Homework 8.

Amath 352
Applied Linear Algebra and Numerical Analysis

Due: 3/3/23 at 11:59pm to Gradescope

Directions:

Complete all component skills exercises as neatly as possible. Up to 2 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to some solutions of the textbook exercises and are encouraged to use them. Note that these solutions are not always correct, so double check your work just in case.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Component Skills Exercises

Exercise 1. (CS4.6)

From Olver and Shakiban, complete Exercise 8.2.1(e).

Hint: $\lambda = 1$ is a root of the characteristic polynomial.

Exercise 2. (CS4.6)

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

a. Find the eigenvalues of $\mathbf{A}$.

b. Find the corresponding eigenvectors of $\mathbf{A}$.

Exercise 3. (CS4.6)

Consider the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

a. Find the eigenvalues of $\mathbf{A}$.

b. What is the algebraic multiplicity of each eigenvalue?

c. What is the geometric multiplicity of each eigenvalue?

d. Based on (b) and (c), is $\mathbf{A}$ diagonalizable?

Exercise 4. (CS4.6)

Prove that $\mathbf{A}^T$ and $\mathbf{A}$ have the same eigenvalues with the same algebraic multiplicity.

Hint: Consider the characteristic polynomial of $\mathbf{A}^T$.

Exercise 5. (CS4.7)

Obtain a spectral decomposition of the matrix in Exercise 2.

Exercise 6. (CS4.8)

Obtain a spectral decomposition of $\mathbf{A} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ in the form $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{Q}$ is orthogonal.

Multi-Step Problem

Note: This problem requires programming in MATLAB. Please attach your MATLAB code to the end of your homework. Include your code's output!

On cold winter days, Sally the Cat spends her afternoons deciding between which of her three beds to snuggle in for some warmth and much needed rest. Today, Sally is having a particularly hard time making decisions. Every time she chooses a bed, she almost immediately gets up and heads to a different bed, creating an endless cycle of indecision.

One can imagine there is a bit of randomness to Sally's whereabouts. One minute, she's on her plush, pink bed, and the next minute she's on her feather bed with her scratching post. Rather than tracking Sally's exact movements, we can use probability to describe the likelihood of where Sally may be.

Let's denote Sally's position after she has made n decisions by a vector

$$\mathbf{x}^{(n)} = \begin{pmatrix} x_1^{(n)} & x_2^{(n)} & x_3^{(n)} \end{pmatrix}^T,$$

where $x_1^{(n)}$, $x_2^{(n)}$, and $x_3^{(n)}$ represent the probability that Sally is at her first, second, and third bed, respectively. As an example, if we knew that Sally was on her first bed before she made any decisions, then $\mathbf{x}^{(0)} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$.

When Sally makes a decision, $\mathbf{x}^{(n)} \to \mathbf{x}^{(n+1)}$, where $\mathbf{x}^{(n+1)} \neq \mathbf{x}^{(n)}$ in general, but how exactly does one obtain $\mathbf{x}^{(n+1)}$ from $\mathbf{x}^{(n)}$? Let's suppose we have the following probabilities at our disposal:

- $p_{2,1}$, the probability Sally goes to her second bed given she is on her first bed.
- $p_{3,1}$, the probability Sally goes to her third bed given she is on her first bed.
- $p_{1,2}$, the probability Sally goes to her first bed given she is on her second bed.
- $p_{3,2}$, the probability Sally goes to her third bed given she is on her second bed.
- $p_{1,3}$, the probability Sally goes to her first bed given she is on her third bed.
- $p_{2,3}$, the probability Sally goes to her second bed given she is on her third bed.

Then, the following equations hold by basic probability rules:

$$x_1^{(n+1)} = (1 - p_{2,1} - p_{3,1})x_1^{(n)} + p_{1,2}x_2^{(n)} + p_{1,3}x_3^{(n)}$$

$$x_2^{(n+1)} = p_{2,1}x_1^{(n)} + (1 - p_{1,2} - p_{3,2})x_2^{(n)} + p_{2,3}x_3^{(n)}$$

$$x_3^{(n+1)} = p_{3,1}x_1^{(n)} + p_{3,2}x_2^{(n)} + (1 - p_{1,3} - p_{2,3})x_3^{(n)}.$$

We can express these rules in matrix-vector notation

$$\mathbf{x}^{(n+1)} = \begin{pmatrix} 1 - p_{2,1} - p_{3,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & 1 - p_{1,2} - p_{3,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & 1 - p_{1,3} - p_{2,3} \end{pmatrix} \mathbf{x}^{(n)} = \mathcal{P}\mathbf{x}^{(n)}.$$

The matrix $\mathcal{P}$ is called a Markov transition matrix: multiplication by $\mathcal{P}$ transitions your state from $\mathbf{x}^{(n)}$ to $\mathbf{x}^{(n+1)}$. By iterating the equation above n times, we determine that Sally's probabilistic position vector after she makes n decisions is

$$\mathbf{x}^{(n)} = \mathcal{P}^n \mathbf{x}^{(0)}.$$

a. Verify that $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T$ is an eigenvector of $\mathcal{P}^T$. What is its corresponding eigenvalue?

   Hint: Use the definition of an eigenvector. Don't use the characteristic polynomial.

b. In light of (a), you should know one eigenvalue of $\mathcal{P}$ without calculating its characteristic polynomial. What is this eigenvalue, and how do you know that this is an eigenvalue of $\mathcal{P}$ without calculating the characteristic polynomial?

c. Suppose we are given the following probabilities:

   - $p_{2,1} = 0.2$.
   - $p_{3,1} = 0.3$.
   - $p_{1,2} = 0.2$.
   - $p_{3,2} = 0.4$.
   - $p_{1,3} = 0.5$.
   - $p_{2,3} = 0.4$.

If Sally is initially located in her first bed, what is her probabilistic position vector after she has made (i) one decision, (ii) two decisions, (iii) five decisions, and (iv) ten decisions? Use MATLAB. Round your answers to five decimal places.

d. Using the eig command in MATLAB, compute the eigenvalues of $\mathcal{P}$ given the probabilities above. Round your answers to five decimals.

   Remark: One can show that a Markov transition matrix has eigenvalues with absolute value between 0 and 1.

e. For the probabilities given above, $\mathcal{P}$ is diagonalizable, as all the eigenvalues are distinct. Thus, we have the spectral decomposition $\mathcal{P} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, where we agree to arrange the eigenvalues of $\mathcal{P}$ from greatest to least along the diagonal of $\mathbf{\Lambda}$. With the help of the eig command from MATLAB, determine $\mathbf{\Lambda}$ and $\mathbf{V}$. Round your answers to five decimal places.

f. Recall that Sally's position after n decisions is

$$\mathbf{x}^{(n)} = \mathcal{P}^n \mathbf{x}^{(0)}.$$

   Using the spectral decomposition obtained in (e), we get

$$\mathbf{x}^{(n)} = \mathbf{V}\mathbf{\Lambda}^n\mathbf{V}^{-1}\mathbf{x}^{(0)}.$$

   As seen in (d), the largest eigenvalue of $\mathcal{P}$ is 1. Hence, the first diagonal entry of Λ is 1. The remaining eigenvalues on the diagonal entries of Λ have absolute values smaller than 1. Thus,

$$\lim_{n\to\infty} \mathbf{\Lambda}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Therefore,

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{V}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{V}^{-1}\mathbf{x}^{(0)}.$$

   Using the formula above and your results in (e), calculate $\lim_{n\to\infty} \mathbf{x}^{(n)}$ in MATLAB assuming Sally is initially in her first bed. Based on this answer, which bed is Sally most likely to snuggle in after making infinitely many decisions?