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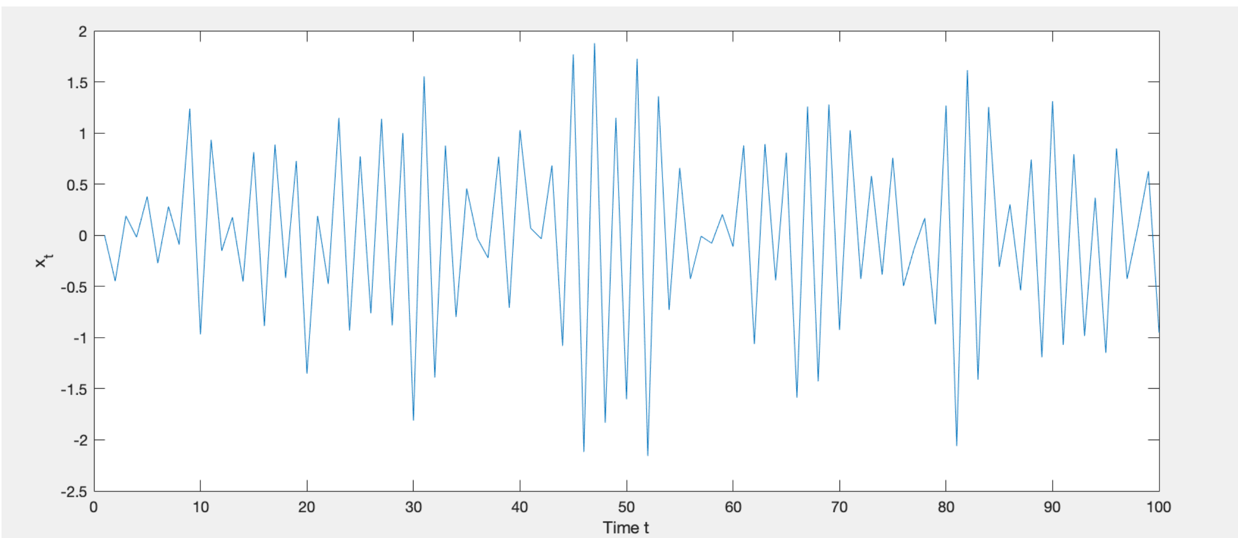
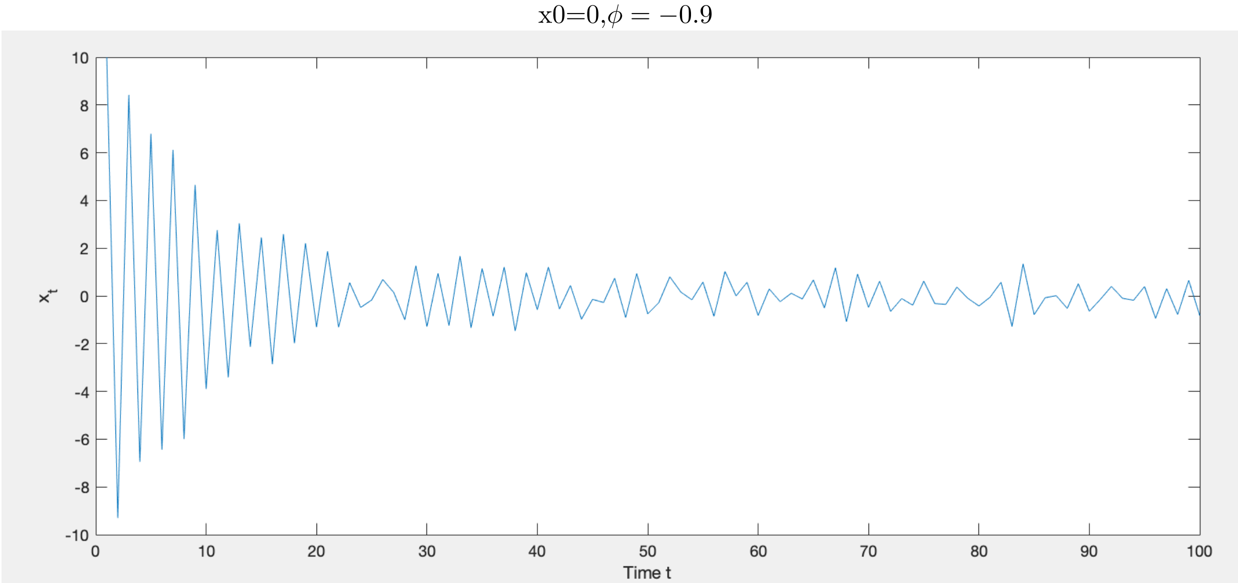
1 Q3

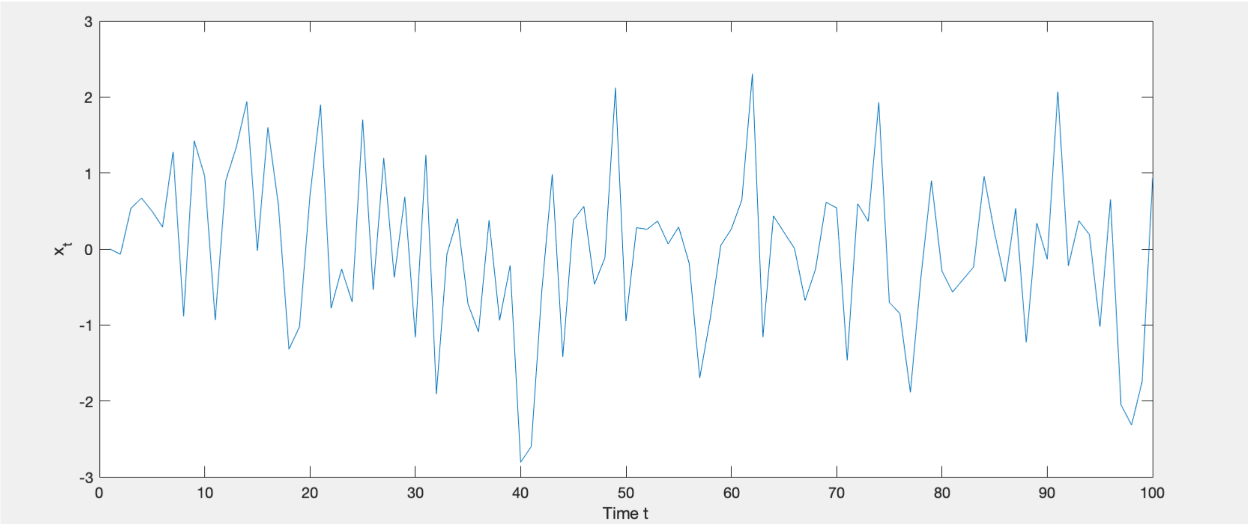
I write a new simulation function with addition of initial value, x0.



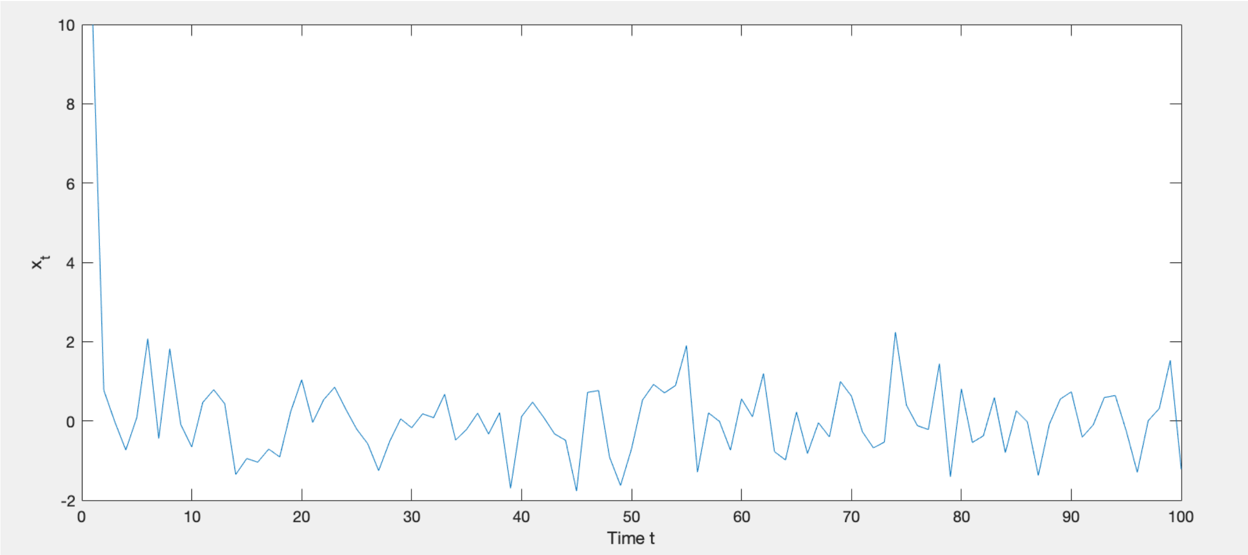
x0=0,$\phi = -0.9$



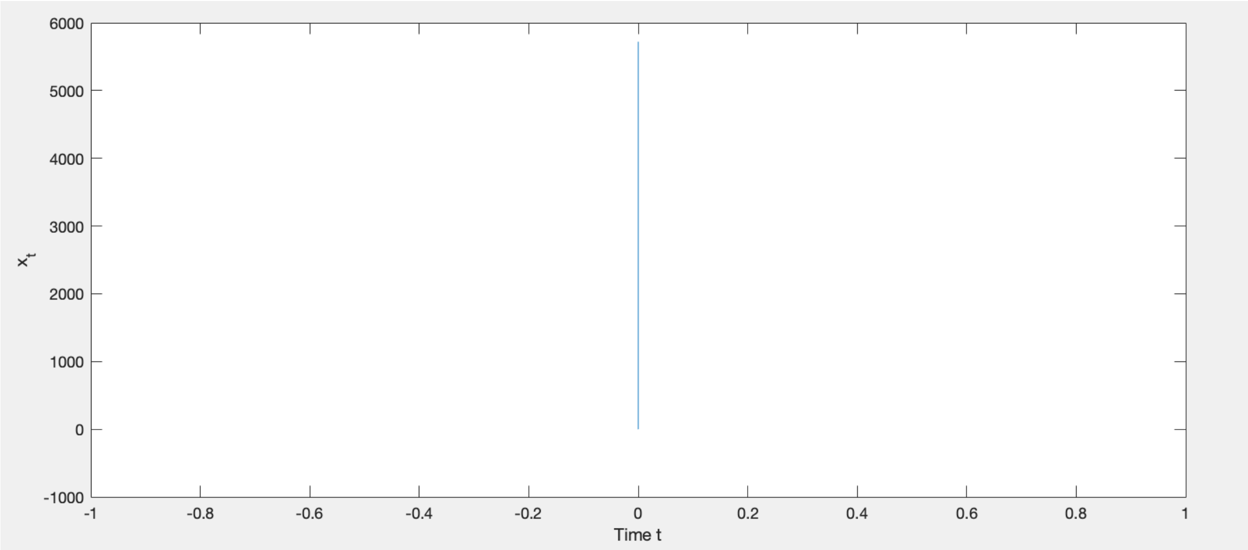
x0=10,$\phi = -0.9$



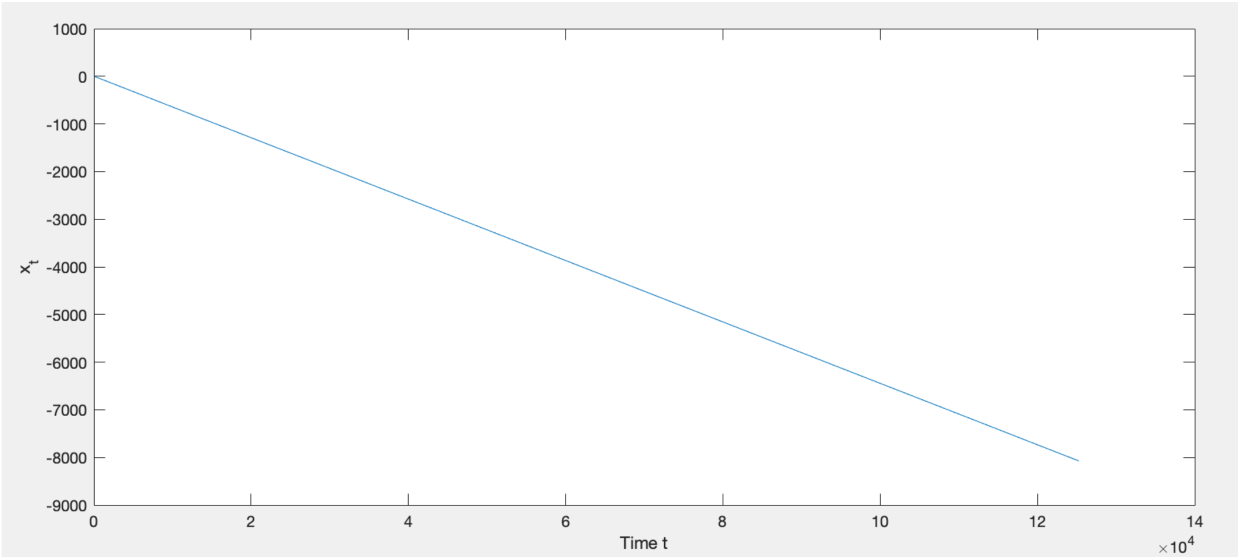
x0=0,$\phi = 0.2$



x0=10,$\phi = 0.2$



x0=0,$\phi = 1.1$



x0=10,$\phi = 1.1$

When ϕ is smaller than 0, with any x0, the series will approach to 0; the larger the ϕ, the slow the converge. When ϕ is greater than 1, the series will diverge. Negative phi will be more oscillate than positive ϕ. The initial value does not matter after the mixing of series.

The above algorithm is a realization of decomposing the joint distribution to conditional densities conditioning on last value and a density of first value. It needs the earlier value for generating the later. The results of the algorithm and the stationary joint distribution are the same, but stationary joint distribution might has distribution such as multivariate normal which can sample multiple values at once.

2 Q4

Before start Q4, I want to proof some facts about normal-gamma distribution which I can apply latter. Suppose (X,T) are random variables from a normal-gamma distribution(u, λ, a, b). The joint pdf will be $f(x,r) = \frac{b^a\sqrt{(\lambda)}}{\Gamma(a)\sqrt{2\pi}}\gamma^{a-1/2}exp(-(b+\lambda(x-u)^2)r/2)$. Claim: The marginal of x is T(2a,u,b/λa).
Proof:

$$
\begin{aligned}
&p_X(x|r,u,\lambda,a,b)\\
&= \int p(x,r|u,\lambda,a,b)dr\\
&= \frac{b^a\sqrt{\lambda}}{\Gamma a\sqrt{2\pi}}\int r^{a-1/2}exp(-(b+\lambda(x-u)^2/2)r)dr\\
&= \frac{b^a\sqrt{\lambda}\Gamma(a+1/2)}{\Gamma a\sqrt{2\pi}(b+\lambda(x-u)^2/2)^{a+1/2}}\\
&= \frac{\Gamma(a+1/2)\lambda a}{\Gamma(a)\sqrt{2\pi a}b}(1+\frac{(x-u)^2b^2}{2a^3\lambda})^{-(a+1/2)}\\
&= T(x|2a,u,b/\lambda a)
\end{aligned}
$$

For Q4, given the full conditional distribution from lecture notes, we have $\phi|x,v \sim N(b, vB^{-1})$, and $v|x \sim invGa(n/2-1, Q(b)/2)$. Apply above fact. We have that $\phi|x \sim T(n-2, b, \frac{Q(b)}{B(n-2)})$

(a)

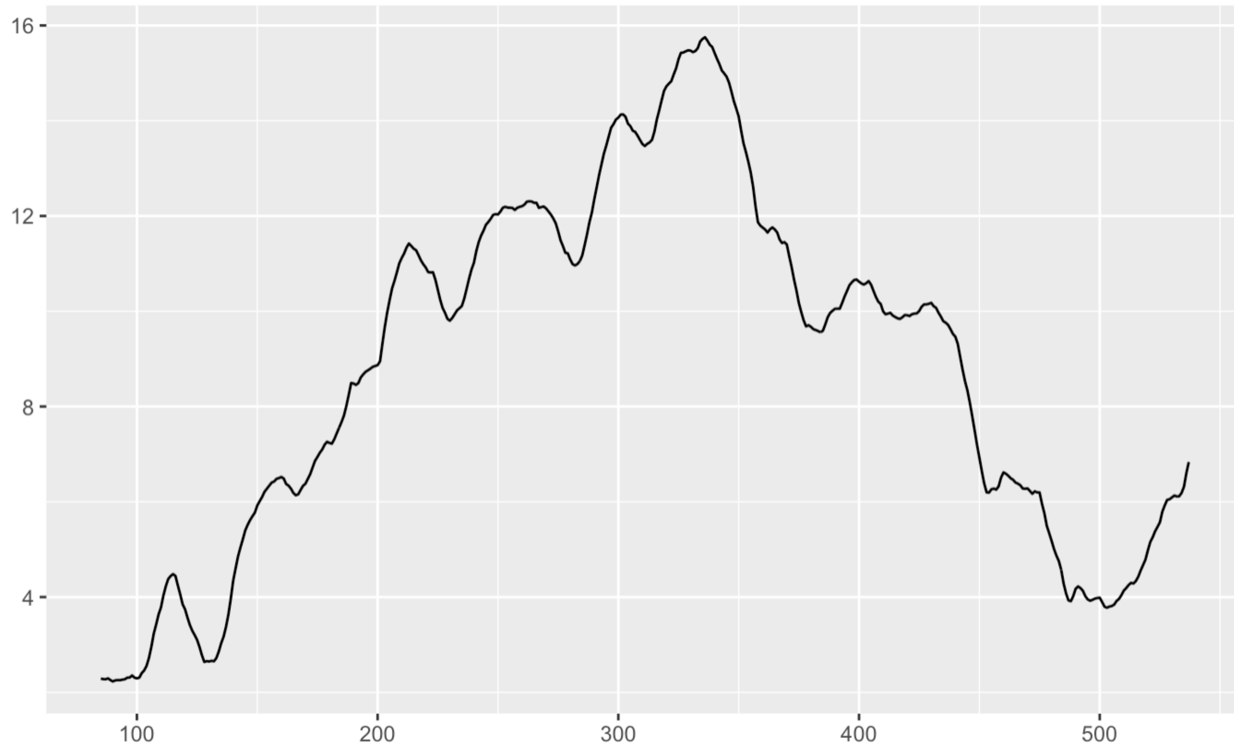
credible interval

The plot has a V-shape, which shows a decreasing and increasing session at the middle of the period. At the begin and the end, ϕ is between 0.9 and 0.95 for probability of 0.9. At the middle, the credible interval of ϕ decreased down to 0.8 and 0.87. Decrease of *phi* implies lower correlation.

(b)

Yes, I do. The Credible intervals are volatile which means that the AR1 model has unstable ϕ over time. This is a sign of change in monthly dependency.

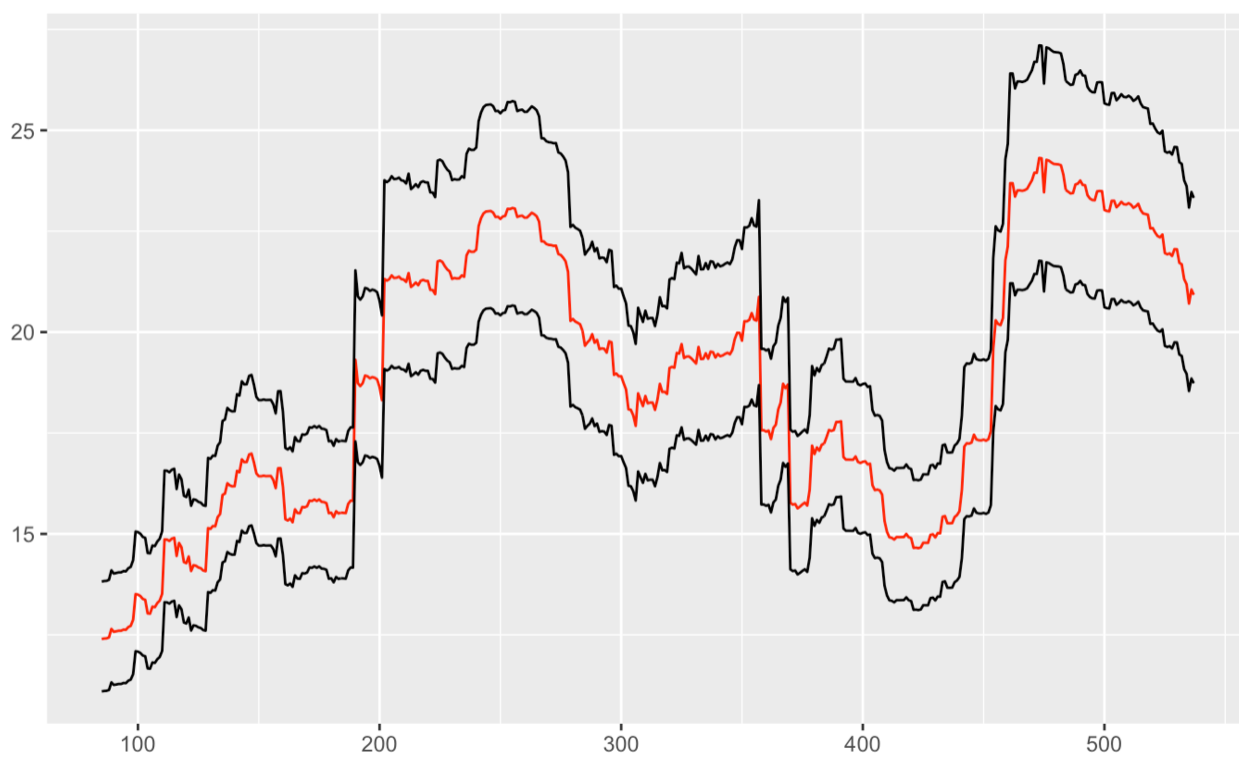
(c)



mean

Based on the plot, the mean is not stable. It is increasing for the first half of the period and then decrease to approximately the original level.

(d)



posterior expectation of innovation

The innovation is gradually increasing. It is generally high at the middle of the period and reach the peak at the end.

(e)

We could first use AR(p) with p greater than 1. We could also use a moving average or ARMA.

3 Q5

(a)

We apply the fact in Q4. Note that $u = m_{t-1}, \lambda = \frac{S_{t-1}}{C_{t-1}}, a = n_{t-1}/2, b = n_{t-1}S_{t-1}/2$. We have T$(\phi|n_{t-1}, m_{t-1}, C_{t-1})$

(b)

$X_t = \phi X_{t-1} + \epsilon$, ϵ follows N(0,ν). Note that X_t is sum of two normal random variables, X_t must be normal distributed.

E[X$_t|\nu, D_{t-1}] = E[\phi X_{t-1} + \epsilon] = m_{t-1}X_{t-1}$
$Var(X_t|\nu, D_{t-1}) = X_{t-1}^2 Var(\phi) + Var(\epsilon) = X_{t-1}^2 \frac{C_{t-1}\nu}{S_{t-1}} + \nu$
Thus we have $x_t|\nu, D_{t-1}$ follows $Normal(f_t, q_t\nu/s_{t-1})$ defined as in question.

(c)

p(x$_t|D_{t-1})$
$= \int p(x_t|v, D_{t-1})p(v|D_{t-1})dv$
$= \int 1/(\sqrt{2\pi q_t/S_{t-1}})v^{-1/2}exp(-(x_t - f_t)^2/(2q_t v/S_{t-1}))\frac{(n_{t-1}S_{t-1}/2)^(n_{t-1}/2)}{\Gamma(n_{t-1}/2)}v^{-(n_{t-1}-1)/2}exp(-(n_{t-1}S_{t-1}/2)v^{-1})$
$= \frac{(n_{t-1}S_{n-1}/2)^{n_{t-1}/2}\Gamma(n_{t-1}-2)/2}{\sqrt{2\pi q_t/S_{t-1}}\Gamma(n_{t-1}/2)\frac{(x_t-f_t)^2+n_{t-1}q_t}{2q_t/S_{t-1}}}$
(d)

We update v first and then ϕ.
p(v—D$_{t-1}, x_t)$
$= p(x_t|v, D_{t-1})p(v|D_{t-1})$
$exp(S_{t-1}(x_t - f_t)^2 v^{-1}/2q_t)v^{(-n_{t-1}+2)/2}exp(-((n_{t-1}s_{t-1}/(2q_t)) + (s_{t-1}(x_t - f_t)^2)/(2q_t)))$
$= v^{-(n_{t-1}/2+1)}exp(-(s_{t-1}(n_{t-1} - (e_t^2)/q_t)/2)v^{-1})$ This is a invGa distribution with updated parameter n and s given by the question.

For next part, I did not compute the whole version due to the time constraint. But the attempt form is as below.

p($\phi, v|D_{t-1}, x_t)$
$p(x_t|v, D_{t-1}, \phi)p(\phi|v, D_{t-1})p(v|D_{t-1})$
I would have done that if I had more time.
(e)

i.

m_t and $C - t$ represents the mean and variance of ϕ. m_t is updated by the sum of m_{t-1} and the product of 1-step forecast error and adaptive coefficient. For C_t, it inherits some portion of variance from C_{t-1}.

ii.

For m_t, the role of adaptive coefficient is the weight of 1-step forcast error that will vary the mean, m_t. For C_t, adaptive coefficient is the weight of variance changed by the additional observations on the variance of ϕ, it also reminds me the backward operation, B.

iii.

As we collect a new observation, x_t, the degree of freedom is intuitively increasing by 1. For s_t, it updates by first computing e_t^2/q_t, which represents the variance of x_t, and then averaging the additional variance with previous data.

(f)

In this case, r_t will almost surely increase from r_{t-1}, which will exponentially increase updated s_t each time. This is resulting in great innovation. In addition, if e_t is large, C_t is also likely to keep increasing, which will cause large variance for ϕ.