# Redividing the uniform interval cake

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## 1 Abstract

An interval has already been divided unfairly. We want to find a way to redivide said interval more fairly while giving every agent single interval pieces. A paper has already introduced an interesting re-division fairness metric. In this paper, I will talk about the feasibility of the property.

## 2 Background and related work

Fair division is a well-studied area in social choice theory and has applications in many real-world settings, such as dividing inheritance or allocating resources among organizations. The problem of dividing a cake among n agents has been extensively studied in the literature. In this variant of the cake-cutting problem, the goal is to allocate a cake that has already been divided among n agents in a fair way that satisfies specific properties.

One popular fairness criterion in cake cutting is proportionality, which ensures that every agent receives a piece of cake that they value at least as much as any other piece. However, in some cases, proportionality may not be achievable, and other fairness criteria are considered instead.

One such criterion is r-proportionality, which is a relaxation of proportionality. In this variant, every agent receives at least an $r/n$ fraction of the total cake, where $r$ is a constant independent of n. Another important criterion is democratic ownership. Formally democratic ownership is satisfied when for every integer $d \in \{1, ..., n-1\}$, at least n-d agents receive more than a fraction $1/n/d$ of their previous value. In "Redividing the Cake" Erel Segal Halevi proposed this as a relaxation of w-ownership where every agent received at least a w-fraction of their previous valuation.

In the paper "Redividing the Cake" by Erel Segal Halevi, the problem of redividing a cake that has already been divided among n agents is considered. Halevi proposed an algorithm that guarantees 1/2-proportionality and democratic ownership, even when the original allocation of the cake was not fair.

In this paper, I will argue that democratic ownership isn't a good fairness property since it can be satisfied by proportionality and thus there is no ownership trade-off.

### 2.1 Proposition 1

Proportionality is always compatible with democratic ownership.

### 2.2 Theorem 1

There is an algorithm that redivides an allocation with 1-proportionality and is compatible with democratic ownership.

The way I will show this is using valuation functions where every agent views the whole cake as a uniform distribution

# 3 Model

The cake $C$ is an interval that has to divided among $n \geqslant 1$ *agents*. Each agent $i \in [n]$ has a *value-density* function $v_i$, which is integrable. The value of an interval $X_i$ to agent i is $V_i(X_i) = \int_{x \in X_i} v_i(x)dx$.

Values are queried with *eval queries* and *mark queries* (same as cut queries but previous paper uses this notation).

Allocations are denoted by $X = (X_1, X_2...)$ where each $X_i$ is the interval agent i gets and all the $X_i$ s are disjoint and are a subset of C.

R-proportionality is when every agent receives at least, $r/n$ of the total cake: $\forall i \in [n] V_i(X_i) \geqslant (r/n) * V_i(C)$, $r \in (0, 1)$

W-ownership is when each agent gets at-least $w$ times their old allocation, $w \in (0, 1)$. Formally for a new allocation X and old allocation Z: $V_i(X_i) \geqslant w * V_i(Z_i)$

Democratic ownership is when for every integer $d \in \{1, ..., n-1\}$, at least n-d agents receive more than a fraction $1/n/d$ of their previous value.

# 4 Proofs

## 4.1 Proof of Proposition 1

The following is the linear program that given an allocation Z represented by a sorted vector of length n finds the highest r compatible with democratic ownership:

$$\begin{aligned} &\max \quad r \\ \text{s.t.} \quad &\sum X_i \leqslant \sum Z_i \text{ (normalising allocations)} \\ &\mathbb{X}_i \geqslant \frac{r}{n} \sum Z_i, \forall i \text{ (r-proportionality)} \\ &\mathbb{X}_1 \geqslant 0 \\ &\mathbb{X}_j \geqslant \frac{1}{\lceil \frac{n}{j-1} \rceil} Z_j, \forall j \in 2, .., n \text{ (democratic ownership)} \\ &X_i \geqslant 0, \forall i \end{aligned}$$

This program has an input Z which is a list of numbers. We only ask that the numbers are positive. We have n variables other than r which represent the individual entitlements in the new allocation. We have constraints that normalize the new allocations according to the size of the cake in the original allocation. the second set of constraints is r proportionality and the third set of constraints is democratic ownership. This program maximizes the r in r-proportionality such that it agrees with the constraints described.

**Claim**: These democratic ownership constraints describe an allocation with democratic ownership
Proof:For every $k \in 1, ...n-1$, $X_j$s from k+1 through n are guaranteed at least $\frac{1}{\lceil \frac{n}{k} \rceil}$ of their previous valuation. From the definition this is valid.

So we have a linear program that will find a very specific kind of democratic allocation but we don't know if it will always find one.

Maybe looking at the dual of this equation will give us some insight into this problem.

### 4.2 Dual of the problem

Let's separate the variables of the dual as follows $y_{1i}$ is a variable corresponding to the r-proportionality constraint. $y_{2i}$ corresponds to the democratic ownership constraint. Finally, $y_{2n+1}$ is the last variable corresponding to the normalization constraint. So the dual looks like this:

$$\begin{array}{ll} \text{minimize} & \sum_{2}^{n} \frac{1}{\lceil \frac{n}{i-1} \rceil} Z_i y_{2i} + \sum Z_i y_{2n+1} \\ & y_{2n+1} \geqslant y_{1i} + y_{2i} \forall i \in [n] \\ \text{subject to} & \frac{1}{n} \sum Z_i \sum y_{1i} \geqslant 1 \forall i \in [n] \\ & y_i \geqslant 0 \forall i \in [n] \end{array} \tag{1}$$

Looking at it it is clear that we can give all the $y_{2i}$s 0 without losing anything. Then we are minimizing $\sum Z_i y_{2n+1}$. Now we can add all of the first n constraints to get:

$$n y_{2n+1} \geqslant \sum y_{1i} + \sum y_{2i} \geqslant \sum y_{1i} + 0$$

$$n y_{2n+1} \geqslant \sum y_{1i} \geqslant n / \sum Z_i$$

$$y_{2n+1} \geqslant 1 / \sum Z_i$$

So we know the dual's minimum is the primal's maximum which is 1. Thus 1-proportionality is compatible with democratic allocation for identical uniform valuations. 1-proportionality is compatible with democratic allocation for any input vector Z.

### 4.3 Proof of Theorem 1

Since we know that every agent receiving a 1/n proportion of the whole cake is compatible with democratic ownership so we can use Dubins-Spanier to cut a proportional allocation.

## 5 Conclusions

This means that proportionality is compatible with democratic ownership even in the case where there are varied valuations since there are algorithms that can cut proportional allocations in those scenarios. Unfortunately, it means that w-ownership remains a better property to study trade-offs between ownership rights and proportionality. The linear program for w-ownership did not yield similar results but "Redividing the cake" has interesting results..

## 6 References

1. Segal-Halevi, Erel. "Redividing the cake." Autonomous Agents and Multi-Agent Systems 36.1 (2022): 14.

2. Dubins, Lester E., and Edwin H. Spanier. "How to cut a cake fairly." The American Mathematical Monthly 68.1P1 (1961): 1-17.

3. Segal-Halevi, Erel. "Fair multi-cake cutting." Discrete Applied Mathematics 291 (2021): 15-35.