STA642 hw5

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Q4

(a)

$$E(\phi_1|\phi_0) = \frac{\phi_0 E[\eta]}{\beta} = \phi_0$$

(b)

$$E(\phi_0) = \frac{a}{b}$$
$$E(\phi_1) = E(E(\phi_1|\phi_0)) = \frac{a}{b}$$

(c)

$$P(\phi_0, \eta) = Ga(\phi_0|a, b)Be(\eta|\beta a, (1-\beta)a)$$
$$g^{-1}(\phi_0, \phi_1) = (\phi_0, \frac{\phi_1\beta}{\phi_0})$$
$$|J_{g^{-1}}| = det(\begin{bmatrix} 1 & 0 \\ -\phi_1\beta\phi_0^{-1} & \frac{\beta}{\phi_0} \end{bmatrix}) = \frac{\beta}{\phi_0}$$
$$p_{\phi_0,\phi_1}(\phi_0, \phi_1) = p_{\phi_0,\eta}(\phi_0, \eta)|J_{g^{-1}}|$$
$$= ce^{-b\phi_0}\phi_0^{a-1}\frac{\phi_1}{\phi_0}^{\beta a-1}(1 - \frac{\phi_1\beta}{\phi_0})^{(1-\beta)a-1}\frac{\beta}{\phi_0}$$
$$= ce^{-b\phi_0}\phi_0^{a-1-(1-\beta)a+1-\beta a}\phi_1^{\beta a-1}(\phi_0 - \phi_1\beta)^{(1-\beta)a-1}$$
$$= ce^{-b\phi_0}\phi_1^{\beta a-1}(\phi_0 - \phi_1\beta)^{(1-\beta)a-1}$$

Note that $\frac{\beta\phi_1}{\phi_0}$ is from Beta distribution, thus $1 > \frac{\beta\phi_1}{\phi_0} > 0$, thus $0 < \phi_1 < \phi_0/\beta$.

(d)

$$
\begin{aligned}
&p(\phi_1) = \int p(\phi_0, \phi_1) d\phi_0 \\
&\propto \phi_1^{\beta a-1} \int e^{-b\phi_0} (\phi_0 - \beta\phi_1)^{(1-\beta)a-1} \\
&\propto \phi_1^{\beta a-1} e^{-b\beta\phi_1} \int e^{-b(\phi_0-\beta\phi_1)} (\phi_0 - \beta\phi_1)^{(1-\beta)a-1} d(\phi_0 - \beta\phi_1) \\
&\propto \phi_1^{\beta a-1} e^{-b\beta\phi_1} \\
&\phi_1 \sim Ga(\beta a, \beta b)
\end{aligned}
$$

(e)

Note that in above derivation we have:

$$(\phi_0 - \beta\phi_1) \sim Ga((1-\beta)a, b)$$

Rewrite this by letting $\gamma = \phi_0 - \beta\phi_1$, we get the desired form.

Q5

We have $\phi_t = \frac{\phi_{t-1}\gamma_t}{\beta}$. Apply Q4, suppose $(\phi_{t-1}|D_{t-1}) \sim Ga(\frac{n_{t-1}}{2}, \frac{d_{t-1}}{2})$, then $(\phi_t|D_{t-1} \sim Ga(\frac{\beta n_{t-1}}{2}, \frac{\beta d_{t-1}}{2})$.

Q6

(a)

Apply Q4(e) by replacing a by $a_{t-1}/2$, b by $d_{t-1}/2$, we obtain $\nu_{t-1}^* = \phi_{t-1} - \beta\phi_t \sim Ga((1-\beta)n_{t-1}/2, d_{t-1}/2)$.

(b)

Since $\phi_{t-1} = \beta\phi_t + \nu_{t-1}^*$ depends on ϕ_t and ν_{t-1}^*, ν_{t-1}^* depends on D_{t-1} through n_{t-1}, d_{t-1}. Thus $p(\phi_{t-1}|\phi_t, D_T) \equiv p(\phi_{t-1}|\phi_t, D_{t-1})$ for $T \geq t$. This can also be shown in graphic model.

(c)

Note that $E[\nu_{t-1}^*|D_{t-1}] = \frac{(1-\beta)n_{t-1}}{d_{t-1}}$. Given $E[\phi_t|D_t] = \frac{n_t}{d_t}$, then $E[\phi_{t-1}|D_T, \phi_t] = E[\phi_{t-1}|D_{t-1}, \phi_t] = \beta E[\phi_t|D_t] + E[\nu_{t-1}^*|D_{t-1}] = \frac{\beta n_t}{d_t} + \frac{(1-\beta)n_{t-1}}{d_{t-1}}$.

(d)

Sample $\phi_T|D_T\ from\ Ga(\frac{n_T}{2}, \frac{d_T}{2})$, then sample recursively downwards from $t = T-1$. Suppose we have ϕ_t, sample $\nu^*_{t-1}\ from\ Ga((1-\beta)n_{t-1}/2, d_{t-1}/2)$, let $\phi_{t-1} = \beta\phi_t + \nu^*_{t-1}$.