Solution for A 2D Heat Conduction Model

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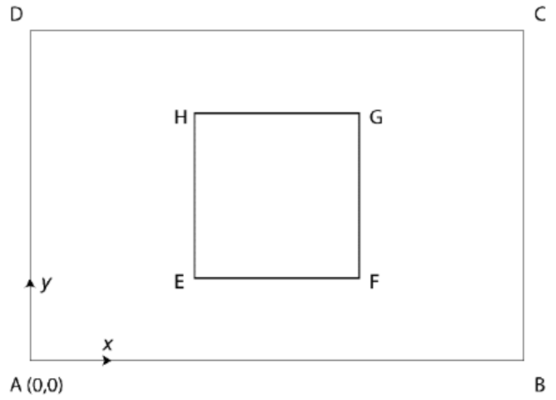
Question Description:



Consider the above plate $ABCD$ with a central opening $EFGH$. The dimensions are $AB = CD = 6m$, $AD = BC = 4m$ and $EF = FG = GH = EH = 2m$. The origin is located at A.

At $t = 0$, the plate has a temperature of $\phi = 100°\text{C}$ everywhere.

A cooling element is located at the central opening such that

- $\phi = (50 + 50e^{-0.2t})°\text{C}$ along EF, FG, GH, EH

The remaining boundary conditions are:

- $\phi = 100°\text{C}$ along AD and BC
- Zero flux, i.e. $\frac{\partial \phi}{\partial y} = 0$, along AB, CD

An internal point a is located at (1,2).

Making suitable simplifications such that a smaller problem can be considered with less nodes. Elaborate on the simplifications made. Use this simplified problem for the scenarios below.

1) Consider a steady state situation (no further changes in temperature in the entire plate). Use a spatial grid size of 1m x 1m. Determine the temperature at point a at steady state. Use the Central Difference Method. Write down the steps clearly. Use of Matlab is optional for this problem.

2) Solve the transient heat problem for $0 \le t \le 100$. Use a Forward Time Central Space strategy. Elaborate on the numerical implementation. How do you decide that the solution looks "correct"? Discuss and demonstrate the conditional stability of the numerical scheme. Plot the temperature at point a with respect to t.

Solution 1): steady state heat conduction problem

Because the cooling element contributes the main changes in temperature at the whole plate, considering a steady state situation means to compute stable temperature at t→∞ along the hole sides. When t=100s, T≈50°C stably, that is the most important condition in Q1.

The method to get the temperature at point a is follow:

a) Formulation of Governing Differential Equation

At a steady state situation, there are no further changes in temperature in the whole plate *ABCD*:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1-1}$$

b) Domain Discretization.

Divide the plate with a spatial grid size of 1m*1m: $\Delta x = \Delta y = 1$m. Create a matrix to describe the central opening area *EFGH*. And the corresponding location description is as follows:

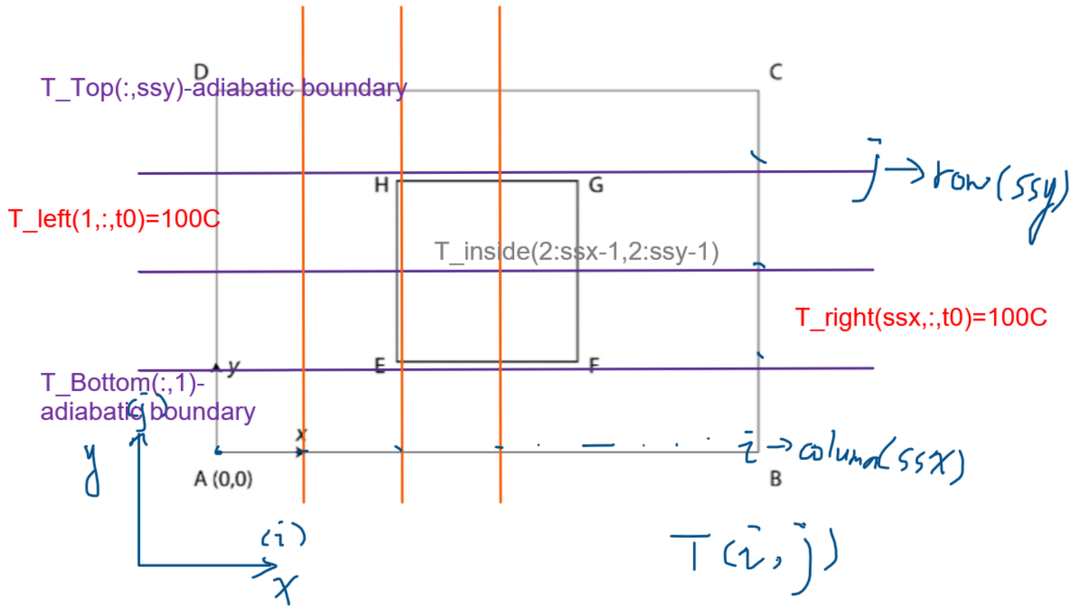


Fig. 1. location description (ssx and ssy mean number of spatial steps)

Without the time element, just consider x and y direction. Set $\phi_{i,j}$, where (*i, j*) are indices in the *x* and *y* direction, respectively. Along AB and CD, *i* = 0 to 6m; along BC and AD, *j* = 0 to 4m.

For the hole inside the plate, its location is ($2 \le X \le 4$, $1 \le Y \le 3$), and the corresponding temperature matrix is T(2/dx+1: 4/dx+1, 1/dy+1: 3/dy+1) = 50°C, that is its BCs.

c) Convert differential equation into algebraic equation.

Using the central difference formula, obtain:

$$\frac{\partial^2 \phi(\mathrm{i,j})}{\partial x^2} = \frac{1}{\Delta x^2}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j})$$

$$\frac{\partial^2\phi(\mathrm{i},\mathrm{j})}{\partial y^2}=\frac{1}{\Delta y^2}(\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1})$$

The finite difference form of Eq.(1.1) is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\frac{1}{\Delta x^2}\left(\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}\right)+\frac{1}{\Delta y^2}\left(\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}\right)=0$$

As mesh lengths in x, y direction are same, $\Delta\mathrm{x}=\Delta\mathrm{y}$, the central-difference approximation of the second derivative in space is

$$\phi_{i+1,j}+\phi_{i,j+1}-4\phi_{i,j}+\phi_{i-1,j}+\phi_{i,j-1}=0$$

This equation is applied at each interior grid point of the meshed structure and a set of algebraic equations is obtained after applying proper boundary conditions.[1] And it is the main equation to compute $\phi_{i,j}$ during the loop:

$$\phi_{i,j}=\tfrac{1}{4}(\phi_{i+1,j}+\phi_{i,j+1}+\phi_{i-1,j}+\phi_{i,j-1}) \qquad (1\text{-}2)$$

d) IC and BC

Based on the stable state, the given conditions about t=0 can be ignored in this solution, and only consider the stable condition and assume the initial condition about the plate.

① Because the main change in temperature happens around the hole sides that locate inside the plate, at t=100s, T≈50℃:

$$\phi(2\le \mathrm{x}\le 4,1\le \mathrm{y}\le 3)=50$$

② Zero flux along AB and CD: $\frac{\partial\phi}{\partial y}_{(y=0)}=\frac{\partial\phi}{\partial y}_{(y=4)}=0$. It means the heat conduction along AB and CD is effectively blocked or insulated, that can be viewed as the plate has two adiabatic boundary conditions.

③ Considering the total initial temperature is 100℃ in the plate at t=0, assume temperature of the whole plate except the hole is 100℃.

$$\phi_{i,j}=100\ \text{(except the hole)}$$

④ The left and right sides remain $\phi(\mathrm{x}=0,\mathrm{y})=\phi(\mathrm{x}=6,\mathrm{y})=100$.

Therefore, these BCs at stable state can be described by algebraic equations:

For left and right sides: $\phi_{i,j}=100$;

For the top side:

$$\frac{\partial\phi}{\partial y}=\frac{\phi_{i,j}-\phi_{i,j-1}}{\Delta y}=0\Rightarrow\phi_{i,j}=\phi_{i,j-1}$$

For the bottom side:

$$\frac{\partial\phi}{\partial y}=\frac{\phi_{i,j+1}-\phi_{i,j}}{\Delta y}=0\Rightarrow\phi_{i,j}=\phi_{i,j+1}$$

For interior nodes:

$$\phi_{i,j}=\frac{1}{4}(\phi_{i+1,j}+\phi_{i,j+1}+\phi_{i-1,j}+\phi_{i,j-1})$$

For the hole sides: $\phi_{i,j}=50$.

e) Conditional statement in loop

During the loop for Eq.(1.2), set a conditional statement to ensure the hole and its area remain the state T= 50°C.

f) Convergence check

Create a variable *tol* used to check the convergence during iteration calculation.

g) Computing results

Use *Plot_Q1.m* to check the plotting figure. *stable_CDM.m* and *U0.m* are the relevant function files.

The results and figure about the stable state is as follows(Fig. 2):

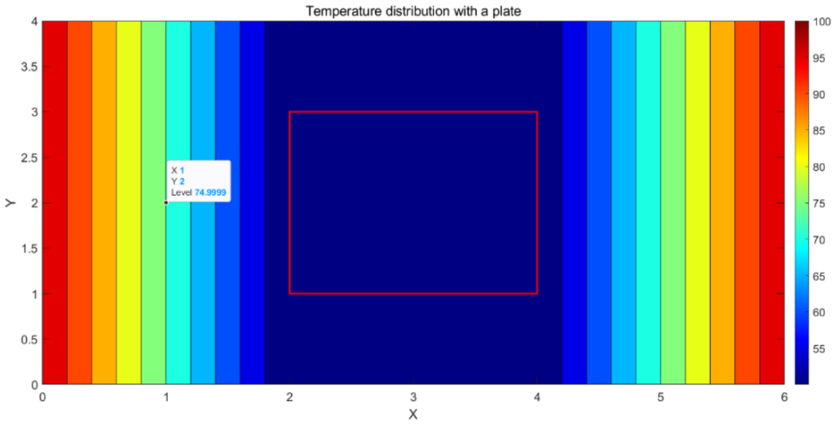


Fig. 2. Temperature at a stable state

Convergence reached, and the number of iterations is 25 (dx=dy=1m). Temperature at point a(1,2) at steady state: 75.00 °C.

If with a smaller spatial step size or with more iteration, the approximating results of temperature changes will be more smooth, and the temperature at point a(1,2) is 75.84°C(maxI=1663, dx=dy=0.1m) in Fig. 3.

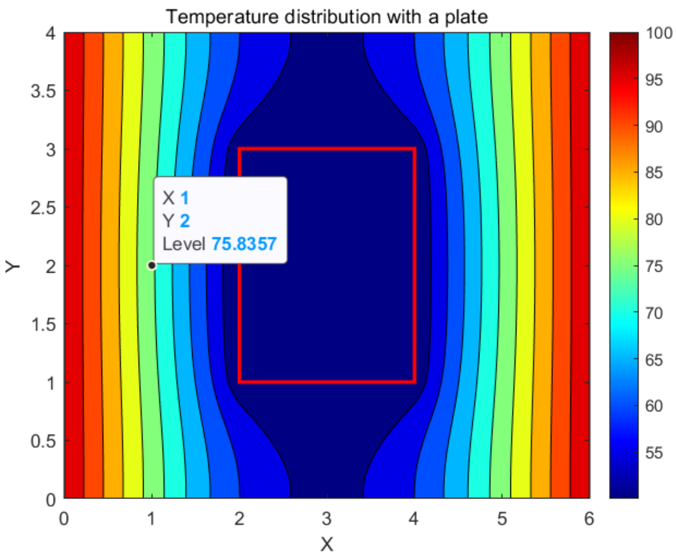


Fig. 3. Temperature at a stable state with smaller spatial size(dx=0.1)

Solution 2): transient heat conduction problem

In the transient heat conduction problem, it is necessary to consider the time element. Based on the symbol setting in Q1, create a 3-dimension matrix to store temperature respect to x, y and time: T(x,y,p) or $\phi_{i,j}^{p}$, where p means time.

The derivation process expands based on Q1:

a) Formulation of Governing Differential Equation

The transient heat conduction equation can be written as:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \tag{2-1}$$

b) Domain Discretization.

Add a dimension Time to describe the changes of temperature over time.

c) Convert differential equation into algebraic equation.[1]

Use a Forward Time Central Space strategy, Eq.(2-1) can be expanded in the following form:

$$\frac{\partial \phi}{\partial t} = \rho C A_c \frac{\phi_{i,j}^{p+1} - \phi_{i,j}^{p}}{\Delta t}$$

$$\frac{\partial^2 \phi(\mathrm{i,j})}{\partial x^2} = \frac{c A_c}{\Delta x^2}(\phi_{i+1,j}^{p} - 2\phi_{i,j}^{p} + \phi_{i-1,j}^{p})$$

$$\frac{\partial^2 \phi(\mathrm{i,j})}{\partial y^2} = \frac{c A_c}{\Delta y^2}(\phi_{i,j+1}^{p} - 2\phi_{i,j}^{p} + \phi_{i,j-1}^{p})$$

Let $\Delta x = \Delta y$, equation can be simplified is as follows:

$$\phi_{i,j}^{p+1} - \phi_{i,j}^{p} = \frac{k\Delta t}{\rho C(\Delta x)^2}[\phi_{i+1,j}^{p} + \phi_{i-1,j}^{p} + \phi_{i,j+1}^{p} + \phi_{i,j-1}^{p} - 4\phi_{i,j}^{p}]$$

$$\Rightarrow \phi_{i,j}^{p+1} - \phi_{i,j}^{p} = F[\phi_{i+1,j}^{p} + \phi_{i-1,j}^{p} + \phi_{i,j+1}^{p} + \phi_{i,j-1}^{p} - 4\phi_{i,j}^{p}]$$

Where $F = \frac{c\Delta t}{\rho C(\Delta x)^2}$, *c* is thermal conductivity, *ρ* is the density of a material, *C* is the specific heat capacity, *Δt* is the time step in second, *Δx* and *Δy* is the mesh length in the *x, y* direction, A_c is the cross-sectional area of the plate; *p* is the time number.

Then, the final form of the transient heat conduction equation is as follows:

$$\phi_{i,j}^{p+1} = \phi_{i,j}^{p}(1-4F) + F[\phi_{i+1,j}^{p} + \phi_{i-1,j}^{p} + \phi_{i,j+1}^{p} + \phi_{i,j-1}^{p}] \tag{2-2}$$

d) IC and BCs

Referring to BCs in Q1, the BCs along the hole sides cannot be described as a constant, but the top and bottom sides are still adiabatic boundaries.

① For the hole sides: as the hole's BCs respect to time: $\phi = 50 + 50\mathrm{e}^{-0.2t}$, update its BCs over time: $\phi^{p}(2 \le \mathrm{x} \le 4, 1 \le \mathrm{y} \le 3) = 50 + 50\mathrm{e}^{-0.2p}$.

② For the top and bottom sides as adiabatic boundaries: $\phi_{i,j}^{p} = \phi_{i,j-1}^{p}\ or\ \phi_{i,j+1}^{p}$

③ For interior nodes:

$$\phi_{i,j}^{p+1} = \phi_{i,j}^{p}(1-4F) + F[\phi_{i+1,j}^{p} + \phi_{i-1,j}^{p} + \phi_{i,j+1}^{p} + \phi_{i,j-1}^{p}]$$

e) Loop and Convergence check

Since add a time dimension, check the convergence at every time step. However, the number of iterations is 3 with this method, that is less than solution_1.

f) Conditional stability of the numerical scheme

As the coefficient *(1-4F)* in Eq(2-2), this method is stable if **F ≤ 0.25**.

g) Computing results and Plot the temperature at point a with respect to t

Use *Plot_Q2.m* to check the plotting figure. transient_*FDM.m* and *U0.m* are the relevant function files.

① %% Q2_1: Δx=Δy=0.1, F=0.25, total time=100s, Δt=1, sst=101

Figures follow(Fig. 4~Fig. 6):

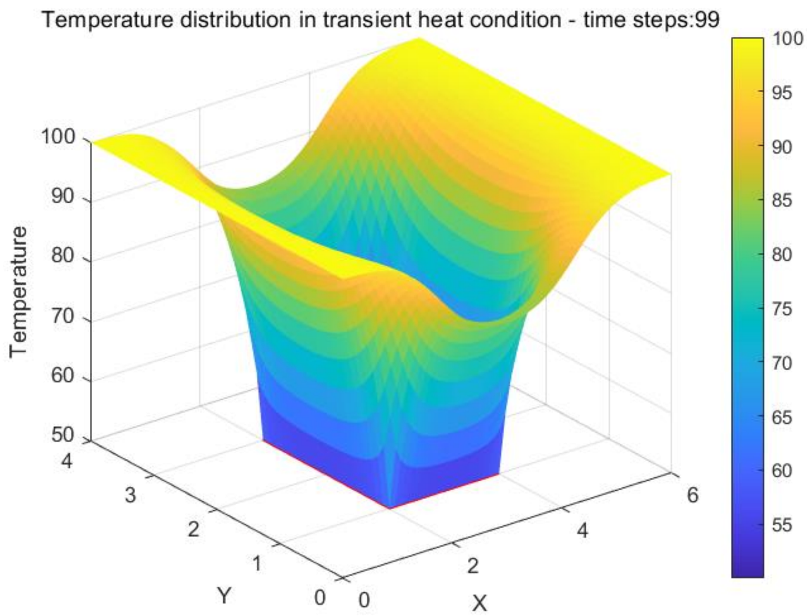


Fig. 4. sst=101, 3D figure about T

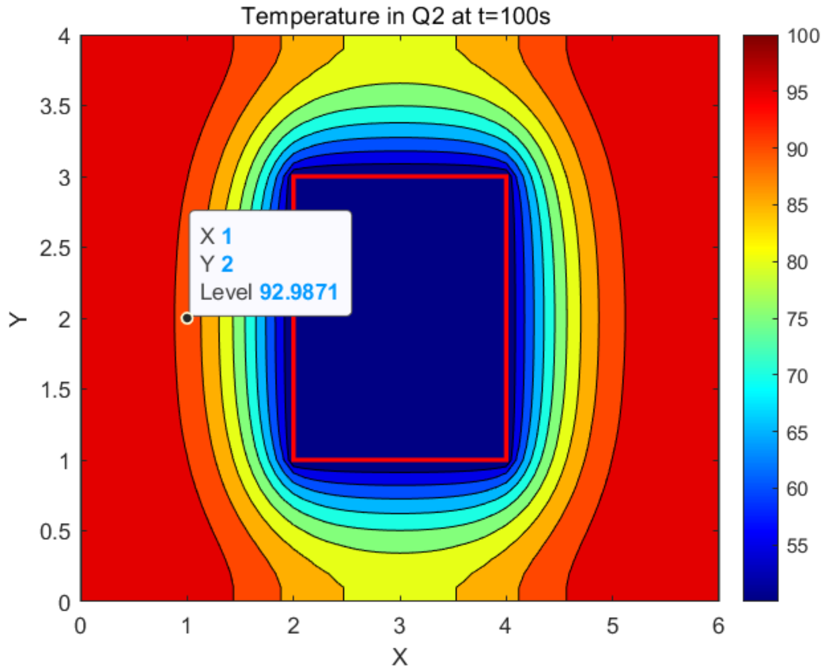


Fig. 5. 2d T at t=100s with time steps=101

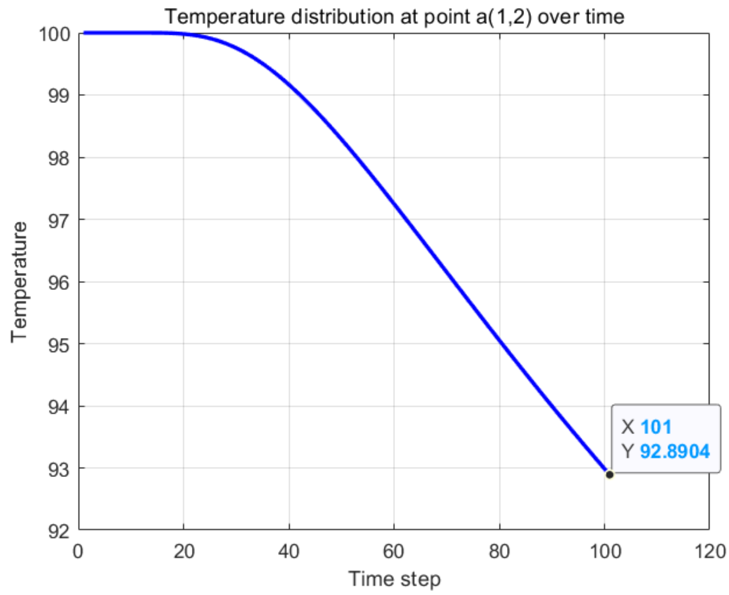


Fig. 6. Plot T at point a(1,2) with time steps=101

② %% Q2_2: Δx=Δy=0.1, F=0.1, total time=100s, Δt=0.1, sst=1010
Figures follow(Fig. 7~Fig. 9):

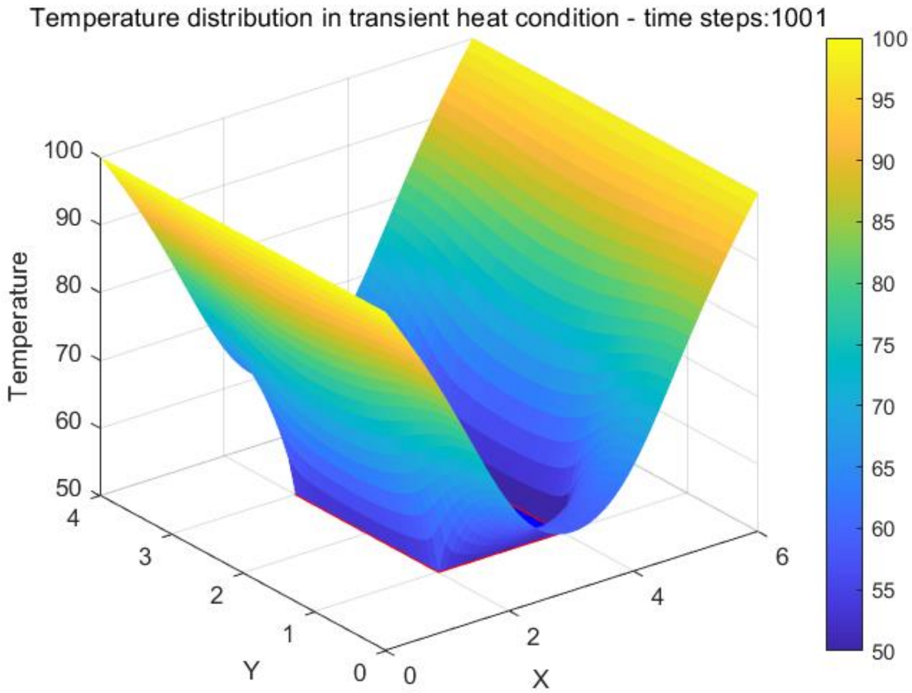


Fig. 7. sst=1010, 3D figure about T

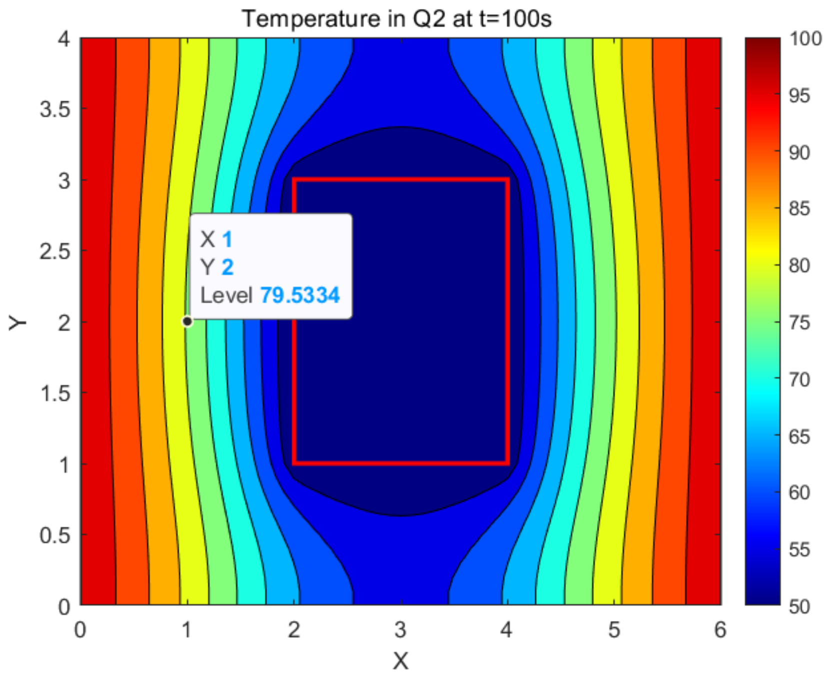


Fig. 8. 2d T at t=100s with time steps=1010

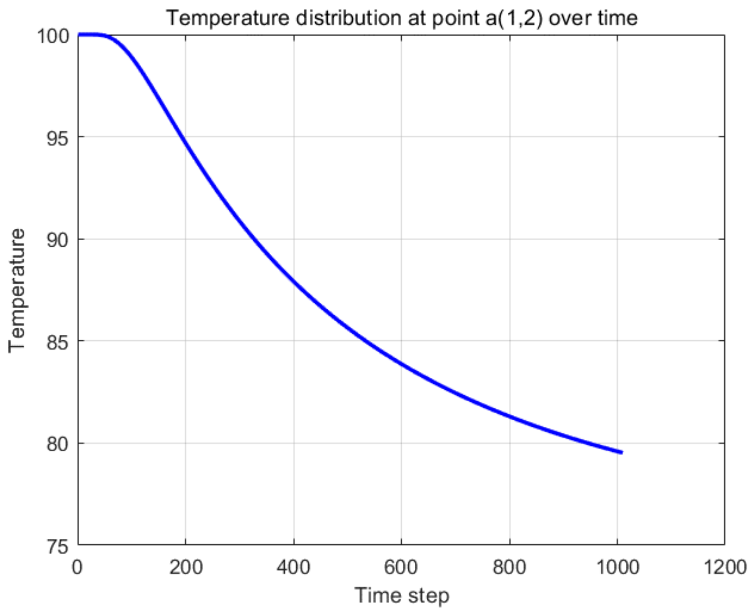


Fig. 9. Plot T at point a(1,2) with time steps=1010 (T=79.53℃)

h) Comment on results

To judge if the solution looks "correct", compared the final results with 2D temperature distribution figure by this method to the solution in Q1 that is stable by CDM as above figures. In the Q1 stable results at t=100s, the temperature change trend is: in the vertical direction of the center, the central hole and top and bottom boundaries are affected by the cooling element whose temperature approaches 50℃, which is stable temperature; for the left and right sides, the temperature in the horizontal direction diverges from the center to both sides and tends towards 100℃ (the initial temperature at the left and right boundaries).

There are some conclusions about CDM and FTCS methods:

Forward Time Central Space strategy(FTCS) needs more calculation to get stable results, like more time steps.

Take the stable temperature at point a(1,2) as an example. For the number of iterations, CDM method requires more iterations to get a stable result: 75.84℃ (maxI=1663, dx=dy=0.1m); and FTCS needs less iterations at every time step results but more time steps: 79.53℃ (maxI =3/per, dx=dy=0.1m, sst=1010). If with smaller time steps, FTCS may compute an unstable result: 92.89℃ (maxI =3/per, dx=dy=1m, sst=101). If calculate total iterations with total time, FTCS also needs more iterations: 3*1010=3030.

Only with more time steps, can FTCS compute more smooth temperature change as CDM with the same smaller spatial sizes, it can be illustrated by 2D isotherm figures as Fig. 3 and Fig. 8.

References:

[1] S. K. Deb Nath and N. K. Peyada, ‘Numerical Study Of The Heat Transfer Phenomenon Of A Rectangular Plate Including Void, Notch Using Finite Difference Technique’, *International Journal of Applied Mechanics and Engineering*, vol. 20, no. 4, pp. 733–756, Dec. 2015, doi: 10.1515/ijame-2015-0048.