# — QUIZ 3 —

## AMATH 352:
## Applied Linear Algebra & Numerical Analysis



Due to Gradescope **February 21, 2023** at 11:59pm PST

This quiz should take $\sim$ 1 hour to complete, but you have all day to turn it in. **Follow all directions carefully**. Credit will not be given for work that does not follow directions. You are **not** allowed to work with other students or use your notes, homework assignments, MATLAB, or external sources. Please submit your work to Quiz 3 on Gradescope. Only .pdf files will be accepted.

Sally wishes you the best of luck!

Name: ________________________________________

Student ID: ____________________________________

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 5 | |

| | |
|---|---|
| Total | |

1. ( 10 points total) Answer the following true-false questions. (Think carefully!)

(a) (1 point) Given any $\mathbf{u} \in \mathbb{R}^n$, $||\mathbf{u}||_\infty \leq ||\mathbf{u}||_1$.

True ☐ False ☐

(b) (1 point) Given any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||_2 ||\mathbf{v}||_2$.

True ☐ False ☐

(c) (1 point) If a set of vectors is orthonormal, then the vectors are linearly independent.

True ☐ False ☐

(d) (1 point) There exists an orthonormal basis for every subspace of $\mathbb{R}^n$ except the trivial subspace.

True ☐ False ☐

(e) (1 point) If $\mathbf{Q}$ is a $n \times n$ orthogonal matrix, then $(\mathbf{Q}\mathbf{u}) \cdot (\mathbf{Q}\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

True ☐ False ☐

(f) (1 point) If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors in $\mathbb{R}^n$, then $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are orthogonal.

True ☐ False ☐

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For (g)-(j), consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}.$$

(g) (1 point) The columns of $\mathbf{A}$ form a basis for a 3-dimensional subspace of $\mathbb{R}^4$.

True ☐ False ☐

(h) (1 point) The matrix $\mathbf{A}^T\mathbf{A}$ is invertible.

True ☐ False ☐

(i) (1 point) The Gram-Schmidt algorithm can be applied to the columns of $\mathbf{A}$ without dividing by zero.

True ☐ False ☐

(j) (1 point) For the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, there is a unique $\mathbf{x}$ that minimizes $||\mathbf{A}\mathbf{x} - \mathbf{b}||_2$.

True ☐ False ☐

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2. ( 5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

(a) (4 points) Obtain the QR factorization of $\mathbf{A}$. Show each step of Gram-Schmidt!

$$\boxed{\mathbf{Q} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}}$$

(b) (1 point) Use your QR factorization to solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 5 & 5 \end{pmatrix}^T$.

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3. ( 5 points total) Consider the set of all vectors $\mathbf{x} = (x, y, z, w)^T \in \mathbb{R}^4$ that lie on the hyperplane

$$x - w = 0.$$

(a) (2 points) Obtain three vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ that form a basis for this hyperplane. These vectors need not be orthonormal. (Be sure to check your vectors satisfy the definition of a basis.)

$$\boxed{\mathbf{p}_1 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}}$$

(b) (3 points) Using your answer from (a), obtain vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ that form an orthonormal basis for this hyperplane.

$$\boxed{\mathbf{q}_1 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}, \quad \mathbf{q}_3 = \begin{pmatrix} \square \\ \square \\ \square \\ \square \end{pmatrix}}$$

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4. ( 5 points total) Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 3 \\ 2 \\ -\sqrt{3} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

(a) (2 points) Find $\mathbf{x}$ that solves the least-squares problem using the normal equation.

$$\boxed{\mathbf{x} = \left(\square\right)}$$

(b) (2 points) Find $\mathbf{x}$ that solves the least-squares problem using the QR equation.

$$\boxed{\mathbf{x} = \left(\square\right)}$$

(c) (1 point) Using the normal equations to solve least-squares problems is less expensive than using the QR equation to solve least-squares problems.

True ☐ False ☐

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5. ( 5 points total) The temperature on a chilly Seattle day is modeled by

$$T(t) = a \sin(\pi t) + b,$$

where the coefficients $a$ and $b$ are to be determined and $t$ is time. Suppose we have the following data:

| $t$ | $T$ |
|---|---|
| 0 | 1 |
| 1/6 | −1/2 |
| 1/2 | 0 |

(a) (1 point) Based on the data above, construct a $3 \times 2$ linear system for $a$ and $b$.

(b) (4 points) Find the least-squares solution of your system in (a).

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