STA642 hw2

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Q2

(a)

$$
\begin{aligned}
y_t &= x_t + \nu_t \\
&= \phi x_{t-1} + \epsilon_t + \nu_t \\
&= \phi(y_{t-1} - \nu_{t-1}) + \epsilon_t + \nu_t \\
&= \phi y_{t-1} + \eta_t \\
&\text{where } \eta = \epsilon_t + \nu_t - \phi\nu_{t-1}
\end{aligned}
$$

(b)

Note that $Var(\eta_t) = Var(\epsilon_t) + Var(\nu_t) + \phi^2 Var(\nu_{t-1})$ and $C(\eta_t, \eta_{t-1}) = C(\epsilon_t + \nu_t - \phi\nu_{t-1}, \epsilon_{t-1} + \nu_{t-1} - \phi\nu_{t-2}) = -\phi\omega +$bunch of 0's (by iid). Thus we have $Corr(\eta_t, \eta_{t-1}) = \frac{-\phi\omega}{\omega(1+\phi^2)+\nu}$.

(c)

Note that $Var(y_t) = s + \omega = q$. For $C(y_t, y_{t+k})$:

$$
\begin{aligned}
C(y_t, y_{t+k}) &= C(x_t + \nu_t, \phi(x_t + \nu_t) + \sum_{i=1}^{k-1} \phi^i(\epsilon_{t+k-i} + \nu_{t+k-i} - \phi\nu_{t+k-i-1})) \\
&= C(x_t, \phi^k x_t) \text{ by iid} \\
&= \phi^k s
\end{aligned}
$$

Thus $Corr(y_t, y_{t+k}) = \frac{s}{q}\phi^k$.

Note that the correlation is positively related to the signal:noise ratio, this is intuitive. If *omega* is large relatively to s (small s/q), then more variation will be added to the AR(1) model, x_t, to form y_t. Then the auto-correlation of y_t will be small relatively. Note that ϕ^k is the lag-k auto-correlation of x_t.

(d)

I think y_t is not AR(1) because η_t is not iid. y_t is Markov since $y_t = \phi y_{t-1} + \eta_t$ only depends on y_{t-1}.

Q3

For 2 by 2 matrix case:

(a)

$AG = \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{bmatrix}$
AG is symmetric.

(b)

(i)

Suppose $|\phi_p| \neq 0$, then $\|A| = det(\begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix}) = \phi_2 \neq 0$, thus A is non-singular.

(ii)

$G'A = \begin{bmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & 0 \end{bmatrix} = AG$

(iii)

$AF = \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = F$

(c)

It suffices to show that $G'z_{t-1} + F\epsilon_t = Ax_t$. Apply (b), Note that since A is non-singular, A^{-1} exists.

$$
\begin{aligned}
& G'z_{t-1} + F\epsilon_t = \\
& = AGA^{-1}z_{t-1} + AF\epsilon_t \\
& = A(GA^{-1}z_{t-1} + F\epsilon_t) \\
& = A(GA^{-1}Ax_{t-1} + F\epsilon_t) \\
& = A(Gx_{t-1} + F\epsilon_t) \\
& = Ax_t
\end{aligned}
$$

(d)

$$z_t = Ax_t = \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} y_t \\ \phi_2 y_{t-1} \end{bmatrix}$$

The first element of z_t is y_t, the current state. For the rest of the elements, the elements are the weighted sum of $y_{t-1}, .., y_{t-p+1}$ with weights of ϕ's. Note that G and G' are similar, thus x_t and z_t form the same forecast function.