive indication of the reason this result is true comes from "patching" Ω together from domains to which Proposition 20.21 applies and also from a similar patching of domains to which Proposition 20.22 applies and then observing that the line integrals along paths that are common boundaries to two patches cancel out. It is quite a different matter to provide a rigorous proof; a completely precise proof involves considerable technical detail. We will not prove the general result here. The area formula (20.24) is what motivates the assumption on the parametrization that (20.25) holds. In fact, (20.25) is taken to be the analytic definition of what it means for a simple closed piecewise smooth parametrized path $\gamma : [a, b] \to \mathbb{R}^2$ to be a counterclockwise parametrization.

Suppose that Ω is a Green's domain with boundary $\mathbf{\Gamma}$, having an associated Green's parametrization by arclength $\gamma : I \to \mathbb{R}^2$. Fix a parameter value s at which γ is differentiable. We define $\boldsymbol{\eta}(s) = (\gamma_2'(s), -\gamma_1'(s))$. Observe that

$$\|\gamma'(s)\| = 1, \qquad \|\boldsymbol{\eta}(s)\| = 1, \qquad \text{and} \qquad \langle \gamma'(s), \boldsymbol{\eta}(s)\rangle = 0,$$

so the vector $\mathbf{N} \equiv \boldsymbol{\eta}(s)$ is a unit vector perpendicular to the unit tangent vector $\mathbf{T} \equiv \gamma'(s)$. The direction of this normal vector is determined by the parametrization γ, and in the specific examples we have seen, its geometric meaning is that it is the "outward-pointing" normal. It is convenient to refer to this parametrization of normals as being associated with the Green's parametrization of the boundary of Ω. For a function $w : \mathcal{N} \to \mathbb{R}$ continuously differentiable in a neighborhood $\mathcal{N}$ of the point $\gamma(s) \equiv \mathbf{p}$, recall that in Section 13.3 we established the Directional Derivative Lemma, which provided the following formula for directional derivatives of the function w at the point $\mathbf{p}$ in the direction $\mathbf{q}$:

$$\frac{\partial w}{\partial \mathbf{q}}(\mathbf{p}) \equiv \lim_{t\to 0} \frac{w(\mathbf{p} + t\mathbf{q}) - w(\mathbf{p})}{t} = \langle \nabla w(\mathbf{p}), \mathbf{q}\rangle.$$

Thus, we have the following formula for the directional derivatives in the directions $\mathbf{T}$ and $\mathbf{N}$, which are called, respectively, the tangential and normal derivatives of the function $w : \mathcal{N} \to \mathbb{R}$ at the point $\mathbf{p}$:

$$\frac{\partial w}{\partial \mathbf{T}}(\mathbf{p}) = \langle \nabla w(\mathbf{p}), \mathbf{T}\rangle \qquad \text{and} \qquad \frac{\partial w}{\partial \mathbf{N}}(\mathbf{p}) = \langle \nabla w(\mathbf{p}), \mathbf{N}\rangle.$$

It is customary to denote the normal derivative by $\partial w/\partial \boldsymbol{\eta}$, so that, in particular,

$$\frac{\partial x}{\partial \boldsymbol{\eta}} = \gamma_2' \qquad \text{and} \qquad \frac{\partial y}{\partial \boldsymbol{\eta}} = -\gamma_1'.$$

The First Fundamental Theorem of Calculus (Integrating Derivatives) and the product formula for differentiation provided the integration by parts formula for functions of a single variable. We now use Green's Formula and the product formula for differentiation to provide the following integration by parts formula for functions of two variables.

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Corollary 20.27 Integration by Parts Let Ω be a Green's domain with boundary Γ. Suppose that the functions $a(x, y)$, $b(x, y)$, $u(x, y)$, and $v(x, y)$ have continuous bounded partial derivatives on an open set $\mathcal{O}$ containing $\Omega \cup \Gamma$. Then

$$\iint_\Omega \left[a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial y}\right] dx\,dy = \int_\Gamma \left[au\frac{\partial x}{\partial \eta} + bv\frac{\partial y}{\partial \eta}\right] ds - \iint_\Omega \left[\frac{\partial a}{\partial x}u + \frac{\partial b}{\partial y}v\right] dx\,dy,$$

where the normal parametrization is that associated with the Green's parametrization of the boundary of Ω.

Proof

Define

$$M(x, y) = a(x, y)u(x, y) \quad \text{and} \quad N(x, y) = -b(x, y)v(x, y) \quad \text{for } (x, y) \text{ in } \mathcal{O}.$$

By Green's Formula, we have

$$\begin{aligned}\iint_\Omega &\left[\frac{\partial(au)}{\partial x}(x, y) + \frac{\partial(bv)}{\partial y}(x, y)\right] dx\,dy \\ &= \int_\Gamma [a(x, y)u(x, y)\,dy - b(x, y)v(x, y)\,dx].\end{aligned}$$

Using the product rule for differentiation, the left-hand side of this formula becomes

$$\iint_\Omega \left[a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial y}\right] dx\,dy + \iint_\Omega \left[\frac{\partial a}{\partial x}u + \frac{\partial b}{\partial y}v\right] dx\,dy,$$

whereas for the right-hand side, by the definition of the line integral and the choice of normal parametrization, we have

$$\gamma_2' = \frac{\partial x}{\partial \eta} \quad \text{and} \quad -\gamma_1' = \frac{\partial y}{\partial \eta},$$

so that

$$\begin{aligned}\int_\gamma &[a(x, y)u(x, y)\,dy - b(x, y)v(x, y)\,dx] \\ &= \int_0^\ell [a(\gamma(s))u(\gamma(s))\gamma_2'(s) - b(\gamma(s))v(\gamma(s))\gamma_1'(s)]\,ds \\ &\equiv \int_\gamma \left[au\frac{\partial x}{\partial \eta} + bv\frac{\partial y}{\partial \eta}\right] ds.\end{aligned}$$

From these two formulas follows the integration by parts formula. ∎

We now raise Green's Formula from a surface $\mathcal{S} = \{(x, y, 0) \mid (x, y) \text{ in } \Omega\}$ that is a Green's domain in the plane $\mathbb{R}^2$ to a surface $\mathcal{S}$ in $\mathbb{R}^3$ that is parametrized by a

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Green's domain; the extension is called *Stokes's Formula.* In order to provide a geometric description of the integrals occurring in Stokes's Formula, it is useful to introduce the concept of a *vector field,* which is a geometric way of considering mappings from a subset of $\mathbb{R}^3$ into $\mathbb{R}^3$: For a subset D of $\mathbb{R}^3$ and a mapping $\mathbf{F}: D \to \mathbb{R}^3$, for each point $\mathbf{p}$ in D we can interpret $\mathbf{F}(\mathbf{p})$ as representing the vector associated with the segment from $\mathbf{p}$ to $\mathbf{p} + \mathbf{F}(\mathbf{p})$. Furthermore, if $\mathbf{u}: D \to \mathbb{R}^3$ is another vector field having the property that $\|\mathbf{u}(\mathbf{p})\| = 1$ for all $\mathbf{p}$ in D, then for each point $\mathbf{p}$ in D, it follows from the geometric description of the scalar product that the number $\langle \mathbf{F}(\mathbf{p}),\ \mathbf{u}(\mathbf{p})\rangle$ is the component of the vector $\mathbf{F}(\mathbf{p})$ in the direction $\mathbf{u}(\mathbf{p})$. Line integrals and surface integrals of functions of the form $\mathbf{p} \mapsto \langle \mathbf{F}(\mathbf{p}),\ \mathbf{u}(\mathbf{p})\rangle$ are essential ingredients of the extension of Green's Formula to surfaces in $\mathbb{R}^3$.

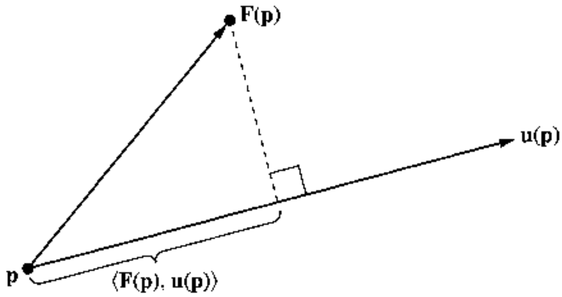

FIGURE 20.8 The projection of $\mathbf{F}$ along $\mathbf{p}$.

Definition For a piecewise smooth parametrized path $\gamma : [a,\ b] \to \mathbb{R}^3$ having image $\mathbf{\Gamma}$ and a continuous mapping $\mathbf{F}: \mathbf{\Gamma} \to \mathbb{R}^3$, we define

$$\int_{\Gamma} \langle \mathbf{F},\ \mathbf{T}\rangle\, ds \equiv \int_a^b \langle \mathbf{F}(\gamma(t)),\ \gamma'(t)\rangle\, dt. \tag{20.26}$$

Observe that if for each parameter value t in $(a,\ b)$ (except possibly at a finite number of parameter values) we define

$$\mathbf{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|},$$

then $\mathbf{T}(t)$ is a unit tangent to the path $\mathbf{\Gamma}$ at the point $\gamma(t)$ and

$$\int_a^b \langle \mathbf{F}(\gamma(t)),\ \gamma'(t)\rangle\, dt = \int_a^b \langle \mathbf{F}(\gamma(t)),\ \mathbf{T}(t)\rangle\, \|\gamma'(t)\|\, dt.$$

So this integral is the line integral of the tangential component of $\mathbf{F}$, where the choice of direction of the tangent is determined by the choice of parametrization of the path. The sign of the integral on the right-hand side of (20.26) depends on the choice of parametrization.

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Definition For a parametrized surface $\mathbf{r}:\mathcal{R}\to\mathbb{R}^3$ having image $\mathcal{S}$ and a continuous mapping $\mathbf{G}:\mathcal{S}\to\mathbb{R}^3$, we define

$$\iint_{\mathcal{S}}\langle \mathbf{G},\eta\rangle\, d\sigma \equiv \iint_{\mathcal{R}}\left\langle \mathbf{G}(\mathbf{r}(u,v)),\ \frac{\partial \mathbf{r}}{\partial u}(u,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,v)\right\rangle du\,dv. \tag{20.27}$$

Observe that if for each parameter value (u, v) in $\mathcal{R}$ we define

$$\eta(u,v)=\frac{\partial\mathbf{r}/\partial u(u,v)\times\partial\mathbf{r}/\partial v(u,v)}{\|\partial\mathbf{r}/\partial u(u,v)\times\partial\mathbf{r}/\partial v(u,v)\|},$$

then $\eta(u, v)$ is a unit normal to the surface $\mathcal{S}$ at the point $\mathbf{r}(u, v)$ and

$$\iint_{\mathcal{S}}\langle \mathbf{G},\eta\rangle\, d\sigma = \iint_{\mathcal{R}}\langle \mathbf{G}(\mathbf{r}(u,v)),\ \eta(u,v)\rangle\left\|\frac{\partial \mathbf{r}}{\partial u}(u,v)\times\frac{\partial \mathbf{r}}{\partial v}(u,v)\right\| du\,dv,$$

so the integral (20.27) is the surface integral over the surface $\mathcal{S}$ of the normal component of the vector field $\mathbf{G}$. Again, in this case, the sign of the integral on the right-hand side of (20.27) depends on the choice of parametrization.

Definition Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$ and suppose that the mapping $\mathbf{F}:\mathcal{U}\to\mathbb{R}^3$ is continuously differentiable. An associated mapping, called the *curl* of $\mathbf{F}$ and denoted by $\operatorname{curl}\mathbf{F}:\mathcal{U}\to\mathbb{R}^3$, is defined as follows: If we express $\mathbf{F}:\mathcal{U}\to\mathbb{R}^3$ in components as

$$\mathbf{F}(x,y,z)=(f(x,y,z),\,g(x,y,z),\,h(x,y,z))\qquad\text{for all } (x,y,z) \text{ in } \mathcal{O},$$

then

$$\operatorname{curl}\mathbf{F}(x,y,z)\equiv\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z},\ \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x},\ \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\qquad\text{for all } (x,y,z) \text{ in } \mathcal{O}.$$

It is not immediately apparent why the curl of a mapping should be of interest. We will soon see that the curl is an essential ingredient in lifting Green's Formula out of the plane.[2]

Example 20.28 Suppose that the mappings $M:\mathbb{R}^2\to\mathbb{R}$ and $N:\mathbb{R}^2\to\mathbb{R}$ are continuously differentiable. Define

$$\mathbf{F}(x,y,z)=(N(x,y),\,M(x,y),\,0)\qquad\text{for all } (x,y,z) \text{ in } \mathbb{R}^3.$$

Then it follows directly from the definition that

$$\operatorname{curl}\mathbf{F}(x,y,z)=\left(0,\,0,\,\frac{\partial M}{\partial x}(x,y)-\frac{\partial N}{\partial y}(x,y)\right)\qquad\text{for all } (x,y,z) \text{ in } \mathbb{R}^3.$$ ■

[2] In the study of physics and engineering, the curl is also an important operator that has specific physical significance in a number of different contexts. For instance, one of the most remarkable statements of physics, Maxwell's Equations, consists of assertions about the relationship between the curls of electric and magnetic vector fields.

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Directly from the definition of curl and the linearity of differentiation, it follows that the curl acts linearly; that is, for $\mathcal{U}$ an open subset of $\mathbb{R}^3$, two continuously differentiable mappings $\mathbf{F}:\mathcal{U} \to \mathbb{R}^3$ and $\mathbf{H}:\mathcal{U} \to \mathbb{R}^3$, and any two numbers α and β, we have

$$\text{curl}\,[\alpha\mathbf{F} + \beta\mathbf{H}] = \alpha\,\text{curl}\,\mathbf{F} + \beta\,\text{curl}\,\mathbf{H}.$$

The lifting of Green's Formula to surfaces in $\mathbb{R}^3$ relies on Green's Formula itself, together with the following identity.

Lemma 20.29 Stokes's Identity For $\mathcal{O}$ an open subset of the plane $\mathbb{R}^2$, suppose that the components of the mapping $\mathbf{r}:\mathcal{O} \to \mathbb{R}^3$ have continuous second-order partial derivatives. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$ containing the image of $\mathbf{r}:\mathcal{O} \to \mathbb{R}^3$ and suppose that the mapping $\mathbf{F}:\mathcal{U} \to \mathbb{R}^3$ is continuously differentiable. Then at each point (u, v) in $\mathcal{O}$, we have

$$\left\langle \text{curl}\,\mathbf{F}(\mathbf{r}(u, v)), \frac{\partial \mathbf{r}}{\partial u}(u, v) \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\rangle$$
$$= \frac{\partial}{\partial u}\left[\left\langle \mathbf{F}(\mathbf{r}(u, v)), \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\rangle\right] - \frac{\partial}{\partial v}\left[\left\langle \mathbf{F}(\mathbf{r}(u, v)), \frac{\partial \mathbf{r}}{\partial u}(u, v) \right\rangle\right].$$

Proof

First observe that both the left- and right-hand sides of Stokes's Identity depend linearly on the vector field $\mathbf{F}$. Thus, to verify the identity, it suffices to consider the three cases that occur when two of the component functions of the mapping $\mathbf{F}:\mathcal{U} \to \mathbb{R}^3$ are identically equal to 0. We verify the identity in the case where

$$\mathbf{F}(x, y, z) = (g(x, y, z), 0, 0) \qquad \text{for all } (x, y, z) \text{ in } \mathcal{U}. \tag{20.28}$$

The verification in the other two cases is entirely similar.

A direct computation shows that for $\mathbf{F}$ of the form (20.28),

$$\text{curl}\,\mathbf{F}(\mathbf{r}(u, v)) = \left(0, \frac{\partial g}{\partial z}(\mathbf{r}(u, v)), -\frac{\partial g}{\partial y}(\mathbf{r}(u, v))\right) \qquad \text{for all } (u, v) \text{ in } \mathcal{O}.$$

Therefore, writing $\partial\mathbf{r}/\partial u(u, v) \times \partial\mathbf{r}/\partial v(u, v)$ in components and taking the scalar product, the left-hand side of Stokes's Identity becomes

$$\frac{\partial g}{\partial z}(\mathbf{r}(u, v)) \left[\frac{\partial r_3}{\partial u}\frac{\partial r_1}{\partial v} - \frac{\partial r_1}{\partial u}\frac{\partial r_3}{\partial v}\right] - \frac{\partial g}{\partial y}(\mathbf{r}(u, v)) \left[\frac{\partial r_1}{\partial u}\frac{\partial r_2}{\partial v} - \frac{\partial r_2}{\partial u}\frac{\partial r_1}{\partial v}\right]. \tag{20.29}$$

To evaluate the right-hand side of Stokes's Identity, we first observe that since the mixed second-order partial derivatives of $r_1(u, v)$ are equal, the right-hand side equals

$$\frac{\partial}{\partial u}[g(\mathbf{r}(u, v))]\frac{\partial r_1}{\partial v}(u, v) - \frac{\partial}{\partial v}[g(\mathbf{r}(u, v))]\frac{\partial r_1}{\partial u}(u, v). \tag{20.30}$$

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Thus, to prove Stokes's Identity, we have to show that (20.29) equals (20.30). To do this, observe that by the Chain Rule,

$$\frac{\partial}{\partial u}[g(\mathbf{r}(u,v))] = \frac{\partial g}{\partial x}(\mathbf{r}(u,v))\frac{\partial r_1}{\partial u} + \frac{\partial g}{\partial y}(\mathbf{r}(u,v))\frac{\partial r_2}{\partial u} + \frac{\partial g}{\partial z}(\mathbf{r}(u,v))\frac{\partial r_3}{\partial u}$$

and

$$\frac{\partial}{\partial v}[g(\mathbf{r}(u,v))] = \frac{\partial g}{\partial x}(\mathbf{r}(u,v))\frac{\partial r_1}{\partial v} + \frac{\partial g}{\partial y}(\mathbf{r}(u,v))\frac{\partial r_2}{\partial v} + \frac{\partial g}{\partial z}(\mathbf{r}(u,v))\frac{\partial r_3}{\partial v}.$$

Substituting each of these partial derivatives in (20.30), it follows that (20.29) is indeed equal to (20.30). Therefore, Stokes's Identity is verified. ■

Lemma 20.30 For $\mathcal{O}$ an open subset of the plane $\mathbb{R}^2$, suppose that the mapping $\mathbf{r}:\mathcal{O}\to\mathbb{R}^3$ is continuously differentiable. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$ containing the image of $\mathbf{r}:\mathcal{O}\to\mathbb{R}^3$ and suppose that the mapping $\mathbf{F}:\mathcal{U}\to\mathbb{R}^3$ is also continuously differentiable. Let $\boldsymbol{\beta}:\mathrm{I}\to\mathbb{R}^2$ be a piecewise smooth parametrized path whose image, the path C, lies in $\mathcal{O}$ and consider the curve γ defined by the parametrized composition $\gamma\equiv\mathbf{r}\circ\boldsymbol{\beta}:\mathrm{I}\to\mathbb{R}^3$. Then

$$\int_\Gamma \langle \mathbf{F},\mathbf{T}\rangle\,ds = \int_C \left\langle \mathbf{F}(\mathbf{r}(u,v)), \frac{\partial \mathbf{r}}{\partial u}(u,v)\right\rangle du + \left\langle \mathbf{F}(\mathbf{r}(u,v)), \frac{\partial \mathbf{r}}{\partial v}(u,v)\right\rangle dv. \tag{20.31}$$

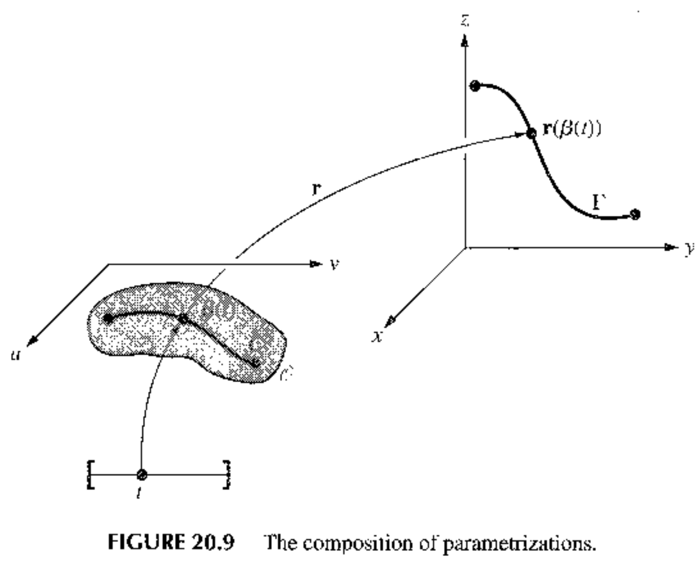

FIGURE 20.9 The composition of parametrizations.

Proof

By the Chain Rule,

$$\frac{d}{dt}(\gamma(t)) = \frac{d}{dt}(\mathbf{r}(\boldsymbol{\beta}(t))) = \frac{\partial \mathbf{r}}{\partial u}(\boldsymbol{\beta}(t))\frac{d\beta_1}{dt}(t) + \frac{\partial \mathbf{r}}{\partial v}(\boldsymbol{\beta}(t))\frac{d\beta_2}{dt}(t).$$

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Hence

$$\begin{aligned}\int_\Gamma \langle \mathbf{F}, \mathbf{T}\rangle\, ds &= \int_I \langle \mathbf{F}(\gamma(t)),\ \gamma'(t)\rangle\, dt\\ &= \int_I \Big[\Big\langle (\mathbf{F}\circ\mathbf{r})(\boldsymbol{\beta}(t)),\ \frac{\partial \mathbf{r}}{\partial u}(\boldsymbol{\beta}(t))\Big\rangle \frac{d\beta_1}{dt}(t)\\ &\qquad + \Big\langle (\mathbf{F}\circ\mathbf{r})(\boldsymbol{\beta}(t)),\ \frac{\partial \mathbf{r}}{\partial v}(\boldsymbol{\beta}(t))\Big\rangle \frac{d\beta_2}{dt}(t)\Big]\, dt\\ &= \int_C \Big\langle \mathbf{F}(\mathbf{r}(u, v)),\ \frac{\partial \mathbf{r}}{\partial u}(u,\ v)\Big\rangle\, du + \Big\langle \mathbf{F}(\mathbf{r}(u, v)),\ \frac{\partial \mathbf{r}}{\partial v}(u,\ v)\Big\rangle\, dv. \qquad \blacksquare\end{aligned}$$

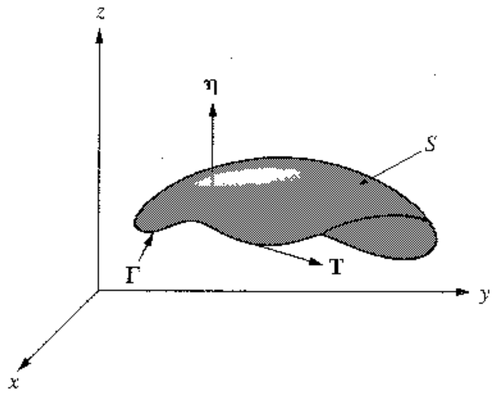

FIGURE 20.10 Stokes's Formula.

Theorem 20.31 Stokes's Formula For $\mathcal{O}$ an open subset of the plane $\mathbb{R}^2$, suppose that the components of the mapping $\mathbf{r}: \mathcal{O} \to \mathbb{R}^3$ have continuous second-order partial derivatives. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^3$ containing the image of $\mathbf{r}: \mathcal{O} \to \mathbb{R}^3$ and suppose that the mapping $\mathbf{F}: \mathcal{U} \to \mathbb{R}^3$ is continuously differentiable. Suppose that $\mathcal{R}$ is a Green's domain such that both $\mathcal{R}$ and its boundary are contained in $\mathcal{O}$. Let $\mathcal{S}$ be the surface defined by the parametrized surface $\mathbf{r}: \mathcal{R} \to \mathbb{R}^3$ and let $\boldsymbol{\Gamma}$ be the path that is the image of the parametrized path defined by the composition of $\mathbf{r}$ with the Green's parametrization of the boundary of $\mathcal{R}$. Then

$$\iint_{\mathcal{S}} \langle \operatorname{curl} \mathbf{F},\ \boldsymbol{\eta}\rangle\, d\boldsymbol{\sigma} = \int_{\boldsymbol{\Gamma}} \langle \mathbf{F},\ \mathbf{T}\rangle\, ds.$$

Proof

By the very definition of a surface integral,

$$\iint_{\mathcal{S}} \langle \operatorname{curl} \mathbf{F},\ \boldsymbol{\eta}\rangle\, d\boldsymbol{\sigma} = \iint_{\mathcal{R}} \Big\langle \operatorname{curl} \mathbf{F}(\mathbf{r}(u, v)),\ \frac{\partial \mathbf{r}}{\partial u}(u,\ v) \times \frac{\partial \mathbf{r}}{\partial v}(u,\ v)\Big\rangle\, du\, dv.$$

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Therefore, by Stokes's Identity,

$$\iint_S \langle \operatorname{curl} \mathbf{F}, \boldsymbol{\eta} \rangle \, d\sigma = \iint_{\mathcal{R}} \left\{ \frac{\partial}{\partial u} \left[\left\langle \mathbf{F} \circ \mathbf{r}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle \right] - \frac{\partial}{\partial v} \left[\left\langle \mathbf{F} \circ \mathbf{r}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle \right] \right\} du \, dv.$$

Thus, by Green's Formula applied to the integral on the right-hand side,

$$\iint_S \langle \operatorname{curl} \mathbf{F}, \ \boldsymbol{\eta} \rangle \, d\sigma = \int_C \left\langle \mathbf{F} \circ \mathbf{r}, \frac{\partial \mathbf{r}}{\partial u} \right\rangle du + \left\langle \mathbf{F} \circ \mathbf{r}, \frac{\partial \mathbf{r}}{\partial v} \right\rangle dv,$$

where C is the boundary path of $\mathcal{R}$. Finally, by the integral formula (20.31) applied to the right-hand side,

$$\iint_S \langle \operatorname{curl} \mathbf{F}, \ \boldsymbol{\eta} \rangle \, d\sigma = \int_{\mathbf{r}} \langle \mathbf{F}, \ \mathbf{T} \rangle \, ds. \qquad \blacksquare$$

In the language of vector fields, Stokes's Formula asserts that provided the parametrizations are appropriately chosen, the integral over a surface of the normal component of the curl of a vector field equals the integral along the boundary of the surface of the tangential component of the vector field.

EXERCISES FOR SECTION 20.3

1. Verify Green's Formula in the case where $M = x^2 - y^2$, $N = 2xy$, and Ω is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(1, 1)$.
2. Verify Green's Formula in the case where $M = x^2 - xy^3$, $N = y^2 - 2xy$, and Ω is the square with opposite vertices $(0, 0)$ and $(2, 2)$.
3. Use the area formula (20.24) to find the area of the triangle bounded by the line $x + y = 4$ and the coordinate axes.
4. Find simple closed parametrizations of the boundary of the set Ω defined in Example 20.20 in the cases where $g(c) = h(c)$ and/or $g(d) = h(d)$.
5. Let $\mathbf{p}$ and $\mathbf{q}$ be points in $\mathbb{R}^3$. Prove that

$$\operatorname{curl} [(\mathbf{p} - \mathbf{q}) \times ((x, \ y, \ z) - \mathbf{q})] = 2\mathbf{p} - 2\mathbf{q}.$$

6. Verify Stokes's Formula for $\mathbf{F}(x, \ y, \ z) = (3y, \ -xz, \ yz^2)$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by the plane $z = 2$.
7. Verify Stokes's Formula in the case where $\mathbf{F}(x, \ y, \ z) = (z, \ x, \ y)$ and S is the upper hemisphere of radius 1 centered at the origin.
8. Evaluate

$$\iint_S \langle \mathbf{F}, \boldsymbol{\eta} \rangle \, d\sigma,$$

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where $\mathbf{F}(x, y, z) = (xz, yz, z^2)$ and S is the upper hemisphere of radius 1 centered at the origin.

9. For a continuous function $f : [a, b] \to \mathbb{R}$ that has a continuous bounded derivative on the open interval (a, b), define $\Omega = \{(x, y) \mid a < x < b, 0 < y < 1\}$ and define $M(x, y) = f(x)$ and $N(x, y) = 0$, for (x, y) in Ω. Show that Green's Formula reduces to the formula $\int_a^b f'(t)\,dt = f(b) - f(a)$.

10. Show that Green's Formula is a special case of Stokes's Formula.

11. Suppose that Ω is a Green's domain with boundary $\mathbf{\Gamma}$. For functions $u(x, y)$ and $v(x, y)$ that are twice continuously differentiable on an open set containing $\Omega \cup \mathbf{\Gamma}$, show that

$$\int_{\mathbf{\Gamma}} \left[u\frac{\partial v}{\partial x}\,dx + u\frac{\partial v}{\partial y}\,dy\right] = \iint_{\Omega} \det \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}\,dx\,dy.$$

12. For continuously differentiable vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$, show that

$$\frac{\partial}{\partial x}[\mathbf{F} \times \mathbf{G}] = \frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x}.$$

13. Suppose that $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ are continuously differentiable. Show that

$$\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \nabla f \times \mathbf{F}.$$

14. For a function $f : \mathbb{R}^3 \to \mathbb{R}$ that has continuous second-order partial derivatives, show that

$$\operatorname{curl} \nabla f = \mathbf{0}.$$

15. For continuously differentiable vector fields $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{H} : \mathbb{R}^3 \to \mathbb{R}^3$, show that

$$\nabla(\mathbf{E} \times \mathbf{H}) = \langle \mathbf{E}, \operatorname{curl} \mathbf{H} \rangle - \langle \mathbf{H}, \operatorname{curl} \mathbf{E} \rangle.$$

16. Under the assumptions of the integration by parts formula stated in Corollary 20.27, suppose that the functions u and v have continuous second-order partial derivatives on an open set containing $\Omega \cup \mathbf{\Gamma}$. Use the integration by parts formula to obtain Green's First Identity:

$$\iint_{\mathcal{R}} \langle \nabla u, \nabla v \rangle\,dx\,dy = \int_{\mathbf{\Gamma}} u\frac{\partial v}{\partial \eta}\,ds - \iint_{\mathcal{R}} u \Delta v\,dx\,dy.$$

By subtracting, obtain Green's Second Identity:

$$\iint_{\mathcal{R}} [u\Delta v - v\Delta u]\,dx\,dy = \int_{\Gamma} \left[u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}\right]\,ds.$$

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(Recall that Δw, called the *Laplacian* of w, is defined by

$$\Delta w(x, y) \equiv \frac{\partial^2 w}{\partial x^2}(x, y) + \frac{\partial^2 w}{\partial y^2}(x, y).$$

17. Use Green's First Identity, from Exercise 16, to show that if the function $u(x, y)$ has continuous second-order partial derivatives on an open set containing $\{(x, y) \mid x^2 + y^2 \leq 1\}$, then the function that is identically 0 is the only function having the properties that

$$\begin{cases} \Delta u(x, y) = 0 & \text{for } x^2 + y^2 < 1 \\ u(x, y) = 0 & \text{for } x^2 + y^2 = 1. \end{cases}$$

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APPENDIX A

CONSEQUENCES OF THE FIELD AND POSITIVITY AXIOMS

In the Preliminaries, we stated the Field Axioms and the Positivity Axioms for the real numbers and made various assertions regarding elementary consequences of these axioms. In this appendix, we verify some of these assertions.

A.1 THE FIELD AXIOMS AND THEIR CONSEQUENCES

For convenience, we restate the Field Axioms. For each pair of real numbers a and b, a real number called the *sum* of a and b is defined and is denoted by $a + b$, and a real number called the *product* of a and b is defined and is denoted by ab. These operations satisfy the following collection of axioms:

Commutativity of Addition: For all real numbers a and b,

$$a + b = b + a.$$

Associativity of Addition: For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

The Additive Identity: There is a real number, denoted by 0, such that

$$0 + a = a \qquad \text{for all real numbers } a.$$

The Additive Inverse: For each real number a, there is a real number b such that

$$a + b = 0.$$

Commutativity of Multiplication: For all real numbers a and b,

$$ab = ba.$$

Associativity of Multiplication: For all real numbers a, b, and c,

$$(ab)c = a(bc).$$

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The Multiplicative Identity: There is a real number, denoted by 1, such that

$$1a = a \qquad \text{for all real numbers } a.$$

The Multiplicative Inverse: For each real number $a \neq 0$, there is a real number b such that

$$ab = 1.$$

The Distributive Property: For all real numbers a, b, and c,

$$a(b + c) = ab + ac.$$

The Nontriviality Assumption:

$$1 \neq 0.$$

First, observe that there is only one number that has the property asserted in the additive identity axiom. Indeed, if $0'$ also has the property that

$$0' + a = a \qquad \text{for all real numbers } a,$$

then in particular we have

$$0' + 0 = 0.$$

But by the commutative property of addition and the definition of 0 as an additive identity,

$$0' + 0 = 0 + 0' = 0'.$$

Thus, $0 = 0'$, so there is only one additive identity.

Proposition A.1 For each real number a,

$$a0 = 0a = 0.$$

Proof
Observe that

$$0 + 0 = 0.$$

Thus, by the Distributive Axiom,

$$0a + 0a = 0a.$$

If we add the additive inverse of $0a$ to each side and use the associativity of addition, we obtain $0a = 0$, and from this and the commutative property of multiplication it also follows that $a0 = 0$. ■

Proposition A.2 For any pair of real numbers a and b, if

$$ab = 0,$$

then $a = 0$ or $b = 0$.

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Proof
If $a = 0$, the proof is complete. So suppose that $a \neq 0$. We must show that $b = 0$. Since $a \neq 0$, by the multiplicative inverse axiom we can select a number d such that $da = 1$. Since $ab = 0$, it follows from Proposition 1 that

$$d(ab) = d0 = 0.$$

On the other hand, by the associative and commutative properties of multiplication and by the definition of 1 as the multiplicative identity, it follows that

$$d(ab) = (da)b = 1b = b.$$

Thus, $b = 0$. ∎

The additive inverse axiom asserts that for each number a, there is a number b such that $a + b = 0$. In fact, there is only one such number; it is called the *additive inverse* of a. To see why there is only one such number, suppose that b' also has the property that $a + b' = 0$. Then

$$\begin{aligned}
b' &= 0 + b' && \\
&= (a + b) + b' && \text{by the choice of } b\\
&= (b + a) + b' && \text{by commutativity of addition}\\
&= b + (a + b') && \text{by associativity of addition}\\
&= b + 0 && \text{by the choice of } b'\\
&= 0 + b && \text{by commutativity of addition}\\
&= b && \text{by the definition of } 0.
\end{aligned}$$

Thus, $b = b'$. Of course, we denote the additive inverse of a by $-a$. The additive inverse possesses the following familiar properties.

Proposition A.3 For all real numbers a and b,

i. $-(-a) = a$
ii. $-a = (-1)a$
iii. $-ab = (-a)b$
iv. $ab = (-a)(-b)$
v. $1 = (-1)(-1)$.

Proof
Part (i) follows from the fact that $-(-a)$ is the unique number that when added to $-a$ equals 0 and from the observation that a has this property. To verify (ii), we must show that

$$a + (-1)a = 0.$$

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But since 1 is the multiplicative identity,

$$\begin{aligned} a+(-1)a &= 1a+(-1)a \\ &= (1+(-1))a && \text{by the distributive property} \\ &= 0a && \text{since } -1 \text{ is the additive inverse of } 1 \\ &= 0 && \text{by Proposition A.1.} \end{aligned}$$

To verify (iii), observe that

$$\begin{aligned} -ab &= (-1)ab && \text{by (ii)} \\ &= ((-1)a)b && \text{by associativity of multiplication} \\ &= (-a)b && \text{again by (ii)} \end{aligned}$$

To verify (iv), observe that

$$\begin{aligned} ab &= -(-ab) && \text{by (i)} \\ &= -((-a)b) && \text{by (iii)} \\ &= -(b(-a)) && \text{by commutativity of multiplication} \\ &= (-b)(-a) && \text{by (iii)} \\ &= (-a)(-b) && \text{by commutativity of multiplication.} \end{aligned}$$

Finally, observe that (v) follows from (iv) when we set $a = b = 1$. ■

For numbers a and b, we define the *difference* $a - b$ by

$$a - b \equiv a + (-b).$$

Using the preceding proposition, it is not difficult to verify that for any numbers a, b, and c,

$$a(b-c) = ab - ac \qquad \text{and} \qquad -(b-c) = -b + c.$$

Let us now examine some consequences of the multiplication axioms. Just as we have shown that the additive identity is unique, a similar argument shows that the multiplicative identity is unique. Also, an argument similar to the one showing that the additive inverse is unique shows that for a nonzero number a, its multiplicative inverse is unique; the multiplicative inverse of a is denoted by a^{-1}. The multiplicative inverse possesses the following familiar properties.

Proposition A.4 For any nonzero real numbers a and b,

i. $(a^{-1})^{-1} = a$
ii. $(-a)^{-1} = -a^{-1}$
iii. $(ab)^{-1} = a^{-1}b^{-1}$.

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Proof
To verify (i), observe that $(a^{-1})^{-1}$ is the unique number that has the property that when it is multiplied by a^{-1}, the product is 1, and that the number a has this property. To verify (ii), we must show that

$$(-a)(-a^{-1}) = 1.$$

However, by part (iv) of Proposition A.3, we have

$$(-a)(-a^{-1}) = (a)(a^{-1}) = 1.$$

Finally, to verify (iii), we must show that

$$(ab)(a^{-1}b^{-1}) = 1.$$

However, by the commutative and associative properties of multiplication,

$$(ab)(a^{-1}b^{-1}) = (aa^{-1})(bb^{-1}) = 1 \cdot 1 = 1. \qquad \blacksquare$$

For any two numbers a and b, with $b \neq 0$, we define

$$\frac{a}{b} = ab^{-1}.$$

Directly from the definition of division and the distributive property, it follows that for any numbers a, b, and c, with $c \neq 0$,

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

A.2 THE POSITIVITY AXIOMS AND THEIR CONSEQUENCES

In the real numbers, there is a natural notion of order—that is, *greater than, less than,* and so forth. A convenient way to codify these properties is by specifying axioms that are satisfied by the set of positive numbers.

There is a set of real numbers denoted by $\mathcal{P}$, called the set of *positive numbers,* that has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are also positive.

P2 For a real number a, exactly one of the following three alternatives is true:

$$a \text{ is positive}, \qquad -a \text{ is positive}, \qquad a = 0.$$

The positivity axioms lead in a natural way to an ordering of the real numbers: For real numbers a and b, we define $a > b$ to mean that $a - b$ is positive, and $a \geq b$ to mean that $a > b$ or $a = b$. We then define $a < b$ to mean that $b > a$, and $a \leq b$ to mean that $b \geq a$.

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Proposition A.5 For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$.

Proof

Since $a \neq 0$, it follows from the second positivity axiom that either a or $-a$ is positive. If a is positive, then since the product of positive numbers is again positive, a^2 is positive. Similarly, if $-a$ is positive, so is $(-a)(-a)$. But by part (iv) of Proposition A.3, $(-a)(-a) = a^2$. Thus again, in this case, a^2 is positive. In particular, since by the nontriviality axiom $1 \neq 0$, $1 = 1 \cdot 1$ is positive. ∎

Proposition A.6 For each positive number a, its multiplicative inverse a^{-1} is also positive.

Proof

Since $a \cdot a^{-1} = 1 \neq 0$, it follows from Proposition A.2 that $a^{-1} \neq 0$. By the first positivity axiom, either a^{-1} or $-a^{-1}$ is positive. But it is not possible for $-a^{-1}$ to be positive since then $a \cdot (-a^{-1}) = -1$ would also be positive and this contradicts Proposition A.5. Thus, a^{-1} is positive. ∎

Proposition A.7 If $a > b$, then

$$ac > bc \qquad \text{if } c > 0$$

and

$$ac < bc \quad \text{if } c < 0.$$

Proof

The number $a - b$ is positive. If c is positive, then the product $(a - b)c = ab - ac$ is also positive; that is, $ac > bc$. On the other hand, if $c < 0$, then $-c$ is positive, so $(a - b)(-c)$ also is positive. However, $(a - b)(-c) = bc - ac$, so $ac < bc$. ∎

EXERCISES FOR APPENDIX A

1. Prove that for any numbers a, b, and c,

$$a(b - c) = ab - ac \qquad \text{and} \qquad -(b - c) = -b + c.$$

2. Prove that the multiplicative identity is unique.
3. Prove that each number $a \neq 0$ has a unique multiplicative inverse.
4. Prove that for any numbers a and b, with $b \neq 0$,

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

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APPENDIX B

LINEAR ALGEBRA

In the analysis of differentiation and integration of functions of several variables and of mappings between Euclidean spaces, we are principally concerned with the case where the functions, or mappings, are nonlinear. However, underlying the study of these nonlinear functions or mappings is an understanding of linear functions and mappings. The body of knowledge related to the study of linear functions and mappings is called *linear algebra*. The concepts of the vector sum of two vectors, the product of a number and a vector, and the scalar product of two vectors were discussed in Chapter 10. The correspondence between linear mappings and matrices was established in Section 15.1, and in the same section we defined and described various properties of the determinant of a square matrix.

A full treatment of linear algebra is outside the scope of this book. In this appendix, we first define some general concepts of linear algebra and state some general results. Then we prove the stated results in the important special case of $\mathbb{R}^3$. Beyond the simplification this provides, it also permits us to take a more geometric viewpoint that arises from the geometric properties of the scalar product and the cross-product of two vectors in $\mathbb{R}^3$.

Given k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in Euclidean space $\mathbb{R}^n$, a vector $\mathbf{v}$ of the form

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k,$$

where $\lambda_1, \ldots, \lambda_k$ are numbers, is said to be a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Definition The k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are said to be *linearly dependent* provided that there are numbers $\lambda_1, \ldots, \lambda_k$, not all of which are 0, such that

$$\lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k = \mathbf{0}.$$

If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are not linearly dependent, they are said to be *linearly independent*.

It is easy to see that the k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly dependent if and only if one of these vectors is a linear combination of the remaining $k-1$ vectors. Moreover, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent and the vector $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$,

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then there are *unique* numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k.$$

Definition The k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are said to *span* $\mathbb{R}^n$ provided that every vector $\mathbf{v}$ in $\mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Definition The k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are said to be a *basis* for $\mathbb{R}^n$ provided that for each vector $\mathbf{v}$ in $\mathbb{R}^n$, there are unique numbers $\lambda_1, \ldots, \lambda_k$ such that

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k.$$

The definition of linear independence can be restated by asserting that the k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are linearly independent provided that the only numbers $\lambda_1, \ldots, \lambda_k$ having the property that

$$\mathbf{0} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$$

are $\lambda_1 = 0, \ldots, \lambda_k = 0$. It follows that if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are a basis for $\mathbb{R}^n$, then they must be linearly independent. The converse is not true. There is, however, the following important theorem.

Theorem B.1 For n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$, the following three assertions are equivalent:

i. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis for $\mathbb{R}^n$.
ii. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.
iii. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

The above theorem has an immediate interpretation in terms of $n \times n$ systems of linear equations. Recall that an $n \times n$ *matrix* is a rectangular array of real numbers consisting of n rows and n columns. If such an $n \times n$ matrix is denoted by $\mathbf{A}$, we write

$$\mathbf{A} = [a_{ij}],$$

where for each pair of indices i and j such that $1 \le i \le n$ and $1 \le j \le n$, a_{ij} denotes the number in the ith row and jth column of the matrix $\mathbf{A}$. For a point $\mathbf{x}$ in $\mathbb{R}^n$, by the symbol $\mathbf{Ax}$ we denote the point in $\mathbb{R}^n$ that, for each index i such that $1 \le i \le n$, has an ith component equal to the scalar product of the ith row of $\mathbf{A}$ with $\mathbf{x}$. Thus,

$$\mathbf{Ax} \equiv \mathbf{y},$$

where

$$y_i \equiv \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for each index } i \text{ such that } 1 \le i \le n.$$

Now, for each index i with $1 \le i \le n$, define the vector $\mathbf{v}_i$ by

$$\mathbf{v}_i = (a_{1i}, \ldots, a_{ni}),$$

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so that $\mathbf{v}_i$ corresponds to the ith column of the matrix $\mathbf{A}$. Then, by the very definition of $\mathbf{Ax}$, it follows immediately that for vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$,

$$\mathbf{Ax} = \mathbf{y} \qquad \text{if and only if } x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{y}.$$

This equivalence allows us to restate Theorem B.1 as follows.

Theorem B.2 For an $n \times n$ matrix $\mathbf{A}$ and the $n \times n$ system of linear equations

$$\mathbf{Ax} = \mathbf{y}, \tag{B.1}$$

the following three assertions are equivalent:

i. For each $\mathbf{y}$ in $\mathbb{R}^n$, the system of linear equations (B.1) has a unique solution $\mathbf{x}$.
ii. For each $\mathbf{y}$ in $\mathbb{R}^n$, the system of linear equations (B.1) has a solution $\mathbf{x}$.
iii. For $\mathbf{y} = \mathbf{0}$, the only solution of the system of linear equations (B.1) is $\mathbf{x} = \mathbf{0}$.

Finally, Theorem B.1 (and hence also Theorem B.2) has an interpretation in terms of linear mappings. The correspondence between linear mappings $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ and $n \times n$ matrices is completely described in Section 15.1 and will not be repeated here. It should be noted that for a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ associated with the $n \times n$ matrix $\mathbf{A}$ by

$$\mathbf{T}(\mathbf{x}) = \mathbf{Ax} \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n,$$

since $\mathbf{T}(\mathbf{u}) = \mathbf{T}(\mathbf{v})$ if and only if $\mathbf{T}(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, we see that such a mapping is one-to-one if and only if whenever $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. This is the observation needed to see that Theorem B.2 is equivalent to the following theorem.

Theorem B.3 For a linear mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$, the following three assertions are equivalent:

i. The mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and has image equal to $\mathbb{R}^n$; that is, it is invertible.
ii. The mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ has image equal to $\mathbb{R}^n$.
iii. The mapping $\mathbf{T}: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one.

Although we now have the equivalence of Theorems B.1, B.2, and B.3, we do not have a proof of any one of them. Moreover, we also lack any explicit criterion for determining when n vectors in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$ or, equivalently, for determining when a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ is invertible. There is a number called the *determinant*, which can be associated with any ordered n-tuple of vectors in $\mathbb{R}^n$; the vectors are a basis if and only if the determinant is nonzero. Equivalently, the determinant can be associated with the matrix that represents a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$; the determinant is nonzero if and only if the mapping is invertible. When the determinant is nonzero, its magnitude has an interpretation as a measure of volume.

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As we have mentioned, we will not prove the above assertions in general Euclidean space $\mathbb{R}^n$. Rather, we will provide all details for the special but very important case of $\mathbb{R}^3$.

Not all Euclidean spaces are created equal. Of course $\mathbb{R}^1$ is special since it is the set of real numbers, which have been described in the Preliminaries and in Chapter 1 by the Field Axioms, the Positivity Axioms, and the Completeness Axiom. The plane $\mathbb{R}^2$ also is special since it turns out that there is a concept of product called the *complex product* (or complex multiplication) that associates with a pair of points in $\mathbb{R}^2$ another point in $\mathbb{R}^2$, called the complex product of the pair. With the usual concept of sum, and with multiplication replaced by the complex product, the Field Axioms are satisfied (Exercises 6 and 7). This has far-reaching consequences and is the basis of the subject called *complex analysis*. This topic, however, lies outside the scope of this book.[1] Here, we study geometry and algebra in $\mathbb{R}^3$ by introducing a construction called the *cross-product*. In contrast to the scalar product, which associates with any pair of vectors in $\mathbb{R}^3$ a *number*, the cross-product associates with any ordered pair of vectors in $\mathbb{R}^3$ another *vector* in $\mathbb{R}^3$. By using both the scalar product and the cross-product, we obtain interesting geometric and algebraic results that have intuitive geometric interpretations.

For a point $\mathbf{p}$ and a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$, the line ℓ through $\mathbf{p}$ parallel to $\mathbf{v}$ is defined to be the set of points of the form $\mathbf{p} + t\mathbf{v}$, where t is any number. For another point $\mathbf{u}$ in $\mathbb{R}^3$, it is often useful to find the point on the line ℓ closest to $\mathbf{u}$. The distance from $\mathbf{u}$ to this point is called the *distance* from the point $\mathbf{u}$ to the line ℓ. We now provide a formula for this point that reveals the geometric significance of the scalar product.

Recall that by the linearity and symmetry of the scalar product, for two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$,

$$\langle \mathbf{u}+\mathbf{v},\ \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle,$$

so that we have the following.

The Pythagorean Identity

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{if and only if } \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}$$

Theorem B.4 For a nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$, let ℓ be the line through the origin parallel to $\mathbf{v}$. For a point $\mathbf{u}$ in $\mathbb{R}^3$, set $\lambda = \langle \mathbf{u}, \mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle$. Then

i. the vector $\mathbf{u} - \lambda\mathbf{v}$ is orthogonal to the vector $\mathbf{v}$.
ii. $\lambda\mathbf{v}$ is the point on the line ℓ closest to $\mathbf{u}$, so the distance from $\mathbf{u}$ to ℓ is $\|\mathbf{u} - \lambda\mathbf{v}\|$.

Proof

By the linearity of the scalar product and the definition of λ,

$$\langle \mathbf{u} - \lambda\mathbf{v},\ \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \lambda\langle \mathbf{v}, \mathbf{v}\rangle = 0,$$

so (i) is verified. To verify (ii), it is necessary to show that

$$\|\mathbf{u} - t\mathbf{v}\| \geq \|\mathbf{u} - \lambda\mathbf{v}\| \qquad \text{for all } t \text{ in } \mathbb{R}. \tag{B.2}$$

[1] See, for instance, R. V. Churchill and T. A. Ward, *Complex Variables and Applications*, 5th ed. (New York: McGraw-Hill, 1990).

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We write $\mathbf{u} - t\mathbf{v} = (\mathbf{u} - \lambda\mathbf{v}) + (\lambda - t)\mathbf{v}$. By (i), $\mathbf{u} - \lambda\mathbf{v}$ is orthogonal to $\mathbf{v}$, and hence is also orthogonal to $(\lambda - t)\mathbf{v}$. Thus, by the Pythagorean Identity,

$$\|\mathbf{u} - t\mathbf{v}\|^2 = \|(\mathbf{u} - \lambda\mathbf{v}) + (\lambda - t)\mathbf{v}\|^2 = \|\mathbf{u} - \lambda\mathbf{v}\|^2 + \|(\lambda - t)\mathbf{v}\|^2 \geq \|\mathbf{u} - \lambda\mathbf{v}\|^2,$$

so the inequality (B.2) holds. ∎

Definition For two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, the cross-product of $\mathbf{u}$ with $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector in $\mathbb{R}^3$ defined by the formula

$$\mathbf{u} \times \mathbf{v} \equiv (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

At first glance, the geometric significance of the cross-product is not at all apparent. In preparation for the discussion of its geometric significance, we first collect some algebraic properties of the cross-product.

Proposition B.5 For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}, \qquad (\textit{Antisymmetry})$$

and if α and β are any numbers,

$$[\alpha\mathbf{u} + \beta\mathbf{w}] \times \mathbf{v} = \alpha[\mathbf{u} \times \mathbf{v}] + \beta[\mathbf{w} \times \mathbf{v}]. \qquad (\textit{Linearity})$$

Proof

The proof of these identities is by inspection. By definition,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

and

$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2,\ v_3u_1 - v_1u_3,\ v_1u_2 - v_2u_1),$$

so $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. Similarly, we verify linearity.

Define $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. For each point $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$,

$$\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

so it is clear that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ form a basis for $\mathbb{R}^3$. The basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is called the *standard basis* for $\mathbb{R}^3$. Observe that for the basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

∎

The following theorem explains the significance of the length of the cross-product of two vectors.

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Theorem B.6 For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ such that $\mathbf{v} \neq \mathbf{0}$, set $\lambda = \langle \mathbf{u}, \mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle$. Then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{v}\| \cdot \|\mathbf{u} - \lambda \mathbf{v}\|; \tag{B.3}$$

that is, the length of $\mathbf{u} \times \mathbf{v}$ is the length of the vector $\mathbf{v}$ times the distance from the point $\mathbf{u}$ to the line through the origin parallel to the vector $\mathbf{v}$.

Proof
By part (i) of Theorem B.4 the vector $\mathbf{u} - \lambda\mathbf{v}$ is orthogonal to the vector $\mathbf{v}$, and hence is also orthogonal to $\lambda\mathbf{v}$. To verify (B.3) we square the left-hand side and compute:

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2 - \langle \mathbf{u}, \mathbf{v}\rangle^2 \\
&= \|\mathbf{v}\|^2 \left\{ \|\mathbf{u}\|^2 - \frac{\langle \mathbf{u}, \mathbf{v}\rangle^2}{\|\mathbf{v}\|^2} \right\} \\
&= \|\mathbf{v}\|^2 \langle \mathbf{u} - \lambda\mathbf{v}, \mathbf{u}\rangle && \text{by the definition of } \lambda \\
&= \|\mathbf{v}\|^2 \langle \mathbf{u} - \lambda\mathbf{v}, \mathbf{u} - \lambda\mathbf{v}\rangle && \text{since } \langle \mathbf{u} - \lambda\mathbf{v}, \lambda\mathbf{v}\rangle = 0 \\
&= \|\mathbf{v}\|^2 \cdot \|\mathbf{u} - \lambda\mathbf{v}\|^2.
\end{aligned}$$

Thus formula (B.3) holds. The last remark in the statement of the theorem follows from part (ii) of Theorem B.4. ■

Theorem B.7 For two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the following assertions are equivalent:

i. The vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.
ii. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Proof
First, suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. Then one of the vectors is a scalar multiple of the other, say $\mathbf{v} = \alpha\mathbf{u}$. By the antisymmetry of the cross-product, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Thus, using the linearity property of the cross-product,

$$\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \alpha\mathbf{u} = \alpha(\mathbf{u} \times \mathbf{u}) = \alpha\mathbf{0} = \mathbf{0}.$$

To prove the converse, suppose that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. If $\mathbf{v} = \mathbf{0}$, then of course $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent since $\mathbf{v} = 0\mathbf{u}$. If $\mathbf{v} \neq \mathbf{0}$, then from formula (B.3) we conclude that $\|\mathbf{u} - \lambda\mathbf{v}\| = 0$, and again $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent since we now have $\mathbf{u} = \lambda\mathbf{v}$. ■

The following theorem partially reveals the significance of the direction of the cross-product.

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Theorem B.8 Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. Then

i. The cross-product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
ii. Moreover, in the case where $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, if $\mathbf{w}$ is any vector in $\mathbb{R}^3$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{w}$ is a scalar multiple of $\mathbf{u} \times \mathbf{v}$; that is, there is a number γ such that $\mathbf{w} = \gamma(\mathbf{u} \times \mathbf{v})$.

Proof
To verify (i), it is necessary to show that

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{u}\rangle = \langle \mathbf{u} \times \mathbf{v}, \mathbf{v}\rangle = 0.$$

But by the very definition of cross-product,

$$\langle \mathbf{u} \times \mathbf{v}, \mathbf{u}\rangle = (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 = 0,$$

and similarly, $\langle \mathbf{u} \times \mathbf{v}, \mathbf{v}\rangle = 0$. We now verify (ii). Suppose that the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent and let $\mathbf{w}$ be orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. We write out the orthogonality assumptions as the following system of equations:

$$\begin{aligned} u_1w_1 + u_2w_2 + u_3w_3 &= 0 \\ v_1w_1 + v_2w_2 + v_3w_3 &= 0. \end{aligned} \tag{B.4}$$

We first eliminate w_1 from this system of equations by multiplying the first equation by v_1, multiplying the second equation by $-u_1$, and adding the resulting equations. Similarly, we eliminate w_2 and then w_3. The new system of equations is

$$\begin{aligned} w_2(u_2v_1 - u_1v_2) + w_3(u_3v_1 - u_1v_3) &= 0 \\ w_1(u_1v_2 - u_2v_1) + w_3(u_3v_2 - u_2v_3) &= 0 \\ w_1(u_1v_3 - u_3v_1) + w_2(u_2v_3 - u_3v_2) &= 0. \end{aligned} \tag{B.5}$$

Now we have assumed that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Thus, by Theorem B.7, $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, and we suppose it is the last component of $\mathbf{u} \times \mathbf{v}$ that is nonzero; that is, $u_1v_2 - u_2v_1 \neq 0$. Define $\gamma = w_3/(u_1v_2 - u_2v_1)$. Then, by definition, $w_3 = \gamma(u_1v_2 - u_2v_1)$. The first equation in (B.5) gives $w_2 = \gamma(u_3v_1 - u_1v_3)$; the second equation in (B.5) gives $w_1 = \gamma(u_2v_3 - u_3v_2)$. Consequently,

$$w_1 = \gamma(u_2v_3 - u_3v_2), \qquad w_2 = \gamma(u_3v_1 - u_1v_3), \qquad \text{and} \qquad w_3 = \gamma(u_1v_2 - u_2v_1);$$

that is, $\mathbf{w} = \gamma(\mathbf{u} \times \mathbf{v})$. ■

Theorem B.9 Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^3$. Then the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are a basis for $\mathbb{R}^3$.

Proof
Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. Define $\lambda = \langle \mathbf{u}, \mathbf{v}\rangle/\langle \mathbf{v}, \mathbf{v}\rangle$ and then define $\mathbf{u}' = \mathbf{u} - \lambda\mathbf{v}$. Part (i) of Theorem B.4 asserts that $\mathbf{u}'$ is orthogonal to $\mathbf{v}$. Moreover, $\mathbf{u}' \neq \mathbf{0}$ since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

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Choose $\mathbf{p}$ to be a vector in $\mathbb{R}^3$. Define $\alpha' = \langle \mathbf{p}, \mathbf{u}' \rangle / \langle \mathbf{u}', \mathbf{u}' \rangle$ and $\beta' = \langle \mathbf{p}, \mathbf{v} \rangle / \langle \mathbf{v}, \mathbf{v} \rangle$. By the linearity of the scalar product and the orthogonality of $\mathbf{u}'$ and $\mathbf{v}$,

$$\langle \mathbf{p} - (\alpha'\mathbf{u}' + \beta'\mathbf{v}), \mathbf{u}' \rangle = 0 \qquad \text{and} \qquad \langle \mathbf{p} - (\alpha'\mathbf{u}' + \beta'\mathbf{v}), \mathbf{v} \rangle = 0;$$

that is, the vector $\mathbf{p} - (\alpha'\mathbf{u}' + \beta'\mathbf{v})$ is orthogonal to both $\mathbf{u}'$ and $\mathbf{v}$. Since $\mathbf{u}'$ and $\mathbf{v}$ are nonzero and orthogonal, they are linearly independent. By part (ii) of Theorem B.8, there is a number γ' such that $\mathbf{p} - (\alpha'\mathbf{u}' + \beta'\mathbf{v}) = \gamma'(\mathbf{u}' \times \mathbf{v})$; that is,

$$\mathbf{p} = \alpha'\mathbf{u}' + \beta'\mathbf{v} + \gamma'(\mathbf{u}' \times \mathbf{v}).$$

Substituting $\mathbf{u}' = \mathbf{u} - \lambda\mathbf{v}$ in the above expression, since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, we can regroup the coefficients to find numbers α, β, and γ such that

$$\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v}). \tag{B.6}$$

It remains to verify that the numbers α, β, and γ are unique. We suppose otherwise and derive a contradiction. Indeed, if there were two distinct triples of numbers for which (B.6) holds, then by subtraction there would be numbers α, β, and γ, not all equal to 0, such that

$$\mathbf{0} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v}).$$

Taking the scalar product of both sides with $\mathbf{u} \times \mathbf{v}$, we conclude that

$$0 = \gamma \|\mathbf{u} \times \mathbf{v}\|^2.$$

But $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, so by Theorem B.7, $\|\mathbf{u} \times \mathbf{v}\| \neq 0$. Thus, $\gamma = 0$. But then

$$\mathbf{0} = \alpha\mathbf{u} + \beta\mathbf{v},$$

and either α or β is nonzero. This contradicts the linear independence of $\mathbf{u}$ and $\mathbf{v}$. ■

With the properties of the cross-product that we have so far established, we can now provide a proof of Theorem B.1 (and hence also of Theorems B.2 and B.3) in the case where $n = 3$.

Theorem B.10 For the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, the following three assertions are equivalent:

i. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are a basis for $\mathbb{R}^3$.
ii. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ span $\mathbb{R}^3$.
iii. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

Proof
By the very definition of basis, (i) implies (ii). Now suppose (ii) holds. We argue by contradiction to show that (iii) holds; that is, the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent. Indeed, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent, then one of these vectors is a linear combination of the other two, which implies that just two of these vectors

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span $\mathbb{R}^3$. Suppose it is $\mathbf{u}$ and $\mathbf{v}$ that span $\mathbb{R}^3$. If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, it follows from Theorem B.7 that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, which implies that $\mathbb{R}^3$ is spanned by one of them, say $\mathbf{u}$. But this is impossible since we can easily find a nonzero vector that is orthogonal to $\mathbf{u}$, so such a vector is certainly not a multiple of $\mathbf{u}$. It follows that $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. This too is impossible since $\mathbf{u} \times \mathbf{v}$ would then be a nonzero vector that is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Consequently, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent.

Now suppose that (iii) holds. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, the two vectors $\mathbf{u}$ and $\mathbf{v}$ are also linearly independent. By Theorem B.9, the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are a basis for $\mathbb{R}^3$. So there are real numbers α, β, and γ such that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v}).$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, it follows that γ is nonzero, so we have

$$\mathbf{u} \times \mathbf{v} = \frac{1}{\gamma}(\mathbf{w} - \alpha\mathbf{u} - \beta\mathbf{v}).$$

Now let $\mathbf{p}$ be a vector in $\mathbb{R}^3$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are a basis for $\mathbb{R}^3$, there are real numbers α', β', and γ' such that

$$\mathbf{p} = \alpha'\mathbf{u} + \beta'\mathbf{v} + \gamma'(\mathbf{u} \times \mathbf{v}).$$

Then

$$\begin{aligned}\mathbf{p} &= \alpha'\mathbf{u} + \beta'\mathbf{v} + \frac{\gamma'}{\gamma}(\mathbf{w} - \alpha\mathbf{u} - \beta\mathbf{v}) \\ &= \left(\alpha' - \frac{\gamma'}{\gamma}\alpha\right)\mathbf{u} + \left(\beta' - \frac{\gamma'}{\gamma}\beta\right)\mathbf{v} + \frac{\gamma'}{\gamma}\mathbf{w},\end{aligned}$$

and hence $\mathbf{p}$ can be written as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Furthermore, this representation is unique since if

$$\mathbf{p} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3\mathbf{w} = \lambda_1'\mathbf{u} + \lambda_2'\mathbf{v} + \lambda_3'\mathbf{w},$$

then we have

$$\mathbf{0} = (\lambda_1 - \lambda_1')\mathbf{u} + (\lambda_2 - \lambda_2')\mathbf{v} + (\lambda_3 - \lambda_3')\mathbf{w},$$

and since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, it follows that

$$\lambda_1 = \lambda_1', \qquad \lambda_2 = \lambda_2', \qquad \text{and} \qquad \lambda_3 = \lambda_3'.$$

Hence the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are a basis for $\mathbb{R}^3$. ∎

It is useful to have a criterion for detecting when a given triple of vectors is a basis for $\mathbb{R}^3$. We show that the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ are a basis for $\mathbb{R}^3$ if and only if

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \neq 0.$$

The number $\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle$ is called the *triple product* of the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The dependence of the triple product on the order of the three vectors is described by the following proposition.

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Proposition B.11 For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle &= \langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{w}\times\mathbf{u}\rangle \\ \langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle &= -\langle \mathbf{v}, \mathbf{u}\times\mathbf{w}\rangle = -\langle \mathbf{w}, \mathbf{v}\times\mathbf{u}\rangle.\end{aligned} \tag{B.7}$$

Proof

The proof is by inspection. Indeed, by definition of the scalar product and the cross-product,

$$\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$$

and

$$\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = w_1(u_2v_3 - u_3v_2) + w_2(u_3v_1 - u_1v_3) + w_3(u_1v_2 - u_2v_1).$$

Observe that the right-hand sides of the two above identities are equal. Similarly, it follows that $\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{w}\times\mathbf{u}\rangle$. Thus, the first line of (B.7) is verified. From the antisymmetry of the cross-product and the inequalities on the first line of (B.7), the equalities on the second line of (B.7) follow; that is, the triple product of an ordered triple of vectors changes sign when two of the vectors are interchanged. ■

Theorem B.12 For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$, the following two assertions are equivalent:

i. $\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle \neq 0$.
ii. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a basis for $\mathbb{R}^3$.

Moreover, when either (and hence both) of these assertions holds, each vector $\mathbf{p}$ in $\mathbb{R}^3$ can be expressed as

$$\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w},$$

where the coefficients α, β, and γ are given by the formulas

$$\alpha = \frac{\langle \mathbf{p}, \mathbf{v}\times\mathbf{w}\rangle}{\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle} \qquad \beta = \frac{\langle \mathbf{u}, \mathbf{p}\times\mathbf{w}\rangle}{\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle} \qquad \text{and} \qquad \gamma = \frac{\langle \mathbf{u}, \mathbf{v}\times\mathbf{p}\rangle}{\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle}. \tag{B.8}$$

Proof

First, suppose that (i) holds. Then $\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle \neq 0$, so $\mathbf{u}\times\mathbf{v} \neq \mathbf{0}$. By Theorem B.9, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ are a basis for $\mathbb{R}^3$. Thus, there are numbers α, β, and γ such that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u}\times\mathbf{v}). \tag{B.9}$$

Taking the scalar product of each side and $\mathbf{u}\times\mathbf{v}$, we conclude that

$$\langle \mathbf{w}, \mathbf{u}\times\mathbf{v}\rangle = \gamma\,\|\mathbf{u}\times\mathbf{v}\|^2.$$

Thus, $\gamma \neq 0$ since $\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle \neq 0$. Divide equation (B.9) by γ. We see that $\mathbf{u}\times\mathbf{v}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Consequently, since every vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$, every vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. It is necessary to show that each

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vector in $\mathbb{R}^3$ can be expressed *uniquely* as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Let $\mathbf{p}$ be a point in $\mathbb{R}^3$ and suppose that α, β, and γ are numbers such that

$$\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}. \tag{B.10}$$

Then taking the scalar product of each side of (B.10), first with $\mathbf{v} \times \mathbf{w}$, then with $\mathbf{w} \times \mathbf{u}$, and finally with $\mathbf{u} \times \mathbf{v}$, and using the fact that the triple product is zero if two vectors in the triple are equal, we have

$$\begin{aligned}\langle \mathbf{p}, \mathbf{v} \times \mathbf{w}\rangle &= \alpha\langle \mathbf{u}, \mathbf{v} \times \mathbf{w}\rangle\\ \langle \mathbf{p}, \mathbf{w} \times \mathbf{u}\rangle &= \beta\langle \mathbf{v}, \mathbf{w} \times \mathbf{u}\rangle\\ \langle \mathbf{p}, \mathbf{u} \times \mathbf{v}\rangle &= \gamma\langle \mathbf{w}, \mathbf{u} \times \mathbf{v}\rangle.\end{aligned}$$

From this, using the reordering property (B.7) of the triple product, we obtain (B.8). Thus we have proved that α, β, and γ are unique and, at the same time, have established formula (B.8) for the coefficients α, β, and γ.

It remains to prove that (ii) implies (i). Indeed, suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are a basis for $\mathbb{R}^3$. Observe that the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent since otherwise there would be two distinct ways of expressing the zero vector as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Hence, by Theorem B.7, $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. Furthermore, by Theorem B.9, the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are a basis for $\mathbb{R}^3$. In particular, there are numbers α, β, and γ such that

$$\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma(\mathbf{u} \times \mathbf{v}). \tag{B.11}$$

The component $\gamma \neq 0$ since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are a basis for $\mathbb{R}^3$. Taking the scalar product of each side of (B.11) and $\mathbf{u} \times \mathbf{v}$, we have

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w}\rangle = \langle \mathbf{w}, \mathbf{u} \times \mathbf{v}\rangle = \gamma\|\mathbf{u} \times \mathbf{v}\|^2 \neq 0. \qquad \blacksquare$$

Theorem B.12 has an interesting interpretation in terms of 3×3 systems of linear equations. For a 3×3 matrix $\mathbf{A} = [a_{ij}]$, consider the 3×3 systems of linear equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= y_1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= y_2\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= y_3,\end{aligned} \tag{B.12}$$

where the triple of numbers (y_1, y_2, y_3) is given and we seek a triple of numbers (x_1, x_2, x_3) for which the above system of equations is satisfied. We define the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ by

$$\mathbf{u} = (a_{11}, a_{21}, a_{31}), \qquad \mathbf{v} = (a_{12}, a_{22}, a_{32}), \qquad \mathbf{w} = (a_{13}, a_{23}, a_{33})$$

and observe that for a given vector $\mathbf{y} = (y_1, y_2, y_3)$, the linear system of equations is equivalent to

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{y}. \tag{B.13}$$

Now, directly from the definition of what it means for a triple of vectors to be a basis, we see that the assertion that the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are a basis for $\mathbb{R}^3$ is equivalent to the assertion that for every triple of numbers (y_1, y_2, y_3) there is a unique solution (x_1, x_2, x_3) of the system of equations (B.12). However, Theorem B.12 provides a

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necessary and sufficient condition for three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ to be a basis for $\mathbb{R}^3$, namely, that $\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \neq 0$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ correspond to the first, second, and third columns, respectively, of the matrix $\mathbf{A}$, we are led to define the *determinant* of a 3×3 matrix of real numbers as follows.

Definition For a 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we define the determinant of $\mathbf{A}$, denoted by *det* $\mathbf{A}$, by the formula

$$\det \mathbf{A} \equiv a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}).$$

Theorem B.13 For the system of linear equations (B.12) determined by the 3×3 matrix $\mathbf{A}$, the following two assertions are equivalent:

i. $\det \mathbf{A} \neq 0$.
ii. For each triple of numbers (y_1, y_2, y_3), there is a unique solution (x_1, x_2, x_3) of the system.

Moreover, when either (and hence both) of these assertions is true, for a given triple (y_1, y_2, y_3), the unique solution (x_1, x_2, x_3) of the system of equations (B.12) is given by

$$x_1 = \frac{1}{D} \det \begin{pmatrix} y_1 & a_{12} & a_{13} \\ y_2 & a_{22} & a_{23} \\ y_3 & a_{32} & a_{33} \end{pmatrix} \qquad x_2 = \frac{1}{D} \det \begin{pmatrix} a_{11} & y_1 & a_{13} \\ a_{21} & y_2 & a_{23} \\ a_{31} & y_3 & a_{33} \end{pmatrix}$$

$$x_3 = \frac{1}{D} \det \begin{pmatrix} a_{11} & a_{12} & y_1 \\ a_{21} & a_{22} & y_2 \\ a_{31} & a_{32} & y_3 \end{pmatrix},$$

where $D = \det \mathbf{A}$.

Proof
As above, define the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ by

$$\mathbf{u} = (a_{11}, a_{21}, a_{31}), \qquad \mathbf{v} = (a_{12}, a_{22}, a_{32}), \qquad \mathbf{w} = (a_{13}\, a_{23}\, a_{33}),$$

and for a given vector $\mathbf{y} = (y_1, y_2, y_3)$, observe that the linear system of equations is equivalent to

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{y}.$$

By definition, $\det \mathbf{A} = \langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle$. The theorem now follows from Theorem B.12.

■

From the antisymmetry of the cross-product, it follows that the determinant of a 3×3 matrix changes sign if two columns are interchanged; from the linearity of the

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scalar product and the cross-product, it follows that if two columns are fixed, then the determinant depends linearly on the remaining column.

Now that we understand the significance of the triple product being *nonzero,* we turn to a description of the significance, with respect to certain volume calculations, of the *magnitude* of the triple product. For a point $\mathbf{p}$ in $\mathbb{R}^3$ and two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the *parallelogram* based at $\mathbf{p}$ and bounded by the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be the set $\mathcal{S} = \{\mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v} \mid 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1\}$; the *area* of $\mathcal{S}$ is defined to be the length of the vector $\mathbf{v}$ times the distance from the point $\mathbf{p} + \mathbf{u}$ to the line through the point $\mathbf{p}$ parallel to the vector $\mathbf{v}$. It follows immediately from Theorems B.4 and B.6 that

$$\text{area}\,\mathcal{S} = \|\mathbf{u} \times \mathbf{v}\|. \tag{B.14}$$

For a point $\mathbf{p}$ in $\mathbb{R}^3$ and a nonzero vector $\boldsymbol{\eta}$ in $\mathbb{R}^3$, the *plane* through $\mathbf{p}$ that is normal to $\boldsymbol{\eta}$ is defined to be the set $\mathcal{P}$ of points $\mathbf{p} + \mathbf{w}$ such that the vector $\mathbf{w}$ is orthogonal to $\boldsymbol{\eta}$. Given two points $\mathbf{p} + \mathbf{u}$ and $\mathbf{p} + \mathbf{v}$ in this plane such that the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, it follows from part (ii) of Theorem B.8 that there is a scalar γ such that $\boldsymbol{\eta} = \gamma(\mathbf{u} \times \mathbf{v})$. Also, since the triple of vectors $\mathbf{u},\ \mathbf{v},\ \mathbf{u} \times \mathbf{v}$ is a basis for $\mathbb{R}^3$, it follows that $\mathcal{P} = \{\mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v} \mid \alpha \text{ in } \mathbb{R},\ \beta \text{ in } \mathbb{R}\}$. The distance from a point $\mathbf{q}$ to a plane $\mathcal{P}$ is defined to be the distance from $\mathbf{q}$ to the point in $\mathcal{P}$ closest to $\mathbf{q}$.

Proposition B.14 For a vector $\boldsymbol{\eta}$ in $\mathbb{R}^3$ of length 1, let $\mathcal{P}$ be the plane through the origin that is normal to $\boldsymbol{\eta}$. Then for any point $\mathbf{p}$ in $\mathbb{R}^3$, the distance from $\mathbf{p}$ to the plane $\mathcal{P}$ is equal to $|\langle \mathbf{p},\ \boldsymbol{\eta}\rangle|$.

Proof

Since by assumption $\langle \boldsymbol{\eta},\ \boldsymbol{\eta}\rangle = 1$, by the linearity of the scalar product,

$$\langle \mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta},\ \boldsymbol{\eta}\rangle = \langle \mathbf{p},\ \boldsymbol{\eta}\rangle - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\langle \boldsymbol{\eta},\ \boldsymbol{\eta}\rangle = 0;$$

that is, the vector $\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}$ is orthogonal to $\boldsymbol{\eta}$ and hence the point $\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}$ lies in $\mathcal{P}$. The distance between the point $\mathbf{p}$ and $\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}$ is

$$\|\langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}\| = |\langle \mathbf{p},\ \boldsymbol{\eta}\rangle|\|\boldsymbol{\eta}\| = |\langle \mathbf{p},\ \boldsymbol{\eta}\rangle|.$$

Thus, to prove the proposition, we must show that

$$\|\mathbf{p} - \mathbf{u}\| \geq |\langle \mathbf{p},\ \boldsymbol{\eta}\rangle| \qquad \text{for every point } \mathbf{u} \text{ in } \mathcal{P}.$$

However, for a point $\mathbf{u}$ in $\mathcal{P}$, write $\mathbf{p} - \mathbf{u} = (\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta} - \mathbf{u}) + \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}$. Since the vectors $\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta} - \mathbf{u}$ and $\langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}$ are orthogonal, it follows from the Pythagorean Identity that

$$\|\mathbf{p} - \mathbf{u}\|^2 = \|\mathbf{p} - \langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta} - \mathbf{u}\|^2 + \|\langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}\|^2 \geq \|\langle \mathbf{p},\ \boldsymbol{\eta}\rangle\boldsymbol{\eta}\|^2 = |\langle \mathbf{p},\ \boldsymbol{\eta}\rangle|^2,$$

so $\|\mathbf{p} - \mathbf{u}\| \geq |\langle \mathbf{p},\ \boldsymbol{\eta}\rangle|$. ∎

Now suppose that the triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are a basis for $\mathbb{R}^3$. The *parallelepiped* based at the point $\mathbf{p}$ and bounded by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is defined to be the set $\mathcal{V} = \{\mathbf{p} + \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \mid 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma \leq 1\}$; the *volume* of $\mathcal{V}$ is defined to be the area of the parallelogram based at $\mathbf{p}$ and bounded by the

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vectors $\mathbf{u}$ and $\mathbf{v}$ times the distance from the point $\mathbf{p}+\mathbf{w}$ to the plane containing $\mathbf{p},\ \mathbf{p}+\mathbf{u}$, and $\mathbf{p}+\mathbf{v}$. Observe that the plane containing $\mathbf{p},\ \mathbf{p}+\mathbf{u}$, and $\mathbf{p}+\mathbf{v}$ is the plane through the point $\mathbf{p}$ that is normal to the vector $\mathbf{u}\times\mathbf{v}$.

Theorem B.15 Let the vectors $\mathbf{u},\ \mathbf{v}$, and $\mathbf{w}$ be a basis for $\mathbb{R}^3$ and let $\mathcal{V}$ be the parallelepiped based at the origin and bounded by the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. Then the volume of $\mathcal{V}$ is given by the formula

$$\text{vol}\ \mathcal{V} = |\langle \mathbf{u},\ \mathbf{v}\times\mathbf{w}\rangle| \tag{B.15}$$

Proof

The vector $\mathbf{u}\times\mathbf{v}$ is a normal to the plane containing the origin, $\mathbf{u}$, and $\mathbf{v}$. Thus, $\eta = \mathbf{u}\times\mathbf{v}/\|\mathbf{u}\times\mathbf{v}\|$ is a normal to the plane that has length 1. Observe that

$$\langle \mathbf{w},\ \eta\rangle = \left\langle \mathbf{w},\ \frac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|}\right\rangle = \frac{\langle \mathbf{w},\ \mathbf{u}\times\mathbf{v}\rangle}{\|\mathbf{u}\times\mathbf{v}\|},$$

so that by Proposition B.14, the distance from the point $\mathbf{w}$ to the plane $\mathcal{P}$ equals

$$|\langle \mathbf{w},\ \eta\rangle| = \frac{|\langle \mathbf{w},\ \mathbf{u}\times\mathbf{v}\rangle|}{\|\mathbf{u}\times\mathbf{v}\|}.$$

On the other hand, by Theorem B.6, the area of the parallelogram based at the origin and bounded by the vectors $\mathbf{u}$ and $\mathbf{v}$ equals $\|\mathbf{u}\times\mathbf{v}\|$. Thus, by the very definition of volume,

$$\text{vol}\ \mathcal{V} = \frac{|\langle \mathbf{w},\ \mathbf{u}\times\mathbf{v}\rangle|}{\|\mathbf{u}\times\mathbf{v}\|}\cdot\|\mathbf{u}\times\mathbf{v}\| = |\langle \mathbf{w},\ \mathbf{u}\times\mathbf{v}\rangle| = |\langle \mathbf{u},\ \mathbf{v}\times\mathbf{w}\rangle|. \qquad \blacksquare$$

Corollary B.16 Let $\mathcal{V}$ be the parallelepiped based at the origin that is spanned by the standard basis vectors $\mathbf{e}_1,\ \mathbf{e}_2$, and $\mathbf{e}_3$. Then for an invertible linear mapping $\mathbf{T}:\mathbb{R}^3\to\mathbb{R}^3$ associated with the 3×3 matrix $\mathbf{A}$, the volume of the image $\mathbf{T}(\mathcal{V})$ is given by the formula

$$\text{vol}\ \mathbf{T}(\mathcal{V}) = |\det \mathbf{A}|\ \text{vol}\ \mathcal{V}. \tag{B.16}$$

Proof

Define $\mathbf{u} = \mathbf{T}(\mathbf{e}_1),\ \mathbf{v} = \mathbf{T}(\mathbf{e}_2)$, and $\mathbf{w} = \mathbf{T}(\mathbf{e}_3)$. Then if the point $(\alpha,\ \beta,\ \gamma)$ is in $\mathcal{V}$,

$$\mathbf{T}(\alpha,\ \beta,\ \gamma) = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}.$$

Hence $\mathbf{T}(\mathcal{V})$ is the parallelepiped bounded by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Thus,

$$\text{vol}\ \mathbf{T}(\mathcal{V}) = |\langle \mathbf{u},\ \mathbf{v}\times\mathbf{w}\rangle|. \tag{B.17}$$

On the other hand, by the very way in which the matrix $\mathbf{A}$ is associated with the linear mapping $\mathbf{T}:\mathbb{R}^3\to\mathbb{R}^3$, it follows that the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ correspond to the first, second, and third columns of the matrix $\mathbf{A}$. Consequently, by the definition of determinant,

$$\det \mathbf{A} = \langle \mathbf{u},\ \mathbf{v}\times\mathbf{w}\rangle. \tag{B.18}$$

The volume formula (B.16) follows from the two preceding equalities since it is clear that vol $\mathcal{V} = 1$. $\blacksquare$

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The triple product $\langle \mathbf{u},\ \mathbf{v} \times \mathbf{w} \rangle$ of the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is nonzero if and only if these vectors are a basis for $\mathbb{R}^3$, and when the triple product is nonzero, Theorem B.15 describes the significance of the absolute value of the triple product. It is natural to inquire as to the significance of the *sign* of the triple product. The ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is defined to be *positively oriented* provided that $\langle \mathbf{u},\ \mathbf{v} \times \mathbf{w} \rangle > 0$. Observe that the standard ordered basis $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is positively oriented since $\langle \mathbf{i},\ \mathbf{j} \times \mathbf{k} \rangle = 1$. Moreover, the geometric significance of being positively oriented is that if the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is positively oriented, then this basis can be continuously deformed into the standard ordered basis in the following precise sense: There are three parametrized paths $\alpha : [0, 1] \to \mathbb{R}^3$, $\beta : [0, 1] \to \mathbb{R}^3$, and $\gamma : [0, 1] \to \mathbb{R}^3$ such that

$$\begin{array}{llll} \alpha(0) = \mathbf{u}, & \beta(0) = \mathbf{v}, & \text{and} & \gamma(0) = \mathbf{w} \\ \alpha(1) = \mathbf{i}, & \beta(1) = \mathbf{j}, & \text{and} & \gamma(1) = \mathbf{k}, \end{array}$$

and for each parameter value t in $[0, 1]$, the vectors $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are a basis for $\mathbb{R}^3$.

Observe that if the vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ is positively oriented since

$$\langle \mathbf{u},\ \mathbf{v} \times (\mathbf{u} \times \mathbf{v}) \rangle = \langle \mathbf{u} \times \mathbf{v},\ \mathbf{u} \times \mathbf{v} \rangle > 0.$$

The fact that the ordered basis $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is positively oriented is what is informally described in elementary courses as the "right-hand rule."

The determinant can be defined for any $n \times n$ matrix; it has the same algebraic significance as it does in the 3×3 case. In Section 15.1, we provide a definition by induction. By inspection, we see that in the case of 3×3 matrices, this definition is consistent with the definition given in this appendix.[2]

EXERCISES FOR APPENDIX B

1. Find the equation of the line ℓ through the origin in $\mathbb{R}^3$ parallel to the vector $(1, 0, 2)$. Find the distance from the point $(0, 2, 4)$ to this line.
2. Show that the set $\{\mathbf{u} = (x, y, z) \mid 2x + 3y - z = 0\}$ is the plane through the origin that is normal to $\eta = (2, 3, -1)$. Find the distance from the point $(1, 1, 0)$ to this plane.
3. Show that the triple of vectors $\mathbf{u} = (1, 0, 1/2)$, $\mathbf{v} = (0, 2, 1)$, and $\mathbf{w} = (-4, 0, 0)$ is a basis for $\mathbb{R}^3$. Use formula (B.8) to write the point $(1, 0, 0)$ in this basis and also to write the point $(0, 1, 0)$ in this basis.
4. For a nonzero vector $\mathbf{u}$ in $\mathbb{R}^3$, find a nonzero vector $\mathbf{v}$ that is orthogonal to $\mathbf{u}$ and then find a nonzero vector $\mathbf{w}$ such that the triple $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a basis for $\mathbb{R}^3$.
5. Find the area of the parallelogram based at the origin and bounded by the vectors $(1, 0, 2)$ and $(0, 0, 1)$.

[2] A clear exposition of elementary linear algebra can be found in the book by David C. Lay, *Linear Algebra and Its Applications* (Boston: Addison Wesley, 2002). A more advanced exposition can be found in the book by Peter D. Lax, *Linear Algebra* (New York: John Wiley, 1996).

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6. For two points (x_1, y_1) and (x_2, y_2) in the plane $\mathbb{R}^2$, define the complex product, denoted by $(x_1, y_1)(x_2, y_2)$, by the formula

$$(x_1, y_1)(x_2, y_2) \equiv (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

Define the sum of two points to be the usual sum.

a. Show that $(1, 0)(x, y) = (x, y)$ for every point (x, y) in $\mathbb{R}^2$; that is, show that the point $(1, 0)$ is the multiplicative identity.

b. Show that $(0, 0) + (x, y) = (x, y)$ for every point (x, y) in $\mathbb{R}^2$; that is, show that the point $(0, 0)$ is the additive identity.

c. Finally, show that with the usual definition of sum and with the product being the complex product, the Field Axioms are satisfied.

7. Express the two points (x_1, y_1) and (x_2, y_2) in the plane $\mathbb{R}^2$ in polar coordinates as $(x_1, y_1) = (r_1 \cos\theta_1, r_1 \sin\theta_1)$ and $(x_2, y_2) = (r_2 \cos\theta_2, r_2 \sin\theta_2)$. Use the cosine and sine addition formulas to show that the complex product defined in Exercise 6 can be written as

$$(x_1, y_1)(x_2, y_2) = (r_1r_2 \cos(\theta_1 + \theta_2), r_1r_2 \sin(\theta_1 + \theta_2)).$$

a. Use this formula to provide a geometric interpretation of the complex product.

b. Find a geometric interpretation of the complex multiplicative inverse of a point $(x_1, y_1) \neq (0, 0)$ in $\mathbb{R}^2$.

8. Let the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ be orthogonal. Show that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

9. Show that there is no vector $\mathbf{u}$ that has the property that $\mathbf{u} \times \mathbf{v} = \mathbf{v}$ for every vector $\mathbf{v}$ in $\mathbb{R}^3$.

10. Given vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, under what conditions is there a vector $\mathbf{w}$ such that $\mathbf{u} \times \mathbf{w} = \mathbf{v}$?

11. Suppose that the triple of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is a basis for $\mathbb{R}^3$. For any pair of numbers α and β, show that the parallelepiped based at the origin and bounded by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ has the same volume as the parallelepiped based at the origin and bounded by $\mathbf{u}$, $\mathbf{v}$, and $\alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{w}$. Interpret this result geometrically.

12. Show that the system of equations

$$a_{11}x_1 + a_{12}x_2 = y_1$$
$$a_{21}x_1 + a_{22}x_2 = y_2$$

has a unique solution (x_1, x_2) for each pair of numbers (y_1, y_2) if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. *Hint:* Show that the above system of equations is equivalent to the following system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + 0x_3 &= y_1 \\ a_{21}x_1 + a_{22}x_2 + 0x_3 &= y_2 \\ 0x_1 + 0x_2 + 1x_3 &= y_3. \end{aligned}$$

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