6.2

dériver

$h(10) = 25 \quad h'(10) = -3 \quad h''(10) = 1$ et $h'''(10) = \frac{3}{5}$

$f(x) = t_n(x) = \sum_{n=0}^{n} C_n (x-a)^n$

au voisinage de a

$h(x) = \sum_{k=0}^{n} \underbrace{\frac{h^{(k)}(a_0)}{n!}}_{C_n} (x-a)^k \simeq t_3(x) \quad , a = 10$

avec $C_k = \frac{f^{(k)}(a)}{n!}$

$t_1(x) = \underset{f(a)}{C_0} + \underset{\frac{f'(a)}{1!} = f'(a)}{C_1} (x-a)$

$T_3(x) = h(10) + \frac{h'(10)}{1!}(x-10) + \frac{h^{(2)}(10)}{2!}(x-10)^2 + \frac{h^{(3)}(10)}{3!}(x-10)^3$

$= 25 - 3(x-10) + \frac{1}{2}(x-10)^2 + \frac{3/5}{6}(x-10)^3$

$h(11) = t_3(x) = 25 - 3(1) + \frac{1}{2}(1)^2 + \frac{3/5}{6}(1)^3 = \frac{113}{5}$

$= 22,6$

6.4

$t_4(x) = 15 + 2x - 2x^2 + \frac{x^4}{8}$

$t_4(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + C_4(x-a)^4$

$a = 0$
$C_0 = 5$
$C_1 = 2$
$C_2 = -2$
$C_3 = 0$
$C_4 = \frac{1}{8}$

avec $C_n = \frac{f^{(n)}(a)}{n!}$

On a $C_n = \frac{f^{(n)}(a)}{n!}$ donc $f^{(n)}(a) = n!\, C_n$

$f(0) = C_0 = 5$

$f'(0) = 1!\, C_1 = 2$

$f''(0) = 2!\, C_2 = -4$

$f'''(0) = 3!\, C_3 = 0$

$f''''(0) = 4!\, C_4 = 24 \cdot \frac{1}{8} = 3$

6.5

b) $f(x) = \sin 4x$

$t_6 = \sum_{n=0}^{6} C_n (x-0)^n$ avec $C_n = \frac{f^{(n)}(0)}{n!}$

t_i

$f(x) := \sin(4 \cdot x)$

$\frac{d^1}{dx^1}(f(x)) \mid x = 0$

f) $f(x) = 3x \cdot \ln(1+3x)$ avec $a = 0$

$t_6 = \sum_{n=1}^{6} C_n x^n$ avec $C_n = \frac{f^{(n)}(0)}{n!}$

$f(0) = 0 \Rightarrow C_0 = 0$

$f'(0) = 0 \Rightarrow C_1 = 0$

$f''(0) = 18 \Rightarrow C_2 = \frac{f^{(2)}(0)}{2!} = 9$

$f^{(6)}(0) = 164926 \Rightarrow C_6 = \frac{f^{(6)}(0)}{6!} = \frac{729}{5}$

6.11

a)

on a $u_n = \frac{2^n (x-5)^n}{n+1}$

$R = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = |2(x-5)|$

$u_{n+1} = \frac{2^{n+1} (x-5)^{n+1}}{n+2}$

on a $R = \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{2^{n+1}(x-5)^{n+1}}{n+2}}{\frac{2^n (x-5)^n}{n+1}} \right| = \left| 2(x-5) \cdot \frac{n+1}{n+2} \right|$

Avec la ti

$u(n) := \frac{2^n (x-5)^n}{n+1}$

$u(n+1)$

$\frac{2(x-5)(2(x-5))^n}{n+2}$

$S(x)$ converge si $|2(x-5)| < 1$

donc $-1 < 2(x-5) < 1$

$-\frac{1}{2} < x-5 < \frac{1}{2}$

$\frac{9}{2} < x < \frac{11}{2}$

$I_c = \left] \frac{9}{2}, \frac{11}{2} \right[$

$\left| \frac{u(n+1)}{u(n)} \right|$

$2 \left| \frac{(n+1)(x-5)}{n+2} \right|$

$\lim_{n \to \infty} \left(\left| \frac{u(n+1)}{u(n)} \right| \right)$

$2|x-5|$

solve$(2|x-5| < 1, x)$

c)

$u(n) = \frac{(-1)^n \cdot x^{2n+1}}{2n+1}$

$u(n) := \frac{(-1)^n x^{2n+1}}{2n+1}$

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = x^2$

$u(n+1)$

$u(n+1) = \frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1} = \frac{(-1)^{n+1} x^{2n+3}}{2n+3}$

$R < 1 \iff x^2 < 1$

$\iff -1 < x < 1$

d'où $I_c =]-1, 1[$

$\left| \frac{u(n+1)}{u(n)} \right|$

$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{(-1)^{n+1} \cdot x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right| = \left| x^2 \frac{2n+1}{2n+3} \right|$

$\lim_{n \to \infty} \left(\left| \frac{u(n+1)}{u(n)} \right| \right)$

solve$(x^2 < 1, x)$

d)

$u(n) = \frac{n! x^n}{100^n}$

$u(n+1) = \frac{(n+1)! x^{n+1}}{100^{n+1}}$

$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\frac{(n+1)! x^{n+1}}{100^{n+1}}}{\frac{n! x^n}{100^n}} \right| = \left| \frac{n+1}{100} x \right|$

on $(n+1)! = (n+1) n!$

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \infty$ donc $S(x)$ diverge partout sauf $x = 0$

6.14

i) $f(x) = 3x \ln(1+3x)$

on a $\ln(1+u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{n+1}}{n+1}$ | ou bien $= u - \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4} + \frac{u^5}{5} + \dots \to \infty$

$u \in]-1,1]$

On a $\ln(1+3x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{n+1}}{n+1}$ $\quad = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \frac{(3x)^5}{5} \dots$ avec $-1 < 3x < 1 \Leftrightarrow -1/3 < x < 1/3$

On a $3x \ln(1+3x) = \sum_{n=0}^{\infty} (-1)^n \frac{3x\,(3x)^{n+1}}{n+1}$ $\quad = (3x)^2 - \frac{(3x)^3}{2} + \frac{(3x)^4}{3} - \frac{(3x)^5}{4} + \frac{(3x)^6}{6}$

$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{n+2}}{n+1}$

g) $f(x) = \frac{1}{1+2x^2}$

on a $f(x) = \sum_{n=0}^{\infty} (-2x^2)^n$

ou bien

$f(x) = 1 + (-2x^2)^2 + (-2x^2)^2 + (-2x^2)^3 + (-2x^2)^4 + \dots$

$= 1 - 2x^2 - 2^2x^4 - 2^3x^6 + 2^4x^8 - 2^5x^{10} + \dots$

$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty} = 1 + u + u^2 + u^3 - \dots$

$u = -2x^2$ $\quad -1 < -2x^2 < 1$

$-1/2 < x^2 < 1/2$

$\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

6.21

b) $I = \int_0^{1/2} x^2 e^{-x^2} dx$

$I = \int_0^{1/2} x^2 e^{-x^2} dx = \int_0^{1/2} \left(x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \dots\right)$

$= \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{2!\,7} + \dots\right]_0^{1/2}$

On a $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$

$e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots$

$x^2 e^{-x^2} = x^2 - \frac{x^4}{1!} + \frac{x^6}{2!} - \frac{x^8}{3!} + \frac{x^{10}}{4!}$

6.24

c) $1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots$

$= a + ar + ar^2 + ar^3 + \dots$

$a = 1$

$r = 2/3$

On a $|r| < 1$, donc la série converge

$S = \frac{a}{1-r}$

$= \frac{1}{1-2/3} = 3$

d) $\sum_{n=0}^{\infty} \frac{3^{n+1}}{4^n} = \frac{3}{4^0} + \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n}$

$= 3 + \sum_{n=1}^{\infty} \left(\frac{9}{4}\right)\left(\frac{3}{4}\right)^{n-1}$

$S = \sum_{n=1}^{\infty} a r^{n-1}$

si $|r| < 1 \Rightarrow S = \frac{a}{1-r}$

sinon S diverge

On a $a = \frac{9}{4}$ $\quad r = \frac{3}{4}$

$|r| < 1$ donc la série converge

et $S = 3 + \frac{a}{1-r} = 3 + \frac{9/4}{1-3/4} = 12$

1)

$$\sum_{n=1}^{\infty} \frac{4^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \frac{4^n}{6^n} + \sum_{n=1}^{\infty} \frac{2^n}{6^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

$$S_1 = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} 2/3 \, (2/3)^{n-1} = \frac{a}{1-r}$$

$a = 2/3 \quad r = 2/3$

$$S_1 = \frac{2/3}{1-2/3} = 2$$

$$S_2 = \sum_{n=1}^{\infty} (1/3)^n = \sum_{n=1}^{\infty} 1/3 \, (1/3)^{n-1}$$

$a = 1/3 \,,\, r = 1/3 < 1$

donc $S_2 = \frac{a}{1-r} = \frac{1/3}{1-1/3} = 1/2$

donc $S = S_1 + S_2 = 2 + 1/2 = 5/2$

6.25

$$d = B + cB + c^2B + c^3B + \dots$$

$$\sum_{n=1}^{\infty} Bc^{n-1} = \frac{B}{1-c}$$