Measuring causal influence with back-to-back regression: the linear case - supplementary material

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A Theorem - detailed proof

Theorem 1 (B2B consistency - general case)**.** *Consider the B2B model from Equation*

$$Y = (XS + N)F$$

N *centered and full rank noise.*

If F *and* X *are full-rank on* $Img(S)$*, then, the solution of B2B,* $\hat{H}$*, will minimize*

$$\min_H \|X - XH\|^2 + \|NH\|^2$$

and satisfy

$$S\hat{H} = \hat{H}$$

Proof. Let $\hat{G}$ and $\hat{H}$ be the solutions of the first and second regressions of B2B.

Since $\hat{G}$ is the least square estimator of X from Y

$$\hat{G} = \arg\min_G \mathbb{S}[\|YG - X\|^2]$$

Replacing Y by its model definition $Y = (XS + N)F$, we have

$$\hat{G} = \arg\min_G \mathbb{S}[\|X - (XS + N)FG\|^2] = \arg\min_G \mathbb{S}[\|X - XSFG + NFG\|^2]$$

Since N is centered and independent of X, we have

$$\hat{G} = \arg\min_G \|X - XSFG\|^2 + \|NFG\|^2 \tag{1}$$

Samely, for $\hat{H}$, we have

$$\begin{aligned}\hat{H} = \arg\min_H \mathbb{S}[\|XH - Y\hat{G}\|^2] &= \arg\min_H \mathbb{S}[\|XH - (XS + N)F\hat{G}\|^2] \\ &= \arg\min_H \mathbb{S}[\|X(H - SF\hat{G})\|^2] + \mathbb{S}[\|NF\hat{G}\|^2] \\ &= \arg\min_H \mathbb{S}[\|X(H - SF\hat{G})\|^2]\end{aligned}$$

a positive quantity which reaches a minimum (zero) for

$$\hat{H} = SF\hat{G} \tag{2}$$

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Let us now prove that $SF\hat{G} = F\hat{G}$.

Let $F^\dagger$ be the pseudo inverse of F, and $Z = F^\dagger SF\hat{G}$, we have $FZ = FF^\dagger SF\hat{G}$

Since F is full rank on $Img(S)$, we have $FF^\dagger S = S$, and $FZ = SF\hat{G}$

As S is a binary diagonal matrix, it is an orthogonal projection and therefore a contraction, thus

$$\|NSF\hat{G}\|^2 \leq \|NF\hat{G}\|^2$$

and

$$\|X - XSFZ\|^2 + \|NFZ\|^2 = \|X - XSF\hat{G}\|^2 + \|NSF\hat{G}\|^2 \leq \|X - XSF\hat{G}\|^2 + \|NF\hat{G}\|^2$$

But since $\hat{G} = \arg\min_G \|X - XSFG\|^2 + \|NFG\|^2$, we also have

$$\left\|X - XSF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 \leq \|X - XSFZ\|^2 + \|NFZ\|^2$$

Summarizing the above,

$$\left\|X - XSF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 \leq \|X - XSF\hat{G}\|^2 + \|NSF\hat{G}\|^2 \leq \|X - XSF\hat{G}\|^2 + \|NF\hat{G}\|^2$$

$$\left\|X - XSF\hat{G}\right\|^2 + \left\|NF\hat{G}\right\|^2 = \|X - XSF\hat{G}\|^2 + \|NSF\hat{G}\|^2$$

$$\left\|NF\hat{G}\right\|^2 = \|NSF\hat{G}\|^2$$

N being full rank, this yields $SF\hat{G} = F\hat{G}$.

Replacing into (1), and setting $H = SFG$, we have

$$\begin{aligned}
\hat{G} &= \arg\min_G \|X - XSFG\|^2 + \|NFG\|^2 \\
&= \arg\min_G \|X - XSFG\|^2 + \|NSFG\|^2 \\
\hat{H} &= \arg\min_H \|X - XH\|^2 + \|NH\|^2
\end{aligned}$$

Finally, $S\hat{H} = SSF\hat{G} = SF\hat{G} = \hat{H}$, since S, a binary diagonal matrix, is involutive. This completes the proof. $\square$