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Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates

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ABSTRACT

In this paper, the exact analytical solution of the Susceptible-Infected-Recovered (SIR) epidemic model is obtained in a parametric form. By using the exact solution we investigate some explicit models corresponding to fixed values of the parameters, and show that the numerical solution reproduces exactly the analytical solution. We also show that the generalization of the SIR model, including births and deaths, described by a nonlinear system of differential equations, can be reduced to an Abel type equation. The reduction of the complex SIR model with vital dynamics to an Abel type equation can greatly simplify the analysis of its properties. The general solution of the Abel equation is obtained by using a perturbative approach, in a power series form, and it is shown that the general solution of the SIR model with vital dynamics can be represented in an exact parametric form.

1. Introduction

The outbreak and spread of diseases has been questioned and studied for many years. In fact, Graunt was the first scientist who attempted to quantify causes of death systematically [1], and his analysis of causes of death ended up with a theory that is now well established among modern epidemiologists. Bernoulli was the first mathematician to propose a mathematical model describing an infectious disease. In 1760 he modeled the spread of smallpox [2], which was prevalent at the time, and argued the advantages of variolation [3]. A simple deterministic (compartmental) model predicting the behavior of epidemic outbreaks was formulated by McKendrick and Kermack in 1927 [4]. In their mathematical epidemic model, called the Susceptible-Infected-Recovered (SIR) model, or the *xyz* model, to describe the spread of diseases, McKendrick and Kermack proposed the following nonlinear system of ordinary differential equations [4]

$$\frac{dx}{dt} = -\beta x(t)y(t), \tag{1}$$

$$\frac{dy}{dt} = \beta x(t)y(t) - \gamma y(t) \tag{2}$$

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and

$$\frac{dz}{dt} = \gamma y(t), \tag{3}$$

respectively, with the initial conditions $x(0) = N_1 \geqslant 0,\ y(0) = N_2 \geqslant 0$ and $z(0) = N_3 \geqslant 0,\ N_i \in \Re,\ i = 1,2,3$, and where the infection rate β and the mean recovery rate γ are positive constants. In this model a fixed population with only three compartments is considered: susceptible (S) $x(t)$, infected (I) $y(t)$, and recovered (R) $z(t)$, respectively. The compartments used for this model consist of three classes:

(a) the function $x(t)$ represents the number of individuals not yet infected with disease at time t or those susceptible to the disease,
(b) the function $y(t)$ represents the number of individuals who have been infected with the disease, and are capable of spreading the disease to those in the susceptible category, and
(c) the function $z(t)$ is the compartment of the individuals who have been infected and then recovered from the disease. Those in this category are not able to be infected again, or to transmit the infection to others. For this model the initial conditions $x(0) = N_1,\ y(0) = N_2$ and $z(0) = N_3$ are not independent, since they must satisfy the condition $N_1 + N_2 + N_3 = N$, where N is the total fixed number of the individuals in the given population. The constants β and γ give the transition rates between compartments. The transition rate between S (Susceptible) and I (infected) is β, where β is the contact rate, which takes into account the probability of getting the disease in a contact between a susceptible and an infectious subject [5–8]. The transition rate between I (Infected) and R (recovered), is γ, which has the meaning of the rate of recovery or death. If the duration of the infection is denoted D, then $\gamma = 1/D$, since an individual experiences one recovery in D units of time [5–8]. Since β and γ are interpreted as transition rates (probabilities), their range is $0 \leqslant \beta \leqslant 1$ and $0 \leqslant \gamma \leqslant 1$, respectively.

In many infectious diseases, such as in the case of measles, there is an arrival of new susceptible individuals into the population. For this type of situation deaths must be included in the model. By considering a population characterized by a death rate μ and a birth rate equal to the death rate, the non-linear system of the differential equations representing this epidemic model is given by [5–8],

$$\frac{dx}{dt} = -\beta xy + \mu(N - x), \tag{4}$$

$$\frac{dy}{dt} = \beta xy - (\gamma + \mu)y, \tag{5}$$

$$\frac{dz}{dt} = \gamma y - \mu z \tag{6}$$

and it must be considered with the initial conditions $x(0) = N_1,\ y(0) = N_2$ and $z(0) = N_3$, with the constants $N_i,\ i = 1,2,3$ satisfying the condition $\sum_{i=1}^{3} N_i = N$.

The non-linear differential equations systems, given by Eqs. (1)–(3) and (4)–(6) represent modified three-dimensional competitive Lotka–Volterra type models [5,7]. These systems can also be related to the so-called T systems, introduced recently in [9], and which have the form

$$\frac{dx}{dt} = a_0(y - x), \tag{7}$$

$$\frac{dy}{dt} = (c_0 - a_0)x - a_0xz, \tag{8}$$

$$\frac{dz}{dt} = -b_0z + xy, \tag{9}$$

which is chaotic for $a_0 = 2.1,\ b_0 = 0.6$ and $c_0 = 30$. The mathematical properties of the T-system were studied in [10–12].

In recent years, the mathematical epidemic models given by Eqs. (1)–(3) and (4)–(6) were investigated numerically in a number of papers, with the use of a wide variety of methods and techniques, namely, the Adomian decomposition method [13], the variational iteration method [14], the homotopy perturbation method [15], and the differential transformation method [16], respectively. Very recently, a stochastic epidemic-type model with enhanced connectivity was analyzed in [17], and an exact solution of the model was obtained. With the use of a quantum mechanical approach the master equation was transformed via a quantum spin operator formulation. The time-dependent density of infected, recovered and susceptible populations for random initial conditions was calculated exactly. An exact solution of a particular case of the SIR and Susceptible–Infected–Susceptible (SIS) epidemic models was presented in [18]. Note that in the latter SIS model, the infected return to the susceptible class on recovery because the disease confers no immunity against reinfection. A SIR epidemic model

with nonlinear incidence rate and time delay was investigated in [19], while an age-structured SIR epidemic model with time periodic coefficients was studied in [20]. In fact, the standard pair approximation equations for the Susceptible-Infective-Recovered-Susceptible (SIRS) model of infection spread on a network of homogeneous degree k predict a thin phase of sustained oscillations for parameter values that correspond to diseases that confer long lasting immunity. Indeed, the latter SIRS model has been thoroughly studied, with the results strongly suggesting that its stochastic Markovian version does not yield sustained oscillations [21]. A stochastic model of infection dynamics based on the Susceptible-Infective-Recovered model, where the distribution of the recovery times can be tuned, interpolating between exponentially distributed recovery times, as in the standard SIR model, and recovery after a fixed infectious period, was investigated in [22]. For large populations, the spectrum of fluctuations around the deterministic limit of the model was obtained analytically.

An epidemic model with stage structure was introduced in [23], with the period of infection partitioned into the early and later stages according to the developing process of infection. The basic reproduction number of this model is determined by the method of next generation matrix. The global stability of the disease-free equilibrium and the local stability of the endemic equilibrium have been obtained, with the global stability of the endemic equilibrium is determined under the condition that the infection is not fatal. Lyapunov functions for classical SIR and SIS epidemiological models were introduced in [24], and the global stability of the endemic equilibrium states of the models were thereby established. A new Lyapunov function for a variety of SIR and SIRS models in epidemiology was introduced in [25]. Traveling wave trains in generalized two-species predator–prey models and two-component reaction–diffusion equations were considered in [26], and the stability of the fixed points of the traveling wave ordinary differential equations in the usual "spatial" variable was analyzed. For general functional forms of the nonlinear prey birthrate/prey death rate or reaction terms, a Hopf bifurcation occurs at two different critical values of the traveling wave speed. Subcritical Hopf bifurcations yield more complex post-bifurcation dynamics in the wavetrains of the system of the partial differential equations. In order to investigate the post-bifurcation dynamics all the models were integrated numerically, and chaotic regimes were characterized by computing power spectra, autocorrelation functions, and fractal dimensions, respectively.

It is the purpose of this paper to present the exact analytical solution of the SIR epidemic model. The solution is obtained in an exact parametric form. The generalization of the SIR model including births deaths, described by Eqs. (4)–(6), is also considered, and we show that the nonlinear system of differential equations governing the generalized SIR model can be reduced to an Abel type equation. The general solution of the Abel equations is obtained by using an iterative method and, once the solution of this ordinary differential equation is known, the general solution of the SIR model with vital dynamics can be obtained, similarly to the standard SIR model, in an exact parametric form.

The present paper is organized as follows. The exact solution of the SIR epidemic model is presented in Section 2. The nonlinear system of differential equations governing the SIR model with deaths is reduced to an Abel type equation, and the general solution of the model equations is obtained in an exact parametric form in Section 3. We conclude our results in Section 4.

2. The exact solution of the SIR epidemic model

By adding Eqs. (1)–(3), yields the following differential equation,

$$\frac{d}{dt}[x(t)+y(t)+z(t)]=0, \tag{10}$$

which can be immediately integrated to give

$$x(t)+y(t)+z(t)=N, \quad \forall t \geqslant 0, \tag{11}$$

where $x(t)>0,\ y(t)>0$ and $z(t)>0,\ \forall t \geqslant 0$. Hence, the total population $N=N_1+N_2+N_3$ must be an arbitrary positive integration constant. This is consistent with the model in which only a fixed population N with only three compartments is considered.

2.1. The general evolution equation for the SIR model

As a next step in our study, we differentiate Eq. (1) with respect to the time t, thus obtaining the following second order differential equation,

$$\frac{dy}{dt}=-\frac{1}{\beta}\left[\frac{x''}{x}-\left(\frac{x'}{x}\right)^2\right], \tag{12}$$

where the prime represents the derivative with respect to time t. By inserting Eqs. (1) and (12) into Eq. (2), the latter is transformed into

$$\frac{x''}{x}-\left(\frac{x'}{x}\right)^2+\gamma\frac{x'}{x}-\beta x'=0. \tag{13}$$

By eliminating y from Eqs. (1) and (3) yields

$$\frac{dz}{dt} = -\frac{\gamma}{\beta}\left(\frac{x'}{x}\right). \tag{14}$$

which can be integrated to give

$$x = x_0 e^{-\frac{\beta}{\gamma}z}, \tag{15}$$

where x_0 is a positive integration constant. By estimating Eq. (15) at $t = 0$ provides the following value for the integration constant

$$x_0 = N_1 e^{\frac{\beta}{\gamma}N_3}. \tag{16}$$

From Eq. (15), it is easy to obtain the relation

$$x' = -\frac{x_0\beta}{\gamma}z' e^{-\frac{\beta}{\gamma}z}. \tag{17}$$

Now, differentiation of Eq. (14) with respect to the time t leads to the second order differential equation

$$z'' = -\frac{\gamma}{\beta}\left[\frac{x''}{x} - \left(\frac{x'}{x}\right)^2\right]. \tag{18}$$

By inserting Eqs. (14), (17) and (18) into Eq. (13), the latter becomes the basic differential equation describing the spread of a non-fatal disease in a given population

$$z'' = x_0\beta z' e^{-\frac{\beta}{\gamma}z} - \gamma z'. \tag{19}$$

Eq. (19) is equivalent to the system of differential equations Eqs. (1)–(3), respectively.

2.2. The general solution of the evolution equation of the SIR model

In order to solve the nonlinear differential equation Eq. (19), we introduce a new function $u(t)$ defined as

$$u = e^{-\frac{\beta}{\gamma}z}. \tag{20}$$

At $t = 0$, u has the initial value

$$u(0) = u_0 = e^{-\frac{\beta}{\gamma}N_3}. \tag{21}$$

Substituting Eq. (20) into Eq. (19), we obtain the following second order differential equation for u,

$$u\frac{d^2u}{dt^2} - \left(\frac{du}{dt}\right)^2 + (\gamma - x_0\beta u)\, u\frac{du}{dt} = 0. \tag{22}$$

Next we introduce the new function ϕ, defined as

$$\phi = \frac{dt}{du}. \tag{23}$$

With the help of the transformation given by Eq. (23), Eq. (22) becomes a Bernoulli type differential equation,

$$\frac{d\phi}{du} + \frac{1}{u}\phi = (\gamma - x_0\beta u)\phi^2 \tag{24}$$

with the general solution given by

$$\phi = \frac{1}{u(C_1 - \gamma \ln u + x_0\beta u)}, \tag{25}$$

where C_1 is an arbitrary integration constant. In view of Eqs. (23) and (25), we obtain the integral representation of the time as

$$t - t_0 = \int_{u_0}^{u} \frac{d\xi}{\xi(C_1 - \gamma \ln \xi + x_0\beta\xi)}, \tag{26}$$

where t_0 is an arbitrary integration constant, and one may choose $t_0 = 0$, without loss of generality. Hence we have obtained the complete exact solution of the system of Eqs. (1)–(3), describing the SIR epidemic model, given in a parametric form by

$$x = x_0 u, \tag{27}$$

$$y = \frac{\gamma}{\beta}\ln u - x_0 u - \frac{C_1}{\beta}, \tag{28}$$

$$z = -\frac{\gamma}{\beta}\ln u \tag{29}$$

with u taken as a parameter. Now adding Eqs. (27)–(29), we obtain

$$x + y + z = -\frac{C_1}{\beta}. \tag{30}$$

Comparing Eq. (30) with Eq. (11), we have

$$C_1 = -\beta N \tag{31}$$

and hence C_1 is a negative integration constant. Eqs. (26)–(31) give the exact parametric solution of the SIR system of three differential equations, with u taken as parameter. The solution describes exactly the dynamical evolution of the SIR system for any given initial conditions $x(0) = N_1,\ y(0) = N_2$ and $z(0) = N_3$, and for arbitrary values of β and γ. The numerical values of the two constants in the solution, u_0 and C_1 are determined by the model parameters and the initial conditions. Any change in the numerical values of the initial conditions and/or of the rate parameters will not affect the validity of the solution.

In order to compare the results of the present analytical solution with the results of the numerical integration of the system of differential equations Eqs. (1)–(3) we adopt the initial values and the numerical values for the coefficients considered in [16]. Hence we take $N_1 = 20$, $N_2 = 15$, and $N_3 = 10$, respectively. For the parameter β we take the value $\beta = 0.01$, while $\gamma = 0.02$. The variations of $x(t)$, $y(t)$ and $z(t)$ obtained by both numerical integration and the use of the analytical solution are represented, as a function of time, in Fig. 1.

As one can see from the figure, the analytical solution perfectly reproduces the results of the numerical integration. The exact solution we found here is also in complete agreement with the numerical results obtained in [13–16]. The variation of $z(t)$ for different initial conditions $x(0), y(0)$ and $z(0)$ is represented in Fig. 2.

In the present Section, after a brief discussion of the general properties of the SIR model with births and deaths, we will show that the time evolution of the SIR model with equal birth and death rates can be obtained from the study of a single first order Abel type differential equation. An iterative approach for solving this equation is also presented.

3. The SIR model with equal death and birth rates

In the present Section we consider the extension of the simple SIR model given by Eqs. (1)–(3) by including equal rates of births and deaths. In this case the system of differential equations we are going to consider is given by Eqs. (4)–(6).

It is easy to see that by adding Eqs. (4)–(6), and integrating the resulting equation yields the following result

$$x(t) + y(t) + z(t) = N + N_0 e^{-\mu t} \tag{32}$$

for this model, where N_0 is an arbitrary integration constant. In order that to total number of individuals is a constant,

$$x(t) + y(t) + z(t) = N, \quad \forall t \geqslant 0, \tag{33}$$

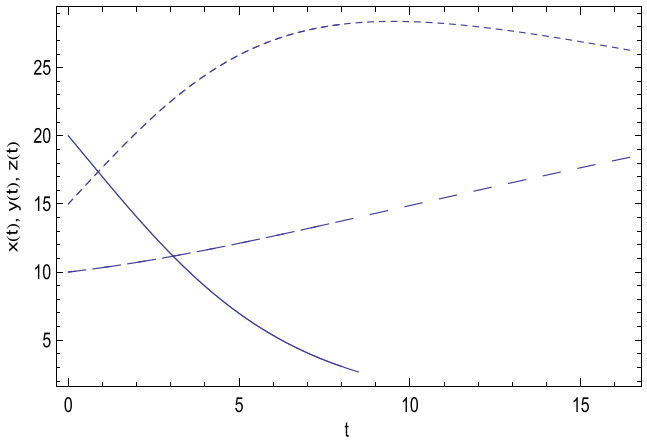

Fig. 1. Variation of $x(t)$ (solid curve), $y(t)$ (dotted curve) and $z(t)$ (dashed curve), obtained by the numerical integration of the differential equations Eqs. (1)–(3), and with the use of the analytical solution, for $\beta = 0.01$ and $\gamma = 0.02$. The initial conditions are $x(0) = N_1 = 20,\ y(0) = N_2 = 15$, and $z(0) = N_3 = 10$, respectively. The numerical and the analytical solutions completely overlap.

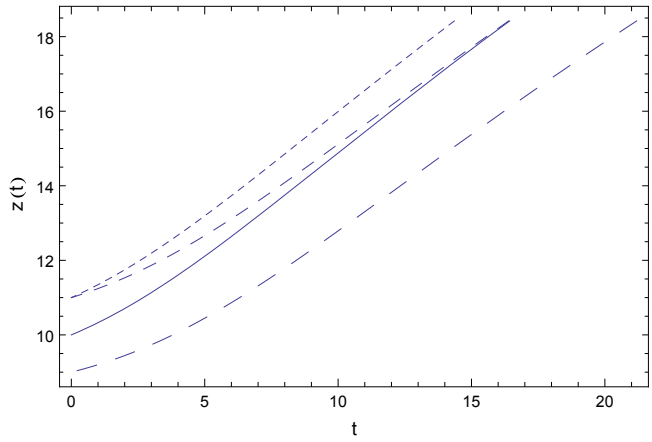

Fig. 2. Time variation of $z(t)$ obtained with the use of the analytical solution, for $\beta = 0.01$ and $\gamma = 0.02$, and for different initial conditions: $x(0) = N_1 = 20,\ y(0) = N_2 = 15$, and $z(0) = N_3 = 10$ (solid curve), $x(0) = N_1 = 19,\ y(0) = N_2 = 16$, and $z(0) = N_3 = 11$ (dotted curve), $x(0) = N_1 = 22,\ y(0) = N_2 = 11$, and $z(0) = N_3 = 11$ (dashed curve), and $x(0) = N_1 = 24,\ y(0) = N_2 = 9$, and $z(0) = N_3 = 9$ (long dashed curve), respectively.

we must fix the arbitrary integration constant N_0 as zero, $N_0 = 0$.

The variation of $x(t),\ y(t)$ and $z(t)$ for the SIR model with vital dynamics is represented in Fig. 3.

3.1. Qualitative properties of the SIR model with vital dynamics

Both the simple SIR model ($\mu = 0$) and the SIR model with vital dynamics ($\mu \neq 0$) are two dimensional dynamical systems in the $x + y + z = N$ invariant plane. There is no chaotic behavior in the plane – essentially because the existence and unicity theorem prevents (in dimension 2) the existence of transversal homoclinic points. The dynamics of both systems is simple, and well understood, including the bifurcation that takes place at $\beta N = \gamma + \mu$. There is also no chaotic behavior for $\mu \neq 0$ outside the $x + y + z = N$ invariant plane, because, as one can see from Eqs. (4)–(6), for the time evolution of $x + y + z$, trajectories with initial conditions out of the invariant plane tend exponentially fast to the invariant plane – there can be no attractor or invariant set, chaotic or otherwise, outside the invariant plane. For $\mu = 0,\ y = 0$ is a line of degenerate equilibria, for all parameter values.

For $\mu \neq 0$ and $\beta < 0,\ x = N,\ y = 0,\ z = 0$ is a global attractor – a stable node –, while for $\beta > 0$,

$$x^* = \frac{\gamma + \mu}{\beta}, \tag{34}$$

$$y^* = \frac{\mu}{\beta}\left(\frac{N\beta}{\gamma + \mu} - 1\right) \tag{35}$$

and

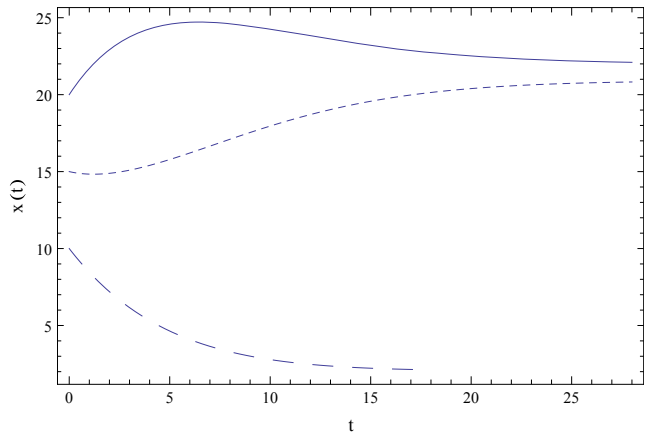

Fig. 3. Variation of $x(t)$ (solid curve), $y(t)$ (dotted curve) and $z(t)$ (dashed curve), obtained by the numerical integration of the differential equations Eqs. (4)–(6) of the SIR model with equal death and birth rates for $\beta = 0.01,\ \gamma = 0.02$, and $\mu = 0.20$. The initial conditions are $x(0) = N_1 = 20,\ y(0) = N_2 = 15$, and $z(0) = N_3 = 10$, respectively.

$$z^* = \frac{\gamma}{\beta}\left(\frac{N\beta}{\gamma+\mu} - 1\right) \tag{36}$$

is a global attractor – a stable node/focus. In the two-dimensional invariant plane $x + y + z = N$ the basic equations of the SIR model with deaths and births are

$$\frac{dx}{dt} = \mu N - \beta xy - \mu x \tag{37}$$

and

$$\frac{dy}{dt} = \beta xy - (\gamma + \mu)y, \tag{38}$$

respectively. Let x^* and y^* the equilibrium points of the system. In the following we rigorously show that the equilibrium (x^*, y^*) is globally asymptotically stable, i.e., all initial conditions with $x(0) > 0$ and $y(0) > 0$ give solutions that converge onto this equilibrium point. We will prove this result by using the Lyapounov direct method. As a first step we scale the variables by population size, so that $x \to x/N$, and $y \to y/N$, respectively. Next we introduce the function $L(x, y)$ defined as [8]

$$L(x, y) = x - x^* \ln x + y - y^* \ln y, \quad x, y \in (0, 1). \tag{39}$$

Then, with the use of Eqs. (37) and (38) it immediately follows that

$$\frac{dL(x, y)}{dt} = \nabla V(x, y) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) < 0, \quad x, y \in (0, 1). \tag{40}$$

$L(x, y)$ is therefore a Lyapunov function for the basic SIR model with vital dynamics. The existence of a Lyapounov function L ensures the global asymptotic stability of the equilibrium point (x^*, y^*) [8].

3.2. The evolution equation of the SIR models with vital dynamics

We derive now the basic differential equation describing the dynamics of the SIR model with equal birth and death rates. We differentiate Eq. (6) with respect to the time t and obtain first the second order differential equation

$$y' = \frac{1}{\gamma}(z'' + \mu z'). \tag{41}$$

By inserting Eq. (6) into Eq. (4) leads to the following differential equation,

$$x' = -\frac{\beta}{\gamma}x(z' + \mu z) + \mu(N - x). \tag{42}$$

Now, substituting Eqs. (6) and (41) into Eq. (5) yields the differential equation:

$$\beta x = \frac{z'' + \mu z'}{z' + \mu z} + \gamma + \mu. \tag{43}$$

Then, by differentiating Eq. (43) with respect to the time t gives the third order differential equation,

$$\beta x' = \frac{z''' + \mu z''}{z' + \mu z} - \left(\frac{z'' + \mu z'}{z' + \mu z}\right)^2. \tag{44}$$

Finally, by substituting Eqs. (43) and (44) into Eq. (42) gives the basic differential equation describing the SIR model with equal rate of deaths and births,

$$\frac{z''' + \mu z''}{z' + \mu z} - \left(\frac{z'' + \mu z'}{z' + \mu z}\right)^2 = -\left(\frac{z'' + \mu z'}{z' + \mu z} + \gamma + \mu\right)\left[\frac{\beta}{\gamma}(z' + \mu z) + \mu\right] + \beta\mu N. \tag{45}$$

Eq. (45) must be integrated by taking into account the initial conditions of the SIR model with equal rates of deaths and births, given by $z(0) = N_3$, $z'(0) = \gamma N_2 - \mu N_3$, and $z''(0) = \beta\gamma N_1 N_2 - \gamma(\gamma + 2\mu)N_2 + \mu^2 N_3$, respectively.

3.3. Reduction of the evolution equation for the SIR model with vital dynamics to an Abel type equation

In order to simplify Eq. (45), we introduce a set of transformations defined as

$$\psi = z' + \mu z, \tag{46}$$

$$\psi' = z'' + \mu z', \tag{47}$$

$$\psi'' = z''' + \mu z''. \tag{48}$$

By substituting Eqs. (46)–(48) into Eq. (45) leads to a second order differential equation

$$\left(\frac{\psi'}{\psi}\right)^2-\frac{\psi''}{\psi}=\frac{\beta}{\gamma}\psi'+\mu\frac{\psi'}{\psi}+\beta\left(\frac{\mu}{\gamma}+1\right)\psi+\mu(\mu+\gamma-\beta N), \tag{49}$$

which can be rewritten as

$$\left(\frac{d\psi}{dt}\right)^2-\psi\frac{d^2\psi}{dt^2}=(\mu\psi+c\psi^2)\frac{d\psi}{dt}+b\psi^2+a\psi^3, \tag{50}$$

where we have denoted

$$a=\beta\left(\frac{\mu}{\gamma}+1\right), \tag{51}$$

$$b=\mu(\mu+\gamma-\beta N) \tag{52}$$

and

$$c=\frac{\beta}{\gamma}, \tag{53}$$

respectively. The initial conditions for the integration of Eq. (50) are $\psi(0)=\gamma N_2$, and $\psi'(0)=\gamma[\beta N_1-(\gamma+\mu)]N_2$.

With the help of the transformation defined as

$$w=\frac{dt}{d\psi}=\frac{1}{\psi'}, \tag{54}$$

Eq. (50) takes the form of the standard Abel type first order differential equation of the first kind,

$$\frac{dw}{d\psi}=(a\psi^2+b\psi)w^3+(c\psi+\mu)w^2-\frac{1}{\psi}w \tag{55}$$

with the corresponding initial condition given by $w(\gamma N_2)=1/\gamma[\beta N_1-(\gamma+\mu)]N_2$.

By introducing a new function v defined as

$$v=w\psi=\frac{\psi}{\psi'}, \tag{56}$$

the Abel Eq. (55) reduces to the form

$$\frac{dv}{d\psi}=\left(a+\frac{b}{\psi}\right)v^3+\left(c+\frac{\mu}{\psi}\right)v^2, \tag{57}$$

which is equivalent to the non-linear system of Eqs. (4)–(6), and must be integrated with the initial condition $v(\gamma N_2)=1/[\beta N_1-(\gamma+\mu)]$.

The mathematical properties of the Abel type equation, and its applications, have been intensively investigated in a series of papers [27–31]. Note that when the average death rate μ is zero, in view of Eq. (52) then $b=0$, and the Abel equation (57) becomes a separate variable type differential equation of the form

$$\frac{dv}{d\psi}=av^3+cv^2. \tag{58}$$

Eq. (58) is equivalent to the system of differential equations Eqs. (1)–(3) describing the SIR epidemic model without deaths. We shall not present the simple solution of Eq. (58) here since we have already presented the complete exact solution of Eqs. (1)–(3) in Section 2.

3.4. The iterative solution of the Abel equation

By introducing a new independent variable Ψ defined as

$$\Psi=\ln v, \tag{59}$$

Eq. (57) takes the form

$$\frac{d\Psi}{d\psi}=\left(a+\frac{b}{\psi}\right)e^{2\Psi}+\left(c+\frac{\mu}{\psi}\right)e^{\Psi} \tag{60}$$

or, equivalently,

$$\frac{d\Psi}{d\psi}=\left(a+\frac{b}{\psi}\right)\left[1+2\Psi+\frac{(2\Psi)^2}{2!}+\frac{(2\Psi)^3}{3!}+\cdots\right]+\left(c+\frac{\mu}{\psi}\right)\left[1+\Psi+\frac{\Psi^2}{2!}+\frac{\Psi^3}{3!}+\cdots\right] \tag{61}$$

and must be integrated with the initial condition given by $\Psi(\gamma N_2)=-\ln|\beta N_1-(\gamma+\mu)|$.

In the limit of small Ψ, in the zero order of approximation Eq. (61) becomes a first order differential equation of the form

$$\frac{d\Psi_0}{d\psi}=\left(2a+c+\frac{2b+\mu}{\psi}\right)\Psi_0+a+c+\frac{b+\mu}{\psi}, \tag{62}$$

with the general solution given by

$$\begin{aligned}\Psi_0(\psi)&=e^{(2a+c)\psi}\psi^{2b+\mu}\left[C_0+\int e^{-(2a+c)\psi}\psi^{-(2b+\mu)}\left(a+c+\frac{b+\mu}{\psi}\right)d\psi\right]\\&=\frac{(\gamma N_2)^{-2b-\mu}e^{-(2a+c)(\gamma N_2-\psi)}}{2a+c}\left\{\psi^{2b+\mu}\left[-(bc-a\mu)e^{\gamma N_2(2a+c)}E_{2b+\mu+1}((2a+c)N_2\gamma)-(2a+c)\ln|\beta N_1-\gamma-\mu|+a+c\right]\right.\\&\left.-(\gamma N_2)^{2b+\mu}e^{(2a+c)(\gamma N_2-\psi)}\times\left[a+c-e^{\psi(2a+c)}(bc-a\mu)E_{2b+\mu+1}((2a+c)\psi)\right]\right\},\end{aligned} \tag{63}$$

where $E_n(x)$ is the exponential integral $E_n(x)=\int_1^\infty e^{-xt}/t^n dt$, and C_0 is an arbitrary constant of integration, which has been determined from the initial condition.

In order to obtain the solution of the Abel equation in the next order of approximation, we write Eq. (61) as

$$\frac{d\Psi}{d\psi}=\left(2a+c+\frac{2b+\mu}{\psi}\right)\Psi+a+c+\frac{b+\mu}{\psi}+\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi^k}{k!}. \tag{64}$$

To obtain the first order approximation Ψ_1 of the solution of the Abel equation, we substitute in Eq. (64) the non-linear terms containing Ψ with Ψ_0. Therefore the first order approximation Ψ_1 for Eq. (61) or Eq. (64) satisfies the following linear differential equation,

$$\frac{d\Psi_1}{d\psi}=\left(2a+c+\frac{2b+\mu}{\psi}\right)\Psi_1+a+c+\frac{b+\mu}{\psi}+\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_0^k}{k!} \tag{65}$$

with the general solution given by

$$\begin{aligned}\Psi_1(\psi)&=e^{(2a+c)\psi}\psi^{2b+\mu}\left\{C_0+\int e^{-(2a+c)\psi}\psi^{-(2b+\mu)}\left[a+c+\frac{b+\mu}{\psi}+\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_0^k}{k!}\right]d\psi\right\}\\&=\Psi_0(\psi)+e^{(2a+c)\psi}\psi^{2b+\mu}\int e^{-(2a+c)\psi}\psi^{-(2b+\mu)}\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_0^k}{k!}d\psi.\end{aligned} \tag{66}$$

The nth order of approximation of the Abel equation satisfies the following linear differential equation,

$$\frac{d\Psi_n}{d\psi}=\left(2a+c+\frac{2b+\mu}{\psi}\right)\Psi_n+a+c+\frac{b+\mu}{\psi}+\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_{n-1}^k}{k!} \tag{67}$$

with the general iterative solution given by

$$\begin{aligned}\Psi_n(\psi)&=e^{(2a+c)\psi}\psi^{2b+\mu}\left\{C_0+\int e^{-(2a+c)\psi}\psi^{-(2b+\mu)}\left[a+c+\frac{b+\mu}{\psi}+\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_{n-1}^k}{k!}\right]d\psi\right\}\\&=\Psi_0(\psi)+e^{(2a+c)\psi}\psi^{2b+\mu}\int e^{-(2a+c)\psi}\psi^{-(2b+\mu)}\sum_{k=2}^{\infty}\left[2^k\left(a+\frac{b}{\psi}\right)+\left(c+\frac{\mu}{\psi}\right)\right]\frac{\Psi_{n-1}^k}{k!}d\psi.\end{aligned} \tag{68}$$

Therefore the general solution of the Abel equation can be obtained as

$$\Psi(\psi)=\lim_{n\to\infty}\Psi_n(\psi). \tag{69}$$

Once the function Ψ is known, we obtain immediately $v(\psi)=e^{\Psi(\psi)}$, and $w(\psi)=e^{\Psi(\psi)}/\psi$, respectively. Therefore the time evolution of the SIR model with deaths can be obtained, as a function of the parameter ψ, as

$$t-t_0=\int w d\psi=\int\frac{e^{\Psi(\psi)}}{\psi}d\psi. \tag{70}$$

In terms of the variable ψ, Eq. (42) for x becomes

$$\frac{dx}{d\psi}=-e^{\Psi(\psi)}\left(c+\frac{\mu}{\psi}\right)x+\mu N\frac{e^{\Psi(\psi)}}{\psi} \tag{71}$$

and must be integrated with the initial condition $x(\gamma N_2)=N_1$. Eq. (71) has the general solution given by

$$x(\psi)=e^{-\int_1^\psi\frac{e^{\Psi(\xi)}(c\xi+\mu)}{\xi}d\xi}\left[\int_1^\psi\frac{\mu N e^{\Psi(\chi)+\int_1^\chi\frac{e^{\Psi(\xi)}(c\xi+\mu)}{\xi}d\xi}}{\chi}d\chi-\int_1^{\gamma N_2}\frac{\mu N e^{\Psi(\chi)+\int_1^\chi\frac{e^{\Psi(\xi)}(c\xi+\mu)}{\xi}d\xi}}{\chi}d\chi+N_1 e^{\int_1^{\gamma N_2}\frac{e^{\Psi(\xi)}(c\xi+\mu)}{\xi}d\xi}\right]. \tag{72}$$

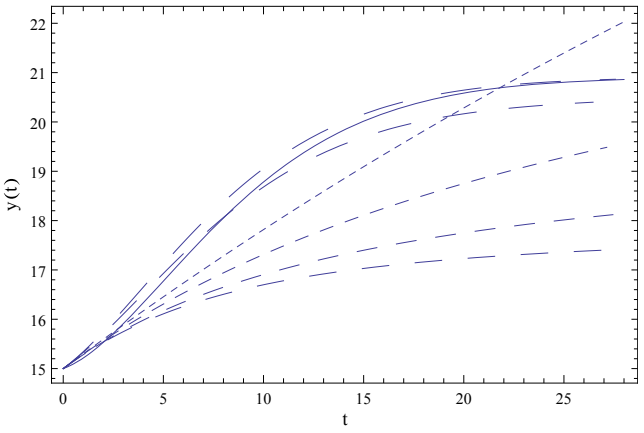

Fig. 4. Comparison of $y(t)$, obtained from the numerical integration of the equations of the SIR model with vital dynamics (solid curve), and of $y_n(t)$, obtained by iteratively solving the Abel equation (61), for different orders of approximations: $n = 1$ (dotted curve), $n = 2$ (small dashed curve), $n = 3$ (medium dashed curve), $n = 5$ (dashed curve), $n = 10$ (long dashed curve), and $n = 20$ (ultra-long dashed curve). The initial value of $y(t)$ is $y(0) = 15$.

With the use of Eq. (47), Eq. (41) for y takes the form

$$y' = \frac{1}{\gamma}\psi' \tag{73}$$

with the general solution

$$y(\psi) = \frac{1}{\gamma}\psi + Y_0, \tag{74}$$

where Y_0 is an arbitrary constant of integration. By estimating Eq. (74) at $t = 0$, corresponding to $\psi|_{t=0} = \gamma N_2$, we obtain

$$y(\gamma N_2) = y(t = 0) = N_2 = N_2 + Y_0, \tag{75}$$

a condition that fixes the integration constant Y_0 as $Y_0 = 0$.

Finally, in the new variable ψ, Eq. (46) for z takes the form

$$\psi = \frac{dz}{d\psi}\frac{d\psi}{dt} + \mu z \tag{76}$$

or, equivalently,

$$\frac{dz(\psi)}{d\psi} = -\mu\frac{e^{\Psi(\psi)}}{\psi}z(\psi) + e^{\Psi(\psi)} \tag{77}$$

with the initial condition $z(\gamma N_2) = N_3$. The general solution of Eq. (77) is provided by

$$z(\psi) = e^{-\mu\int_1^{\psi}\frac{e^{\Psi(\xi)}}{\xi}d\xi}\left[\int_1^{\psi} e^{\Psi(\chi)+\mu\int_1^{\chi}\frac{e^{\Psi(\xi)}}{\xi}d\xi}d\chi - \int_1^{\gamma N_2} e^{\Psi(\chi)+\mu\int_1^{\chi}\frac{e^{\Psi(\xi)}}{\xi}d\xi}d\chi + N_3 e^{\mu\int_1^{\gamma N_2}\frac{e^{\Psi(\xi)}}{\xi}d\xi}\right]. \tag{78}$$

Eqs. (70), (72), (74) and (78) give the general solution of the SIR model with vital dynamics, in a parametric form, with ψ taken as a parameter.

In Fig. 4 we present the comparison of the exact numerical solution for $y(t)$ with the different order approximations obtained by iteratively solving the Abel equation (61). After twenty steps the iterative and the numerical solution approximately overlap.

4. Conclusions

In the present paper we have considered two versions of the SIR model, describing the spread of an epidemic in a given population. For the SIR model without births and deaths the exact analytical solution was obtained in a parametric form. The main properties of the exact solution were investigated numerically, and it was shown that it reproduces exactly the numerical solution of the model equations.

For the SIR model with births and deaths we have shown that the non-linear system of differential equations governing it can be reduced to the Abel equation (57). This Abel equation can be easily studied by means of semi-analytical/numerical methods, thus leading to a significant simplification in the study of the model. Once the general solution of the Abel equation is known, the general solution of the SIR epidemic model with deaths can be obtained in an exact parametric form.

The exact solution is important because biologists could use it to run experiments to observe the spread of infectious diseases by introducing natural initial conditions. Through these experiments one can learn the ways on how to control the spread of epidemics.

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