Measuring causal influence with back-to-back regression: the linear case - supplementary material

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A Theorem - detailed proof

Theorem 1 (B2B consistency - general case)**.** *Consider the B2B model from Equation*

$$Y = (XE + N)F$$

N centred and full rank noise.

If F and X are full-rank on $Img(E)$, then, the solution of B2B, $\hat{H}$, will minimise

$$\min_H \|X - XH\|^2 + \|NH\|^2$$

and satisfy

$$E\hat{H} = \hat{H}$$

Proof. Let $\hat{G}$ and $\hat{H}$ be the solutions of the first and second regressions of B2B.

Since $\hat{G}$ is the least square estimator of X from Y

$$\hat{G} = \arg\min_G \mathbb{E}[\|YG - X\|^2]$$

Replacing Y by its model definition $Y = (XE + N)F$, we have

$$\hat{G} = \arg\min_G \mathbb{E}[\|X - (XE + N)FG\|^2] = \arg\min_G \mathbb{E}[\|X - XEFG + NFG\|^2]$$

Since N is centered and independent of X, we have

$$\hat{G} = \arg\min_G \|X - XEFG\|^2 + \|NFG\|^2 \tag{1}$$

Samely, for $\hat{H}$, we have

$$\begin{aligned}\hat{H} = \arg\min_H \mathbb{E}[\|XH - Y\hat{G}\|^2] &= \arg\min_H \mathbb{E}[\|XH - (XE + N)F\hat{G}\|^2] \\ &= \arg\min_H \mathbb{E}[\|X(H - EF\hat{G})\|^2] + \mathbb{E}[\|NF\hat{G}\|^2] \\ &= \arg\min_H \mathbb{E}[\|X(H - EF\hat{G})\|^2]\end{aligned}$$

a positive quantity which reaches a minimum (zero) for

$$\hat{H} = EF\hat{G} \tag{2}$$

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Let us now prove that $EF\hat{G} = F\hat{G}$.

Let $F^\dagger$ be the pseudo inverse of F, and $Z = F^\dagger EF\hat{G}$, we have $FZ = FF^\dagger EF\hat{G}$

Since F is full rank on $Img(E)$, we have $FF^\dagger E = E$, and $FZ = EF\hat{G}$

As E is a binary diagonal matrix, it is an orthogonal projection and therefore a contraction, thus

$$\|NEF\hat{G}\|^2 \leq \|NF\hat{G}\|^2$$

and

$$\|X - XEFZ\|^2 + \|NFZ\|^2 = \|X - XEF\hat{G}\|^2 + \|NEF\hat{G}\|^2 \leq \|X - XEF\hat{G}\|^2 + \|NF\hat{G}\|^2$$

But since $\hat{G} = \arg\min_G \|X - XEFG\|^2 + \|NFG\|^2$, we also have

$$\left\|X - XEF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 \leq \|X - XEFZ\|^2 + \|NFZ\|^2$$

Summarizing the above,

$$\left\|X - XEF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 \leq \|X - XEF\hat{G}\|^2 + \|NEF\hat{G}\|^2 \leq \|X - XEF\hat{G}\|^2 + \|NF\hat{G}\|^2$$

$$\left\|X - XEF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 = \|X - XEF\hat{G}\|^2 + \|NEF\hat{G}\|^2$$

$$\left\|NF\hat{G}\right\|^2 = \|NEF\hat{G}\|^2$$

N being full rank, this yields $EF\hat{G} = F\hat{G}$.

Replacing into (1), and setting $H = EFG$, we have

$$\begin{aligned}
\hat{G} &= \arg\min_G \|X - XEFG\|^2 + \|NFG\|^2 \\
&= \arg\min_G \|X - XEFG\|^2 + \|NEFG\|^2 \\
\hat{H} &= \arg\min_H \|X - XH\|^2 + \|NH\|^2
\end{aligned}$$

Finally, $E\hat{H} = EEF\hat{G} = EF\hat{G} = \hat{H}$, since E, a binary diagonal matrix, is involutive. This completes the proof. $\square$