NST1A:
Mathematics I (Course B)

9:00, Tuesday, Thursday Saturday,
Michaelmas 2024

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Lectures presented *in person*
in the Bristol-Myers Squibb Lecture Theatre,
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Lecture notes on Moodle (https://vle.cam.ac.uk)

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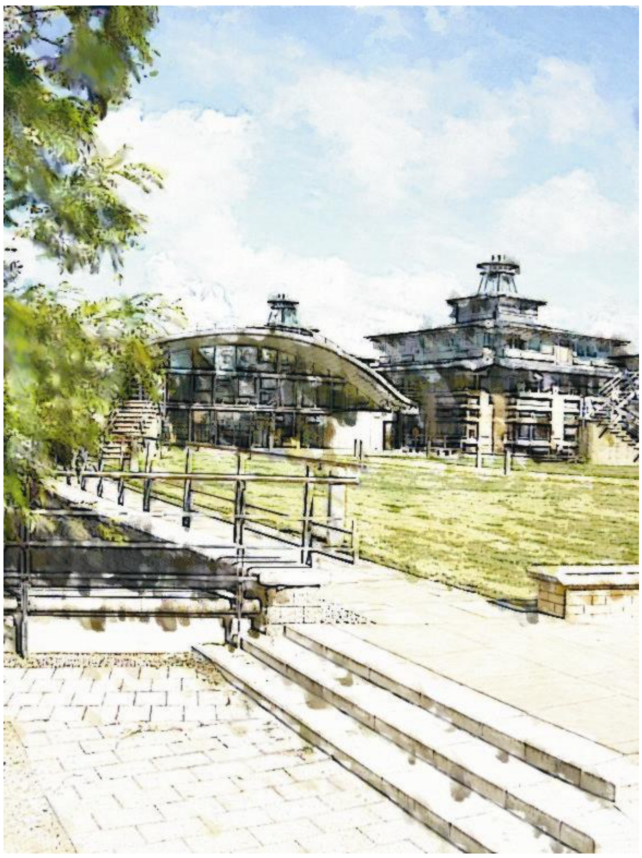
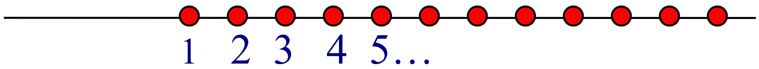


2. Complex numbers

2.1 History of numbers

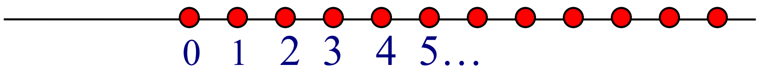
Counting numbers: 1, 2, 3, … ? When? Base 10 from ~3,100 BC

Natural numbers: $i \in \bullet$ - does this include zero?



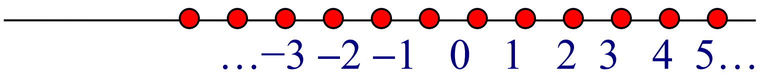
Zero: India ~500 BC, Americas ~50 BC, Europe ~130 AD

Whole numbers? Non-negative integers i ∈ W, i ∈ N_0



Negative numbers: China ~100 BC, Greece ~250AD, Europe ~17th C

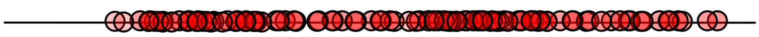
Integers: $i \in \mathbb{Z}$ (from 'zahlen', German for "numbers")



Rational numbers: before ~3,000 BC; r ∈ ℚ; r= a/b; a,b ∈ ℤ; not continuous

Irrational numbers: India 800~500 BC; sometimes r ∈ 𝕁; not continuous

$\sqrt{2},\ \pi,\ e$



Real numbers (rational + irrational); r ∈ ℝ = ℚ ∪ 𝕁 continuous

Infinity: Greece ~500 BC, India ~400 BC, Europe ~17th C

Complex numbers: Gerolamo Cardano (Italy) ~1545

Quaternion: William Rowan Hamilton (Ireland) ~1843
Can be viewed as an extension to complex numbers and arise from the solutions to $i^2 = j^2 = k^2 = ijk = -1$. We will not discuss these here.

2.2 Definitions and properties

2.2.1 Parts of a complex value

Imaginary unit

We define $i^2 = -1$, so

$$i \equiv \sqrt{-1}. \tag{59}$$

Whether we take the 'positive' or 'negative' root is immaterial as everything will ultimately work out the same, provided we are consistent.

Real magnitudes

We then write the complex number z as

$$z = x + iy\,, \tag{60}$$

where $x, y \in {}^{\sim}$.

> For electrical engineering and some other disciplines, the symbol j is sometimes adopted instead of i for the square root of -1.

We also define $\Re(z) \equiv \mathrm{Re}(z)$ and $\Im(z) \equiv \mathrm{Im}(z)$ such that

$$\begin{aligned} x &= \Re(z) \equiv \mathrm{Re}(z) && \text{the real part of } z, \\ y &= \Im(z) \equiv \mathrm{Im}(z) && \text{the imaginary part of } z\,. \end{aligned} \tag{61}$$

If

$$z = x + iy \text{ for } x, y \in {}^{\sim} \quad \Rightarrow \quad \mathrm{Re}(z) = x \quad \text{and} \quad \mathrm{Im}(z) = y \in {}^{\sim}$$

then
$$\mathrm{Re}(\mathrm{Re}(z)) = \mathrm{Re}(z) = x$$

$$\mathrm{Re}(\mathrm{Im}(z)) = \mathrm{Im}(z) = y \text{ since } \mathrm{Im}(z) \in {}^{\sim}$$

but
$$\mathrm{Im}(\mathrm{Re}(z)) = 0 = \mathrm{Im}(\mathrm{Im}(z)).$$

It is easy to factorise a quadratic once we know its roots. For the example, if $f(z)=z^2+2z+2=0$ then using the usual formula

$$az^2+bz+c=0 \quad \to \quad z=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{62}$$

then

$$f(z)=z^2+2z+2=(z+1+i)(z+1-i)\,.$$

Note that there are two solutions to $f(z)=0$, but both are complex.

Equality

If $z_1=a+ib$ and $z_2=c+id$,
then $z_1=z_2$ if and only if (iff) $a=c$ and $b=d$.

Zero: If $z=x+iy$, then $z=0 \Rightarrow x=0$ and $y=0$.

2.2.2 Argand diagram

The real and imaginary parts of z are really independent quantities, so we can think of z as being a bit like a vector in two-dimensional space, with components (x,y). This allows us to represent a complex number graphically as an ***Argand diagram.*** instead of as a number line.

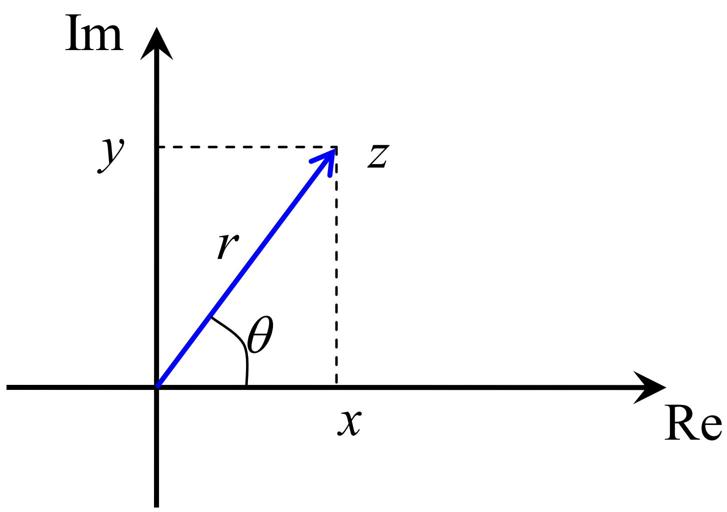


Figure 21: Argand diagram showing real and imaginary parts of complex number z

Addition

If $z_1 = a + ib, \quad z_2 = c + id$ then

$$z_1 + z_2 = (a+ib) + (c+id) \equiv (a+c) + i(b+d)$$

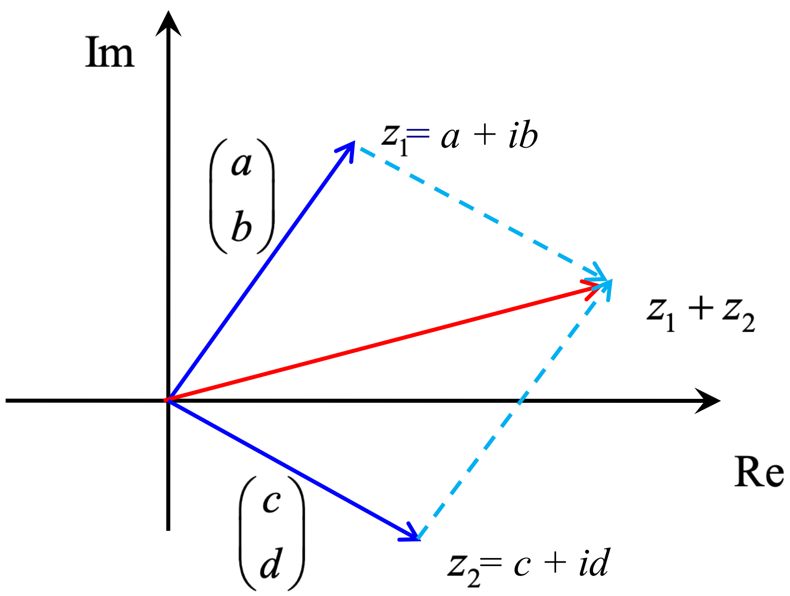
$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$$

$$\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

$\Rightarrow$ Add real components together, and add imaginary components together

Commutative: $\quad z_1 + z_2 = z_2 + z_1$

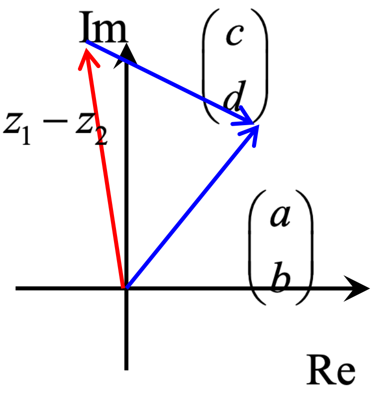
Associative: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 = z_1 + z_2 + z_3$



Subtraction

Again, work with Re(..) and Im(..) as though they are two components of a vector:

$$z_1 - z_2 = (a+ib) - (c+id) \equiv (a-c) + i(b-d)$$



Multiplication

$$\begin{aligned}
i^2 &= -1 \\
i^3 &= i^2 \times i = -i \\
i^4 &= i^3 \times i = -i \times i = 1 \\
i^5 &= i^4 \times i = i \\
&\vdots
\end{aligned} \tag{63}$$

More generally, recalling that $i \equiv \sqrt{-1}$, then for integer m and n,

$$i^n = \begin{cases} i^{2m} & n = 2m, \\ i^{2m+1} & n = 2m+1, \end{cases} = \begin{cases} (-1)^m & n = 2m, \\ (-1)^m i & n = 2m+1. \end{cases} \tag{64}$$

If $z_1 = a + ib,\ z_2 = c + id$ then their product is

$$\begin{aligned}
z_1 z_2 &= (a+ib)(c+id) \\
&= ac + iad + ibc + i^2 bd \\
&= (ac - bd) + i(ad + bc)
\end{aligned}$$

Properties:

(i) Commutative $z_1 z_2 = z_2 z_1$

(ii) Associative $(z_1 z_2) z_3 = z_1 (z_2 z_3) = z_1 z_2 z_3$

(iii) Distributive over addition $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 24: Basic arithmetic

If $z_1 = 3+i$ and $z_2 = 1-i$, calculate z_1+z_2, z_1-z_2 and $z_1 z_2$.

$$z_1+z_2 = (3+i)+(1-i) = (3+1)+(1-1)i = 4$$

$$z_1-z_2 = (3+i)-(1-i) = (3-1)+(1+1)i = 2+2i = 2(1+i)$$

$$\begin{aligned} z_1 z_2 &= (3+i)(1-i) \\ &= 3\times 1 + 3\times(-i) + i\times 1 + i\times(-i) \\ &= 3-3i+i-i^2 \\ &= 3-2i-(-1) \\ &= 4-2i = 2(2-i) \end{aligned}$$

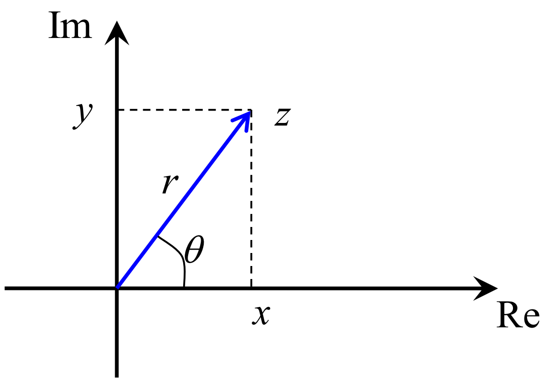
2.2.3 Modulus and argument

As an alternative way of expressing a complex number, we can use plane polar coordinates in the complex plane (Argand diagram).

$$z = x+iy$$

$$r = \sqrt{x^2+y^2} = |z| \equiv \operatorname{mod}(z)$$

$$\theta = \tan^{-1}\frac{y}{x} = \arg(z) \qquad (65)$$



Note that when determining θ you must also consider the signs of both x and y to ensure you select the correct value for θ.

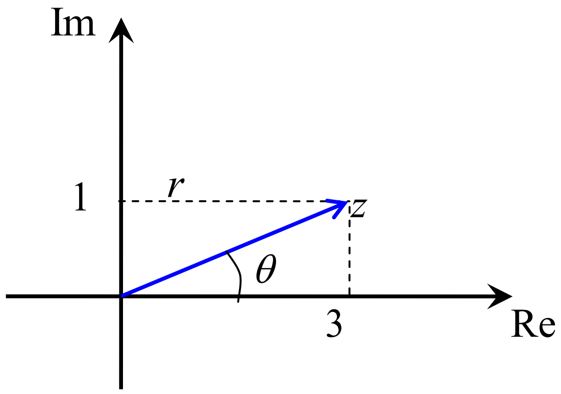
The quantity r is the ***modulus*** of z and θ is the ***argument***. For complex numbers it is common for us to restrict the argument to the range $-\pi < \theta \le \pi$, which is the called ***principal argument***.

Sometimes the principal argument is taken in the range $0 \le \theta < 2\pi$. The choice is generally arbitrary, so long as it is used consistently. More on this later, and much more in NST1B.

Example 25: Modulus and argument

Calculate the modulus and argument for z_1 and z_2 of the previous example.

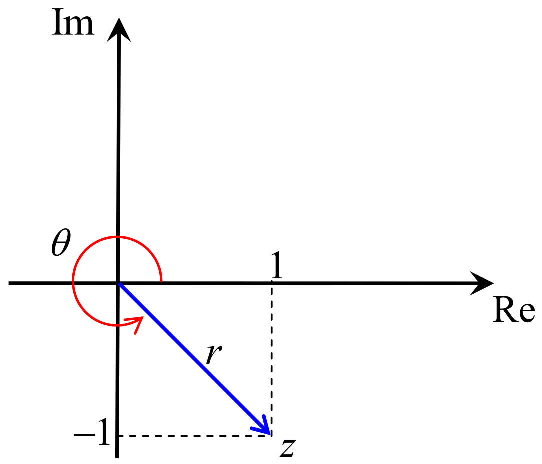
$z_1 = 3 + i$



Modulus $\qquad |z_1| = \sqrt{3^2 + 1^2} = \sqrt{10}$,

Argument $\qquad \theta_1 = \arg(z_1) = \tan^{-1}\frac{1}{3} \approx 0.3218\,\mathrm{rad}$

$z_2 = 1 - i$ (taking arg(z) ∈ $[0,2\pi)$))



Modulus $\qquad |z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$,

Argument $\qquad \theta_2 = \arg(z_2) = \tan^{-1}\frac{-1}{1} = \frac{7\pi}{4}\,\mathrm{rad}$

Note that $\theta_2 \neq \frac{3\pi}{4}$ despite $\tan\frac{3\pi}{4} = -1$!

Example 26: Argand diagram for equation

What shape in the Argand diagram is described by the equations (a) $3|z|=|z-i|$ and (b) $|z|=|z-i|$?

(a) $$3|z|=|z-i|$$

$\Rightarrow$ $$9|z|^2=|z-i|^2$$

$\Rightarrow$ $$9|x+iy|^2=|x+i(y-1)|^2$$

$\Rightarrow$ $$9\left(x^2+y^2\right)=x^2+(y-1)^2=x^2+y^2-2y+1$$

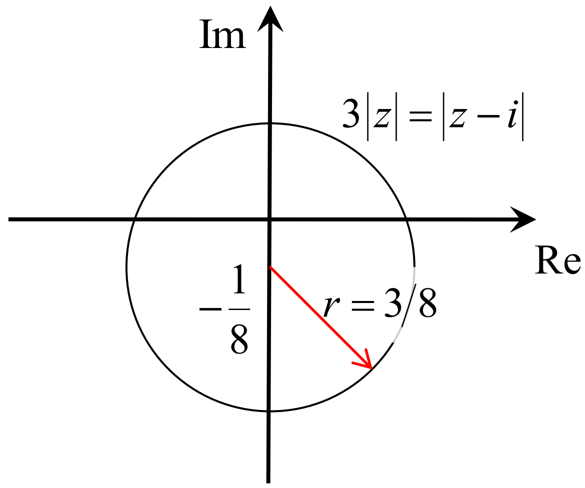
$\Rightarrow$ $$8x^2+8y^2+2y=1$$

$\Rightarrow$ $$8x^2+8\left(y^2+2\frac{1}{8}y+\left(\frac{1}{8}\right)^2\right)=1+\frac{1}{8}$$

$\Rightarrow$ $$8x^2+8\left(y+\frac{1}{8}\right)^2=\frac{9}{8}$$

$\Rightarrow$ $$x^2+\left(y+\frac{1}{8}\right)^2=\frac{9}{64}=\left(\frac{3}{8}\right)^2$$

This is the equation for a circle of radius $3/8$ centred on $z=-\frac{i}{8}$



2.2.4 Complex multiplication (again)

By looking at the Argand diagram (figure 21), we can write z in terms of r and θ quite easily using trigonometry. Specifically,

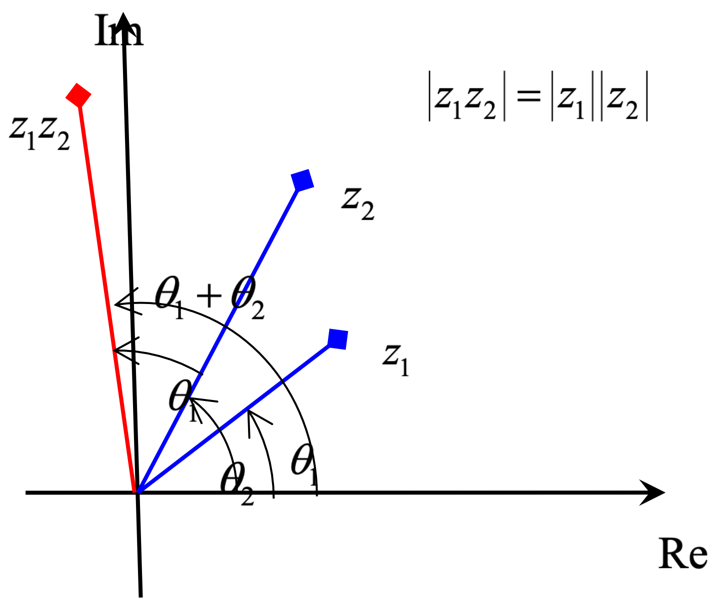
$$\begin{aligned} x &= r\cos\theta = |z|\cos(\arg(z)), \\ y &= r\sin\theta = |z|\sin(\arg(z)) . \end{aligned} \tag{66}$$

(This is really just the same as the plane polar coordinates for vectors seen in equation (52).) It now follows from (66) that

$$z = x + iy = |z|(\cos\theta + i\sin\theta) . \tag{67}$$

> Sometimes $\cos\theta + i\sin\theta$ is written as $\mathrm{cis}\,\theta$, so that $z = |z|\,\mathrm{cis}\,\theta$.
> Later, we will see that $\cos\theta + i\sin\theta = e^{i\theta}$.

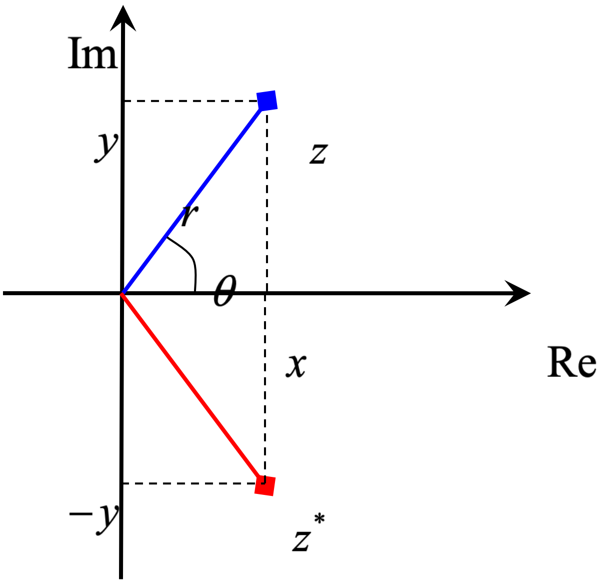
Multiplication: $z_1 z_2$



2.2.5 Complex conjugate

The complex conjugate of $z = x + iy$ is z^* and is defined as

$$z^* = x - iy\,. \tag{68}$$



Sometimes the complex conjugate is denoted by an over bar: $\bar{z} = x - iy$ rather than z^*.

From (67) it follows that

$$z^* = |z|(\cos\theta - i\sin\theta) = |z|(\cos(-\theta) + i\sin(-\theta))\,. \tag{69}$$

Note that if we multiply a complex number by its complex conjugate,

$$zz^* = (x+iy)(x-iy) = (x)^2 - (iy)^2 = x^2 + y^2 = |\,z\,|^2\,, \tag{70}$$

we recover the square of the modulus, which is real.

2.2.6 Complex division

We want z_1/z_2 in form of $x+iy$:

Example 27: Division of complex numbers

Calculate z_1/z_2, where $z_1 = 3+i$ and $z_2 = 1-i$.

Want to calculate

$$\frac{z_1}{z_2} = \frac{3+i}{1-i},$$

but rather than divide by a complex number, we will multiply through by $z_2^*/z_2^* = 1$ and calculate

$$\frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{3+i}{1-i}\frac{1+i}{1+i} = \frac{3+3i+i+i^2}{1+i-i-i^2}$$

$$= \frac{3+4i-1}{1-(-1)} = \frac{2+4i}{2} = 1+2i$$

2.2.7 Complex exponential

Euler's identity (1748):

$$e^{i\pi} + 1 = 0,$$

is often considered "the most beautiful equation".

We will prove in §**6.3** that

Multiplication and division

If $z_1 = |z_1|\exp(i\theta_1)$ and $z_2 = |z_2|\exp(i\theta_2)$ then

$$z_1 z_2 = |z_1||z_2|\exp\left(i[\theta_1 + \theta_2]\right). \tag{73}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\exp\left(i[\theta_1 - \theta_2]\right), \tag{74}$$

$$\begin{aligned} z_1^* &= |z_1|\left(\cos\theta_1 - i\sin\theta_1\right) \\ &= |z_1|\left(\cos\left(-\theta_1\right) + i\sin\left(-\theta_1\right)\right) \\ &= |z_1|\exp\left(-i\theta_1\right). \end{aligned} \tag{75}$$

$$z_1 z_1^* = |z_1|e^{i\theta_1}|z_1|e^{-i\theta_1} = |z_1|^2 = |z_1^*|^2 \tag{76}$$

These recover the expressions we had in §2.2.4–2.2.6.

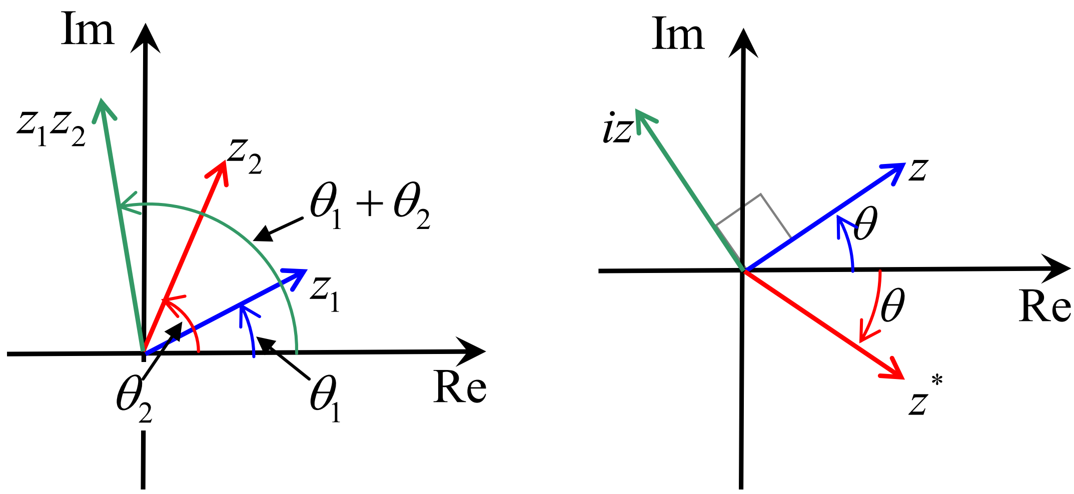


Figure 22: Geometrical interpretation of multiplication shown as an Argand diagram. (left) Multiplication. (right) Complex conjugates (red) and multiplication by i (green).

2.3 Roots of unity

If $z = r\exp\left(i\theta\right)$, then $z^n = r^n \exp(in\theta)$. As $r \equiv \left|z\right|$ then there is no restriction on $n \in \sim$ (provided $z \neq 0$).

Consider $\qquad z^4 = 1$

The roots of $r^n = 1$ are distributed around the unit circle (circle of radius 1 centred on the origin) at regular angles separated by $2\pi / n$, and are shown in figure 19.

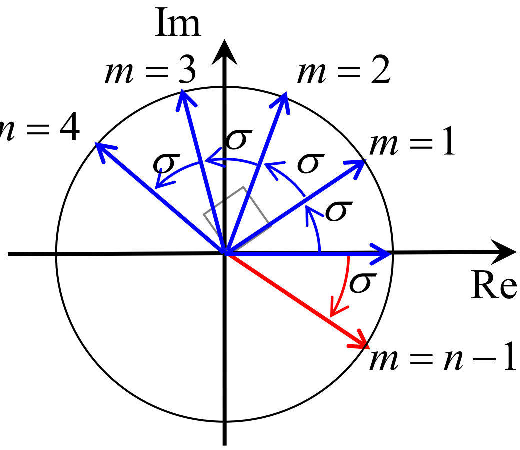


Figure 23: Argand diagram showing the n roots of $z^n = 1$ given by $z = \exp(i2\pi m/n)$. The angle $\sigma = 2\pi/n$.

Indeed, for **any** polynomial of degree n, there will always be n roots in the complex plane.

Example 28: Complex roots

Solve $z^5 = 2i$.

We begin by writing z in complex polar form as

$$z = r\, e^{i\theta},$$

and $2i = 2\exp\left(i\frac{\pi}{2}\right)$. Thus our equation is

$$z^5 = r^5 \exp(i5\theta) = 2i = 2\exp\left(i\left[\frac{\pi}{2} + 2n\pi\right]\right),$$

$\Rightarrow$
$$r = 2^{1/5}$$

and
$$5\theta = \frac{\pi}{2} + 2n\pi$$

$\Rightarrow$ $$\theta = \frac{\pi}{10} + \frac{2}{5}n\pi.$$

Thus $$z = 2^{1/5}\exp\left(i\left[\frac{\pi}{10} + \frac{2}{5}n\pi\right]\right), \qquad n = 0,1,\ldots 4.$$

$n = 0$ $$z = 2^{1/5}\exp\left(i\frac{\pi}{10}\right),$$

$n = 1$ $$z = 2^{1/5}\exp\left(i\frac{5\pi}{10}\right) = 2^{1/5}\exp\left(i\frac{\pi}{2}\right),$$

$n = 2$ $$z = 2^{1/5}\exp\left(i\frac{9\pi}{10}\right),$$

$n = 3$ $$z = 2^{1/5}\exp\left(i\frac{13\pi}{10}\right),$$

$n = 4$ $$z = 2^{1/5}\exp\left(i\frac{17\pi}{10}\right),$$

$n = 5$

2.4 De Moivre's Theorem

If $z = \cos\theta + i\sin\theta = \exp(i\theta)$, then

$$z^n = (\cos\theta + i\sin\theta)^n = \exp(i\theta)^n = \exp(in\theta) = \cos n\theta + i\sin n\theta$$

This is ***De Moivre's*** theorem:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Note that n is not restricted to be integer; indeed, n can be complex.

Example 29: De Moivre's theorem

Use De Moivre's Theorem to determine expressions for $\cos 4\theta$ and $\sin 4\theta$.

$$\cos 4\theta + i\sin 4\theta = \left(\cos\theta + i\sin\theta\right)^4$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta$$
$$+4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$+4i\left(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta\right)$$
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta\left(1-\cos^2\theta\right) + \left(1-\cos^2\theta\right)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$
$$\sin 4\theta = 4\cos\theta\sin\theta\left(\cos^2\theta - \sin^2\theta\right)$$
$$=$$

We can use De Moivre's Theorem in other ways. Taking the complex conjugate of (71) gives

$$\exp\left(-i\theta\right) = \cos\theta - i\sin\theta\,,$$

and adding (71) to this yields

$$\cos\theta = \frac{1}{2}\left(\exp(i\theta) + \exp(-i\theta)\right). \tag{77}$$

Similarly, subtracting rather than adding gives

$$\sin\theta = \frac{1}{2i}\left(\exp(i\theta) - \exp(-i\theta)\right). \tag{78}$$

One use of (77) and (78) is for working out powers of $\sin\theta$ and $\cos\theta$. For instance,

$$\begin{aligned}\cos^3\theta &= \left[\frac{1}{2}\left(\exp(i\theta)+\exp(-i\theta)\right)\right]^3 \\ &= \frac{1}{8}\left[\exp(3i\theta)+3\exp(2i\theta)\exp(-i\theta)\right. \\ &\quad \left.+3\exp(i\theta)\exp(-2i\theta)+\exp(-3i\theta)\right] \\ &= \frac{1}{8}\left[\exp(3i\theta)+\exp(-3i\theta)\right]+\frac{3}{8}\left[\exp(i\theta)+\exp(-i\theta)\right] \\ &= \frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta\end{aligned}$$

Another application of De Moivre's Theorem is to work out sums of trigonometric functions. For example, if we wish to sum the series

$$S_N=\sum_{k=0}^{N-1}\cos k\theta\,,$$

then the thing to do is to write

$$\sum_{k=0}^{N-1}\cos k\theta=\Re\left[\sum_{k=0}^{N-1}\exp(ik\theta)\right]. \tag{79}$$

The series

$$\sum_{k=0}^{N-1}\exp(ik\theta)=\sum_{k=0}^{N-1}\exp(i\theta)^k$$

is a geometric progression with first term 1 and common ratio $\exp(\mathrm{i}\theta)$, for which we can write down the answer, and then the cosine series we want follows by taking the real part. We do this in detail in the next example.

Example 30: De Moivres theorem to sum series

Evaluate $S_N = \sum_{k=0}^{N-1} \cos k\theta$.

$$\sum_{k=0}^{N-1} \cos r\theta = \Re\left[\sum_{k=0}^{N-1} \exp(ik\theta)\right]$$
$$= \Re\left[\sum_{k=0}^{N-1} z^r\right]$$

where $z = \exp(i\theta)$. Now for a geometric series, recall

$$S_N = \sum_{k=0}^{N-1} a\lambda^k = a\frac{1-\lambda^N}{1-\lambda}.$$

Identifying $a = 1$ and $\lambda = z = \exp(i\theta)$, then

$$S_N = \Re\left[\sum_{k=0}^{N-1} \lambda^k\right] = \Re\left[\frac{1-\lambda^N}{1-\lambda}\right]$$
$$= \Re\left[\frac{1-\exp(iN\theta)}{1-\exp(i\theta)}\right]$$
$$= \Re\left[\frac{1-\exp(iN\theta)}{1-\exp(i\theta)}\,\frac{1-\exp(-i\theta)}{1-\exp(-i\theta)}\right]$$
$$= \Re\left[\frac{1-\exp(-i\theta)-\exp(iN\theta)+\exp(i[N-1]\theta)}{2-\exp(i\theta)-\exp(-i\theta)}\right]$$
$$= \frac{1-\cos\theta-\cos(N\theta)+\cos([N-1]\theta)}{2(1-\cos\theta)}$$
$$= \frac{1}{2}+\frac{\cos([N-1]\theta)-\cos(N\theta)}{2(1-\cos\theta)}$$

2.5 Complex logarithms

If $z = x + iy = |z|\exp(i\theta)$ with $x, y, \theta \in \sim$, then what is $\ln z$?

Often the *principal value* of $\ln z$ is defined by choosing just one of these possible values of θ, and the usual convention with $\ln z$ is to choose $-\pi < \theta \le \pi$. For example, on sheet 1 question 10 note that the principal values in (a) and (b) differ by (almost) $2\pi i$. This will be considered further in later maths courses.

Example 31: Complex logarithms

Use complex logarithms to evaluate 2^i and i^i in the form $x+iy$.

Write $z = 2^i \Rightarrow$ $\quad \ln z = \ln 2^i = i \ln 2$.

$$z = \exp(\ln z) = \exp(i \ln 2)$$

$\Rightarrow$
$$= \cos(\ln 2) + i \sin(\ln 2)$$

Write $z = i^i \Rightarrow$ $\quad \ln z = \ln i^i = i \ln i$

Noting that $i = \exp\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right)$ for integer n,

$$\ln z = i \ln\left(\exp\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right)\right)$$

$$= i\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right)$$

$$= -\left(\frac{\pi}{2} + 2n\pi\right)$$

$$z = \exp(\ln z)$$

$\Rightarrow$
$$= \exp\left[-\left(\frac{\pi}{2} + 2n\pi\right)\right]$$

General powers $z_1^{z_2}$

2.6 Oscillation problems

Complex numbers are especially useful in problems which involve oscillatory or periodic motion, such as when describing the motion of a simple pendulum, alternating electrical circuits, or any sort of wave motion in air and water. The differential equations governing these motions will be covered next term; here we simply explore some of the characteristics of these motions.

To be specific, let us consider a simple pendulum swinging under gravity with angular frequency ω. For small amplitude motion, the frequency is related to the length of the pendulum l by $\omega = \sqrt{g/l}$, where g is the gravitational acceleration. The angular displacement, $x(t)$, of the pendulum about the vertical then takes the general form

$$x(t) = a\cos\omega t + b\sin\omega t, \tag{81}$$

where a and b are real constants. This form of motion is often referred to as ***simple harmonic motion***.

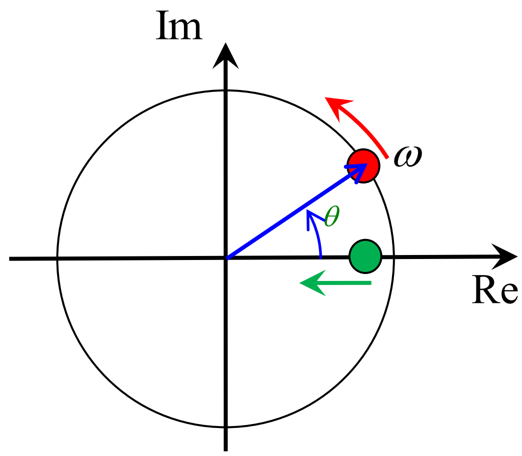


Figure 24: Sketch of an oscillator in the complex plane.

Using complex numbers, we can write (81) as

$$x(t) = \Re\left[A\exp(i\omega t)\right], \tag{82}$$

where A is now a *complex* constant, the ***complex amplitude***. By comparing (81) and (82), we find that

$$A = a - ib.$$

The big advantage of the complex representation (82) is that differentiation is very easy, and is handled in the same way as for real variables. So, for example the velocity $v(t)$ is given by

$$\begin{aligned} v(t) &= \frac{dx}{dt} = \frac{d}{dt}\Re\left[A\exp(i\omega t)\right] \\ &= \Re\left[\frac{d}{dt}A\exp(i\omega t)\right] = \Re\left[i\omega A\exp(i\omega t)\right]. \end{aligned} \tag{83}$$

In other words, to differentiate $e^{i\omega t}$ we simply multiply by $i\omega$ before taking the real part.

If at $t=0$ we have $x(0)=c$ and $v(0)\equiv\dot{x}(0)=\sqrt{3}c\omega$ then

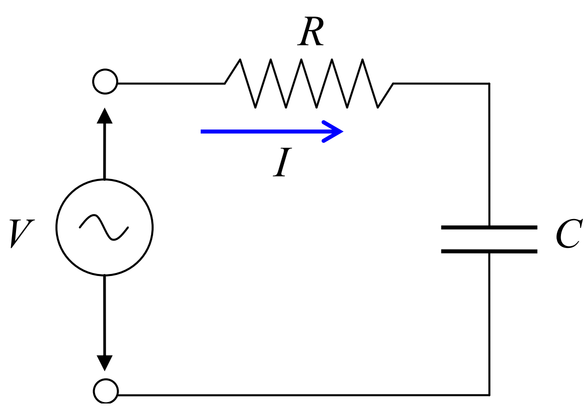
$$x(0)=\Re(A)=c \quad \text{and} \quad \dot{x}(0)=\Re(i\omega A)=\sqrt{3}c\omega$$

$\Rightarrow$
$$A=c(1-i\sqrt{3})=2ce^{-i\pi/3}.$$

This is an oscillation with amplitude $|A|=2c$ and phase $\theta=-\pi/3$:

$$x(t)=2c\cos(\omega t-\tfrac{1}{3}\pi).$$

Example 32: Impedance of AC circuit



Find the complex resistance of a capacitor and resistor connected in series to an applied alternating voltage.

2.7 Fundamental theorem of algebra

If $P(z)$ is a polynomial of degree $n \in \mathbb{Z}$ ($n \geq 1$),

$$P(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad \text{with} \quad a_n \neq 0,$$

then

$$P(z) = 0$$

has n (complex) roots (possibly repeated) for all possible coefficients $a_0, \cdots, a_n \in \mathbb{C}$, $a_n \neq 0$.

This is equivalent to saying that every polynomial equation has at least one complex root.

The statements are the equivalent as if $z = z_1$ is a root of $P(z) = 0$, then

$$P(z) = (z - z_1) Q(z) = 0$$

where $Q(z)$ is a polynomial of degree $n-1$. Now, since $Q(z) = 0$ must also have at least one route, so $P(z) = (z - z_1)(z - z_2) R(z) = 0$, *etc.*

We will not give a formal proof, but simply note that the roots must exist.

> The fundamental theorem of algebra for $a_j \in$ ~ is often attributed to Gauss (1799), but the key elements first appeared earlier in work by Peter Roth (1608). Argand (1806) extended things to $a_j \in \mathbb{C}$ and provided the first complete proof.