Graduate Texts in Contemporary Physics

(continued after index)

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Modeling Complex Systems

With 158 Illustrations

Springer

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Differential Equations

The study of dynamical models formulated in terms of ordinary differential equations began with Newton's attempts to explain the motion of bodies in the solar system. Except in very simple cases, such as the two-body problem, most problems in celestial mechanics proved extremely difficult. At the end of the nineteenth century, Poincaré developed new methods to analyze the qualitative behavior of solutions to nonlinear differential equations. In a paper [287][1] devoted to functions defined as solutions of differential equations, he explains:

> *Malheureusement, il est évident que, dans la grande généralité des cas qui se présentent, on ne peut intégrer ces équations à l'aide des fonctions déjà connues, ...*
>
> *Il est donc nécessaire d'étudier les fonctions définies par des équations différentielles en elles-mêmes et sans chercher à les ramener à des fonctions plus simples, ...*[2]

The modern qualitative theory of differential equations has its origin in this work.

There exists a wide variety of models formulated in terms of differential equations, and some of them are presented in this chapter.

3.1 Flows

Consider a system whose dynamics is described by the differential equation

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[1] This paper is a revision of the work for which Poincaré was awarded a prize offered by the king of Sweden in 1889.

[2] Unfortunately, it is clear that in most cases we cannot solve these equations using known functions, ...

It is therefore necessary to study functions defined by differential equations for themselves without trying to reduce them to simpler functions, ...

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}), \tag{3.1}$$

where $\mathbf{x}$, which represents the state of the system, belongs to the *state* or *phase space* $\mathcal{S}$, and $\mathbf{X}$ is a given *vector field*. Figure 3.1 shows an example of a vector field. We have seen (Chapter 1, Example 1) that a differential equation of order higher than one, autonomous or nonautonomous, can always be written under the above form.

To present the theory, we need to recall some definitions.

Definition 1. *A function* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is* differentiable *at* $\mathbf{x}_0 \in \mathbb{R}^n$ *if there exists a linear transformation* $D\mathbf{f}(\mathbf{x}_0)$ *that satisfies*

$$\lim_{\|\mathbf{h}\|\to 0} \frac{\|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\mathbf{h}\|}{\|\mathbf{h}\|} = 0. \tag{3.2}$$

The linear transformation $D\mathbf{f}(\mathbf{x}_0)$ *is called the* derivative *of* $\mathbf{f}$ *at* $\mathbf{x}_0$.

Instead of the word *function*, many authors use the words *map* or *mapping*. In this text, we shall indifferently use any of these terms.

It is easily verified that the derivative $D\mathbf{f}$ is given by the $n \times n$ Jacobian matrix

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

Let U be an open subset of $\mathbb{R}^n$; the function $\mathbf{f} : U \to \mathbb{R}^n$ is continuously differentiable, or of *class* C^1, in U if all the partial derivatives

$$\frac{\partial f_i}{\partial x_j} \quad (1 \le i \le n,\ 1 \le j \le n)$$

are continuous in U. More generally, $\mathbf{f}$ is of *class* C^k in U if all the partial derivatives

$$\frac{\partial^k f_i}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_k}} \quad (1 \le i \le n,\ 1 \le j_1 \le n,\ 1 \le j_2 \le n,\ \ldots,\ 1 \le j_k \le n)$$

exist and are continuous in U. Continuous but not differentiable functions are referred to as C^0 functions. A function is said to be *smooth* if it is differentiable a sufficient number of times.

Definition 2. *A function* $\mathbf{f} : U \to V$*, where* U *and* V *are open subsets of* $\mathbb{R}^n$*, is said to be a* C^k diffeomorphism *if it is a bijection, and both* $\mathbf{f}$ *and* $\mathbf{f}^{-1}$ *are* C^k *functions. If* $\mathbf{f}$ *and* $\mathbf{f}^{-1}$ *are* C^0*,* $\mathbf{f}$ *is called a* homeomorphism.

In many models, the phase space is not Euclidean. It may have, for instance, the structure of a circle or a sphere. But if, as in both these cases, we can define *local coordinates*, the notions of derivative and diffeomorphism can be easily extended. Phase spaces that have a structure similar to the

structure of a circle or a sphere are called *manifolds*. More precisely, $\mathcal{M}$ is a manifold of dimension n if, for any $\mathbf{x} \in \mathcal{M}$, there exist a neighborhood $N(\mathbf{x}) \subseteq \mathcal{M}$ containing $\mathbf{x}$ and a homeomorphism $\mathbf{h} : N(\mathbf{x}) \to \mathbb{R}^n$ that maps $N(\mathbf{x})$ onto a neighborhood of $\mathbf{h}(\mathbf{x}) \in \mathbb{R}^n$. Since we can define coordinates in $\mathbf{h}(N(\mathbf{x})) \subseteq \mathbb{R}^n$, $\mathbf{h}$ defines local coordinates on $N(\mathbf{x})$. The pair $(\mathbf{h}(N(\mathbf{x})), \mathbf{h})$ is called a *chart*. In order to obtain a global description of $\mathcal{M}$, we cover it with a family of open sets N_i, each associated with a chart $(\mathbf{h}_i(N_i), \mathbf{h}_i)$. The set of all these charts is called an *atlas*. In all the models we shall study, even if the phase space is a manifold, the functions $\mathbf{f}$ will be given in terms of local coordinates. We shall, therefore, never be really involved with charts and atlases. In all definitions and theorems involving "differential manifolds of dimension n," the reader could replace this expression by "open sets of $\mathbb{R}^n$."

If, in Equation (3.1), the vector field $\mathbf{X}$ defined on an open subset U of $\mathbb{R}^n$ is C^k, then given $\mathbf{x}_0 \in U$ and $t_0 \in \mathbb{R}$, for $|t - t_0|$ sufficiently small, there exists a solution of Equation (3.1) through the point $\mathbf{x}_0$ at t_0, denoted $\mathbf{x}(t, t_0, \mathbf{x}_0)$ with $\mathbf{x}(t_0, t_0, \mathbf{x}_0) = \mathbf{x}_0$. This solution is unique and is a C^k function of t, t_0, and $\mathbf{x}_0$. As we already pointed out (page 10, Footnote 16), we shall not give a proof of this fundamental theorem since a differential equation modeling a real system should have a unique evolution for any realizable initial state.[3]

The solution of Equation (3.1) being unique, we have

$$\mathbf{x}(t + s, t_0, \mathbf{x}_0) = \mathbf{x}(s, t + t_0, \mathbf{x}(t, t_0, \mathbf{x}_0)).$$

This property shows that the solutions of (3.1) form a one-parameter group of C^k diffeomorphisms of the phase space. These diffeomorphisms are referred to as a *phase flow* or just a *flow*. The common notation for flows is $\varphi(t, \mathbf{x})$ or $\varphi_t(\mathbf{x})$, and we have

$$\varphi_t \circ \varphi_s = \varphi_{t+s} \tag{3.3}$$

for all t and s in $\mathbb{R}$. Note that φ_0 is the identity and that $(\varphi_t)^{-1}$ exists and is given by φ_{-t}.[4]

In most cases $t_0 = 0$ and, from the definition of φ_t above we have

$$\mathbf{X}(\mathbf{x}) = \frac{d}{dt}\varphi_t(\mathbf{x})\bigg|_{t=0}. \tag{3.4}$$

Given a point $\mathbf{x} \in U \subseteq \mathbb{R}^n$, the *orbit* or *trajectory* of φ passing through $\mathbf{x} \in U$ is the set $\{\varphi_t(\mathbf{x}) \mid t \in \mathbb{R}\}$ oriented in the sense of increasing t. There is only one trajectory of φ passing through any given point $\mathbf{x} \in U$. That is, if two trajectories intersect, they must coincide.

The set of all trajectories of a flow is called its *phase portrait*. A helpful representation of a flow is obtained by plotting typical trajectories (Figure 3.1).

[3] The mathematically oriented reader may consult Hale [163] or Hirsch and Smale [171].

[4] The mapping $t \mapsto \varphi_t$ is an *isomorphism* from the group of real numbers $\mathbb{R}$ to the group $\{\varphi_t \mid t \in \mathbf{R}\}$. This group is called an *action of the group* $\mathbb{R}$ *on the state space* U.

Definition 3. *Let* $\mathbf{X}$ *be a vector field defined on an open set* U *of* $\mathbb{R}^n$*; a point* $\mathbf{x}^* \in U$ *is an* equilibrium point *of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ *if* $\mathbf{X}(\mathbf{x}^*) = 0$.

Note that if $\mathbf{x}^*$ is an equilibrium point, then $\varphi_t(\mathbf{x}^*) = \mathbf{x}^*$ for all $t \in \mathbb{R}$. Thus $\mathbf{x}^*$ is also called a *fixed point* of the flow φ. The orbit of a fixed point is the fixed point itself.

A *closed orbit* of a flow φ is a trajectory that is not a fixed point but is such that $\varphi_\tau(\mathbf{x}) = \mathbf{x}$ for some $\mathbf{x}$ on the trajectory and a nonzero τ. The smallest nonzero value of τ is usually denoted by T and is called the *period of the orbit*. That is, we have $\varphi_T(\mathbf{x}) = \mathbf{x}$, but $\varphi_t(\mathbf{x}) \neq \mathbf{x}$ for $0 < t < T$.

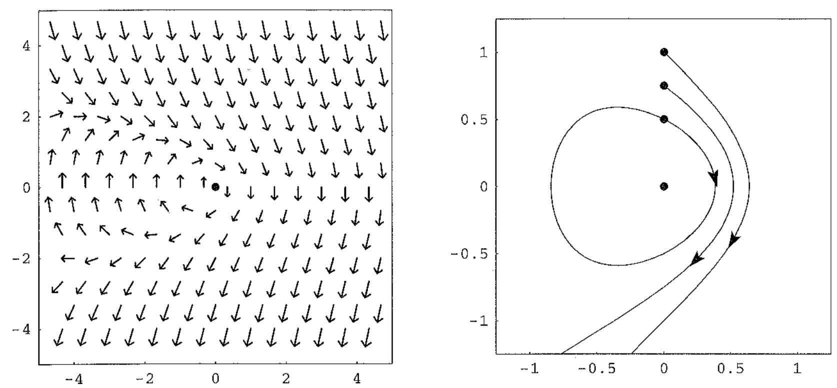

Fig. 3.1. *Vector field (left) and phase portrait (right) of the two-dimensional system* $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - x_2^2$. $(0,0)$ *is a nonhyperbolic equilibrium point.*

Definition 4. *An equilibrium point* $\mathbf{x}^*$ *of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ *is said to be* Lyapunov stable *(or* L-stable*) if, for any given positive* ε*, there exists a positive* δ *(which depends on* ε *only) such that, for all* $\mathbf{x}_0$ *in the neighborhood of* $\mathbf{x}^*$ *defined by* $\|\mathbf{x}_0 - \mathbf{x}^*\| < \delta$*, the solution* $\mathbf{x}(t, 0, \mathbf{x}_0)$ *of the differential equation above satisfying the initial condition* $\mathbf{x}(0, 0, \mathbf{x}_0) = \mathbf{x}_0$ *is such that* $\|\mathbf{x}(t, 0, \mathbf{x}_0) - \mathbf{x}^*\| < \varepsilon$ *for all* $t > 0$*. The equilibrium point is said to be* unstable *if it is not stable.*

An equilibrium point $\mathbf{x}^*$ of a differential equation is stable if the trajectory in the phase space going through a point sufficiently close to the equilibrium point at $t = 0$ remains close to the equilibrium point as t increases. Lyapunov stability does not imply that, as t tends to infinity, the point $\mathbf{x}(t, 0, \mathbf{x}_0)$ tends to $\mathbf{x}^*$. But:

Definition 5. *An equilibrium point* $\mathbf{x}^*$ *of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ *is said to be* asymptotically stable *if it is Lyapunov stable and*

$$\lim_{t \to \infty} \mathbf{x}(t, 0, \mathbf{x}_0) = \mathbf{x}^*.$$

Example 8. Kermack-McKendrick epidemic model. To discuss the spread of an infection within a population, Kermack and McKendrick [182] divide the population into three disjoint groups.

1. *Susceptible* individuals are capable of contracting the disease and becoming infective.
2. *Infective* individuals are capable of transmitting the disease to others.
3. *Removed* individuals have had the disease and are dead, have recovered and are permanently immune, or are isolated until recovery and permanent immunity occur.[5]

Infection and removal are governed by the following rules.

1. The rate of change in the susceptible population is proportional to the number of contacts between susceptible and infective individuals, where the number of contacts is taken to be proportional to the product of the numbers S and I of, respectively, susceptible and infective individuals. The model ignores incubation periods.
2. Infective individuals are removed at a rate proportional to their number I.
3. The total number of individuals $S + I + R$, where R is the number of removed individuals, is constant, that is, the model ignores births, deaths by other causes, immigration, emigration, etc.[6]

Taking into account the rules above yields

$$\begin{aligned} \dot{S} &= -iSI, \\ \dot{I} &= iSI - rI, \\ \dot{R} &= rI, \end{aligned} \tag{3.5}$$

where i and r are positive constants representing infection and the removal rates.

From the first equation, it is clear that S is a nonincreasing function, whereas the second equation implies that $I(t)$ increases with t, if $S(t) > r/i$, and decreases otherwise. Therefore, if, at $t = 0$, the initial number of susceptible individuals S_0 is less than r/i, since $S(t) \leq S_0$, the infection dies out, that is, no epidemic occurs. If, on the contrary, S_0 is greater than the critical value r/i, the epidemic occurs; that is, the number of infective individuals first increases and then decreases when $S(t)$ becomes less than r/i.[7]

Remark 1. This *threshold phenomenon* shows that an epidemic occurs if, and only if, the initial number of susceptible individuals S_0 is greater than a threshold value S_{th}. For this model, $S_{\text{th}} = r/i$; *i.e.*, in the case of a deadly disease, an epidemic has less chance to occur if the death rate due to the disease is high!

[5] Models of this type are called *SIR models*.

[6] Or we could say that birth, death, and migration are in exact balance.

[7] That is, *by definition*, an epidemic occurs if the time derivative of the number of infective individuals $\dot{I}$ *is positive at* $t = 0$.

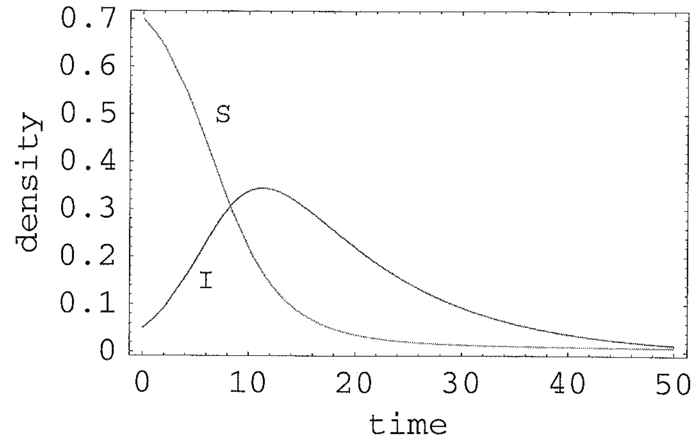

Fig. 3.2. *Kermack-McKendrick epidemic model. Susceptible and infective densities as functions of time for* $i = 0.6$, $r = 0.1$, $S(0) = 0.7$, *and* $I(0) = 0.05$. $S(0)$ *is above the threshold value, and, as a consequence of the epidemic, the total population is seriously reduced.*

Note that $\dot{S}$ is nonincreasing and positive, and $\dot{R}$ is positive and less than or equal to the total population N, therefore, $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} R(t)$ exist. Since $I(t) = N - S(t) - R(t)$, $\lim_{t\to\infty} I(t)$ also exists.[8] Moreover, from the first and third equations of (3.5), it follows that

$$\frac{dS}{dR} = -\frac{i}{r}\,S;$$

that is,

$$\begin{aligned} S &= S_0 \exp\left(-\frac{iR}{r}\right) \\ &\geq S_0 \exp\left(-\frac{iN}{r}\right) > 0. \end{aligned} \tag{3.6}$$

In this model, even in the case of a very serious epidemic, some individuals are not infected. The spread of the disease does not stop for lack of susceptible individuals (see Figure 3.2).

Example 9. Hethcote-York model for the spread of gonorrhea [170]. Gonorrhea is a sexually transmitted disease that presents the following important characteristics that differ from other infections such as measles or mumps:

[8] Equations (3.5) show that in the steady state $I(\infty) = 0$. Then, $R(\infty) = N - S(\infty)$, and Relation (3.6) shows that $S(\infty)$ is the only positive root of $x = S_0 \exp(-i(N-x)/r)$. More details can be found in Waltman [341].

- Gonococcal infection does not confer protective immunity, so individuals are susceptible again as soon as they recover from infection.[9]
- The latent period is very short: 2 days, compared to 12 days for measles.
- The seasonal oscillations in gonorrhea incidence[10] are very small (less than 10%), while the incidence of influenza or measles often varies by a factor of 5 to 50.

If we assume that the infection is transmitted only through heterosexual intercourse, we divide the population into two groups, N_f females and N_m males at risk, each group being divided into two subgroups, $N_f S_f$ (resp. $N_m S_m$) susceptible females (resp. males) and $N_f I_f$ (resp. $N_m I_m$) infective females (resp. males). N_f and N_m are assumed to be constant. The dynamics of gonorrhea is then modeled by the four-dimensional system

$$\begin{aligned} N_f \dot{S}_f &= -\lambda_f S_f N_m I_m + N_f I_f / d_f, \\ N_f \dot{I}_f &= \lambda_f S_f N_m I_m - N_f I_f / d_f, \\ N_m \dot{S}_m &= -\lambda_m S_m N_f I_f + N_m I_m / d_m, \\ N_m \dot{I}_m &= \lambda_m S_m N_f I_f - N_m I_m / d_m, \end{aligned}$$

where λ_f (resp. λ_m) is the rate of infection of susceptible females (resp. males), and d_f (resp. d_m) is the average duration of infection for females (resp. males). The rates λ_f and λ_m are different since the probability of transmission of gonococcal infection during a single sexual exposure from an infectious woman to a susceptible man is estimated to be about 0.2–0.3, while the probability of transmission from an infectious man to a susceptible woman is about 0.5–0.7. The average durations of infection d_f and d_m are also different since 90% of all men who have a gonococcal infection notice symptoms within a few days after exposure and promptly seek medical treatment, while up to 75% of women with gonorrhea fail to have symptoms and remain untreated for some time.

Since $S_f + I_f = 1$ and $S_m + I_m = 1$, the four-dimensional system reduces to the two-dimensional system

$$\begin{aligned} \dot{I}_f &= \frac{\lambda_f}{r}(1 - I_f) I_m - \frac{I_f}{d_f}, \\ \dot{I}_m &= r\lambda_m (1 - I_m) I_f - \frac{I_m}{d_m}, \end{aligned}$$

where $r = N_f/N_m$. The system has two equilibrium points $(I_f, I_m) = (0,0)$ and

$$(I_f,\ I_m) = \left(\frac{d_f d_m \lambda_f \lambda_m - 1}{d_m \lambda_m (r + d_f \lambda_f)},\ \frac{r(d_f d_m \lambda_f \lambda_m - 1)}{d_f \lambda_f (1 + r d_m \lambda_m)}\right).$$

[9] Models of this type are called *SIS models.*

[10] *Incidence* is the number of new cases in a time interval.

Since acceptable solutions should not be negative, we find that the nontrivial equilibrium point exists if

$$d_f d_m \lambda_f \lambda_m > 1.$$

The coefficient λ_f/r (resp. $\lambda_m r$) represents the average fraction of females (resp. males) being infected by one male (resp. female) per unit of time. Since males (resp. females) are infectious during the period d_m (resp. d_f), then the average fraction of females (resp. males) being infected by one infective male (female) during his (resp. her) period of infection is $\lambda_f d_m/r$ (resp. $\lambda_m d_f r$). The condition $\lambda_m d_f \lambda_f d_m > 1$ therefore expresses that the average fraction of females infected by one male will infect, during their period of infection, more than one male. In this case, gonorrhea remains endemic. If the condition is not satisfied, then gonorrhea dies out. As a consequence, for this model, if the nontrivial equilibrium point exists, it is asymptotically stable; if it does not, then the trivial fixed point is asymptotically stable.

Example 10. Leslie's predator-prey model. After the publication of the Lotka-Volterra model (Equations (2.1) and (2.1)), many other predator-prey models were proposed. In 1948, Leslie [201] suggested the system

$$\begin{aligned} \dot{H} &= r_H H\left(1 - \frac{H}{K}\right) - sHP, \\ \dot{P} &= r_P P\left(1 - \frac{P}{cH}\right), \end{aligned} \tag{3.7}$$

where H and P denote, respectively, the prey and predator populations. The equation for the preys is similar to Lotka-Volterra equation (2.1) except that, in the absence of predators, the growth of the preys is modeled by the logistic equation. The equation for the predators is a logistic equation in which the carrying capacity is proportional to the prey population. This model contains five parameters. There is only one nontrivial equilibrium point (H^*, P^*), which is the unique solution of the linear system

$$r_H\left(1 - \frac{H}{K}\right) = sP, \quad P = cH.$$

If we put

$$h = \frac{H}{H^*}, \quad p = \frac{P}{P^*}, \quad \rho = \sqrt{\frac{r_H}{r_P}}, \quad k = \frac{K}{H^*}, \quad \tau = \sqrt{r_H r_P}\, t, \tag{3.8}$$

Equations (3.7) become

$$\begin{aligned} \frac{dh}{d\tau} &= \rho h\left(1 - \frac{h}{k}\right) - \alpha p h, \\ \frac{dp}{d\tau} &= \frac{1}{\rho} p\left(1 - \frac{p}{h}\right). \end{aligned} \tag{3.9}$$

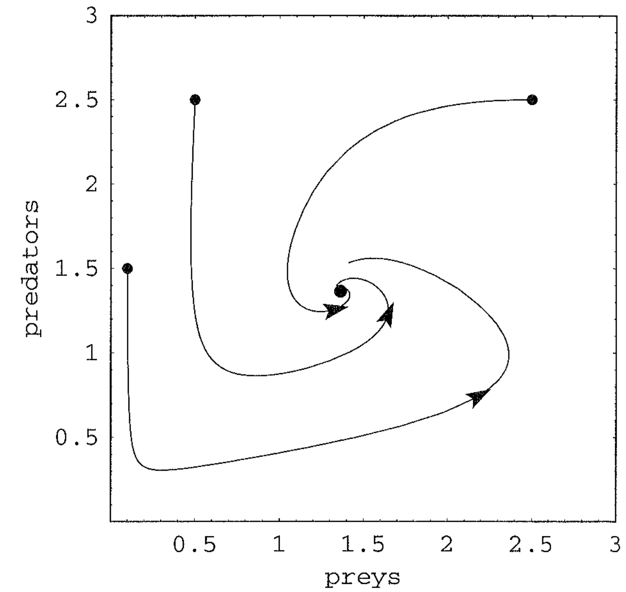

Fig. 3.3. *Phase portrait of the scaled Leslie model for $k = 5$ and $\rho = 1.5$.*

These equations contain two independent parameters, ρ and k, the extra parameter α being given in terms of these by[11]

$$\alpha = \rho\left(1 - \frac{1}{k}\right).$$

The equilibrium points are $(0,0)$, $(k,0)$, and $(1,1)$. A few trajectories converging to the asymptotically stable equilibrium point $(1,1)$ are shown in Figure 3.3.

Definition 6. *Let $\varphi_t : U \to U$ and $\psi_t : W \to W$ be two flows; if there exists a diffeomorphism $\mathbf{h} : U \to W$ such that, for all $t \in \mathbb{R}$,*

$$\mathbf{h} \circ \varphi_t = \psi_t \circ \mathbf{h}, \tag{3.10}$$

the flows φ_t and ψ_t are said to be conjugate.

In other words, the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi_t} & U \\ {\scriptstyle \mathbf{h}}\downarrow & & \downarrow{\scriptstyle \mathbf{h}} \\ W & \xrightarrow{\psi_t} & W \end{array}$$

[11] This result follows from the first of the two equations (3.9) when we put $\dfrac{dh}{d\tau} = 0, \quad h = 1, \quad p = 1.$

is commutative. The purpose of Definition 6 is to provide with a way of characterizing when two flows have qualitatively the same dynamics. Equation (3.10) can also be written

$$\boldsymbol{\psi}_t = \mathbf{h} \circ \boldsymbol{\varphi}_t \circ \mathbf{h}^{-1}.$$

That is, $\mathbf{h}$ takes the orbits of the flow $\boldsymbol{\varphi}_t$ into the orbits of the flow $\boldsymbol{\psi}_t$. In other words, the flow $\boldsymbol{\varphi}_t$ becomes the flow $\boldsymbol{\psi}_t$ under the change of coordinates $\mathbf{h}$.

It is readily verified that conjugacy is an equivalence relation, *i.e.*,

$$(\boldsymbol{\varphi}_t \sim \boldsymbol{\varphi}_t),\quad (\boldsymbol{\varphi}_t \sim \boldsymbol{\psi}_t) \Rightarrow (\boldsymbol{\psi}_t \sim \boldsymbol{\varphi}_t),\quad \text{and}\quad (\boldsymbol{\varphi}_t \sim \boldsymbol{\psi}_t, \boldsymbol{\psi}_t \sim \boldsymbol{\chi}_t) \Rightarrow (\boldsymbol{\varphi}_t \sim \boldsymbol{\chi}_t).$$

If $\mathbf{h}$ is a C^1 function such that $\mathbf{h} \circ \boldsymbol{\varphi}_t = \boldsymbol{\psi}_t \circ \mathbf{h}$, then differentiating both sides with respect to t and evaluating at $t = 0$ yields

$$D\mathbf{h}(\boldsymbol{\varphi}_t(\mathbf{x}))\frac{d}{dt}\boldsymbol{\varphi}_t(\mathbf{x})\bigg|_{t=0} = \frac{d}{dt}\boldsymbol{\psi}_t(\mathbf{h}(\mathbf{x}))\bigg|_{t=0},$$

and taking into account Relation (3.4), we obtain

$$D\mathbf{h}(\mathbf{x})\mathbf{X}(\mathbf{x}) = \mathbf{Y}(\mathbf{h}(\mathbf{x})). \tag{3.11}$$

That is, if $\mathbf{h}$ is a differentiable flow conjugacy of the flows $\boldsymbol{\varphi}_t$ and $\boldsymbol{\psi}_t$, the derivative $D\mathbf{h}(\mathbf{x})$ transforms $\mathbf{X}(\mathbf{x})$ into $\mathbf{Y}(\mathbf{h}(\mathbf{x}))$.

Remark 2. Definition 6 requires conjugacy to preserve the parameter t. If we are required to preserve only the *orientation* along the orbits of $\boldsymbol{\varphi}_t$ and $\boldsymbol{\psi}_t$, we obtain more satisfactory equivalence classes for flows. In that case, Relation (3.10) would have to be replaced by

$$\mathbf{h} \circ \boldsymbol{\varphi}_t = \boldsymbol{\psi}_{\tau(t,\mathbf{x})} \circ \mathbf{h}, \tag{3.12}$$

where, for all $\mathbf{x}$, the function $t \mapsto \tau(t, \mathbf{x})$ is strictly increasing; *i.e.*, its derivative with respect to t has to be positive for all $\mathbf{x}$. If there exist a homeomorphism $\mathbf{h}$ and a differentiable function τ such that Relation (3.12) is satisfied, it is said that the flows $\boldsymbol{\varphi}_t$ and $\boldsymbol{\psi}_t$ are *topologically equivalent.* Here is a simple example. Consider the two one-dimensional flows:

$$\varphi_1(t, x) = e^{-\lambda_1 t}x \quad \text{and} \quad \varphi_2(t, x) = e^{-\lambda_2 t}x,$$

where λ_1 and λ_2 are different positive numbers. Let h be such that

$$h(e^{-\lambda_1 t}x) = e^{-\lambda_2 t}h(x). \tag{3.13}$$

If h is a *diffeomorphism,* then taking the derivative with respect to x of each side of this relation yields

$$e^{-\lambda_1 t}h'(e^{-\lambda_1 t}x) = e^{-\lambda_2 t}h'(x).$$

A diffeomorphism being invertible by definition, $h'(0) \neq 0$, and we obtain $e^{-\lambda_1 t} = e^{-\lambda_2 t}$, which contradicts the assumption $\lambda_1 \neq \lambda_2$. If, on the contrary, we assume that h is not differentiable everywhere (*i.e.*, h is not a diffeomorphism but only a homeomorphism), then the homeomorphism[12]

[12] h is continuous and invertible.

$$h \mapsto \begin{cases} -|x|^{\lambda_2/\lambda_1}, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x^{\lambda_2/\lambda_1}, & \text{if } x > 0 \end{cases}$$

satisfies (3.13) and shows that φ_1 and φ_2 are topologically equivalent.

3.2 Linearization and stability

In order to analyze a model described by a nonlinear differential equation of the form (3.1), we first have to determine its equilibrium points and study the behavior of the system near these points. Under certain conditions, this behavior is qualitatively the same as the behavior of a linear system. We therefore begin this section with a brief study of linear differential equations.

3.2.1 Linear systems

The solution of the one-dimensional linear differential equation

$$\dot{x} = ax,$$

which satisfies the initial condition $x(0) = x_0$, is $x(t) = x_0 e^{at}$.

Question. Let $\mathbf{A}$ be a time-independent linear operator defined on $\mathbb{R}^n$; is it possible to generalize the result above and say that, if $\mathbf{x} \in \mathbb{R}^n$, the solution to the linear differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3.14}$$

which satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$?

The answer is yes, provided we define and show how to express the linear operator $e^{\mathbf{A}t}$.

Definition 7. *Let* $\mathbf{A}$ *be a linear operator defined on* $\mathbb{R}^n$*; the* exponential *of* $\mathbf{A}$ *is the linear operator defined on* $\mathbf{R}^n$ *by*[13]

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}. \tag{3.15}$$

[13] For this definition to make sense, it is necessary to show that the series converges and, therefore, to first define a metric on the space of linear operators on $\mathbb{R}^n$ in order to be able to introduce the notion of limit of a sequence of linear operators. If $\|\mathbf{x}\|$ is the norm of $\mathbf{x} \in \mathbb{R}^n$, the norm of a linear operator $\mathbf{A}$ may be defined as $\|\mathbf{A}\| = \sup_{\|\mathbf{x}\| \le 1} \|\mathbf{A}\mathbf{x}\|$. The distance d between two linear operators $\mathbf{A}$ and $\mathbf{B}$ on $\mathbb{R}^n$ is then defined as $d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} - \mathbf{B}\|$.

Depending on whether the real linear operator $\mathbf{A}$ has real or complex distinct or multiple eigenvalues, the real linear operator $e^{\mathbf{A}t}$ may take different forms, as described below.[14]

1. *All the eigenvalues of* $\mathbf{A}$ *are distinct.*

If $\mathbf{A}$ has distinct real eigenvalues λ_i and corresponding eigenvectors $\mathbf{u}_i$, where $i = 1, 2, \dots, k$ and distinct complex eigenvalues $\lambda_j = \alpha_j + i\beta_j$ and $\overline{\lambda}_j = \alpha_j - i\beta_j$ and corresponding complex eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j$, where $j = k+1, k+2, \dots, \ell$, then the matrix[15]

$$\mathbf{M} = [\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}, \dots, \mathbf{u}_\ell, \mathbf{v}_\ell]$$

is invertible, and

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathrm{diag}[\lambda_1, \dots, \lambda_k, B_{k+1}, \dots, B_\ell].$$

The right-hand side denotes the matrix

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & B_{k+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & B_\ell \end{bmatrix},$$

where, for $j = k+1, k+2, \dots, \ell$, the B_j are 2×2 blocks given by

$$B_j = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix}.$$

In this case, we have

$$e^{\mathbf{A}t} = \mathbf{M}\,\mathrm{diag}[e^{\lambda_1 t}, \dots, e^{\lambda_k t}, E_{k+1}, \dots, E_\ell]\mathbf{M}^{-1}, \tag{3.16}$$

the 2×2 block E_j being given by

$$E_j = e^{\alpha_j t} \begin{bmatrix} \cos\beta_j t & -\sin\beta_j t \\ \sin\beta_j t & \cos\beta_j t \end{bmatrix}.$$

[14] For a simple and rigorous treatment of linear differential systems, see Hirsch and Smale [171].

[15] If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are n independent vectors of $\mathbb{R}^n$, the matrix $[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ denotes the matrix

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}.$$

Note that the dimension of the vector space on which $\mathbf{A}$ and $e^{\mathbf{A}t}$ are defined is $n = 2\ell - k$.

2. $\mathbf{A}$ *has real multiple eigenvalues.*

If $\mathbf{A}$ has real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ repeated according to their multiplicity and if $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$ is a basis of *generalized eigenvectors*,[16] then the matrix

$$\mathbf{M} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n]$$

is invertible, and the operator $\mathbf{A}$ can be written as the sum of two matrices $\mathbf{S} + \mathbf{N}$, where

$$\mathbf{SN} = \mathbf{NS}, \quad \mathbf{M}^{-1}\mathbf{S}\mathbf{M} = \mathrm{diag}[\lambda_1, \lambda_2, \dots, \lambda_n],$$

and $\mathbf{N}$ is *nilpotent of order* $k \le n$.[17]

In this case, we have[18]

$$e^{\mathbf{A}t} = \mathbf{M}\ \mathrm{diag}[e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}]\ \mathbf{M}^{-1} \times \left(\mathbf{I} + \mathbf{N}t + \cdots + \mathbf{N}^{k-1}\frac{t^{k-1}}{(k-1)\,!}\right). \tag{3.17}$$

3. $\mathbf{A}$ *has complex multiple eigenvalues.*

If a real linear operator $\mathbf{A}$, represented by a $2n \times 2n$ matrix, has complex eigenvalues $\lambda_j = \alpha_j + i\beta_j$ and $\overline{\lambda}_j = \alpha_j - i\beta_j$, where $j = 1, 2, \dots, n$, there exists a basis of generalized eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j$ for $\mathbb{C}^n$, $(\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{u}_n, \mathbf{v}_n)$ is a basis for $\mathbb{R}^{2n}$, the $2n \times 2n$ matrix $\mathbf{M} = [\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{u}_n, \mathbf{v}_n]$ is invertible, and the operator $\mathbf{A}$ can be written as the sum of two matrices $\mathbf{S} + \mathbf{N}$, where

$$\mathbf{SN} = \mathbf{NS}, \quad \mathbf{M}^{-1}\mathbf{S}\mathbf{M} = \mathrm{diag}\left[\begin{bmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & \alpha_1 \end{bmatrix}, \begin{bmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & \alpha_2 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_n & -\beta_n \\ \beta_n & \alpha_n \end{bmatrix}\right],$$

and $\mathbf{N}$ is nilpotent of order $k \le 2n$.

In this case, we have

$$e^{\mathbf{A}t} = \mathbf{M}\,\mathrm{diag}\left[e^{\alpha_1 t}\begin{bmatrix} \cos\beta_1 & -\sin\beta_1 \\ \sin\beta_1 & \cos\beta_1 \end{bmatrix}, \dots, e^{\alpha_n t}\begin{bmatrix} \cos\beta_n & -\sin\beta_n \\ \sin\beta_n & \cos\beta_n \end{bmatrix}\right]\mathbf{M}^{-1} \times \left(\mathbf{I} + \mathbf{N}t + \cdots + \mathbf{N}^{k-1}\frac{t^{k-1}}{(k-1)\,!}\right). \tag{3.18}$$

[16] If the real eigenvalue λ has multiplicity $m < n$, then for $k = 1, 2, \dots, m$, any nonzero solution of $(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{u} = 0$ is called a *generalized eigenvector*.

[17] A linear operator $\mathbf{N}$ is *nilpotent of order* k if $\mathbf{N}^{k-1} \ne \mathbf{0}$ and $\mathbf{N}^k = \mathbf{0}$.

[18] If the linear operators $\mathbf{A}$ and $\mathbf{B}$ commute, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$. The series defining the exponential of a nilpotent linear operator of order k is a polynomial of degree $k - 1$.

4. **A** *has both real and complex multiple eigenvalues.*
In this case, we use a combination of the results above to find the expression of linear operator $e^{\mathbf{A}t}$.

Example 11. Classification of two-dimensional linear flows. Let $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$ denote, respectively, the *trace* and the *determinant* of the 2×2 time-independent real matrix **A**. The eigenvalues of **A** are the roots of its *characteristic polynomial*:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - \operatorname{tr}\mathbf{A}\,\lambda + \det\mathbf{A}. \tag{3.19}$$

We may distinguish the following cases:

1. If $\det\mathbf{A} < 0$, the eigenvalues of **A** are real and have opposite sign. The origin is said to be a *saddle*.
2. If $\det\mathbf{A} > 0$ and $(\operatorname{tr}\mathbf{A})^2 \geq 4\det\mathbf{A}$, the eigenvalues of **A** are real and have the same sign. The origin is said to be an *attractive node* if $\operatorname{tr}\mathbf{A} < 0$ and a *repulsive node* if $\operatorname{tr}\mathbf{A} > 0$.
3. If $\operatorname{tr}\mathbf{A} \neq 0$ and $(\operatorname{tr}\mathbf{A})^2 < 4\det\mathbf{A}$, the eigenvalues of **A** are complex. The origin is said to be an *attractive focus* if $\operatorname{tr}\mathbf{A} < 0$ and a *repulsive focus* if $\operatorname{tr}\mathbf{A} > 0$.
4. If $\operatorname{tr}\mathbf{A} = 0$ and $\det\mathbf{A} < 0$, the eigenvalues of **A** are complex with a zero real part. The origin is said to be a *center*.

The phase portraits corresponding to the different cases described above are represented in Figure 3.4. Attractive nodes and attractive foci are asymptotically stable equilibrium points for linear two-dimensional systems. Centers are Lyapunov stable but not asymptotically stable equilibrium points.

To the classification above, we have to add two *degenerate* cases illustrated in Figure 3.5.

1. If $\operatorname{tr}\mathbf{A} \neq 0$ and $\det\mathbf{A} = 0$, one eigenvalue is equal to zero and the other one equal to $\operatorname{tr}\mathbf{A}$. The origin is said to be a *saddle node*.[19] In this case, there exists a basis in which $e^{\mathbf{A}t} = \operatorname{diag}[1, e^{\lambda t}]$, showing that the equations of the orbits are $x = a, y > 0$ and $x = a, y < 0$, with $a \in \mathbf{R}$.
2. If $\operatorname{tr}\mathbf{A} = 0$ and $\det\mathbf{A} = 0$ with $\mathbf{A} \neq \mathbf{0}$, there exists a basis in which **A** is of the form $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$. All the orbits are the straight lines $y = y_0$ traveled at constant velocity ay_0.

We mentioned (Remark 2) that, if we require only the orientation along the orbits to be preserved, we obtain more satisfactory flow equivalence classes. In the case of two-dimensional linear systems it can be shown [164][20] that if the eigenvalues of the two matrices **A** and **B** have nonzero real parts, then the two linear systems $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$ are topologically equivalent if,

[19] Also called a *fold* or a *tangent bifurcation point*. See Section 3.5.
[20] See pp. 238–246.

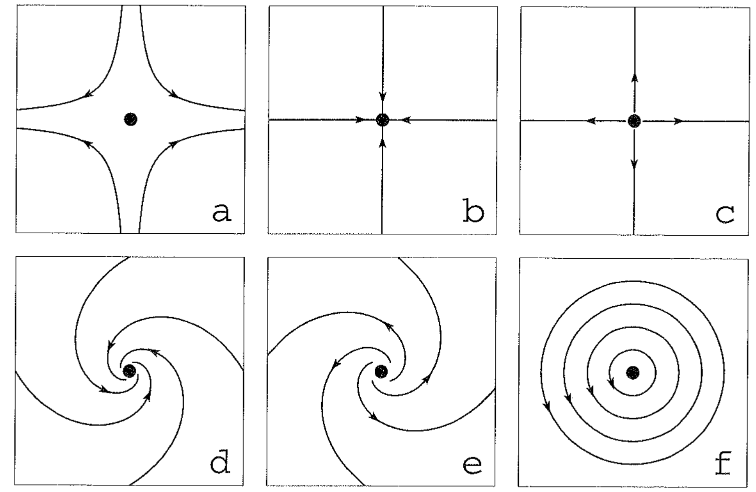

Fig. 3.4. *Two-dimensional linear flows. Phase portraits of the nondegenerate cases. (a) saddle, (b) attractive node, (c) repulsive node, (d) attractive focus, (e) repulsive focus, (f) center.*

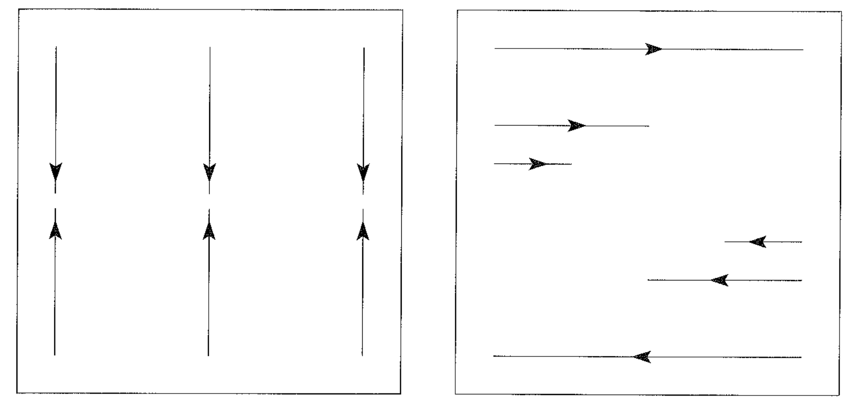

Fig. 3.5. *Two-dimensional linear flows. Phase portraits of the degenerate cases. Left:* $\operatorname{tr}\mathbf{A} \neq 0$ *and* $\det\mathbf{A} = 0$. *Right:* $\operatorname{tr}\mathbf{A} = 0$ *and* $\det\mathbf{A} = 0$ *with* $\mathbf{A} \neq \mathbf{0}$.

and only if, **A** and **B** have the same number of eigenvalues with negative (and hence positive) real parts. Consequently, up to topological equivalence, there are three distinct equivalence classes of hyperbolic[21] two-dimensional linear systems; that is, cases (a), (b), and (c) in Figure 3.4.

If the origin is a hyperbolic equilibrium point of a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the subspace spanned by the (generalized) eigenvectors corresponding to the

[21] That is, whose eigenvalues have nonzero real parts. See Definition 8.

eigenvalues with negative (resp. positive) real parts is called the *stable* (resp. *unstable*) *manifold* of the hyperbolic equilibrium point.

3.2.2 Nonlinear systems

Definition 8. *Let* $\mathbf{x}^* \in U \subseteq \mathbb{R}^n$ *be an equilibrium point of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$*;* $\mathbf{x}^*$ *is said to be* hyperbolic *if all the eigenvalues of the Jacobian matrix* $D\mathbf{X}(\mathbf{x}^*)$ *have nonzero real part. The linear function*

$$\mathbf{x} \mapsto D\mathbf{X}(\mathbf{x}^*)\mathbf{x} \tag{3.20}$$

is called the linear part *of* $\mathbf{X}$ *at* $\mathbf{x}^*$.

Let $\mathbf{x}^*$ be an equilibrium point of Equation (3.1). In order to determine the stability of $\mathbf{x}^*$, we have to understand the nature of the solutions near $\mathbf{x}^*$. Let

$$\mathbf{x} = \mathbf{x}^* + \mathbf{y}.$$

substituting in (3.1) and Taylor expanding about $\mathbf{x}^*$ yields

$$\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y} + O(\|\mathbf{y}\|^2). \tag{3.21}$$

Since the stability of $\mathbf{x}^*$ is determined by the behavior of orbits through points arbitrarily close to $\mathbf{x}^*$, we might think that the stability could be determined by studying the stability of the equilibrium point $\mathbf{y} = \mathbf{0}$ of the linear system

$$\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y}. \tag{3.22}$$

The solution of (3.22) through the point $\mathbf{y}_0 \in \mathbb{R}^n$ at $t = 0$ is

$$\mathbf{y}(t) = \exp\big(D\mathbf{X}(\mathbf{x}^*)t\big)\mathbf{y}_0. \tag{3.23}$$

Thus, the equilibrium point $\mathbf{y} = \mathbf{0}$ is asymptotically stable if all the eigenvalues of $D\mathbf{X}(\mathbf{x}^*)$ have negative real parts.[22]

Question: If all the eigenvalues of $D\mathbf{X}(\mathbf{x}^*)$ have negative real parts, is the equilibrium point $\mathbf{x}^*$ of Equation (3.1) asymptotically stable? The answer is yes. More precisely:

Theorem 1. *If* $\mathbf{x}^*$ *is a hyperbolic equilibrium point of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$*, the flow generated by the vector field* $\mathbf{X}$ *in the neighborhood of* $\mathbf{x}^*$ *is* C^0 *conjugate to the flow generated by* $D\mathbf{X}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)$.

This result is known as the *Hartman-Grobman theorem.*[23] Hence, if $D\mathbf{X}(\mathbf{x}^*)$ has no purely imaginary eigenvalues, the stability of the equilibrium point $\mathbf{x}^*$ of the nonlinear differential equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ can be determined from the study of the linear differential equation $\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y}$, where $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$. If $D\mathbf{X}(\mathbf{x}^*)$ has purely imaginary eigenvalues, this is not the case. The following examples illustrate the various possibilities.

[22] See Subsection 3.2.1.

[23] For a proof, consult Palis and de Melo [276].

Example 12. The damped pendulum. The equation for the damped pendulum is

$$\ddot{\theta} + 2a\dot{\theta} + \omega^2 \sin\theta = 0, \tag{3.24}$$

where θ is the displacement angle from the stable equilibrium position, $a > 0$ is the friction coefficient, and ω^2 is equal to the acceleration of gravity g divided by the pendulum length ℓ. If we put

$$x_1 = \theta, \quad x_2 = \dot{\theta},$$

Equation (3.24) may be written

$$\frac{dx_1}{dt} = x_2,$$
$$\frac{dx_2}{dt} = -\omega^2 \sin x_1 - 2a x_2.$$

The equilibrium points of this system are $(n\pi, 0)$, where n is any integer ($n \in \mathbb{Z}$). The Jacobian of the vector field $\mathbf{X} = (x_2, -\omega^2 \sin x_1 - 2a x_2)$ is

$$D\mathbf{X}(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x_1 & -2a \end{bmatrix}.$$

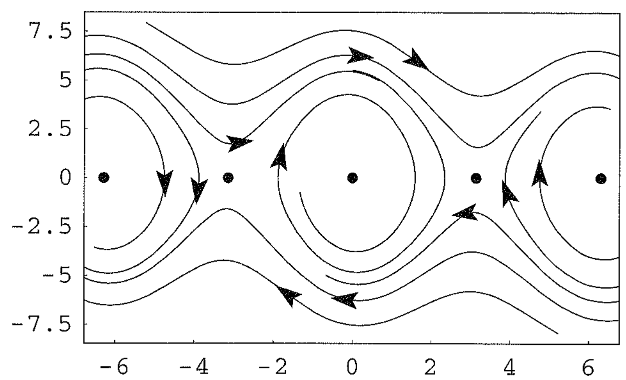

Fig. 3.6. *Phase portrait of a damped pendulum* $\dot{x}_1 = x_2$, $\dot{x}_2 = -\omega^2 \sin x_1 - 2a x_2$ *for* $\omega = 2.8$ *and* $a = 0.1$*. The equilibrium points* 0 *and* $\pm 2\pi$ *are asymptotically stable, while the equilibrium points* $\pm\pi$ *are unstable (saddle points).*

If n is even, the eigenvalues of the Jacobian matrix at $(n\pi, 0)$ are $\lambda_{1,2} = -a \pm \sqrt{a^2 - \omega^2}$. If $\omega \le a$, both eigenvalues are real and negative; if $\omega > a$, the eigenvalues are complex conjugate and their real part is negative (it is equal to $-a$). Therefore, if n is an even integer, the equilibrium point $(n\pi, 0)$ is asymptotically stable (see Figure 3.6).

If n is odd, the eigenvalues of the Jacobian matrix are $\lambda_{1,2} = -a \pm \sqrt{a^2+\omega^2}$, that is, real and of opposite signs. The equilibrium point $(n\pi, 0)$ is therefore unstable (see Figure 3.6).

Remark 3. A hyperbolic equilibrium point $\mathbf{x}^*$ of the differential equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is called a *sink* if all the eigenvalues of $D\mathbf{X}(\mathbf{x}^*)$ have negative real parts; it is called a *source* if all the eigenvalues of $D\mathbf{X}(\mathbf{x}^*)$ have positive real parts; and it is called a *saddle* if it is hyperbolic and $D\mathbf{X}(\mathbf{x}^*)$ has at least one eigenvalue with a negative real part and at least one eigenvalue with a positive real part.

Example 13. A perturbed harmonic oscillator. Consider the system

$$\dot{x}_1 = x_2 + \lambda x_1(x_1^2 + x_2^2),$$
$$\dot{x}_2 = -x_1 + \lambda x_2(x_1^2 + x_2^2),$$

where λ is a parameter. These equations describe the dynamics of a perturbed harmonic oscillator. For all values of λ, the origin is an equilibrium point. The Jacobian at the origin of the vector field $\mathbf{X} = (x_2 + \lambda x_1(x_1^2+x_2^2), -x_1 + \lambda x_2(x_1^2+x_2^2))$ is

$$D\mathbf{X}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -0 \end{bmatrix}.$$

Its eigenvalues are $\pm\mathrm{i}$. The origin is a nonhyperbolic point, and to study its stability we have to analyze the behavior of the orbits close to the origin. If $(x_1(t), x_2(t))$ are the coordinates of a phase point at time t, its distance from the origin will increase or decrease according to the sign of the time derivative

$$\frac{d}{dt}\left(x_1^2(t) + x_2^2(t)\right) = 2x_1(t)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t) = 2\lambda(x_1^2 + x_2^2)^2.$$

Thus, as t tends to infinity, $\|\mathbf{x}(t)\|^2$ tends to zero if $\lambda < 0$, and the origin is an asymptotically stable equilibrium; if $\lambda > 0$ with $\mathbf{x}_0 \neq \mathbf{0}$, $\|\mathbf{x}(t)\|^2$ tends to infinity, showing that, in this case, the origin is an unstable equilibrium.

We could have reached the same conclusion using polar coordinates defined by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta.$$

In terms of the coordinates (r,θ), the system becomes

$$\dot{r} = \lambda r^3, \quad \dot{\theta} = -1.$$

Since $\dot{\theta} = -1$, the orbits spiral monotonically clockwise around the origin, and the stability of the origin is the same as the equilibrium point $r = 0$ of the one-dimensional system $\dot{r} = \lambda r^3$. That is, $r = 0$ is asymptotically stable if $\lambda < 0$ and unstable if $\lambda > 0$ with $r_0 \neq 0$.

Near a hyperbolic equilibrium point $\mathbf{x}^*$ of a nonlinear system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ we can define *(local) stable* and *unstable manifolds*. They are tangent to the respective stable and unstable manifolds of the linear system $\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y}$.[24]

If the equilibrium point $\mathbf{x}^*$ is nonhyperbolic, we can define in a similar manner a *center manifold* tangent to the center subspace spanned by the n_z (generalized) eigenvectors corresponding to the n_z eigenvalues of the linear operator $D\mathbf{X}(\mathbf{x}^*)$ with zero real parts. Note that while the stable and unstable manifolds are unique, the center manifold is not. The essential interest of the center manifold is that it contains all the complicated dynamics in the neighborhood of a nonhyperbolic point. The following classical example illustrates the characteristic features of the center manifold.[25]

Example 14. Consider the system

$$\dot{x}_1 = x_1^2, \quad \dot{x}_2 = -x_2.$$

The origin is a nonhyperbolic fixed point. The solutions are

$$x_1(t) = \frac{x_1(0)}{1 - x_1(0)t} \quad \text{and} \quad x_2(t) = x_2(0)\mathrm{e}^{-t}.$$

Eliminating t, we find that the equations of the trajectories in the (x_1, x_2)-space are

$$x_2(0)\exp\left(\frac{1}{x_1} - \frac{1}{x_1(0)}\right) - x_2 = 0.$$

For $x_1 < 0$, all the trajectories approach the origin with all the derivatives of x_2 with respect to x_1 equal to zero at the origin. For $x_1 \geq 0$, the only trajectory that goes through the origin is $x_2 = 0$. The center manifold, tangent to the eigenvector directed along the x_1-axis, which corresponds to the zero eigenvalue of the linear part of the vector field at the origin is therefore not unique. Except for $x_2 = 0$, all the center manifolds are not C^∞. This example shows that the invariant center manifold, unlike the invariant stable and unstable manifolds, is not necessarily unique and as smooth as the vector field.

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24 More precisely: *Let $\mathbf{x}^*$ be a hyperbolic equilibrium point of the nonlinear system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, where $\mathbf{X}$ is a C^k ($k \geq 1$) vector field on $\mathbb{R}^n$. If the linear operator $D\mathbf{X}(\mathbf{x}^*)$ has n_n eigenvalues with negative real parts and $n_p = n - n_n$ eigenvalues with positive real parts, there exists an n_n-dimensional differentiable manifold W^s_{loc} tangent to the stable subspace of the linear system $\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y}$ at $\mathbf{x}^*$ such that, for all $\mathbf{x}_0 \in W^s_{\text{loc}}$, and all $t > 0$, $\lim_{t\to\infty} \varphi_t(\mathbf{x}_0) = \mathbf{x}^*$, and there exists an n_p-dimensional differentiable manifold W^u_{loc} tangent to the unstable subspace of the linear system $\dot{\mathbf{y}} = D\mathbf{X}(\mathbf{x}^*)\mathbf{y}$ at $\mathbf{x}^*$ such that, for all $\mathbf{x}_0 \in W^u_{\text{loc}}$ and all $t < 0$, $\lim_{t\to-\infty} \varphi_t(\mathbf{x}_0) = \mathbf{x}^*$, where φ_t is the flow generated by the vector field $\mathbf{X}$.* This rather intuitive result is known as the *stable manifold theorem*. For more details on invariant manifolds consult Hirsch, Pugh, and Shub [172].

25 On center manifold theory, see Carr [79].

In order to determine if an equilibrium point $\mathbf{x}^*$ of the differential equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is stable we have to study the behavior of the function $\mathbf{x} \mapsto \|\mathbf{x} - \mathbf{x}^*\|$ in a neighborhood $N(\mathbf{x}^*)$ of $\mathbf{x}^*$. The Lyapunov method introduces more general functions. The essential idea on which the method rests is to determine how an adequately chosen real function varies along the trajectories of the flow φ_t generated by the vector field $\mathbf{X}$.

Definition 9. *Let* $\mathbf{x}^*$ *be an equilibrium point of the differential equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ *on* $U \subseteq \mathbb{R}^n$. *A* C^1 *function* $V : U \to \mathbb{R}$ *is called a* strong Lyapunov function *for the flow* φ_t *on an open neighborhood* $N(\mathbf{x}^*)$ *of* $\mathbf{x}^*$ *provided* $V(\mathbf{x}) > V(\mathbf{x}^*)$ *and*

$$\dot{V}(\mathbf{x}) = \frac{d}{dt} V(\varphi_t(\mathbf{x}))\Big|_{t=0} < 0$$

for all $\mathbf{x} \in N(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}$. *If the condition* $\dot{V}(\mathbf{x}) < 0$ *is replaced by* $\dot{V}(\mathbf{x}) \le 0$, V *is called a* weak Lyapunov function.

It is not difficult to prove that [171], if $\mathbf{x}^*$ is an equilibrium point of the differential equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ on $U \subseteq \mathbb{R}^n$ and there exists a weak Lyapunov function V defined on a neighborhood of $\mathbf{x}^*$, then $\mathbf{x}^*$ is Lyapunov stable. If there exists a strong Lyapunov function V defined on a neighborhood of $\mathbf{x}^*$, then $\mathbf{x}^*$ is asymptotically stable. This result is known as the *Lyapunov theorem*.[26]

The interesting feature of the Lyapunov method is that it is possible to calculate $\dot{V}(\mathbf{x})$ without actually knowing the solutions to the differential equation. To emphasize this particular feature, it is often called the *direct* method of Lyapunov. The inconvenience of the method is that finding a Lyapunov function is a matter of trial and error. In some cases, there are, however, some natural functions to try. As illustrated by the following example, in the case of a mechanical system, energy is often a Lyapunov function.

Example 15. Simple and damped pendulums. We have seen that the Equation (1.10) of the simple pendulum may be written

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell} \sin x_1.$$

The function

[26] Note that $\dot{V}(\mathbf{x})$ is equal to the dot product $\nabla V(\mathbf{x}) \cdot \mathbf{X}(\mathbf{x})$ of the gradient of V with the vector field $\mathbf{X}$ at $\mathbf{x}$. For two-dimensional systems, if $\dot{V}(\mathbf{x}) < 0$, the angle between $\nabla V(\mathbf{x})$ and $\mathbf{X}(\mathbf{x})$ is obtuse. Since the gradient is the outward normal vector to the curve $V(\mathbf{x}) =$ constant at $\mathbf{x}$, this implies that the orbit is crossing the curve from the outside to the inside. Similarly, it could be shown that, if $\dot{V}(\mathbf{x}) = 0$, the orbit is tangent to the curve, and, if $\dot{V}(\mathbf{x}) > 0$, the orbit is crossing the curve from the inside to the outside. These remarks make the Lyapunov theorem quite intuitive.

$$V : (x_1, x_2) \mapsto \frac{1}{2}\,\ell^2 x_2^2 + g\ell(1 - \cos x_1),$$

which represents the energy of the pendulum when the mass of the bob is equal to unity, is a weak Lyapunov function since

$$V(x_1, x_2) > 0 \text{ for } (x_1, x_2) \neq (0,0) \text{ and } \dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \equiv 0.$$

The equilibrium point $(0,0)$ is Lyapunov stable. In the case of the damped pendulum (see Example 12), we have

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{\ell}\sin x_1 - 2a x_2, \quad (a > 0).$$

If here again we consider the function V defined by $V(x_1, x_2) = \frac{1}{2}\ell^2 x_2^2 + g\ell(1 - \cos x_1)$, we find that $\dot{V}(x_1, x_2) = -2a\ell^2 x_2^2$. For the damped pendulum, $\dot{V}(x_1, x_2)$ is a strong Lyapunov function in a neighborhood of the origin. The origin is, therefore, asymptotically stable.

Example 16. The van der Pol oscillator. The differential equation

$$\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0 \tag{3.25}$$

describes the dynamics of the *van der Pol oscillator* [291], which arises in electric circuit theory. It is a harmonic oscillator that includes a nonlinear friction term: $\lambda(x^2 - 1)\dot{x}$. If the amplitude of the oscillations is large, the amplitude-dependent "coefficient" of friction is positive, and the oscillations are damped. As a result, the amplitude of the oscillations decreases, and the amplitude-dependent "coefficient" of friction eventually becomes negative, corresponding to a sort of antidamping.

If we put

$$x_1 = x \quad \text{and} \quad x_2 = \dot{x},$$

Equation (3.25) takes the form

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - \lambda(x_1^2 - 1)x_2. \tag{3.26}$$

The equilibrium point $(x_1, x_2) = (0,0)$ is nonhyperbolic, but we may study its stability using the Lyapunov method. Consider the function

$$V : (x_1, x_2) \mapsto \frac{1}{2}(x_1^2 + x_2^2).$$

It is positive for $(x_1, x_2) \neq (0, 0,)$, and its time derivative

$$\dot{V}(x_1, x_2) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= -\lambda(x_1^2 - 1)x_2^2$$

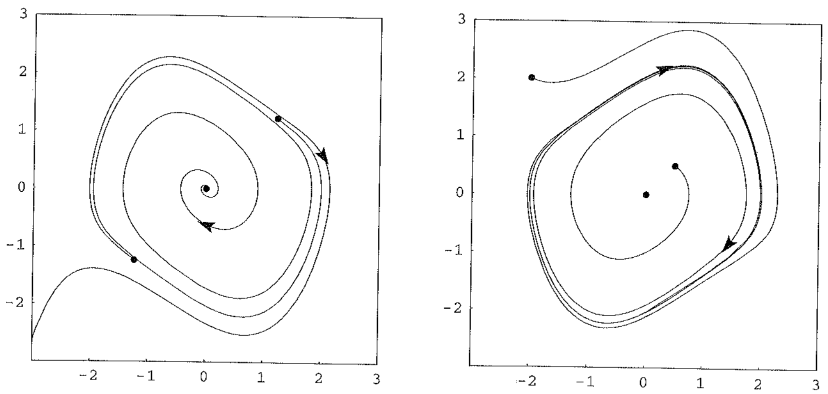

Fig. 3.7. *Phase portraits of the van der Pol equation:* $\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0$. *For* $\lambda = -0.5$, *the origin is an asymptotically stable equilibrium point (left), while for* $\lambda = 0.5$ *the orbits converge to a stable limit cycle (right).*

is negative in a neighborhood of the equilibrium point $(0, 0)$ if λ is negative. Hence, V is a strong Lyapunov function, which proves that $(0, 0)$ is asymptotically stable if $\lambda < 0$. It is an attractive focus. If λ is positive, $(0, 0)$ is unstable. It is a repulsive focus, and, as illustrated in Figure 3.7, trajectories converge to a stable limit cycle.

Arnol'd gives the following simple proof of the existence of a stable limit cycle.[27] Note first that the Lyapunov function V represents the energy of the system, which is a conserved quantity in the case of the harmonic oscillator, *i.e.*, for $\lambda = 0$. If the parameter λ is very small, trajectories are spirals in which the distance between adjacent coils is of the order of λ. To determine if these spirals either approach the origin or recede from it, we may compute an approximate value of the increment ΔV over one revolution around the origin. Since $\dot{V}(x_1, x_2) = -\lambda(x_1^2 - 1)x_2^2$, and, to first order in λ, $x_1(t) = A\cos(t - t_0)$ and $x_2(t) = -A\sin(t - t_0)$, we obtain

$$\begin{aligned}\Delta V &= -\lambda \int_0^{2\pi} \Delta V\big(A\cos(t - t_0), -A\sin(t - t_0)\big) \\ &= \pi\lambda\left(A^2 - \frac{A^4}{4}\right).\end{aligned}$$

- If $\lambda < 0$ and the amplitude A of the oscillations is small, $\Delta V < 0$, *i.e.*, the system gives energy to the external world: the trajectory is a contracting spiral.
- If $\lambda > 0$ and the amplitude A of the oscillations is small, $\Delta V > 0$, *i.e.*, the system receives energy from the external world: the trajectory is an expanding spiral.

[27] See [9], pp. 150–151.

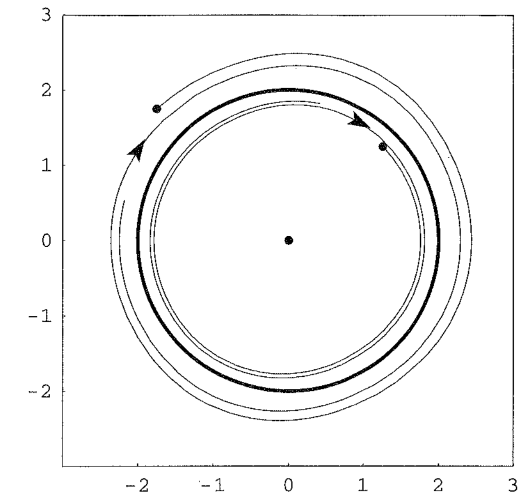

Fig. 3.8. *Phase portrait of the van der Pol equation:* $\ddot{x} + \lambda(x^2 - 1)\dot{x} + x = 0$ *for* $\lambda = 0.05$. *As explained in the text, for a small positive value of* λ, *the stable limit cycle (thick line) is close to the circle of radius 2 centered at the origin.*

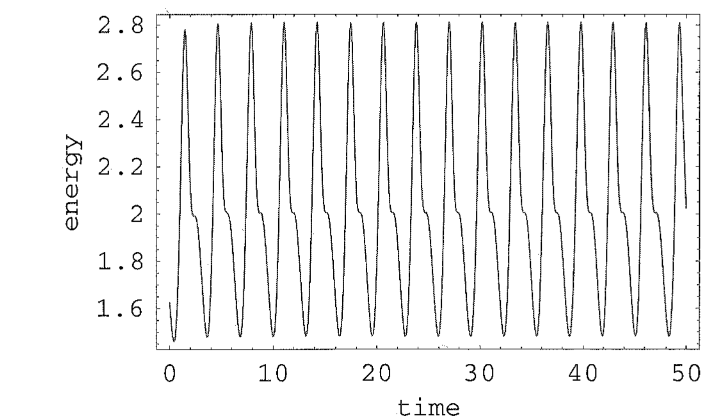

Fig. 3.9. *Energy of the van der Pol oscillator as a function of time along the limit cycle for* $\lambda = 0.5$.

- If $\Delta V = 0$, the energy is conserved, and, for a small positive λ, the trajectory is a cycle close to the circle $x_1^2 + x_2^2 = A^2$, where A is the positive root of $A^2(1 - \frac{1}{4}A^2) = 0$, *i.e.*, $A = 2$.

This result is illustrated in Figure 3.8. Note that for a finite positive value of λ, the energy is not conserved along a stable limit cycle but it varies periodically. As shown in Figure 3.9, the system receives energy from the external world during a part of the cycle and gives it back during the other part.

If $\mathbf{x}^*$ is an asymptotically stable equilibrium point of the differential equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ on $U \subseteq \mathbb{R}^n$, it is of practical importance to determine its *basin of attraction*, that is, the set

$$\{\mathbf{x} \in U \mid \lim_{t\to\infty} \varphi_t(\mathbf{x}) = \mathbf{x}^*\}. \tag{3.27}$$

The method of Lyapunov may be used to obtain estimates of the basin of attraction. The problem is to find the largest subset of U in which $\dot{V}(\mathbf{x})$ is negative.

Remark 4. In the case of the van der Pol equation, the symmetry of the equation, which is invariant under the transformation $t \mapsto -t, \lambda \mapsto -\lambda$, shows that, when $\lambda < 0$, the basin of attraction of the origin is the interior of the closed curve symmetrical, with respect to the $0x_1$-axis, to the stable limit cycle obtained for $|\lambda|$ (see Figure 3.7).

Definition 10. *A point* $\mathbf{y} \in \mathbb{R}^n$ *is an* ω-limit point *for the trajectory* $\{\varphi_t(\mathbf{x}) \mid t \in \mathbb{R}\}$ *through* $\mathbf{x}$ *if there exists a sequence* (t_k) *going to infinity such that* $\lim_{k\to\infty} \varphi_{t_k}(\mathbf{x}) = \mathbf{y}$. *The set of all* ω*-limit points of* $\mathbf{x}$ *is called the* ω-limit set *of* $\mathbf{x}$, *and is denoted by* $L_\omega(\mathbf{x})$.

α-limit points and the α-limit set are defined in the same way but with a sequence (t_k) going to $-\infty$. The α-limit set of $\mathbf{x}$ is denoted $L_\alpha(\mathbf{x})$.[28]

Let $\mathbf{y} \in L_\omega(\mathbf{x})$ and $\mathbf{z} = \varphi_{t_k}(\mathbf{y})$; then $\lim_{k\to\infty} \varphi_{t+t_k}(\mathbf{x}) = \mathbf{z}$, showing that $\mathbf{y}$ and $\mathbf{z}$ belong to the ω-limit set $L_\omega(\mathbf{x})$ of $\mathbf{x}$. The ω-limit set of $\mathbf{x}$ is, therefore, *invariant* under the flow φ_t.[29] Similarly, we could have shown that the α-limit set of $\mathbf{x}$ is invariant under the flow.

If $\mathbf{x}^*$ is an asymptotically stable equilibrium point, it is the ω-limit set of every point in its basin of attraction. A closed orbit is the α-limit and ω-limit set of every point on it. While, in general, limit sets can be quite complicated, for two-dimensional systems the situation is much simpler. The following result known as the *Poincaré-Bendixson theorem* gives a criterion to detect limit cycles (see Definition 11 below) in systems modeled by a two-dimensional differential equation:

Theorem 2. *A nonempty compact limit set*[30] *of a two-dimensional flow defined by a* C^1 *vector field, which contains no fixed point, is a closed orbit.*[31]

Definition 11. *A* limit cycle *is a closed orbit* γ *such that either* $\gamma \subset L_\omega(\mathbf{x})$ *or* $\gamma \subset L_\alpha(\mathbf{x})$ *for some* $\mathbf{x} \notin \gamma$. *In the first case,* γ *is an* ω*-limit cycle; in the second case, it is an* α*-limit cycle.*

[28] The reason for this terminology is that α and ω are, respectively, the first and last letters of the Greek alphabet.

[29] A set M is also invariant under the flow φ_t if, for all $\mathbf{x} \in M$ and all $t \in \mathbb{R}$, $\varphi_t(\mathbf{x}) \in M$. For instance, fixed points and closed orbits are invariant sets.

[30] A set is *compact* if, from any covering by open sets, it is possible to extract a finite covering. Any closed bounded subset of a finite-dimensional metric space is compact.

[31] See Hirsch and Smale [171].

Closed orbits around a center are not limit cycles. A limit cycle is an isolated closed orbit in the sense that there exists an annular neighborhood of the limit cycle that contains no other closed orbits. The stability of limit cycles is studied in the next chapter Section 4.3.

In order to prove the existence of a limit cycle using Theorem 2, one has to find a bounded subset D of $\mathbb{R}^2$ such that, for all $\mathbf{x} \in D$, the trajectories $\{\varphi_t(\mathbf{x}) \mid t > 0\}$ remain in D[32] and show that D does not contain an equilibrium point.

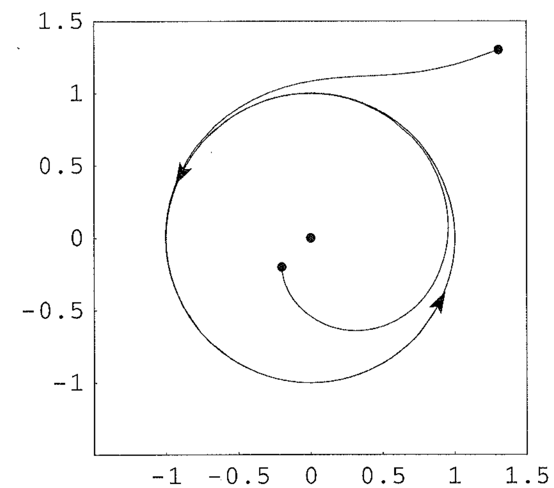

Fig. 3.10. *Phase portrait of the two-dimensional system of Example 17*

Example 17. Perturbed center. Consider the two-dimensional system

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2), \qquad \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2).$$

Using polar coordinates, this system takes the following particularly simple form:

$$\dot{r} = r - r^3, \qquad \dot{\theta} = 1.$$

For $r = 1$, $\dot{r} = 0$; therefore, if r_1 and r_2 are two real numbers such that $0 < r_1 < 1 < r_2$, we verify that, for $r_1 \le r < 1$, $\dot{r} > 0$, and, for $1 < r \le r_2$, $\dot{r} < 0$. Thus, the closed annular set $\{\mathbf{x} \mid r_1 \le (x_1^2 + x_2^2)^{1/2} \le r_2\}$ is positively invariant (see Footnote 32) and does not contain an equilibrium point. According to the Poincaré-Bendixson theorem, this bounded subset contains a limit cycle (see Figure 3.10).

[32] Such a subset D is said to be *positively invariant*.

3.3 Graphical study of two-dimensional systems

There exists a wide variety of models describing the interactions between two populations. If one seeks to incorporate in such a model a minimum of broadly relevant features, the equations describing the model might become difficult to analyze. It is, however, frequently possible to analyze graphically the behavior of the system without entering into specific mathematical details.

The system

$$\dot{N}_1 = N_1 f_1(N_1, N_2), \qquad \dot{N}_2 = N_2 f_2(N_1, N_2), \tag{3.28}$$

represents a model of two interacting populations, whose growth rates $\dot{N}_1/N_1$ and $\dot{N}_2/N_2$ are, respectively, equal to $f_1(N_1, N_2)$ and $f_2(N_1, N_2)$.

The general idea on which rests the qualitative graphical analysis of Equations (3.28) is to:

(i) divide the positive quadrant of the (N_1, N_2)-plane in domains bounded by the sets $\{(N_1, N_2) \mid f_1(N_1, N_2) = 0\}$ and $\{(N_1, N_2) \mid f_2(N_1, N_2) = 0\}$,[33]

(ii) find, in each domain, the sign of both growth rates, which determine the direction of the vector field, and

(iii) represent, in each domain, the direction of the vector field (*i.e.*, the flow) by an arrow.

As shown in the following example, a schematic phase portrait can then be easily obtained.

Example 18. Lotka-Volterra competition model. Assuming that two species compete for a common food supply, their growth could be described by the following simple two-dimensional system:

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \lambda_1 N_1 N_2, \quad \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \lambda_2 N_1 N_2. \tag{3.29}$$

This competition model is usually associated with the names of Lotka and Volterra.

Each population (N_1 and N_2) has a logistic growth but the presence of each reduces the growth rate of the other. The constants r_1, r_2, K_1, K_2, λ_1, and λ_2 are positive. As usual, we define reduced variables writing

$$\tau = \sqrt{r_1 r_2}\, t, \quad \rho = \sqrt{\frac{r_1}{r_2}}, \quad n_1 = \frac{N_1}{K_1}, \quad n_2 = \frac{N_2}{K_2}, \quad \alpha_1 = \frac{\lambda_1 K_2}{\sqrt{r_1 r_2}}, \quad \alpha_2 = \frac{\lambda_2 K_1}{\sqrt{r_1 r_2}},$$

and Equations (3.29) become

$$\frac{dn_1}{d\tau} = \rho n_1(1 - n_1) - \alpha_1 n_1 n_2, \qquad \frac{dn_2}{d\tau} = \frac{1}{\rho} n_2(1 - n_2) - \alpha_2 n_1 n_2. \tag{3.30}$$

[33] These sets, which are the preimages of the point $(0, 0)$ by f_1 and f_2, respectively, are called *null clines*.

To determine under which conditions the two species can coexist, the two straight lines

$$\rho(1 - n_1) - \alpha_1 n_2 = 0, \qquad \frac{1}{\rho}(1 - n_2) - \alpha_2 n_1 = 0,$$

should intersect in the positive quadrant. There are two possibilities represented in Figure 3.11. We find that if $a_1 = \rho/\alpha_1$ and $a_2 = 1/\rho\alpha_2$ are both

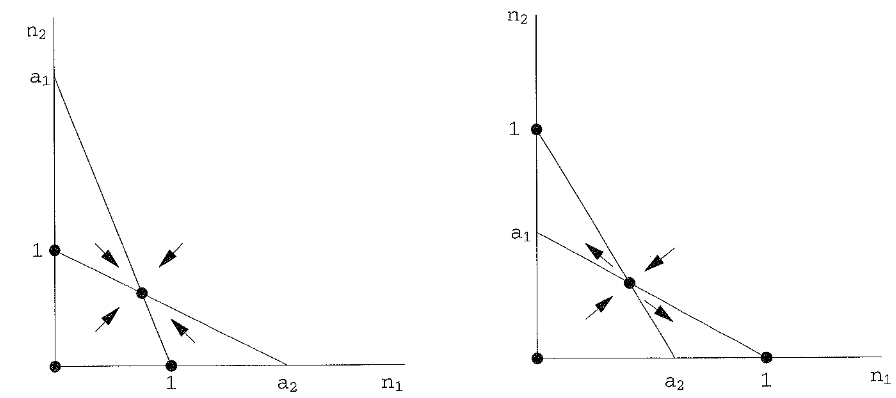

Fig. 3.11. *Lotka-Volterra competition model. Intersecting null clines. Dots show equilibrium points. In each domain, arrows represent the direction of the vector* $(\dot{n}_1, \dot{n}_2)$*, with time derivatives taken with respect to reduced time* τ*. Left:* $a_1 = \rho/\alpha_1 > 1$ *and* $a_2 = 1/\rho\alpha_2 > 1$*. Right:* $a_1 = \rho/\alpha_1 < 1$ *and* $a_2 = 1/\rho\alpha_2 < 1$*.*

greater than 1, the nontrivial equilibrium point is asymptotically stable: the two populations will coexist. If, on the contrary, $a_1 = \rho/\alpha_1$ and $a_2 = 1/\rho\alpha_2$ are both less than 1, the nontrivial equilibrium point is a saddle. The equilibrium points $(0, 1)$ and $(1, 0)$ are stable steady states. The population that will eventually survive depends upon the initial state.

When the null clines do not intersect, as in Figure 3.12, only one population will eventually survive. It is population 1 if $a_1 = \rho/\alpha_1 > 1$ and $a_2 = 1/\rho\alpha_2 < 1$, and population 2 if $a_1 = \rho/\alpha_1 < 1$ and $a_2 = 1/\rho\alpha_2 > 1$. The equilibrium point $(0, 0)$ is, in all cases, unstable.

These results illustrate the so-called *competitive exclusion principle* whereby *two species competing for the same limited resource cannot, in general, coexist.* Note that the species that will eventually survive is the species whose growth rate is less perturbed by the presence of the other.

It is possible to use the graphical method to study more complex models. For instance, Hirsch and Smale [171][34] discuss a large class of competition models. Their equations are of the form

[34] In a much older paper, Kolmogorov [193] had already presented a qualitative study of a general predator-prey system.

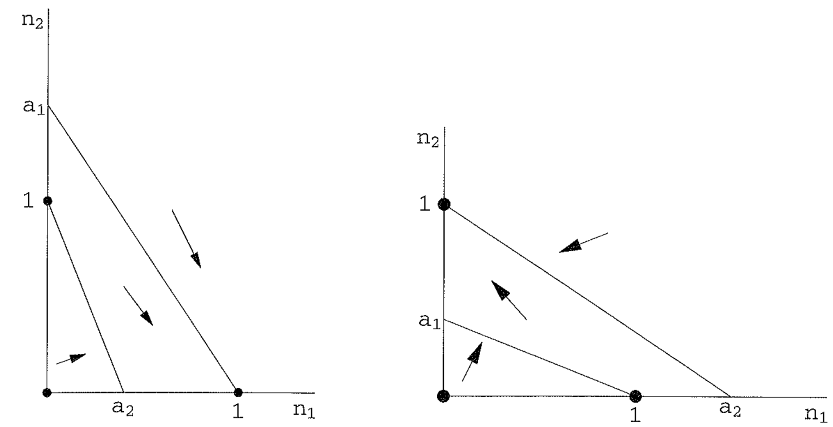

Fig. 3.12. *Lotka-Volterra competition model. Nonintersecting null clines. Dots show equilibrium points. In each domain, arrows represent the direction of the vector* $(\dot{n}_1, \dot{n}_2)$*, with time derivatives taken with respect to reduced time* τ*. Left:* $a_1 = \rho/\alpha_1 > 1$ *and* $a_2 = 1/\rho\alpha_2 < 1$*. Right:* $a_1 = \rho/\alpha_1 < 1$ *and* $a_2 = 1/\rho\alpha_2 > 1$.

$$\dot{N}_1 = N_1 f_1(N_1, N_2), \qquad \dot{N}_2 = N_2 f_2(N_1, N_2),$$

with the following assumptions:

1. If either species increases, the growth rate of the other decreases. Hence,
$$\frac{\partial f_1}{\partial N_2} < 0 \text{ and } \frac{\partial f_2}{\partial N_1} < 0.$$
2. If either population is very large, neither species can multiply. Hence,
$$f_1(N_1, N_2) \leq 0 \text{ and } f_2(N_1, N_2) \leq 0 \quad \text{if either } N_1 > K \text{ or } N_2 > K.$$
3. In the absence of either species, the other has a positive growth rate up to a certain population and a negative growth rate beyond that. There are, therefore, constants $a_1 > 0$ and $a_2 > 0$ such that
$$f_1(N_1, N_2) > 0 \text{ for } N_1 < a_1 \text{ and } f_1(N_1, N_2) < 0 \text{ for } N_1 > a_1$$
and
$$f_2(N_1, N_2) > 0 \text{ for } N_2 < a_2 \text{ and } f_2(N_1, N_2) < 0 \text{ for } N_2 > a_2.$$

Analyzing different possible *generic* shapes of the null clines, Hirsch and Smale show that the ω-limit set of any point in the positive quadrant of the (N_1, N_2)-plane exists and is one of a finite number of equilibria; that is, there are no closed orbits.

3.4 Structural stability

The qualitative properties of a model should not change significantly when the model is slightly modified: a model should be *robust*. To be precise we have to give a definition of what is a "slight modification." That is, in the space $\mathcal{V}(U)$ of all vector fields defined on an open set $U \subseteq \mathbb{R}^n$, we have to define an *appropriate* metric.[35] The metric, we said, has to be appropriate in the sense that, if two vector fields are close for this metric, then the dynamics they generate have the same qualitative properties. Actually, on the space $\mathcal{V}(U)$, we shall first define an appropriate *norm* and associate a metric with that norm.[36]

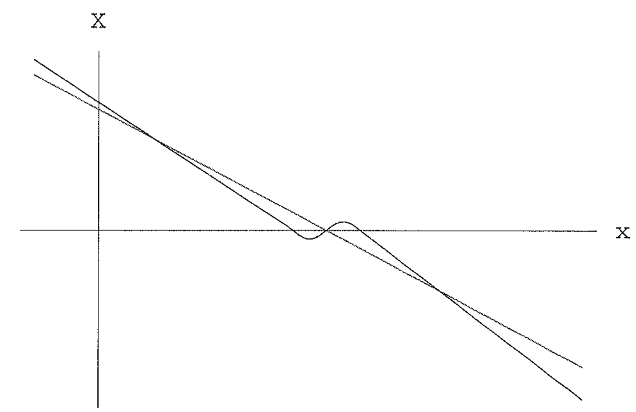

Fig. 3.13. *Two neighboring one-dimensional vector fields in the* C^0 *topology that do not have the same number of equilibrium points.*

If $\mathbf{X} \in \mathcal{V}(U)$, its C^0 norm is defined by

[35] We have already defined the notion of *distance* (or *metric*) (see page 14). A distance d on a space X is a mapping $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a *metric space*.

[36] A mapping $x \mapsto \|x\|$ defined on a vector space X into $\mathbb{R}_+$ is a *norm* if it satisfies the following conditions:

1. $\|x\| = 0 \iff x = 0$.
2. For any $x \in X$ and any scalar λ, $\|\lambda x\| = |\lambda| \|x\|$.
3. For all x and y in X, $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is called a *normed vector space*.

$$\|\mathbf{X}\|_0 = \sup_{\mathbf{x}\in U} \|\mathbf{X}(\mathbf{x})\|,$$

where $\|\mathbf{X}(\mathbf{x})\|$ is the usual norm of the vector $\mathbf{X}(\mathbf{x})$ in $\mathbb{R}^n$. The C^0 distance between two vector fields $\mathbf{X}$ and $\mathbf{Y}$, which belong to $\mathcal{V}(U)$, is then defined by

$$d_0(\mathbf{X},\mathbf{Y}) = \|\mathbf{X}-\mathbf{Y}\|_0.$$

As shown in Figure 3.13, two vector fields that are close in the C^0 topology[37] may not have the same number of hyperbolic equilibrium points. To avoid this undesirable situation, we should define a distance requiring that the vector fields as well as their derivatives be close at all points of U. The C^1 distance between two vector fields $\mathbf{X}$ and $\mathbf{Y}$ belonging to $\mathcal{V}(U)$ is then defined by

$$d_1(\mathbf{X},\mathbf{Y}) = \sup_{\mathbf{x}\in U}\{\|\mathbf{X}(\mathbf{x})-\mathbf{Y}(\mathbf{x})\|, \|D\mathbf{X}(\mathbf{x})-D\mathbf{Y}(\mathbf{x})\|\}.$$

Similarly we could define C^k distances for k greater than 1.

In the space $\mathcal{V}(U)$ of all vector fields defined on an open set $U\subseteq\mathbf{R}^n$, an *ε-neighborhood* of $\mathbf{X}\in\mathcal{V}(U)$ is defined by

$$N_\varepsilon(\mathbf{X}) = \{\mathbf{Y}\in\mathcal{V}(U) \mid \|\mathbf{X}-\mathbf{Y}\|_1 < \varepsilon\}.$$

A vector field $\mathbf{Y}$ that belongs to an *ε-neighborhood* of $\mathbf{X}$ is said to be *ε-C^1-close* to $\mathbf{X}$ or an *ε-C^1-perturbation* of $\mathbf{X}$. In this case, the components $\big(X_1(\mathbf{x}), X_2(\mathbf{x}),\dots,X_n(\mathbf{x})\big)$ and $\big(Y_1(\mathbf{x}), Y_2(\mathbf{x}),\dots,Y_n(\mathbf{x})\big)$ of, respectively, $\mathbf{X}$ and $\mathbf{Y}$ and their first derivatives are close throughout U.

Theorem 3. *Let $\mathbf{x}^*$ be a hyperbolic equilibrium point of the flow φ_t generated by the vector field $\mathbf{X}\in\mathcal{V}(U)$. Then, there exists a neighborhood V of $\mathbf{x}^*$ and a neighborhood N of $\mathbf{X}$ such that each $\mathbf{Y}\in N$ generates a flow ψ_t that has a unique hyperbolic equilibrium point $\mathbf{y}^*\in V$. Moreover, the linear operators $D\mathbf{X}(\mathbf{x}^*)$ and $D\mathbf{Y}(\mathbf{y}^*)$ have the same number of eigenvalues with positive and negative real parts. In this case, the flow φ_t generated by the vector field $\mathbf{X}$ is said to be* locally structurally stable at $\mathbf{x}^*$.

[37] A collection $\mathcal{T}$ of subsets of a set X is said to be a *topology* in X if $\mathcal{T}$ has the following properties:

1. X and $\emptyset$ belong to $\mathcal{T}$.
2. If $\{O_i \mid i\in I\}$ is an arbitrary collection of elements of $\mathcal{T}$, then $\cup_{i\in I}O_i$ belongs to $\mathcal{T}$.
3. If O_1 and O_2 belong to $\mathcal{T}$, then $O_1\bigcap O_2$ belongs to $\mathcal{T}$.

The ordered pair $(X,\mathcal{T})$ is called a *topological space*, and the elements of $\mathcal{T}$ are called *open sets* in X. When no ambiguity is possible, one may speak of the "topological space X."

Since the flows $\exp\big(D\mathbf{X}(\mathbf{x}^*)t\big)$ and $\exp\big(D\mathbf{X}(\mathbf{x}^*)t\big)$ are equivalent, this result follows directly from Theorem 1, which proves that the flows φ_t and ψ_t are conjugate.[38]

The vector field $(x_2, -\omega^2\sin x_1)$ of the undamped pendulum generates a flow that is not structurally stable. Its equilibrium points are nonhyperbolic. The damped pendulum (Example 12), which is obtained by adding to the vector field of the undamped pendulum the perturbation $(0,-2ax_2)$, has hyperbolic equilibrium points, which are either asymptotically stable or unstable (see Figure 3.6).

Similarly, a linear harmonic oscillator whose vector field is $(x_2,-x_1)$ has only one equilibrium point $(0,0)$, which is a center. The flow generated by the vector field is not structurally stable. As shown in Example 13, the perturbation $(\lambda x_1(x_1^2+x_2^2), \lambda x_2(x_1^2+x_2^2))$ generates a qualitatively different flow: its ω-limit set is a stable limit cycle.

3.5 Local bifurcations of vector fields

In Section 2.3 we studied the predator-prey model used by Harrison to explain Luckinbill's experiment with *Didinium* and *Paramecium*. This model, whose dimensionless equations are

$$\frac{dh}{d\tau} = h\left(1-\frac{h}{k}\right) - \frac{\alpha_h ph}{\beta+h},$$
$$\frac{dp}{d\tau} = \frac{\alpha_p ph}{\beta+h} - \gamma p,$$

where

$$\alpha_h = \left(1-\frac{1}{k}\right)(\beta+1) \quad\text{and}\quad \alpha_p = \gamma(\beta+1),$$

exhibits two qualitatively different behaviors. For $k<\beta+2$, the equilibrium point $(1,1)$ is asymptotically stable, but, for $k>\beta+2$, $(1,1)$ is unstable and the ω-limit set is a limit cycle. The change of behavior occurs for $k=\beta+2$; that is, when $(1,1)$ is nonhyperbolic. Such a change in a family of vector fields, which depends upon a finite number of parameters, is referred to as a *bifurcation*.

The van der Pol equation (Example 16) exhibits a similar bifurcation. Here again, at the bifurcation point, the equilibrium point $(0,0)$ is nonhyperbolic.

Like many concepts of the qualitative theory of differential equations, the theory of bifurcations has its origins in the work of Poincaré.[39] Let

[38] The flows have to be restricted on neighborhoods of the respective hyperbolic equilibrium points $\mathbf{x}^*$ and $\mathbf{y}^*$ on which Theorem 1 is valid. For more details, see Arrowsmith and Place [11].

[39] Poincaré was the first to use the French word *bifurcation* in this context [288].

$$(\mathbf{x}, \boldsymbol{\mu}) \mapsto \mathbf{X}(\mathbf{x}, \boldsymbol{\mu}) \qquad (\mathbf{x} \in \mathbb{R}^n, \boldsymbol{\mu} \in \mathbb{R}^r)$$

be a C^k r-parameter family of vector fields.[40] If $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ is an equilibrium point of the flow generated by $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$ (*i.e.*, $\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*) = 0$), we should be able to answer the question: *Is the stability of the equilibrium point affected as $\boldsymbol{\mu}$ is varied?*

If the equilibrium point is hyperbolic (*i.e.*, if all the eigenvalues of the Jacobian matrix[41] $D\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*)$ have nonzero real part), then, from Theorem 3, it follows that the flow generated by $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$, is locally structurally stable at $(\mathbf{x}^*, \boldsymbol{\mu}^*)$. As a result, for values of $\boldsymbol{\mu}$ sufficiently close to $\boldsymbol{\mu}^*$, the stability of the equilibrium point is not affected.

More precisely, since all the eigenvalues of the Jacobian $D\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*)$ have nonzero real part, the Jacobian is invertible, and, from the implicit function theorem,[42] it follows that there exists a unique C^k function $\mathbf{x} : \boldsymbol{\mu} \mapsto \mathbf{x}(\boldsymbol{\mu})$ such that, for $\boldsymbol{\mu}$ sufficiently close to $\boldsymbol{\mu}^*$,

$$\mathbf{X}(\mathbf{x}(\boldsymbol{\mu}), \boldsymbol{\mu}) = \mathbf{0} \quad \text{with} \quad \mathbf{x}(\boldsymbol{\mu}^*) = \boldsymbol{x}^*.$$

Since the eigenvalues of the Jacobian $D\mathbf{X}\big(\mathbf{x}(\boldsymbol{\mu}), \boldsymbol{\mu}\big)$ are continuous functions of $\boldsymbol{\mu}$, for $\boldsymbol{\mu}$ sufficiently close to $\boldsymbol{\mu}^*$, all the eigenvalues of this Jacobian have nonzero real part. Hence, equilibrium points close to $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ are hyperbolic and have the same type of stability as $(\mathbf{x}^*, \boldsymbol{\mu}^*)$.

If the equilibrium point $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ is nonhyperbolic (*i.e.*, if some eigenvalues of $D\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*)$ have zero real part), $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$ is not structurally stable at $(\mathbf{x}^*, \boldsymbol{\mu}^*)$. In this case, for values of $\boldsymbol{\mu}$ close to $\boldsymbol{\mu}^*$, a totally new dynamical behavior can occur.

Our aim in this section is to investigate the simplest bifurcations that occur at nonhyperbolic equilibrium points in one- and two-dimensional systems.

[40] The degree of differentiability has to be as high as needed in order to satisfy the conditions for the family of vector fields to exhibit a given type of bifurcation. See the necessary and sufficient conditions below.

[41] The notation $D\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*)$ means that the derivative is taken with respect to $\mathbf{x}$ at the point $(\mathbf{x}^*, \boldsymbol{\mu}^*)$. If there is a risk of confusion with respect to which variable the derivative is taken, we will write $D_{\mathbf{x}}\mathbf{X}(\mathbf{x}^*, \boldsymbol{\mu}^*)$.

[42] We shall often use this theorem, in particular in bifurcation theory, where it plays an essential role. Here is a simplified version that is sufficient in most cases: Let $f : I_1 \times I_2 \to \mathbb{R}$ be a C^k function of two real variables; if $(x_0, y_0) \in I_1 \times I_2$ and

$$f(x_0, y_0) = 0, \qquad \frac{\partial f}{\partial y}(x_0, y_0) \neq 0,$$

then there exists an open interval I, containing x_0, and a C^k function $\varphi : I \to \mathbb{R}$ such that

$$\varphi(x_0) = y_0, \quad \text{and} \quad f(x, \varphi(x)) = 0, \quad \text{for all } x \in I.$$

For a proof, see Lang [200], pp. 425–429, in which a proof of a more general version of the implicit function theorem is also given.

3.5.1 One-dimensional vector fields

In a one-dimensional system, a nonhyperbolic equilibrium point is necessarily associated with a zero eigenvalue of the derivative of the vector field at the equilibrium point. In this section, we describe the most important types of bifurcations that occur in one-dimensional systems

$$\dot{x} = X(x, \mu) \quad (x \in \mathbb{R},\ \mu \in \mathbb{R}). \tag{3.31}$$

As a simplification, we assume that the bifurcation point (x^*, μ^*) is $(0,0)$, that is,

$$X(0,0) = 0 \quad \text{and} \quad \frac{\partial X}{\partial x}(0,0) = 0. \tag{3.32}$$

In a neighborhood of a bifurcation point, the essential information concerning the bifurcation is captured by the *bifurcation diagram*, which consists of different curves. The locus of stable points is usually represented by a solid curve, while a broken curve represents the locus of unstable points (Figure 3.17).

Saddle-node bifurcation

Consider the equation

$$\dot{x} = \mu - x^2. \tag{3.33}$$

For $\mu = 0$, $x^* = 0$ is the only equilibrium point, and it is nonhyperbolic since $DX(0,0) = 0$. The vector field $X(x,0)$ is not structurally stable and $\mu = 0$ is a bifurcation value. For $\mu < 0$, there are no equilibrium points, while, for $\mu > 0$, there are two hyperbolic equilibrium points $x^* = \pm\sqrt{\mu}$. Since $DX(\pm\sqrt{\mu}, \mu) = \mp 2\sqrt{\mu}$, $\sqrt{\mu}$ is asymptotically stable, and $-\sqrt{\mu}$ is unstable. The phase portraits for Equation (3.33) are shown in Figure 3.14. This type of bifurcation is called a *saddle-node bifurcation*.[43]

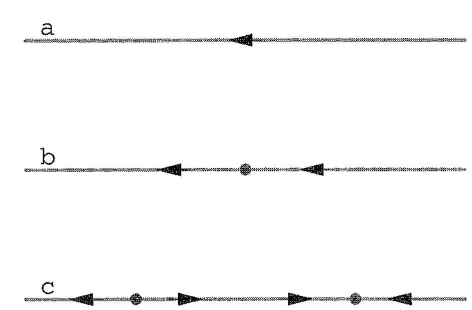

Fig. 3.14. *Saddle-node bifurcation. Phase portraits for the differential equation (3.33). (a) $\mu < 0$, (b) $\mu = 0$, (c) $\mu > 0$.*

[43] Also called *tangent bifurcation*.

Transcritical bifurcation

Consider the equation

$$\dot{x} = \mu x - x^2. \tag{3.34}$$

For $\mu = 0$, $x^* = 0$ is the only equilibrium point, and it is nonhyperbolic since $DX(0,0) = 0$. The vector field $X(x,0)$ is not structurally stable, and $\mu = 0$ is a bifurcation value. For $\mu \neq 0$, there are two equilibrium points, 0 and μ. At the bifurcation point, these two equilibrium points exchange their stability (see Figure 3.15). This type of bifurcation is called a *transcritical bifurcation*.

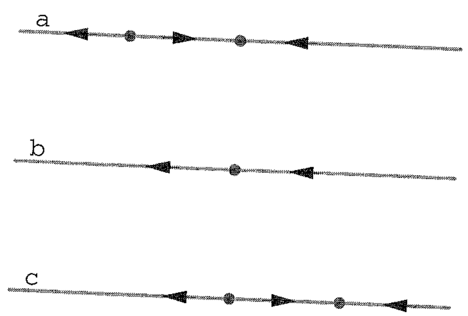

Fig. 3.15. *Transcritical bifurcation. Phase portraits for the differential equation (3.34). (a) $\mu < 0$, (b) $\mu = 0$, (c) $\mu > 0$.*

Pitchfork bifurcation

Consider the equation

$$\dot{x} = \mu x - x^3. \tag{3.35}$$

For $\mu = 0$, $x^* = 0$ is the only equilibrium point. It is nonhyperbolic since $DX(0,0) = 0$. The vector field $X(x,0)$ is not structurally stable and $\mu = 0$ is a bifurcation value. For $\mu \le 0$, 0 is the only equilibrium point, and it is asymptotically stable. For $\mu > 0$, there are three equilibrium points, 0 is unstable, and $\pm\sqrt{\mu}$ are both asymptotically stable. The phase portraits for Equation (3.35) are shown in Figure 3.16. This type of bifurcation is called a *pitchfork bifurcation*.[44]

[44] Sometimes called *symmetry breaking bifurcation* since it is the bifurcation that characterizes the broken symmetry associated with a second-order phase transition in statistical physics.

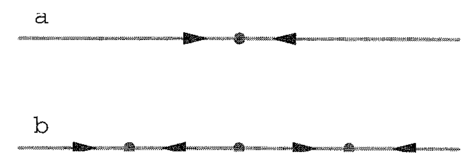

Fig. 3.16. *Pitchfork bifurcation. Phase portraits for the differential equation (3.35). (a) $\mu \le 0$, (b) $\mu > 0$.*

Necessary and sufficient conditions

It is possible to derive necessary and sufficient conditions under which a one-parameter family of one-dimensional vector fields exhibits a bifurcation of one of the types just described. These conditions involve derivatives of the vector field at the bifurcation point.

Saddle-node bifurcation. If the family $X(x,\mu)$ undergoes a saddle-node bifurcation, in the (μ, x)-plane there exists a unique curve of fixed points (see the top panel of Figure 3.17). This curve is tangent to the line $\mu = 0$ at $x = 0$, and it lies entirely to one side of $\mu = 0$. These two properties imply that

$$\frac{d\mu}{dx}(0) = 0, \quad \frac{d^2\mu}{dx^2}(0) \neq 0. \tag{3.36}$$

The bifurcation point is a nonhyperbolic equilibrium; *i.e.*,

$$X(0,0) = 0, \quad \frac{\partial X}{\partial x}(0,0) = 0.$$

If we assume

$$\frac{\partial X}{\partial \mu}(0,0) \neq 0,$$

then, by the implicit function theorem, there exists a unique function

$$\mu : x \mapsto \mu(x), \quad \text{such that} \quad \mu(0) = 0,$$

defined in a neighborhood of $x = 0$ that satisfies the relation $X(x,\mu(x)) = 0$. To express Conditions (3.36), which imply that $(0,0)$ is a nonhyperbolic equilibrium point at which a saddle-node bifurcation occurs, in terms of derivatives of X, we have to differentiate the relation $X(x,\mu(x)) = 0$ with respect to x. We obtain

$$\frac{dX}{dx}(x,\mu(x)) = \frac{\partial X}{\partial x}(x,\mu(x)) + \frac{\partial X}{\partial \mu}(x,\mu(x))\frac{d\mu}{dx}(x) = 0.$$

Hence,

$$\frac{d\mu}{dx}(0) = -\frac{\dfrac{\partial X}{\partial x}(0,0)}{\dfrac{\partial X}{\partial \mu}(0,0)},$$

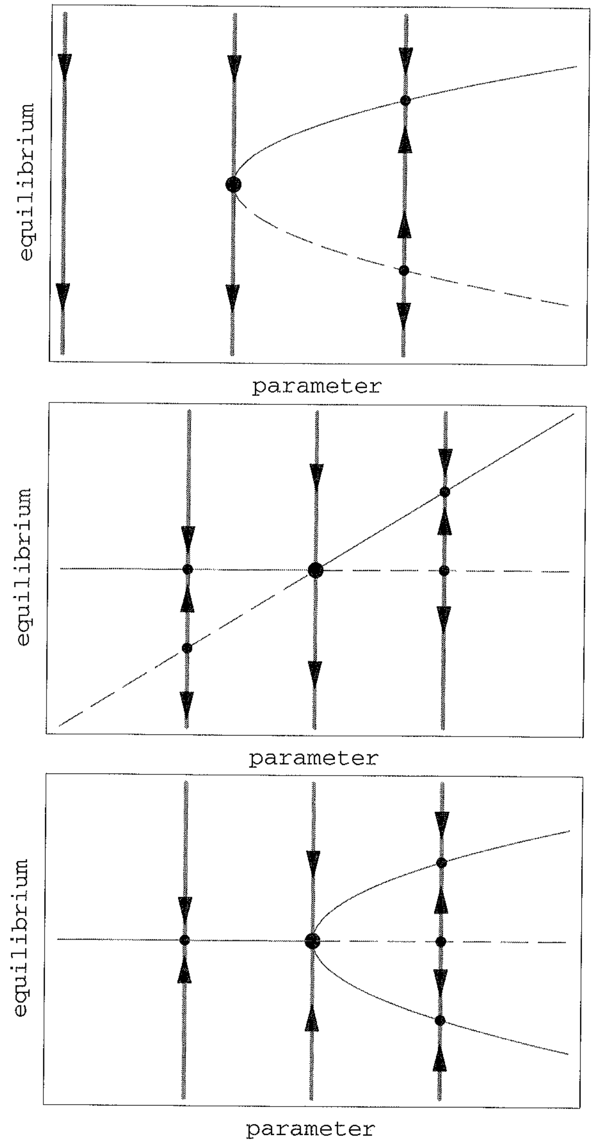

Fig. 3.17. *Bifurcation diagrams with phase portraits. Bifurcation points are represented by • and hyperbolic equilibrium points by •. Stable equilibrium points are on solid curves. Top: saddle-node; middle: transcritical; bottom: pitchfork.*

which shows that

$$\frac{\partial X}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial X}{\partial \mu}(0,0) \neq 0,$$

implies

$$\frac{d\mu}{dx}(0) = 0.$$

If we differentiate $X(x, \mu(x)) = 0$ once more with respect to x, we obtain

$$\begin{aligned}\frac{d^2X}{dx^2}(x,\mu(x)) &= \frac{\partial^2 X}{\partial^2 x}(x,\mu(x)) + 2\frac{\partial^2 X}{\partial x \partial \mu}(x,\mu(x))\frac{d\mu}{dx}(x) \\ &+ \frac{\partial^2 X}{\partial^2 \mu}(x,\mu(x))\left(\frac{d\mu}{dx}(x)\right)^2 + \frac{\partial X}{\partial \mu}(x,\mu(x))\frac{d^2\mu}{dx^2}(x) = 0.\end{aligned}$$

Hence, taking into account the expression of $d\mu/dx(0)$ found above,

$$\frac{d^2X}{dx^2}(0) + \frac{\partial X}{\partial \mu}(0,0)\frac{d^2\mu}{dx^2}(0) = 0,$$

which yields

$$\frac{d^2\mu}{d^2x}(0) = -\frac{\dfrac{\partial^2 X}{\partial x^2}(0,0)}{\dfrac{\partial X}{\partial \mu}(0,0)}.$$

That is,

$$\frac{d^2\mu}{dx^2}(0) \neq 0 \quad \text{if} \quad \frac{\partial^2 X}{\partial x^2}(0,0) \neq 0.$$

In short, the one-parameter family

$$\dot{x} = X(x, \mu)$$

undergoes a saddle-node bifurcation if

$$X(0,0) = 0 \quad \text{and} \quad \frac{\partial X}{\partial x}(0,0) = 0,$$

showing that $(0,0)$ is a nonhyperbolic equilibrium point, and

$$\frac{\partial X}{\partial \mu}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial^2 X}{\partial^2 x}(0,0) \neq 0,$$

which imply the existence of a unique curve of equilibrium points that passes through $(0,0)$ and lies, in a neighborhood of $(0,0)$, on one side of the line $\mu = 0$. In order to exhibit a saddle-node bifurcation, the one-parameter family X has to be C^k with $k \geq 2$.

Transcritical bifurcation. As for the saddle-node bifurcation, the implicit function theorem can be used to characterize the geometry of the bifurcation diagram in the neighborhood of the bifurcation point, assumed to be located at $(0,0)$ in the (μ, x)-plane, in terms of the derivatives of the vector field X. In the case of the transcritical bifurcation, in the neighborhood of the bifurcation point, the bifurcation diagram is characterized (see the middle panel of

Figure 3.17) by the existence of two curves of equilibrium points: $x = \mu$ and $x = 0$. Both curves exist on both sides of $(0,0)$, and the equilibrium points on these curves exchange their stabilities on passing through the bifurcation point.

The bifurcation point $(0,0)$ being nonhyperbolic, we must have

$$X(0,0) = 0, \quad \frac{\partial X}{\partial x}(0,0) = 0.$$

Since *two* curves of equilibrium points pass through this point,

$$\frac{\partial X}{\partial \mu}(0,0) = 0,$$

otherwise the implicit function theorem would imply the existence of *only one* curve passing through $(0,0)$. To be able to use the implicit function theorem, the result obtained in the discussion of the transcritical bifurcation will guide us. If the vector field X is assumed to be of the form $X(x,\mu) = x\,\Xi(x,\mu)$, then $x = 0$ is a curve of equilibrium points passing through $(0,0)$. To obtain the additional curve, Ξ has to satisfy $\Xi(0,0)$. The values of the derivatives of Ξ at $(0,0)$ are determined using the definition of Ξ; *i.e.*,

$$\Xi(x,\mu) = \begin{cases} \dfrac{X(x,\mu)}{x}, & \text{if } x \neq 0, \\[2ex] \dfrac{\partial X}{\partial x}(0,\mu), & \text{if } x = 0. \end{cases}$$

Hence,

$$\frac{\partial \Xi}{\partial x}(0,0) = \frac{\partial^2 X}{\partial^2 x}(0,0), \qquad \frac{\partial^2 \Xi}{\partial^2 x}(0,0) = \frac{\partial^3 X}{\partial^3 x}(0,0),$$

and

$$\frac{\partial \Xi}{\partial \mu}(0,0) = \frac{\partial^2 X}{\partial \mu \partial x}(0,0).$$

If

$$\frac{\partial \Xi}{\partial \mu}(0,0) \neq 0,$$

from the implicit function theorem, it follows that there exists a function $\mu : x \mapsto \mu(x)$, defined in the neighborhood of $x = 0$, that satisfies the relation $\Xi(x,\mu(x)) = 0$. For the curve $\mu = \mu(x)$ not to coincide with the curve $x = 0$ and to be defined on both sides of $(0,0)$, the function μ has to be such that

$$0 < \left|\frac{d\mu}{dx}(0)\right| < \infty.$$

Differentiating with respect to x the relation $\Xi(x,\mu(x)) = 0$, we obtain

$$\frac{d\mu}{dx}(0) = -\frac{\dfrac{\partial \Xi}{\partial x}(0,0)}{\dfrac{\partial \Xi}{\partial \mu}(0,0)} = -\frac{\dfrac{\partial^2 X}{\partial^2 x}(0,0)}{\dfrac{\partial^2 X}{\partial \mu \partial x}(0,0)}.$$

In short, the one-parameter family

$$\dot{x} = X(x,\mu)$$

undergoes a transcritical bifurcation if

$$X(0,0) = 0 \quad \text{and} \quad \frac{\partial X}{\partial x}(0,0) = 0,$$

showing that $(0,0)$ is a nonhyperbolic equilibrium point, and

$$\frac{\partial X}{\partial \mu}(0,0) = 0, \quad \frac{\partial^2 X}{\partial^2 x}(0,0) \neq 0, \quad \text{and} \quad \frac{\partial^2 X}{\partial \mu \partial x}(0,0) \neq 0,$$

which imply the existence of two curves of equilibrium points on both sides $(0,0)$ passing through that point. In order to exhibit a transcritical bifurcation, the one-parameter family X has to be C^k with $k \geq 2$.

Pitchfork bifurcation. The derivation of the necessary and sufficient conditions for a one-parameter family of vector fields $X(x,\mu)$ to exhibit a pitchfork bifurcation is similar to the derivation for such a family to exhibit a transcritical bifurcation.

In the case of a pitchfork bifurcation, the bifurcation diagram is characterized (see the bottom panel of Figure 3.17) by the existence of two curves of equilibrium points: $x = 0$ and $\mu = x^2$. The curve $x = 0$ exists on both sides of the bifurcation point $(0,0)$, and the equilibrium points on this curve change their stabilities on passing through this point. The curve $\mu = x^2$ exists only on one side of $(0,0)$ and at this point is tangent to the line $\mu = 0$.

The point $(0,0)$ being nonhyperbolic, we must have

$$X(0,0) = 0, \quad \frac{\partial X}{\partial x}(0,0) = 0.$$

Since *more than one curve* of equilibrium points passes through this point,

$$\frac{\partial X}{\partial \mu}(0,0) = 0.$$

As in the case of a transcritical bifurcation, we assume the vector field X to be of the form $X(x,\mu) = x\,\Xi(x,\mu)$ to ensure that $x = 0$ is a curve of equilibrium points passing through $(0,0)$. To obtain the additional curve, Ξ has to satisfy $\Xi(0,0)$. The values of the derivatives of Ξ at $(0,0)$ may be determined using the definition of Ξ, *i.e.*,

$$\Xi(x,\mu) = \begin{cases} \dfrac{X(x,\mu)}{x}, & \text{if } x \neq 0, \\ \dfrac{\partial X}{\partial x}(0,\mu), & \text{if } x = 0. \end{cases}$$

Then,

$$\frac{\partial \Xi}{\partial x}(0,0) = \frac{\partial^2 X}{\partial^2 x}(0,0), \qquad \frac{\partial^2 \Xi}{\partial^2 x}(0,0) = \frac{\partial^3 X}{\partial^3 x}(0,0),$$

and

$$\frac{\partial \Xi}{\partial \mu}(0,0) = \frac{\partial^2 X}{\partial \mu \partial x}(0,0).$$

If

$$\frac{\partial \Xi}{\partial \mu}(0,0) \neq 0,$$

from the implicit function theorem, it follows that there exists a function $\mu : x \mapsto \mu(x)$, defined in the neighborhood of $x = 0$, that satisfies the relation $\Xi(x,\mu(x)) = 0$. For the curve $\mu = \mu(x)$ to have, close to $x = 0$, the geometric properties of $\mu = x^2$, it suffices to have

$$\frac{d\mu}{dx}(0) = 0 \quad \text{and} \quad \frac{d^2\mu}{dx^2}(0) \neq 0.$$

Differentiating the relation $\Xi(x,\mu(x)) = 0$, we therefore obtain

$$\frac{d\mu}{dx}(0) = -\frac{\dfrac{\partial \Xi}{\partial x}(0,0)}{\dfrac{\partial \Xi}{\partial \mu}(0,0)} = -\frac{\dfrac{\partial^2 X}{\partial^2 x}(0,0)}{\dfrac{\partial^2 X}{\partial \mu \partial x}(0,0)} = 0,$$

$$\frac{d^2\mu}{dx^2}(0) = -\frac{\dfrac{\partial^{\Xi}}{\partial^2 x}(0,0)}{\dfrac{\partial \Xi}{\partial \mu}(0,0)} = -\frac{\dfrac{\partial^3 X}{\partial^3 x}(0,0)}{\dfrac{\partial^2 X}{\partial \mu \partial x}(0,0)} \neq 0.$$

In short, the one-parameter family

$$\dot{x} = X(x,\mu)$$

undergoes a pitchfork bifurcation if

$$X(0,0) = 0 \quad \text{and} \quad \frac{\partial X}{\partial x}(0,0) = 0,$$

showing that $(0,0)$ is a nonhyperbolic equilibrium point, and

$$\frac{\partial X}{\partial \mu}(0,0) = 0 \quad \frac{\partial^2 X}{\partial^2 x}(0,0) = 0, \quad \frac{\partial^2 X}{\partial \mu \partial x}(0,0) \neq 0, \quad \text{and} \quad \frac{\partial^3 X}{\partial^3 x}(0,0) \neq 0,$$

which imply the existence of two curves of equilibrium points passing through $(0,0)$, one being defined on both sides and the other one only on one side of this point. In order to exhibit a pitchfork bifurcation, the one-parameter family X has to be C^k with $k \geq 3$.

Example 19. Bead sliding on a rotating hoop. Consider a bead of mass m sliding without friction on a hoop of radius R (Figure 3.18). The hoop rotates with angular velocity ω about a vertical axis. The acceleration due to gravity is g. If x is the angle between the vertical downward direction and the position vector of the bead, the Lagrangian of the system is

$$L(x,\dot{x}) = \frac{1}{2} mR^2\dot{x}^2 + \frac{1}{2} m\omega^2 R^2 \sin^2 x + mgR\cos x,$$

and the equation of motion is

$$mR^2\ddot{x} = m\omega^2 R^2 \sin x \cos x - mgR\sin x$$

or

$$R\ddot{x} = \omega^2 R \sin x \cos x - g \sin x.$$

Introducing the reduced variables

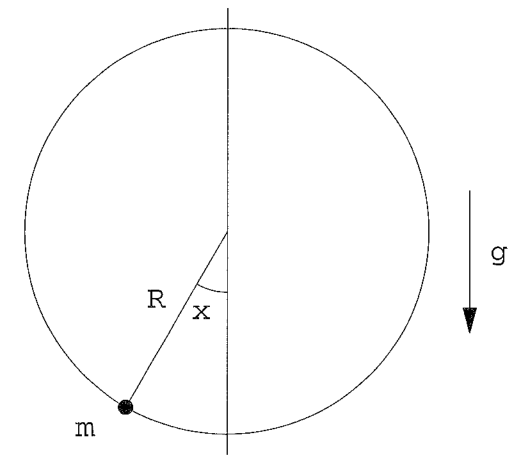

Fig. 3.18. *Bead sliding on a vertical rotating hoop without friction.*

$$\tau = \sqrt{\frac{g}{R}}\, t, \quad \mu = \frac{R}{g}\omega^2,$$

the equation of motion takes the form

$$\frac{d^2x}{d\tau^2} = \sin x(\mu \cos x - 1). \tag{3.37}$$

This equation may be written as a first-order system. If we put $x_1 = x$ and $x_2 = \dot{x}$, where now the dot represents derivation with respect to reduced time τ, Equation (3.37) is replaced by the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \sin x_1(\mu \cos x_1 - 1). \tag{3.38}$$

The state of the system represented by $\mathbf{x} = (x_1, x_2)$ belongs to the cylinder $[0, 2\pi[\times\mathbb{R}$. In this phase space, there exist four equilibrium points

$$(0,0), \quad (\pi, 0), \quad (\pm \arccos \mu^{-1}, 0).$$

At these equilibrium points, the Jacobian of the system $J(x_1^*, x_2^*)$ is given by

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ \mu - 1 & 0 \end{bmatrix}, \quad J(\pi, 0) = \begin{bmatrix} 0 & 1 \\ \mu + 1 & 0 \end{bmatrix},$$

$$J(\pm \arccos \mu^{-1}, 0) = \begin{bmatrix} 0 & 1 \\ \dfrac{1-\mu^2}{\mu} & 0 \end{bmatrix}.$$

If $0 < \mu < 1$, $(0,0)$ is Lyapunov stable but not asymptotically stable, $(\pi, 0)$ is unstable, and $(\pm \arccos \mu^{-1}, 0)$ do not exist. For μ slightly greater than 1, (0, 0) is unstable, and the two equilibrium points $(\pm \arccos \mu^{-1}, 0)$ are stable. The nonhyperbolic equilibrium point $(x, \mu) = (0, 1)$ is a pitchfork bifurcation point. For $0 < \mu < 1$, the bead oscillates around the point $x = 0$, while for μ slightly greater than 1, it oscillates around either $x = \arccos \mu^{-1}$ or $x = -\arccos \mu^{-1}$. Since the Lagrangian is invariant under the transformation $x \to -x$, this system exhibits a symmetry-breaking bifurcation similar to those characterizing second-order phase transitions in some simple magnetic systems.

3.5.2 Equivalent families of vector fields

In a one-dimensional system, Conditions (3.32) are necessary but not sufficient for the system to exhibit a bifurcation. For instance, the equation

$$\dot{x} = \mu - x^3$$

does not exhibit a bifurcation at the equilibrium point $(x^*, \mu^*) = (0, 0)$. A simple analysis shows that, for all real μ, the equilibrium point $x^* = (\mu)^{1/3}$ is always asymptotically stable. In such a case, it is said that the family of vector fields $X(x, \mu)$ does *unfold* the singularity in $X(x, 0)$. More precisely, any local family, $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$, at $(\mathbf{x}^*, \boldsymbol{\mu}^*)$ is said to be an *unfolding* of the vector field $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu}^*)$. When $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu}^*)$ has a singularity at $\mathbf{x} = \mathbf{x}^*$, $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$is referred to as an *unfolding of the singularity*.

Definition 12. *Two local families* $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$ *and* $\mathbf{Y}(\mathbf{y}, \boldsymbol{\nu})$ *are said to be* equivalent *if there exists a continuous mapping* $(\mathbf{x}, \boldsymbol{\mu}) \mapsto \mathbf{h}(\mathbf{x}, \boldsymbol{\mu})$*, defined in a neighborhood of* $(\mathbf{x}^*, \boldsymbol{\mu}^*)$*, satisfying* $\mathbf{h}(\mathbf{x}^*, \boldsymbol{\mu}^*) = \mathbf{y}^*$*, such that for each* $\boldsymbol{\mu}$*,* $\mathbf{x} \mapsto \mathbf{h}(\mathbf{x}, \boldsymbol{\mu})$ *is a homeomorphism that exhibits the topological equivalence*[45] *of the flows generated by* $\mathbf{X}(\mathbf{x}, \boldsymbol{\mu})$ *and* $\mathbf{Y}(\mathbf{y}, \boldsymbol{\nu})$.

[45] On topological equivalence, see Remark 2.

Example 20. The family of vector fields $X(x, \mu) = \mu x - x^2$ can be written

$$X(x, \mu) = \frac{\mu^2}{4} - \left(x - \frac{\mu}{2}\right)^2.$$

If $y = h(x, \mu) = x - \frac{1}{2}\mu$, then, for each μ,

$$\dot{y} = \dot{x} = \frac{\mu^2}{4} - \left(x - \frac{\mu}{2}\right)^2 = \frac{\mu^2}{4} - y^2 = Y(y, \nu),$$

where $\nu = \frac{1}{4}\mu^2$. Hence, the two families $X(x, \mu) = \mu x - x^2$ and $Y(y, \nu) = \nu - y^2$ are topologically equivalent.

3.5.3 Hopf bifurcation

If the differential equation $\dot{x} = X(x, \mu)$ exhibits a bifurcation of one of the types described above, clearly the two-dimensional system

$$\dot{x}_1 = X(x_1, \mu), \quad \dot{x}_2 = -x_2$$

also exhibits the same type of bifurcation. We will not insist. In this section we present a new type of bifurcation that does not exist in a one-dimensional system: the *Hopf bifurcation*. This type of bifurcation appears at a nonhyperbolic equilibrium point of a two-dimensional system whose eigenvalues of its linear part are pure imaginary.

We have already found Hopf bifurcations in two models: the prey-predator model used by Harrison, which predicts the outcome of Luckinbill's experiment with *Didinium* and *Paramecium* qualitatively (Section 2.3), and the van der Pol oscillator (Example 16).

The following theorem indicates under which conditions a two-dimensional system undergoes a Hopf bifurcation.[46]

Theorem 4. *Let* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mu)$ *be a two-dimensional one-parameter family of vector fields* ($\mathbf{x} \in \mathbb{R}^2$, $\mu \in \mathbb{R}$) *such that*

1. $\mathbf{X}(\mathbf{0}, \mu) = \mathbf{0}$,
2. $\mathbf{X}$ *is an analytic function of* $\mathbf{x}$ *and* μ*, and*
3. $D_{\mathbf{x}}\mathbf{X}(\mathbf{0}, \mu)$ *has two complex conjugate eigenvalues* $\alpha(\mu) \pm i\omega(\mu)$ *with* $\alpha(0) = 0$, $\omega(0) \neq 0$*, and* $d\alpha/d\mu|_{\mu=0} \neq 0$;

then, in any neighborhood $U \subset \mathbb{R}^2$ *of the origin and any given* $\mu_0 > 0$ *there is a* $\mu < \mu_0$ *such that the equation* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, \mu)$ *has a nontrivial periodic orbit in* U.

[46] For a proof, see Hassard, Kazarinoff, and Wan [166], who present various proofs and many applications; see also Hale and Koçak [164].

Example 21. The van der Pol system revisited. In the case of the van der Pol system (3.26), we have

$$D_{\mathbf{x}}\mathbf{X}(\mathbf{0},\lambda) = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}.$$

The eigenvalues of the linear part of the van der Pol system are $\frac{1}{2}(\lambda \pm \mathrm{i}\sqrt{4-\lambda^2})$. If $\lambda < 0$, the origin is asymptotically stable, and if $\lambda > 0$ the origin is unstable. Since the real part of both eigenvalues is equal to $\frac{1}{2}\lambda$, all the conditions of Theorem 4 are verified.

Example 22. Section 2.3 model. Harrison found that the model described by the system

$$\dot{H} = r_H H\left(1 - \frac{H}{K}\right) - \frac{a_H PH}{b+H},$$
$$\dot{P} = \frac{a_P PH}{b+H} - cP,$$

exhibits a stable limit cycle and is in good qualitative agreement with Luckinbill's experiment. Using the reduced variables

$$h = \frac{H}{H^*}, \quad p = \frac{P}{P^*}, \quad \tau = r_H t, \quad k = \frac{K}{H^*}, \quad \beta = \frac{b}{H^*}, \quad \gamma = \frac{c}{r},$$

where (H^*, P^*) denotes the nontrivial fixed point, the model can be written

$$\frac{dh}{d\tau} = h\left(1 - \frac{h}{k}\right) - \frac{\alpha_h ph}{\beta + h},$$
$$\frac{dp}{d\tau} = \frac{\alpha_p ph}{\beta + h} - \gamma p,$$

where

$$\alpha_h = \left(1 - \frac{1}{k}\right)(\beta + 1) \quad \text{and} \quad \alpha_p = \gamma(\beta + 1).$$

The Jacobian at the equilibrium point $(1,1)$ is

$$\begin{bmatrix} \dfrac{k-2-\beta}{k(1+\beta)} & -1 + \dfrac{1}{k} \\ \dfrac{\beta\gamma}{1+\beta} & 0 \end{bmatrix}.$$

Its eigenvalues are

$$\frac{k - 2 - \beta \pm \mathrm{i}\sqrt{4(k-1)k^2\beta\gamma - (k-2-\beta)^2}}{2k(1+\beta)}.$$

The equilibrium point is asymptotically stable if $k < 2 + \beta$. Since the derivative of the real part of the eigenvalues with respect to the parameter k is different from zero at the bifurcation point, all the conditions of Theorem 4 are verified: there exists a limit cycle for $k > 2 + \beta$.

It is often desirable to prove that a two-dimensional system does not possess a limit cycle. The following theorem, known as the *Dulac criterion* [111], is often helpful to prove the nonexistence of a limit cycle.

Theorem 5. *Let Ω be a simply connected subset of $\mathbb{R}^2$ and $D : \Omega \to \mathbb{R}$ a C^1 function. If the function*

$$\mathbf{x} \mapsto \nabla D\mathbf{X}(\mathbf{x}) = \frac{\partial DX_1}{\partial x_1} + \frac{\partial DX_2}{\partial x_2}$$

has a constant sign and is not identically zero in Ω, then the two-dimensional system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ has no periodic orbit lying entirely in Ω.

If there exists a closed orbit γ in Ω, Green's theorem implies

$$\oint_\gamma D(\mathbf{x})X_1(\mathbf{x})\,dx_2 - D(\mathbf{x})X_2(\mathbf{x})\,dx_1 = \int_\Gamma \left(\frac{\partial DX_1}{\partial x_1} + \frac{\partial DX_2}{\partial x_2}\right) dx_1\,dx_2 \neq 0,$$

where Γ is the bounded *interior* of γ ($\partial\Gamma = \gamma$).[47] But if γ is an orbit, we also have

$$\oint_\gamma D(\mathbf{x})\big(X_1(\mathbf{x})\,\dot{x}_2 - X_2(\mathbf{x})\,\dot{x}_1\big)\,dt = \oint_\gamma D(\mathbf{x})\big(X_1(\mathbf{x})dx_2 - X_2(\mathbf{x})dx_1\big) = 0,$$

where we have taken into account that $\dot{x}_i = X_i(\mathbf{x})$ for $i = 1, 2$. This contradiction proves the theorem.

The function D is a *Dulac function.* There is no general method for finding an appropriate Dulac function. If $D(x_1, x_2) = 1$, the theorem is referred to as the *Bendixson criterion* [40].

3.5.4 Catastrophes

Catastrophe theory studies abrupt changes associated with smooth modifications of control parameters.[48] In this section, we describe one of the simplest types found in many models: the cusp catastrophe.

[47] Since a closed orbit in $\mathbb{R}^2$ does not intersect itself, it is a closed Jordan's curve and, according to Jordan's theorem, it separates the plane into two disjoint connected components such that only one of these two components is bounded. The closed Jordan's curve is the common boundary of the two components.

[48] On catastrophe theory, one should consult Thom [328]. On the mathematical presentation of the theory, refer to the recent book of Castrigiano and Hayes [82] with a preface by René Thom, the founder of the theory. On early applications of the theory, see Zeeman [354]. Many applications of catastrophe theory have been heavily criticized. For example, here is a quotation from the English translation by Wassermann and Thomas of Arnol'd [10], p. 9:

> *I remark that articles on catastrophe theory are distinguished by a sharp and catastrophic lowering of the level of demands of rigour and also of novelty of published results.*

In [329] Thom expounds his philosophical standpoint and shows that a qualitative approach may offer a subtler explanation than a purely quantitative description.

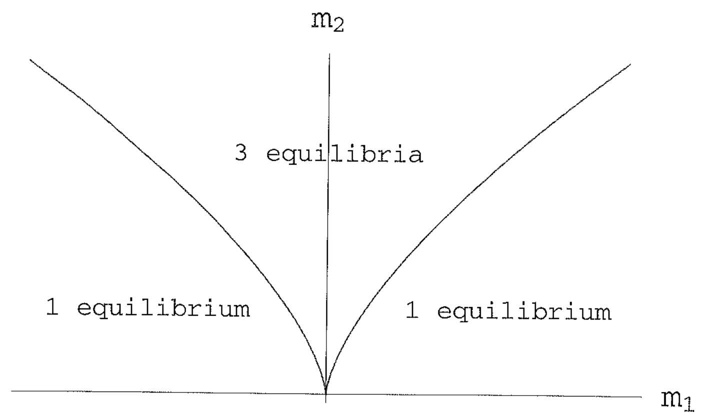

Fig. 3.19. *The bifurcation diagram of the differential equation* $\dot{x} = m_1 + m_2 x - x^3$.

Consider the two-parameter family of maps on $\mathbb{R}$

$$(x, m_1, m_2) \mapsto X(x, m_1, m_2) = m_1 + m_2 x - x^3.$$

Depending upon the values of the parameters, the differential equation $\dot{x} = X(x, m_1, m_2)$ has either one or three equilibrium points. Since at bifurcation points the differential equation must have multiple equilibrium points, bifurcation points are the solutions of the system

$$X(x, m_1, m_2) = 0, \quad \frac{\partial X}{\partial x}(x, m_1, m_2) = 0,$$

that is,

$$m_1 + m_2 x - x^3 = 0, \quad m_2 - 3x^2 = 0.$$

Eliminating x, we find

$$27m_1^2 - 4m_2^3 = 0,$$

which is the equation of the boundary between the domains in the parameter space in which the differential equation has either one or three equilibrium points. It is the equation of a cusp (Figure 3.19).

The bifurcation diagram represented in Figure 3.20 will help us understand the nature of the cusp catastrophe. In this figure, the solid line corresponds to asymptotically stable equilibrium points and the broken line to unstable points. Suppose that the parameter m_2 has a fixed value, say 1. If $m_1 > 2/3\sqrt{3}$, there exists only one equilibrium point, represented by point A. This equilibrium is asymptotically stable. As m_1 decreases, the point representing the equilibrium moves along the curve in the direction of the arrow. It passes point E, where nothing special occurs, to finally reach point B. If we try to further decrease m_1, the state of the system, represented by the x-coordinate of B, jumps to the value of the x-coordinate of C, and, for decreasing values of m_1, the asymptotically stable equilibrium points will be

represented by the points on the solid line below C. If now we start increasing m_1, the point representing the asymptotically stable equilibrium will go back to C and proceed up to point D, where, if we try to further increase m_1, it will jump to E and move up along the solid line towards A as the parameter is varied.

The important fact is that the state of the system has experienced jumps at two different values of the control parameter m_1. The parameter values at which jumps take place depend upon the direction in which the parameter is varied. This phenomenon, well-known in physics, is referred to as *hysteresis*, and the closed path $BCDEB$ is called the *hysteresis loop*. It is important to note that the cusp catastrophe, described here, occurs in dynamical systems in which the vector field X is a *gradient field*; that is,

$$X(x, m_1, m_2) = -\nabla\Phi(x, m_1, m_2),$$

where

$$\Phi(x, m_1, m_2) = \tfrac{1}{4}x^4 - \tfrac{1}{2}m_2 x^2 - m_1 x.$$

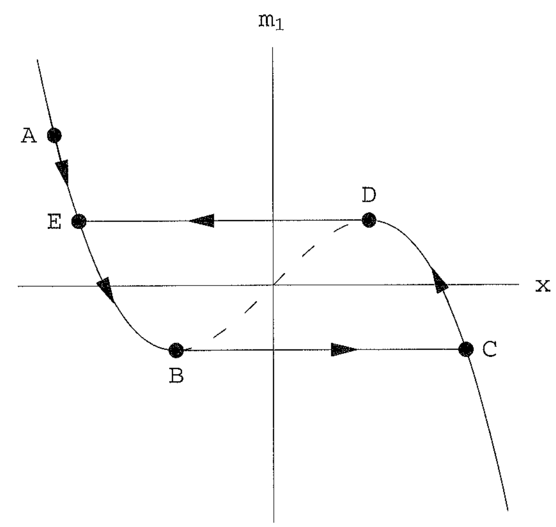

Fig. 3.20. *Hysteresis. The closed path BCDEB is a hysteresis loop.*

Example 23. Street gang control.[49] Street gangs have emerged as tremendously powerful institutions in many communities. In urban ghettos, they may very

[49] I presented a first version of this model at a meeting of the Research Police Forum organized at Starved Rock (IL) by the Police Training Institute of the University of Illinois at Urbana-Champaign in the early 1990s and a more elaborate version a few weeks later at the Criminal Justice Authority (Chicago, IL). It was prepared

well be the most important institutions in the lives of a large proportion of adolescent and young adult males. To model the dynamics of a gang population, assume that the growth rate of the gang population size N is given by

$$\dot{N} = g(N) - p(N),$$

where g is the intrinsic growth function and p the police response function, which describes the amount of resources the police devote at each level of the population. It might be understandably objected that coercive methods are not sufficient; social programs and education also play an important role. The expression $p(N)$ should, therefore, represent the amount of resources devoted by society as a whole.

Using a slight variant of the logistic function, assume that

$$g(N) = r(N + N_0)\left(1 - \frac{N}{K}\right),$$

where r, K, and N_0 are positive constants. Note that the "initial condition" N_0 implies that a zero gang population is not an equilibrium.

For the response function p, assume that, as gang membership in a community grows, the society will devote more resources to the problem. p should, therefore, be monotonically increasing, but since resources are limited, the investment will ultimately approach some maximum level. This pattern can be modeled by

$$p(N) = \frac{aN^\xi}{b + N^\xi},$$

where a, b, and ξ are positive constants. The parameter a is the maximum response level. This maximum is approached faster for decreasing values of b and ξ.[50] The parameter ξ is a measure of how "tough" the society response is.

Introducing the dimensionless variables

$$\tau = rt, \quad n = \frac{N}{K}, \quad \alpha = \frac{a}{rK}, \quad \beta = \frac{b}{K^\xi},$$

the dynamics of the reduced gang population n is modeled by

$$\frac{dn}{d\tau} = (n + n_0)(1 - n) - \frac{\alpha n^\xi}{\beta + n^\xi}. \tag{3.39}$$

The equilibrium points are the solutions to the equation

in collaboration with Jonathan Crane from the Department of Sociology of the University of Illinois at Chicago and the Institute of Government and Public Affairs, Center for Prevention Research and Development.

[50] Note the similarity of this street gang control model with the Ludwig-Holling-Jones model of budworm outbreaks (Equation (1.6)).

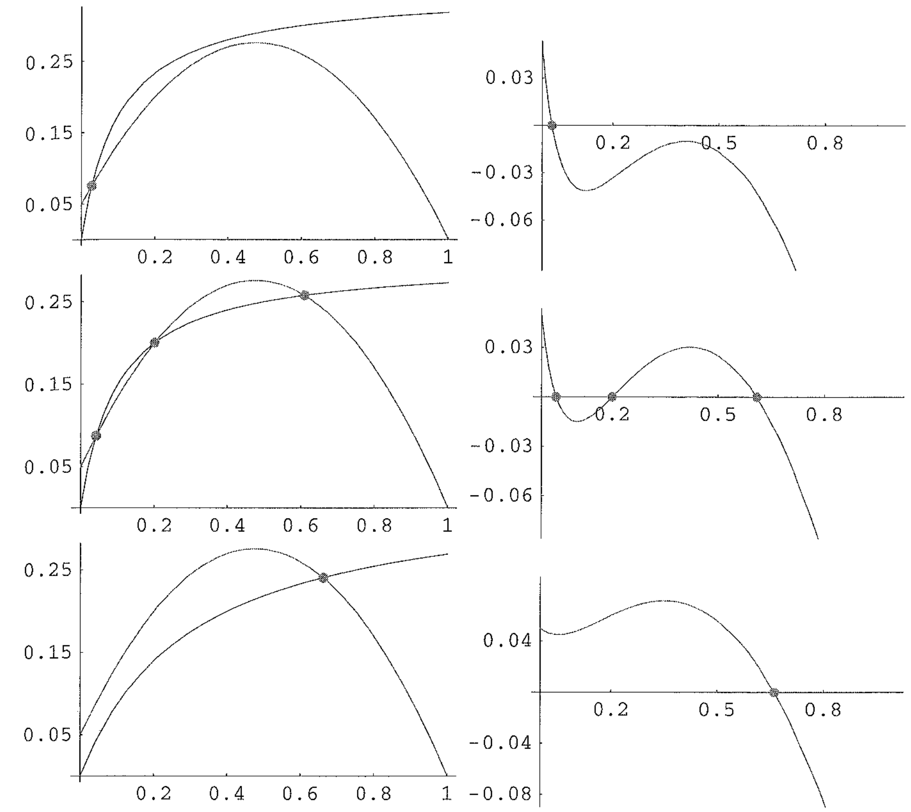

Fig. 3.21. *Equilibrium points of the street gang model. The graphs on the left show the intersections of the graphs of $N \mapsto g(N)$ and $N \mapsto p(N)$; on the right are represented the graphs of $N \mapsto g(n) - p(n)$. Top: one stable low-density equilibrium; middle: two stable equilibria, one at low density and one at high density, separated by one unstable equilibrium at an intermediate density; bottom: one stable equilibrium at high density.*

$$(n + n_0)(1 - n) = \frac{\alpha n^\xi}{\beta + n^\xi}. \tag{3.40}$$

As shown in Figure 3.21, depending upon the values of the parameters, there exist one or three equilibrium points. The equation of the boundary separating the domains in which there exist either one or three equilibrium points is determined by eliminating n between Equation (3.40) and

$$\frac{d}{dn}(n + n_0)(1 - n) = \frac{d}{dn}\frac{\alpha n^\xi}{\beta + n^\xi};$$

that is,

$$1 - n_0 - 2n = \frac{\xi \alpha \beta n^{\xi - 1}}{(\beta + n^\xi)^2}. \tag{3.41}$$

Eliminating α between (3.40) and (3.41), we solve for β, and then, replacing the expression of β in one of the equations, we obtain α. The parametric equation of the boundary is then

$$\alpha = \frac{\xi(n+n_0)^2(1-n)^2}{(2-\xi)n^2 + (\xi-1)(1-n_0)n + \xi n_0},$$
$$\beta = \frac{n^{\xi+1}(1-n_0-2n)}{(2-\xi)n^2 + (\xi-1)(1-n_0)n + \xi n_0}.$$

It is represented in Figure 3.22. It shows the existence of a cusp catastrophe in the model.

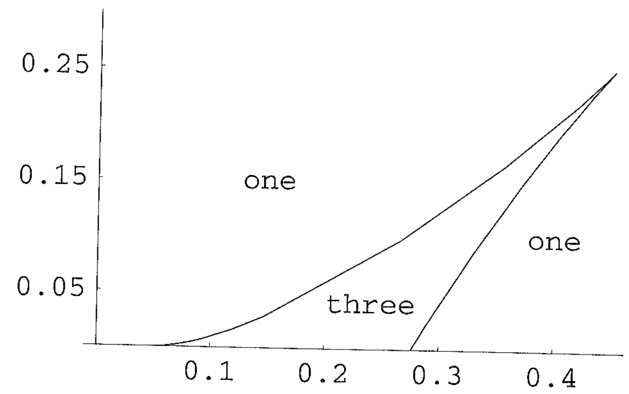

Fig. 3.22. *Boundary between the domains in which there are three equilibrium points and one equilibrium point in the* $(\alpha - \beta)$ *parameter space.*

When dramatic increases in gang populations have occurred, there are usually numerous attempts to try to reverse the rise. Youth employment programs, educational programs in schools, and increases in police resources are common interventions. Such interventions can prevent some individuals from joining the gang, but none of them seem to have had much effect on overall gang populations. This model suggests one possible reason for this.

Due to the existence of the hysteresis effect, small-scale interventions to reduce gang activity may have no permanent effect at all. An intervention may temporarily push gang membership to a slightly lower equilibrium, but unless the intervention is large enough to push the population all the way down to the unstable equilibrium, membership will move back up to the high equilibrium.

While this implication is pessimistic, the model does suggest a strategy that could succeed. If the intervention is large enough to push gang membership below the unstable equilibrium, the gang population will revert to a low equilibrium, and because of the stability of the low equilibrium, the intervention does not have to be continued once the low level is achieved. Thus a short-term but high-intensity intervention might succeed where a long-term, low-intensity strategy would fail.

3.6 Influence of diffusion

In all the models discussed so far, it has always been assumed that the various species were uniformly distributed in space. In other words, over the whole territory available to them, individuals were supposed to mix homogeneously. In a variety of problems, the spatial dimension of the environment cannot, however, be ignored.

In this section, we study the influence of diffusion on the evolution of various populations[51]; that is, we investigate the dynamics of populations assuming that the individuals move at random.[52] We will discover that diffusion can have a profound effect on the dynamics of populations.

3.6.1 Random walk and diffusion

Consider a *random walker* who, in a one-dimensional space, takes a step of length ξ during each time interval τ. Assuming that the steps are taken either in the positive or negative direction with equal probabilities, let $p(t,x)\,dx$ be the probability that the random walker is between x and $x+dx$ at time t. The random walker is between x and $x+dx$ at time t if he was either between $x+\xi$ and $x+\xi+dx$ or between $x-\xi$ and $x-\xi+dx$ at time $t-\tau$. Thus, the function p satisfies the following difference equation

$$p(x,t) = \tfrac{1}{2}(p(x+\xi,t-\tau) + p(x-\xi,t-\tau)). \tag{3.42}$$

If p is continuously differentiable, and if we assume that τ and ξ are small, then

$$p(x\pm\xi,t-\tau) = p(x,t) - \frac{\partial p}{\partial t}(x,t)\tau + O(\tau^2) \pm \frac{\partial p}{\partial x}(x,t)\xi + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x,t)\xi^2 + O(\xi^3),$$

and substituting in (3.42) yields

$$\frac{\partial p}{\partial t}(x,t) = \frac{\xi^2}{2\tau}\frac{\partial^2 p}{\partial x^2}(x,t) + \frac{1}{\tau}(O(\tau^2) + O(\xi^3)).$$

Hence, if there exists a positive constant D such that, when both τ and ξ tend to 0, $\xi^2/2\tau$ tends to a constant D, p satisfies the diffusion equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}. \tag{3.43}$$

[51] The standard text on diffusion in ecology is Okubo [272].

[52] The interest in random dispersal in populations was triggered by the paper of Skellam [314].

If, at $t = 0$, the random walker is at the origin, $p(t, x)$ satisfies the initial condition

$$\lim_{t \to 0} p(x, t) = \delta,$$

where δ is the Dirac distribution, and for $t > 0$, the solution of (3.43) is[53]

$$p(x, t) = \frac{1}{2\sqrt{\pi D t}} \exp -\frac{x^2}{4Dt}. \tag{3.44}$$

The random variable representing the distance of the random walker from the origin is, therefore, *normally distributed*. Its mean value is 0, and its variance $2Dt$ varies *linearly* with time. This result is important. It is indeed a direct consequence of the fact that p is a generalized homogeneous function of t and x.[54]

3.6.2 One-population dynamics with dispersal

As a simple example, consider the influence of random dispersal in a two-dimensional space on the evolution of a Malthusian population. Including a diffusion term in the equation for the population density $\dot{n} = an$, we obtain

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[53] If

$$\widehat{p}(k, t) = \int_{-\infty}^{\infty} p(x, t) e^{ikx}\, dx$$

denotes the Fourier transform of $x \mapsto p(x, t)$, we have

$$\frac{d\widehat{p}}{dt} + Dk^2 \widehat{p} = 0,$$

and $\widehat{p}(k, 0) = 1$. Hence, $\widehat{p}(k, t) = \exp(-Dk^2 t)$. Therefore,

$$p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-Dk^2 t - ikx)\, dk = \frac{1}{\sqrt{2\pi D t}} \exp -\frac{x^2}{4Dt}.$$

This particular solution is known as a *fundamental solution* of the diffusion equation. It can be shown (see Boccara [46]) that, if the initial condition is $p(x, 0) = f(x)$, then the solution of the diffusion equation is simply given by the convolution $g * f$, where g is a fundamental solution. Fundamental solutions are not unique. In physics, fundamental solutions are referred to as *Green's functions*.

[54] $f : \mathbb{R}^n \to \mathbb{R}$ is a *generalized homogeneous function* if, for all $\lambda \in \mathbb{R}$,

$$f(\lambda^{a_1} x_1, \lambda^{a_2} x_2, \ldots, \lambda^{a_n} x_n) \equiv \lambda^r f(x_1, x_2, \ldots, x_n),$$

where $a_1, a_2, \ldots, a_n$ and r are real constants. Since λ can be any real number, we can replace λ by $1/x_1^{1/a_1}$; f may then be written as a function of the $n-1$ reduced variables $x_i / x_1^{a_i/a_1}$ $(i = 2, 3, \ldots, n)$. It is said that x_i scales as $x_1^{a_i/a_1}$. Since the probability density p satisfies the relation $p(\lambda^2 t, \lambda x) \equiv \lambda p(t, x)$, x scales as $\sqrt{t}$.

$$\frac{\partial n}{\partial t} = D\left(\frac{\partial^2 n}{\partial x_1^2} + \frac{\partial^2 n}{\partial x_2^2}\right) + an. \tag{3.45}$$

The exponential growth of a spreading population is an acceptable approximation if, initially, the population consists of very few individuals that spread and reproduce in a habitat where natural enemies, such as competitors and predators, are lacking. If we assume that the dispersal is isotropic, the population dispersal is modeled by

$$\frac{\partial n}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial n}{\partial r}\right) + an, \tag{3.46}$$

where $r = \sqrt{x_1^2 + x_2^2}$.

Introducing the dimensionless variables

$$\tau = at \quad \text{and} \quad \rho = r\sqrt{\frac{a}{D}}, \tag{3.47}$$

Equation (3.46) becomes

$$\frac{\partial n}{\partial \tau} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial n}{\partial \rho}\right) + n. \tag{3.48}$$

Assuming that, at time $t = 0$, there are N_0 individuals concentrated at the origin, the solution of (3.48) is

$$n(\rho, \tau) = \frac{N_0}{4\pi\tau} \exp\left(\tau - \frac{\rho^2}{4\tau}\right). \tag{3.49}$$

The total number of individuals that, at time t, are at a distance greater than R from the origin is given by

$$\begin{aligned} N(R, t) &= \int_R^{\infty} n(r\sqrt{a/D}, at)\, 2\pi r\, dr \\ &= N_0 \exp\left(at - \frac{R^2}{4Dt}\right). \end{aligned}$$

For $R^2 = 4aDt^2$, the number of individuals that, at time t, are outside a circle of radius R, equal to N_0, is negligible compared to the total number of individuals $N_0 e^{at}$. Hence, the radius of an approximate boundary of the habitat occupied by the invading species is proportional to t. According to Skellam [314], this result is in agreement with data on the spread of the muskrat (*Ondatra zibethica*), an American rodent, introduced inadvertently into Central Europe in 1905. The fact that, for the random dispersal of a Malthusian population, R is proportional to t contrasts with simple diffusion, where space scales as the square root of time.

3.6.3 Critical patch size

Consider a *refuge* (*i.e.*, a patch of favorable environment surrounded by a region where survival is impossible). If the population is diffusing, individuals crossing the patch boundary will be lost. The problem is to find the *critical patch size* ℓ_c such that the population cannot sustain itself against losses from individuals crossing the patch boundary if the patch size is less than ℓ_c but can maintain itself indefinitely if the patch size is greater than ℓ_c. As a simplification, we discuss a one-dimensional problem (*i.e.*, we assume that in two dimensions, the patch Σ is an infinite strip of width ℓ):

$$\Sigma = \left\{(x, y) \mid -\tfrac{1}{2}\ell < x < \tfrac{1}{2}\ell, -\infty < y < \infty\right\}.$$

Consider the scaled equation

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} + n, \tag{3.50}$$

where a, the growth rate of the population density, has been absorbed in t and $\sqrt{D/a}$, the square root of the diffusion coefficient divided by the growth rate, in x (see (3.47)). The condition that survival is impossible outside Σ implies

$$n(x, t) = 0 \quad \text{if} \quad x = \pm\tfrac{1}{2}\ell.$$

Assuming that the solution $n(x,t)$ of (3.50), as a function of x, can be represented as a convergent Fourier series, one finds that this solution can be written as

$$n(x,t) = \sum_{k=1}^{\infty} n_k \exp\left(\left(1 - \frac{k^2\pi^2}{\ell^2}\right)t\right) \sin\frac{k\pi}{\ell}\left(x + \tfrac{1}{2}\ell\right). \tag{3.51}$$

If $\ell < \pi$, then, for all positive integers k, $1 - k^2/\pi^2\ell^2 < 0$, and $n(x,t)$ given by (3.51) goes to zero exponentially as t tends to infinity. Therefore, for $\ell < \pi$, the strip Σ is not a refuge. If $\ell > \pi$, then, for any initial population density, Σ is a refuge since the amplitude of the first Fourier component of $n(x,t)$ grows without limit when t tends to infinity. The reduced critical size is then $\ell_c = \pi$. In terms of the original space variable, the critical size is $\sqrt{D/a}\,\pi$.

Remark 5. If, for $-\frac{1}{2}\ell \le x \le \frac{1}{2}\ell$, the initial density $n(x,0)$ is bounded by M, then, for all $t > 0$ and all $|x| \le \frac{1}{2}\ell$,

$$n(x,t) \le \frac{4M}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(\left(1 - \frac{(2k+1)^2\pi^2}{\ell^2}\right)t\right) \times \sin\frac{(2k+1)\pi}{\ell}\left(x + \frac{1}{2}\ell\right). \tag{3.52}$$

If the growth of the population density obeys the reduced logistic equation $\dot{n} = n(1-n)$, Ludwig, Aronson, and Weinberg [216] have shown that it grows less rapidly than if it were Malthusian and, therefore, goes to zero, as $t \to \infty$, if $\ell < \pi$. If $\ell > \pi$, then, as $t \to \infty$, $n(x,t)$ tends to the solution of $u'' + u(1-u) = 0$, satisfying the condition $u(\pm\ell/2) = 0$. In their paper, Ludwig, Aronson, and Weinberg apply their method to the Ludwig-Jones-Holling model of spruce budworm outbreak (Equation (1.4)). They show that, for this system, there exist two critical strip widths. The smaller one gives a lower bound for the strip width that can support a nonzero population. The larger one is the lower bound for the strip width that can support an outbreak.

3.6.4 Diffusion-induced instability

Since diffusion tends to mix the individuals, it seems reasonable to expect that a system eventually evolves to a homogeneous state. Thus, diffusion should be a stabilizing factor. This is not always the case, and the *Turing effect* [334], or *diffusion-induced instability*, is an important exception. Turing's paper, which shows that the reaction (interaction) and diffusion of chemicals can give rise to a spatial structure, suggests that this instability could be a key factor in the formation of biological patterns.[55] Twenty years later, Segel and Jackson [311] showed that diffusive instabilities could also appear in an ecological context.

Consider two populations N_1 and N_2 evolving according to the following one-dimensional diffusion equations[56]:

$$\frac{\partial N_1}{\partial t} = f_1(N_1, N_2) + D_1\frac{\partial^2 N_1}{\partial x^2}, \tag{3.53}$$

$$\frac{\partial N_2}{\partial t} = f_2(N_1, N_2) + D_2\frac{\partial^2 N_2}{\partial x^2}. \tag{3.54}$$

$f_1(N_1, N_2)$ and $f_2(N_1, N_2)$ denote the interaction terms, D_1 and D_2 are the diffusion coefficients, and x is the spatial coordinate. We assume the existence of an asymptotically stable steady state (N_1^*, N_2^*) in the absence of diffusion. To examine the stability of this uniform solution to perturbations, we write

$$N_1(t,x) = N_1^* + n_1(t,x) \quad \text{and} \quad N_2(t,x) = N_2^* + n_2(t,x). \tag{3.55}$$

If $n_1(t,x)$ and $n_2(t,x)$ are small, we can linearize the equations obtained upon substituting (3.55) in Equations (3.53) and (3.54). We obtain[57]

$$\frac{\partial n_1}{\partial t} = a_{11}n_1 + a_{12}n_2 + D_1\frac{\partial^2 n_1}{\partial x^2},$$

$$\frac{\partial n_2}{\partial t} = a_{21}n_1 + a_{22}n_2 + D_2\frac{\partial^2 n_2}{\partial x^2},$$

where the constants a_{ij}, for $i = 1, 2$ and $j = 1, 2$, are given by

[55] A rich variety of models is discussed in Murray [255].

[56] Extension to the more realistic two-dimensional case is straightforward.

[57] N_1^* and N_2^* are such that $f_1(N_1^*, N_2^*) = 0$ and $f_2(N_1^*, N_2^*) = 0$.

$$a_{ij} = \frac{\partial f_i}{\partial N_j}(N_1^*, N_2^*).$$

Solving the system of linear partial differential equations above is a standard application of Fourier transform theory.[58] Let

$$n_1(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{n_1}(t,k)e^{-ikx}\,dk, \tag{3.56}$$

$$n_2(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{n_2}(t,k)e^{-ikx}\,dx. \tag{3.57}$$

Replacing (3.56) and (3.57) in the system of linear partial differential equations yields the following system of ordinary linear differential equations

$$\frac{d\widehat{n_1}}{dt} = a_{11}\widehat{n_1} + a_{12}\widehat{n_2} - D_1k^2\widehat{n_1},$$
$$\frac{d\widehat{n_2}}{dt} = a_{21}\widehat{n_1} + a_{22}\widehat{n_2} - D_2k^2\widehat{n_2}.$$

This system has solutions of the form

$$\widehat{n_1}(t,k) = \widehat{n_{01}}(k)e^{\lambda t},$$
$$\widehat{n_2}(t,k) = \widehat{n_{02}}(k)e^{\lambda t},$$

where λ is an eigenvalue of the 2×2 matrix

$$\begin{bmatrix} a_{11} - D_1k^2 & a_{12} \\ a_{21} & a_{22} - D_2k^2 \end{bmatrix}.$$

For a diffusive instability to set in, at least one of the conditions

$$a_{11} - D_1k^2 + a_{22} - D_2k^2 < 0, \tag{3.58}$$

$$(a_{11} - D_1k^2)(a_{22} - D_2k^2) - a_{12}a_{21} > 0, \tag{3.59}$$

should be violated. Since (N_1^*, N_2^*) is asymptotically stable, the conditions

$$a_{11} + a_{22} < 0, \tag{3.60}$$

$$a_{11}a_{22} - a_{12}a_{21} > 0, \tag{3.61}$$

are satisfied. From Condition (3.60) it follows that (3.58) is always satisfied. Therefore, an instability can occur if, and only if, Condition (3.59) is violated. If $D_1 = D_2 = D$, the left-hand side of (3.59) becomes

$$a_{11}a_{22} - a_{12}a_{21} - Dk^2(a_{11} + a_{22}) + D^2k^4.$$

[58] For discussion of Fourier transform theory and its applications to differential equations, see Boccara [46], Chapter 2, Section 4.

In this case, (3.59) is always satisfied since it is the sum of three positive terms. Thus, *if the diffusion coefficients of the two species are equal, no diffusive instability can occur.*

If $D_1 \neq D_2$, Condition (3.59) may be written

$$H(k^2) = D_1D_2k^4 - (D_1a_{22} + D_2a_{11})k^2 + a_{11}a_{22} - a_{12}a_{21} > 0.$$

Since $D_1D_2 > 0$, the minimum of $H(k^2)$ occurs at $k^2 = k_m^2$, where

$$k_m^2 = \frac{D_1a_{22} + D_2a_{11}}{2D_1D_2} > 0.$$

The condition $H(k_m^2) < 0$, which is equivalent to

$$a_{11}a_{22} - a_{12}a_{21} - \frac{(D_1a_{22} + D_2a_{11})^2}{4D_1D_2} < 0,$$

is a sufficient condition for instability. This criterion may also be written

$$\frac{D_1a_{22} + D_2a_{11}}{(D_1D_2)^{1/2}} > 2(a_{11}a_{22} - a_{12}a_{21})^{1/2} > 0. \tag{3.62}$$

Since the first term on the left-hand side of (3.62) is a homogeneous function of D_1 and D_2, for given interactions between the two species, the occurrence of a diffusion-induced instability depends only on the ratio of the two diffusion coefficients D_1 and D_2.

From (3.60), a_{11} and a_{22} cannot both be positive. If they are both negative, Condition (3.62) is violated, and Condition (3.59) cannot be violated by increasing k; then, necessarily,

$$a_{11}a_{22} < 0 \tag{3.63}$$

and, from (3.61),

$$a_{12}a_{21} < 0. \tag{3.64}$$

If a_{11} (resp. a_{22}) is equal to zero, then, from (3.60), a_{22} (resp. a_{11}) must be negative, and here again Condition (3.59) cannot be violated by increasing k^2. Conditions (3.63)[59] and (3.64), which are strict inequalities, are useful. *An instability can be immediately ruled out if they are not verified.*

[59] This condition also follows from the fact that the expression of k_m^2 has to be positive.

Exercises

Exercise 3.1 *In a careful experimental study of the dynamics of populations of the metazoan* Daphnia magna, *Smith [316] found that his observations did not agree with the predictions of the logistic model. Using the mass M of the population as a measure of its size, he proposed the model*

$$\dot{M} = rM\left(\frac{K-M}{K+aM}\right),$$

where r, K, and a are positive constants. Find the equilibrium points and determine their stabilities.

Exercise 3.2 *The dimensionless Lotka-Volterra equations (2.4 and 2.5) are*

$$\frac{dh}{d\tau} = \rho h(1-p),$$
$$\frac{dp}{d\tau} = -\frac{1}{\rho}p(1-h),$$

where h and p denote, respectively, the scaled prey and predator populations, and ρ is a positive parameter.

(i) Show that there exists a function $(h,p) \mapsto f(h,p)$ that is constant on each trajectory in the (h,p) plane.

(ii) Use the result above to find a Lyapunov function defined in a neighborhood of the equilibrium point $(1,1)$.

Exercise 3.3 *In order to develop a strategy for harvesting[60] a renewable resource, say fish, consider the equation*

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H(N),$$

which is the usual logistic population model with an increase of mortality rate as a result of harvesting. $H(N)$ represents the harvesting yield per unit time.

(i) Assuming $H(N) = CN$, where C is the intrinsic catch rate, find the equilibrium population N^*, and determine the maximum yield.

(ii) If, as an alternative strategy, we consider harvesting with a constant yield $H(N) = H_0$, the model is

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H_0.$$

Determine the stable equilibrium point, and show that when H_0 approaches $\frac{1}{4}rK$ from below, there is a risk for the harvested species to become extinct.

Exercise 3.4 *Assume that the system*

$$\dot{N}_1 = r_1N_1\left(1-\frac{N_1}{K_1}\right) - \lambda_1N_1N_2 - CN_1, \quad \dot{N}_2 = r_2N_2\left(1-\frac{N_2}{K_2}\right) - \lambda_2N_1N_2,$$

[60] For the economics of the sustainable use of biological resources, see Clark [89].

is an acceptable model of two competing fish species in which species 1 is subject to harvesting. For certain values of the parameters, we have seen in Example 18 that for $C = 0$ (no harvesting), this system has an unstable equilibrium point for nonzero values of both populations. In this particular case, what happens when the catching rate C increases from zero?

Exercise 3.5 *The competition between two species for the same resource is described by the two-dimensional system*

$$\dot{N}_1 = N_1f_1(N_1,N_2), \quad \dot{N}_2 = N_2f_2(N_1,N_2),$$

where f_1 and f_2 are differentiable functions.

(i) Show that the slope of the null clines is negative.

(ii) Assuming that there exists only one nontrivial equilibrium point (N_1^*, N_2^*), find the condition under which this equilibrium is asymptotically stable.

Exercise 3.6 *In experiments performed on two species of fruit flies (*Drosophila pseudoobsura *and* D. willistini*), Ayala, Gilpin, and Ehrenfeld [20] tested 10 different models of interspecific competition, including the Lotka-Volterra model, presented in Example 18, as a special case. They found that the model that gave the best fit was the system*

$$\dot{N}_1 = r_1N_1\left(1-\left(\frac{N_1}{K_1}\right)^{\theta_1} - a_{12}\frac{N_2}{K_1}\right), \quad \dot{N}_2 = r_2N_2\left(1-\left(\frac{N_2}{K_2}\right)^{\theta_2} - a_{21}\frac{N_1}{K_2}\right),$$

where r_1, r_2, K_1, K_2, θ_1, θ_2, a_{12}, and a_{21} are positive constants. Under which condition does this model exhibit an asymptotically stable nontrivial equilibrium point?

Exercise 3.7 *Consider the two-dimensional system*

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + x_2(1-3x_1^2-2x_2^2),$$

which describes a perturbed harmonic oscillator. Use the Poincaré-Bendixson theorem to prove the existence of a limit cycle.
Hint: Using polar coordinates, show that there exists an invariant bounded set

$$\{(x_1,x_2)\in\mathbb{R}^2 \mid 0 < r_1 < x_1^2+x_2^2 < r_2\},$$

that does not contain an equilibrium point.

Exercise 3.8 *The second dimensionless Ludwig-Jones-Holling equation modeling budworm outbreaks (Equation (1.6)) reads*

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right) - \frac{x^2}{1+x^2},$$

where x represents the scaled budworm density, and r and k are two positive parameters.

(i) Show that, according to the values of r and k, there exist either two or four equilibrium points and study their stabilities.

(ii) Determine analytically the domains in the (k,r)-space where this equation has either one or three positive equilibrium points. Show that the boundary between the two domains has a cusp point. Find its coordinates.

Solutions

Solution 3.1 *In terms of the dimensionless time and mass variables*

$$\tau = rt \quad \text{and} \quad m = \frac{M}{K},$$

the Smith model takes the form

$$\frac{dm}{d\tau} = m\left(\frac{1-m}{1+am}\right).$$

The parameter a is dimensionless.

The equilibrium points are $m = 0$ and $m = 1$. Since

$$\frac{d}{dm}\left(\frac{m(1-m)}{1+am}\right) = \frac{1-2m-am^2}{(1+am)^2},$$

$m = 0$ is always unstable, and $m = 1$ is asymptotically stable if $a > 0$.

Solution 3.2 *(i) Eliminating $d\tau$ between the two equations yields*

$$\frac{dh}{dp} = -\rho^2 \frac{h(1-p)}{p(1-h)}$$

or

$$\frac{1-h}{h}\,dh = -\rho^2 \frac{1-p}{p}\,dp.$$

Integrating, we find that on any trajectory the function

$$f : (h,p) \mapsto \rho^2(p - \log p) + h - \log h$$

is constant. The constant depends upon the initial values $h(0)$ and $p(0)$.

(ii) Since $f(1,1) = 1 + \rho^2$, define $V(h,p) = f(h,p) - (1+\rho^2)$. It is straightforward to verify that, in the open set $]0,\infty[\times]0,\infty[$, the function V has the following properties

$$V(1,1) = 0,$$
$$V(h,p) > 0 \quad \text{for} \quad (h,p) \neq (1,1),$$
$$\dot V(h,p) \equiv 0.$$

V is, therefore, a weak Lyapunov function, and the equilibrium point $(1,1)$ is, consequently, Lyapunov stable.

Solution 3.3 *(i) It is simpler, as usual, to define dimensionless variables in order to reduce the number of parameters. If*

$$\tau = rt, \quad n = \frac{N}{K}, \quad c = \frac{C}{r},$$

the equation becomes

$$\frac{dn}{d\tau} = n(1-n) - cn.$$

The equilibrium point n^* is the solution of the equation $1 - n - c = 0$ (i.e., $n^* = 1 - c$). This result supposes that $c < 1$ or, in terms of the original parameters, $C < r$. That is, the intrinsic catch rate C must be less than the intrinsic growth rate r for the population not to become extinct. It is easy to check that n^* is asymptotically stable. The scaled yield is $cn^* = c(1-c)$, and the maximum scaled yield, which corresponds to $c = \frac{1}{2}$, is equal to $\frac{1}{4}$, or, in terms of the original parameters, $\frac{1}{4}rK$.

(ii) If the harvesting goal is a constant yield H_0, then, with

$$\tau = rt, \quad n = \frac{N}{K}, \quad h_0 = \frac{H_0}{rK},$$

the equation becomes

$$\frac{dn}{d\tau} = n(1-n) - h_0.$$

The equilibrium points are the solutions of $n(1-n) - h_0 = 0$ (i.e., $n_1^* = \frac{1}{2}(1+\sqrt{1-4h_0})$ and $n_2^* = \frac{1}{2}(1-\sqrt{1-4h_0})$). These solutions are positive numbers if the reduced constant yield h_0 is less than $\frac{1}{4}$. n_1^* and n_2^* are, respectively, asymptotically stable and unstable. When H_0 approaches $\frac{1}{4}rK$, the reduced variable h_0 approaches $\frac{1}{4}$ and the distance $|n_1^* - n_2^*| = \sqrt{1-4h_0}$ between the two equilibrium points becomes very small. Then, as a result of a small perturbation, $n(t)$ might become less than the unstable equilibrium value n_1^* and, in that case, will tend to zero. Therefore, if H_0 is close to $\frac{1}{4}rK$, there is a risk for the harvested species to become extinct.

Solution 3.4 *In terms of the dimensionless variables*

$$\tau = \sqrt{r_1 r_2}\,t, \quad n_1 = \frac{N_1}{K_1}, \quad n_2 = \frac{N_2}{K_2},$$

$$\rho = \sqrt{\frac{r_1}{r_2}}, \quad \alpha_1 = \frac{\lambda_1 K_2}{\sqrt{r_1 r_2}}, \quad \alpha_2 = \frac{\lambda_2 K_1}{\sqrt{r_1 r_2}}, \quad c = \frac{C}{\sqrt{r_1 r_2}},$$

the equations become

$$\frac{dn_1}{d\tau} = \rho n_1(1-n_1) - \alpha_1 n_1 n_2 - cn_1, \quad \frac{dn_2}{d\tau} = \frac{1}{\rho} n_2(1-n_2) - \alpha_2 n_1 n_2.$$

If $C = 0$ (i.e., $c = 0$), referring to Example 18, the nontrivial fixed point

$$\left(\frac{\alpha_1 - \rho}{\rho(\alpha_1\alpha_2 - 1)}, \frac{\rho\alpha_2 - 1}{\alpha_1\alpha_2 - 1}\right)$$

is a saddle if

$$\alpha_1 > \rho \quad \text{and} \quad \rho\alpha_2 > 1.$$

In this case, the equilibrium points $(0,1)$ and $(1,0)$ are both asymptotically stable. Depending upon environmental conditions, either species 1 or species 2 is capable of dominating the natural system. Since, in this model, species 1 is harvested we will assume that species 1 is dominant. As a result of a nonzero c value, the asymptotically stable equilibrium

$$\left(\frac{\rho - c}{\rho}, 0\right),$$

which was equal to $(1,0)$ for $c = 0$, moves along the n_1-axis, as indicated by the arrow on the left in Figure 3.23. For a c value such that

$$\frac{\rho - c}{\rho} = \frac{1}{\rho\alpha_2}, \quad \textit{that is,} \quad c = \rho - \frac{1}{\alpha_2},$$

there is a bifurcation and, for c *slightly greater than* $\rho - 1/\alpha_2$*, as on the right in Figure 3.23, species 1 does not exist anymore. This extinction is, however, not a direct consequence of harvesting (*c *is still less than* ρ*, that is,* $C < r_1$*) but the result of the competitive interaction. For* $c < \rho - 1/\alpha_2$*, the only stable equilibrium point is* $(0,1)$*, so species 2 should become dominant. But, we assumed that, before harvesting, the state of the system was* $n_1^* = 1$ *and* $n_2^* = 0$*. In nature this might not be entirely true, a small population* n_2 *could exist in a refuge and could grow according to the equations of the model once a sufficient increase of the catching rate had changed the values of the parameters.*

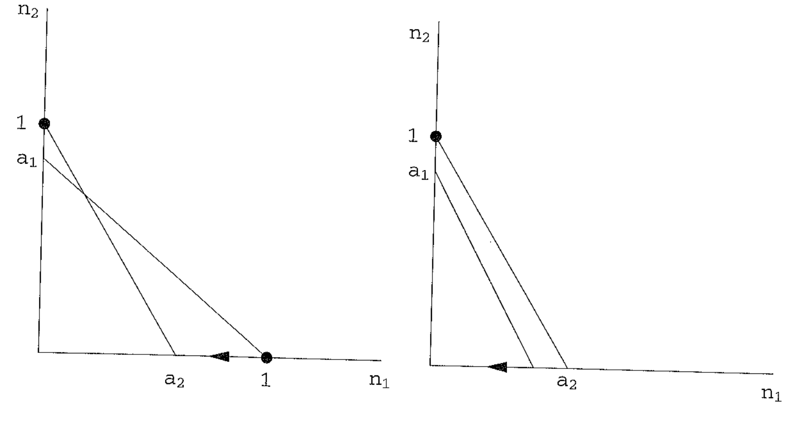

Fig. 3.23. *Modification of the null cline of species 1 as it is harvested.*

This model is a possible explanation of the elimination, in the late 1940s and early 1950s, of the Pacific sardines that have been replaced by an anchovy population.[61]

Solution 3.5 *(i) The equations of the null clines in the* (N_1, N_2)*-plane are*

$$f_1(N_1, N_2) = 0 \quad \textit{and} \quad f_2(N_1, N_2) = 0,$$

and their slopes are given by

$$\frac{dN_2}{dN_1} = -\frac{\dfrac{\partial f_1}{\partial N_1}}{\dfrac{\partial f_1}{\partial N_2}} \quad \textit{for } \dot{N}_1 = 0, \quad \textit{and} \quad \frac{dN_2}{dN_1} = -\frac{\dfrac{\partial f_2}{\partial N_1}}{\dfrac{\partial f_2}{\partial N_2}} \quad \textit{for } \dot{N}_1 = 0.$$

Since all the partial derivatives, which represent limiting effects of each species on itself or on its competitor, are negative in a competition model, both slopes are negative. That is, each population is a decreasing function of the other, as should be the case for a system of two competing species.

[61] See Clark [89].

(ii) The nontrivial equilibrium point (N_1^*, N_2^*)*, if it exists, is the unique solution, in the positive quadrant, of the system*

$$f_1(N_1, N_2) = 0, \quad f_2(N_1, N_2) = 0.$$

This equilibrium point is asymptotically stable if the eigenvalues of the Jacobian matrix

$$J = \begin{bmatrix} N_1^* \dfrac{\partial f_1}{\partial N_1}(N_1^*, N_2^*) & N_1^* \dfrac{\partial f_1}{\partial N_2}(N_1^*, N_2^*) \\ N_2^* \dfrac{\partial f_2}{\partial N_1}(N_1^*, N_2^*) & N_2^* \dfrac{\partial f_2}{\partial N_2}(N_1^*, N_2^*) \end{bmatrix}$$

have negative real parts, that is, if

$$\operatorname{tr} J < 0 \quad \textit{and} \quad \det J > 0.$$

All partial derivatives being negative, the condition on the trace is automatically satisfied. The only condition to be satisfied for the equilibrium point to be asymptotically stable is, therefore,

$$\frac{\partial f_1}{\partial N_1}(N_1^*, N_2^*)\frac{\partial f_2}{\partial N_2}(N_1^*, N_2^*) > \frac{\partial f_1}{\partial N_2}(N_1^*, N_2^*)\frac{\partial f_2}{\partial N_2}(N_1^*, N_2^*).$$

That is, a system of two competitive species exhibits a stable equilibrium if, and only if, the product of the intraspecific growth regulations is greater than the product of the interspecific growth regulations. *This result has been given without proof by Gilpin and Justice [143]. It has been established by Maynard Smith [234] assuming, as suggested by Gilpin and Justice, that, at the point of intersection of the two null clines, the slope of the null cline of the species plotted along the* x*-axis is greater than the slope of the null cline of the species plotted along the* y*-axis. This property of the null clines is a consequence of the position of the stable equilibrium point found by Ayala [19] in his experimental study of two competing species of* Drosophila.

Solution 3.6 *In terms of the dimensionless variables*

$$\tau = \sqrt{r_1 r_2}\,t, \quad \rho = \frac{r_1}{r_2}, \quad n_1 = \frac{N_1}{K_1}, \quad n_2 = \frac{N_2}{K_2}, \quad \alpha_{12} = a_{12}\frac{K_2}{K_1}, \quad \alpha_{21} = a_{21}\frac{K_1}{K_2},$$

the Ayala-Gilpin-Ehrenfeld model becomes

$$\frac{dn_1}{d\tau} = \rho\,(1 - n_1^{\theta_1} - \alpha_{12}n_2), \quad \frac{dn_1}{d\tau} = \frac{1}{\rho}\,(1 - n_2^{\theta_2} - \alpha_{21}n_1).$$

Assuming the existence of a nontrivial equilibrium point, from the general result derived in the preceding exercise, this point is asymptotically stable if the condition

$$\frac{\partial f_1}{\partial n_1}(n_1^*, n_2^*)\frac{\partial f_2}{\partial n_2}(n_1^*, n_2^*) > \frac{\partial f_1}{\partial n_2}(n_1^*, n_2^*)\frac{\partial f_2}{\partial n_2}(n_1^*, n_2^*),$$

where

$$f_1(n_1, n_2) = \rho\,(1 - n_1^{\theta_1} - \alpha_{12}n_2) \quad \textit{and} \quad f_2(n_1, n_2) = \frac{1}{\rho}\,(1 - n_2^{\theta_2} - \alpha_{21}n_1),$$

is satisfied; that is, if

$$\theta_1\theta_2(n_1^*)^{\theta_1 - 1}(n_2^*)^{\theta_2 - 1} > \alpha_{12}\alpha_{21},$$

where n_1^* *and* n_2^* *are the dimensionless coordinates of the equilibrium point.*

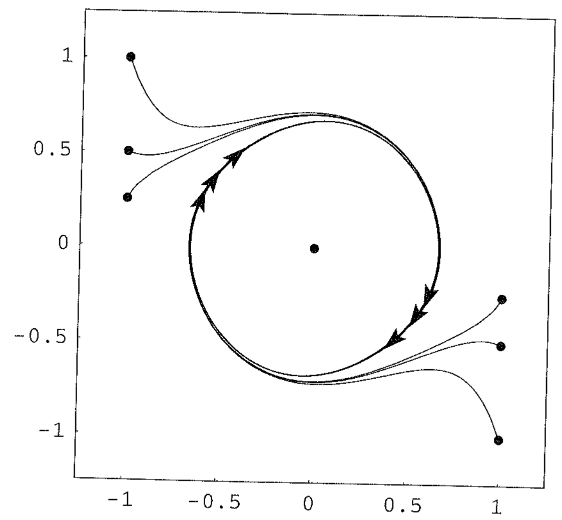

Fig. 3.24. *Phase portrait of the perturbed harmonic oscillator. Dots show initial values.*

Solution 3.7 *In polar coordinates, the equations become*

$$\dot{r} = r\sin^2\theta(1 - 2r^2 - r^2\cos^2\theta),$$
$$\dot{\theta} = -1 + \sin\theta\cos\theta(1 - 2r^2 - r^2\cos^2\theta).$$

For $r = \frac{1}{2}$, the first equation shows that

$$\dot{r} = \tfrac{1}{4}\sin^2\theta(1 - \tfrac{1}{2}\cos\theta) \geq 0$$

with equality only for $\theta = 0$ and $\theta = \pi$. The first equation also implies that

$$\dot{r} \leq r\sin^2(1 - 2r^2).$$

Hence, for $r = 1/\sqrt{2}$, $\dot{r} \leq 0$, with equality only for $\theta = 0$ and $\theta = \pi$. These two results prove that, for any point $\mathbf{x}$ in the annular domain $\left\{\mathbf{x} \in \mathbb{R}^2 \mid \frac{1}{2} < \|\mathbf{x}\| < \frac{1}{\sqrt{2}}\right\}$, the trajectory $\{\varphi_t(\mathbf{x}) \mid t > 0\}$ remains in D. Since the only equilibrium point of the planar system is the origin, which is not in D, D contains a limit cycle. The phase portrait of this system is represented in Figure 3.24.

Solution 3.8 *(i) Equilibrium points are the solutions of the equation*

$$f(x; r, k) = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2} = 0.$$

This is an algebraic equation of degree 4 that has either two or four real solutions. The graph of the function $f : x \mapsto rx(1 - x/k) - x^2/(1 + x^2)$ represented in Figure 3.25 shows that the equilibrium point $x_0 = 0$, which exists for all values of r and k, is always unstable (positive slope). Therefore, if there are only two equilibrium points, the nonzero

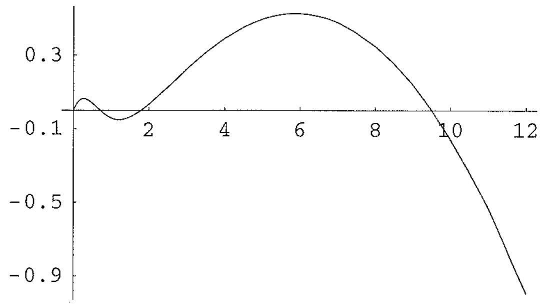

Fig. 3.25. *Typical graph of the function $x \mapsto rx(1 - x/k) - x^2/(1 + x^2)$ when there exist four equilibrium points. For $k = 12$ and $r = 0.5$, the equilibrium points are located at $x_0 = 0$, $x_\ell = 0.704$, $x_u = 1.794$, and $x_h = 9.502$.*

equilibrium point is stable, and if there exist four equilibrium points (as in Figure 3.25) there exist two stable equilibrium points x_ℓ and x_h at, respectively, low and high density separated by an unstable point x_u.

(ii) The boundary between the domains in the (k, r)-space in which there exist either one or three positive equilibrium points is determined by expressing that the equation

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2} \tag{3.65}$$

has a double root. This occurs when

$$r\frac{d}{dx}\left(1 - \frac{x}{k}\right) = \frac{d}{dx}\left(\frac{x}{1 + x^2}\right);$$

that is, if

$$-\frac{r}{k} = \frac{1 - x^2}{(1 + x^2)^2}. \tag{3.66}$$

Solving (3.65) and (3.66) for k and r, we obtain the parametric representation of the boundary of the domain in which Equation (3.65) has either one or three positive solutions (see Figure 3.26):

$$k = \frac{2x^3}{x^2 - 1},$$
$$r = \frac{2x^3}{(1 + x^2)^2}.$$

This model exhibits a cusp catastrophe. At the cusp point, the derivatives of k and r with respect to x both vanish. Its coordinates are $k = 3^{3/2}$ and $r = 3^{3/2}/8$ (i.e., $k = 5.196\ldots$ and $r = 0.650\ldots$). Note that

$$\lim_{x \to \infty} k(x) = \infty, \quad \lim_{x \to \infty} r(x) = 0,$$

and

$$\lim_{x \to 1} k(x) = \infty, \quad \lim_{x \to 1} r(x) = \frac{1}{2}.$$

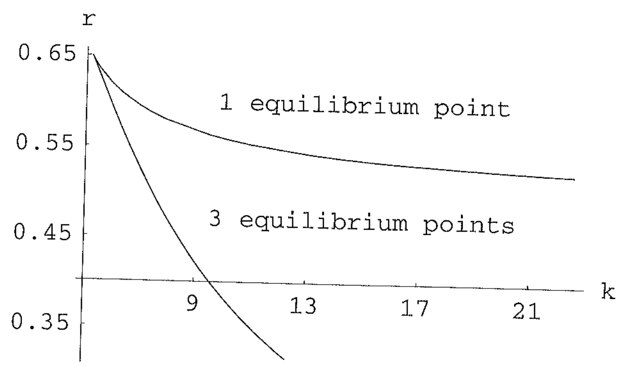

Fig. 3.26. *Boundary between domains in (k, r)-space in which there exist either one or three positive equilibrium points.*

4

Recurrence Equations

In this chapter, we study dynamical models described by recurrence equations of the form

$$\mathbf{x}_{t+1} = \mathbf{f}(\mathbf{x}_t), \tag{4.1}$$

where $\mathbf{x}$, representing the state of the system, belongs to a subset J of $\mathbb{R}^n$, $\mathbf{f} : J \to J$ is a map, and $t \in \mathbb{N}_0$. Since many notions are common to differential equations and recurrence equations, this chapter will be somewhat shorter than the preceding one.

As for differential equations, models formulated in terms of recurrence equations such as (4.1) ignore the short-range character of the interactions between the elements of a complex system.

4.1 Iteration of maps

Let $\mathbf{f}$ be a map defined on a subset J of $\mathbb{R}^n$; in what follows, for the iterates of $\mathbf{f}$, we shall adopt the notation

$$\mathbf{f}^{t+1} = \mathbf{f} \circ \mathbf{f}^t, \quad \text{where} \quad \mathbf{f}^1 = \mathbf{f}$$

and $t \in \mathbb{N}_0$ ($\mathbf{f}^0$ is the identity). If $\mathbf{f}$ is a diffeomorphism, then $\mathbf{f}^{-1}$ is defined and the definition of $\mathbf{f}^t$ above remains valid for all $t \in \mathbb{Z}$. In applications, the map $\mathbf{f}$ is seldom invertible.[1]

In the previous chapter on differential equations, a flow has been defined as a one-parameter group on the phase space. That is, if the phase space is an open set U of $\mathbb{R}^n$, for all $t \in \mathbb{R}$,[2] the flow is a mapping $\varphi_t : U \to U$

[1] In the mathematical literature, diffeomorphisms are favored. Among the different motivations, Smale [315] mentions "its natural beauty" and the fact that "problems in the qualitative theory of differential equations are present in their simplest form in the theory of diffeomorphisms."

[2] Or sometimes only for $t \in \mathbb{R}_+$. In this case, the flow is a one-parameter semi-group.