Homework 4.

Amath 352
Applied Linear Algebra and Numerical Analysis

Due: 2/3/23 at 11:59pm to Gradescope

Directions:

Complete all component skills exercises as neatly as possible. Up to 2 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to some solutions of the textbook exercises and are encouraged to use them. Note that these solutions are not always correct, so double check your work just in case.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Component Skills Exercises

Exercise 1. (CS2.5)

a. Obtain the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ by Gauss-Jordan elimination. Use this inverse to solve $\mathbf{Ax} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

$$\mathrm{A} = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - \frac{3}{2}R_1$
$R_3 = R_3 - R_1$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$R_2 = 2R_2 + R_3$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$R_1 = R_1 - R_2$
$R_1 = R_1 - R_3$
$R_1 = \frac{1}{2}R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\mathbf{Ax} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rightarrow \mathbf{A}^{-1} * \mathbf{Ax} = \mathbf{A}^{-1} * \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rightarrow \mathbf{x} = \mathbf{A}^{-1} * \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rightarrow \mathbf{x} = \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix}$$

b. Show by Gauss-Jordan elimination that the inverse of a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

Remark: You're going to have to consider the case when $a = 0$ separately.

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - \frac{c}{a}R_2$ here if $a = 0$ this does not work see later part in question

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right)$$

$R_2 = aR_2$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right)$$

$R_1 = R_1 - \frac{b}{ad-bc}R_2$

$$\left(\begin{array}{cc|cc} a & 0 & 1 + \frac{b(c)}{ad-bc} & -\frac{ba}{ad-bc} \\ 0 & ad - bc & -c & a \end{array}\right)$$

$R_1 = (ad - bc)R_1$

$$\left(\begin{array}{cc|cc} a(ad-bc) & 0 & ad - bc + bc & -ba \\ 0 & ad - bc & -c & a \end{array}\right)$$

$R_1 = \frac{1}{a}R_1$

$$\left(\begin{array}{cc|cc} ad - bc & 0 & d & -b \\ 0 & ad - bc & -c & a \end{array}\right)$$

Take out $ad - bc$

$$\frac{1}{ad-bc}\left(\begin{array}{cc|cc} 1 & 0 & d & -b \\ 0 & 1 & -c & a \end{array}\right)$$

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $a = 0$ but $b \neq 0$ and $c \neq 0$

$$\left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$R_2 \longleftrightarrow R_1$

$$\left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$

$R_1 = R_1 - \frac{d}{b}R_2$

$$\left(\begin{array}{cc|cc} c & 0 & -\frac{d}{b} & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$

$R_1 = \frac{1}{c}R_1$

$R_2 = \frac{1}{b}R_1$

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{b}{bc} \\ 0 & 1 & \frac{c}{bc} & 0 \end{array}\right)$$

Take out $\frac{1}{bc}$

$$\left(\begin{array}{cc|cc} 1 & 0 & -d & b \\ 0 & 1 & c & 0 \end{array}\right)\frac{1}{bc}$$

If $a = 0$ but $b \neq 0$ and $c \neq 0$ then

$$A^{-1} = -\frac{1}{bc}\begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix}$$

Exercise 2. (CS3.1)

The set $\mathbb{R}^n$ together with the usual vector addition and scalar multiplication operations is one of the most important examples of a vector space, but there are other examples! Complete Exercise 2.1.2 in Olver and Shakiban. Be sure to show that this set, together with its vector addition and scalar multiplication operations, satisfy all eight properties of a vector space.

2.1.2. Show that the positive quadrant $Q = \{(x,y)|x,y > 0\} \subset \mathbb{R}^2$ forms a vector space if we define addition by $(x_1,y_1) + (x_2,y_2) = (x_1x_2, y_1y_2)$ and scalar multiplication by $c(x,y) = (x^c, y^c)$.

Definition 2.1. A *vector space* is a set V equipped with two operations:
(i) *Addition*: adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$;
(ii) *Scalar Multiplication*: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$.
These are subject to the following axioms, valid for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in \mathbb{R}$:
(a) *Commutativity of Addition*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
$(x_1,y_1) + (x_2,y_2) = (x_1x_2, y_1y_2)$
$(x_2,y_2) + (x_1,y_1) = (x_2x_1, y_2y_1) == (x_1x_2, y_1y_2)$ satisfied
(b) *Associativity of Addition*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
$(x_3,y_3) + ((x_1,y_1) + (x_2,y_2)) = (x_3,y_3) + (x_1x_2, y_1y_2) = (x_3x_2x_1, y_3y_2y_1)$
$((x_3,y_3) + (x_1,y_1)) + (x_2,y_2) = (x_3x_2, y_3y_2) + (x_1,y_1) = (x_1x_2x_3, y_1y_2y_3) == (x_3x_2x_1, y_3y_2y_1)$ satisfied
(c) *Additive Identity*: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.
For vector $\mathbf{0} = (1,1)$ Since $x \neq 0, y \neq 0$
Then $(1,1) + (x,y) = (x * 1, y * 1) = (x,y) = (1 * x, 1 * y)$ satisfied
(d) *Additive Inverse*: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.
So $-\mathbf{v} = \mathbf{v}^{-1}$ (Multiplication Rule) If $\mathbf{0} = (1,1)$ then $x_1 + x_1^{-1} = 1$ (Addition rule) which means $(x_1,y_1) + (x_1^{-1}, y_1^{-1}) = (x_1 * x_1^{-1}, y_1 * y_1^{-1}) = (1,1) = (x_1^{-1} * x_1, y_1^{-1} * y_1)$ satisfied
(e) *Distributivity*: $(c+d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, and $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
$(c+d)(x,y) = (x^{c+d}, y^{c+d})$
$(d+c)(x,y) = (x^{d+c}, y^{d+c}) == (x^{c+d}, y^{c+d})$ satisfied
(f) *Associativity of Scalar Multiplication*: $c(d\mathbf{v}) = (cd)\mathbf{v}$.
$c(d(x)) = c(x^d) = (x^d)^c) = x^{dc}$
$d(c(x)) = d(x^c) = (x^c)^d) = x^{cd} == x^{dc}$ satisfied
(g) *Unit for Scalar Multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.
For scalar $c = 1$ then $c(x,y) = 1(x,y) = (x^1, y^1) = (x,y)$ satisfied

Exercise 3. (CS3.1)

From Olver and Shakiban, complete Exercise 2.2.2(a)-(e).

Hint: For 2.2.2(c), think how one could write this single vector as a linear combination of two vectors that don't depend on r or s.

2.2.2 Which of the following are subspaces of $\mathbb{R}^3$? Justify your answers!
2.2.2 (a) The set of all vectors $(x, y, z)^T$ satisfying $x + y + z + 1 = 0$.

> Not a subspace, does not contain zero vector $(0, 0, 0)^T$ as $x + y + z = -1$.
> A subspace needs the zero vector

2.2.2 (b) The set of vectors of the form $(t, -t, 0)^T$ for $t \in R$

> Yes it is a subspace, it is closed under linear combinations as it is equivalent to $t * (1, -1, 0)^T$ which is a line through the origin which is a subspace.

2.2.2 (c) The set of vectors of the form $(r - s, r + 2s, -s)^T$ for $r, s \in R$.

> Yes it is a subspace it is closed under linear combinations as it is equivalent to $r * (-1, 2, -1)^T + s * (1, 1, 0)^T$ which is a two vectors composing a plane through the origin which is a subspace.

2.2.2 (d) The set of vectors whose first component equals 0.

> Yes it is a subspace it is closed under linear combinations as it is equivalent to $c * (0, 1, 0)^T + d * (0, 0, 1)^T$ which is a two vectors composing a plane through the origin which is a subspace.
> For $c, d \in \mathbb{R}$

2.2.2 (e) The set of vectors whose last component equals 1.

> Not a subspace, does not contain zero vector.

Exercise 4. (CS3.1)

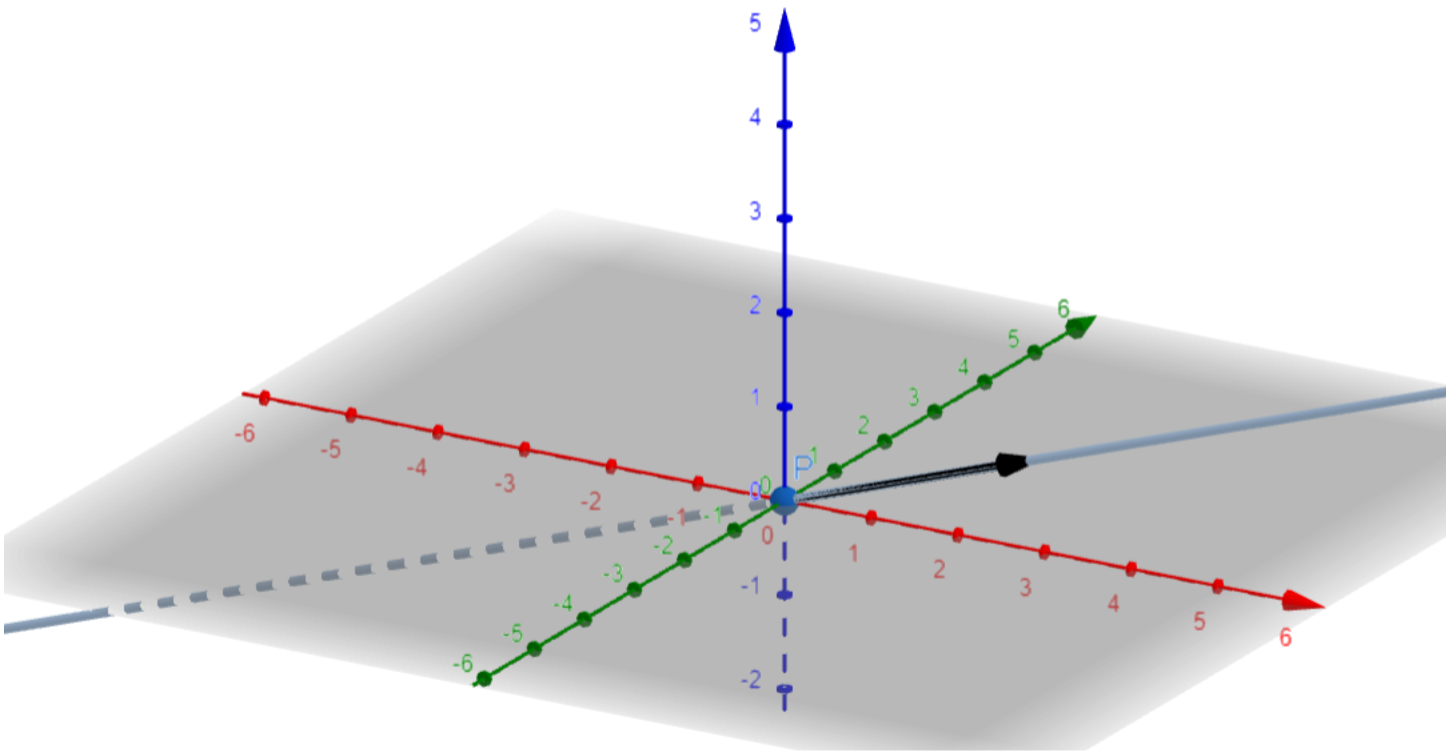
From Olver and Shakiban, complete Exercise 2.3.5(a)-(c).

Note: Feel free to use plotting software of your choice. I recommend GeoGebra. You can also try sketching these subspaces by hand, but that's tricky business.

2.3.5(a) Graph the subspace of $\mathbb{R}^3$ spanned by the vector $\mathbf{v}_1 = (3, 0, 1)^T$

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

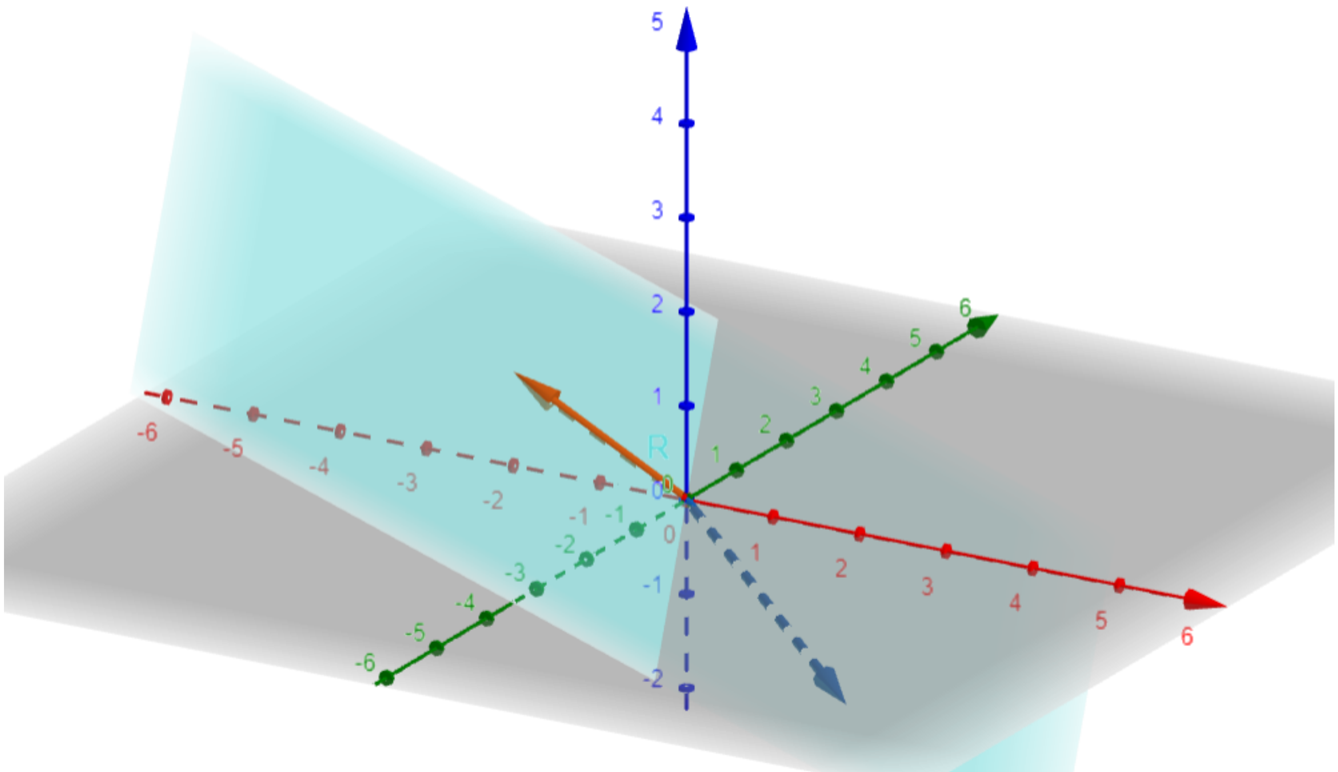
is a space of $\mathbb{R}^1$ and is therefore a line



2.3.5(b) Graph the subspace spanned by the vectors $\mathbf{v}_1 = (3, -2, -1)^T, \mathbf{v}_2 = (-2, 0, -1)^T$

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$$

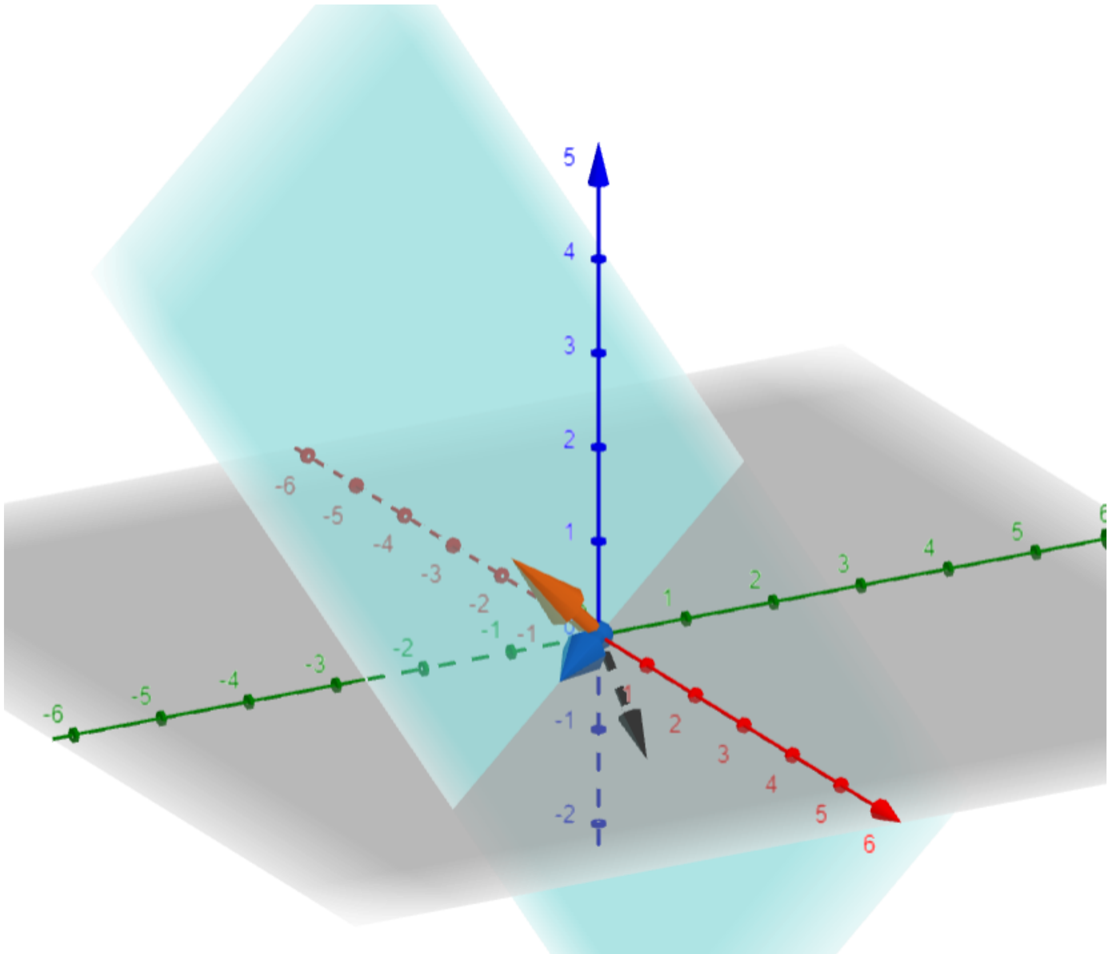
which are two linearly independent vectors and a space of $\mathbb{R}^2$ as a plane



2.3.5(c) Graph the span of $\mathbf{v}_1 = (1,0,-1)^T, \mathbf{v}_2 = (0,-1,1)^T, \mathbf{v}_3 = (1,-1,0)^T$

$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

This has two linearly independent vectors and one that is linearly dependent. So this set of vectors spans $\mathbb{R}^2$ as a plane


5
4
3
2
1
-1
-2
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6

Exercise 5. (CS3.2)

From Olver and Shakiban, complete Exercise 2.3.3(a)-(b).

2.3.3(a) Determine whether $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ is in the span of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

It is a linear combination of the other two vectors so it is in the span

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3 * \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

2.3.3(b) Is $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$?

Can be rewritten as:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & -2 & 3 & -2 \\ 2 & 0 & 4 & -1 \end{array}\right)$$

Gaussian Elimination!

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 1 & -0.4 \end{array}\right)$$

Since it can be solved by Gaussian elimination $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ is in the span of the other vectors.

2.3.3(c) Is $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$?

Can be rewritten as:
$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -2 \end{array}\right)$$
Gaussian Elimination!
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{19}{13} \\ 0 & 1 & 0 & \frac{2}{13} \\ 0 & 0 & 1 & \frac{1}{13} \\ 0 & 0 & 0 & -\frac{46}{13} \end{array}\right)$$

So $-\frac{46}{13} \neq 0$ So the solution does not exist so $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix}$ is not in the span of the other vectors.

Exercise 6. (CS3.2)

From Olver and Shakiban, complete Exercise 2.4.2(b)-(d).
2.4.2 Determine which of the following are bases of $\mathbb{R}^3$:

2.4.2 (b) $\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$?

$$\left(\begin{array}{cc|c} 0 & 1 & -5 \\ -1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right)$$
Gaussian Elimination!
$$\left(\begin{array}{cc|c} 1 & 0 & -18 \\ 0 & 1 & 5 \\ 0 & 0 & 90 \end{array}\right)$$
No solution to the system, so none of the vectors are linear combinations of the others. Thus the set of vectors is a bases for $\mathbb{R}^3$.

2.4.2 (c) $\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -8 \\ 1 \end{pmatrix}$?

$$-2 * \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} + -1 * \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ -8+0 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 1 \end{pmatrix}$$
The third vector is linear dependent on the other two vectors thus this set of vectors are not a bases for $\mathbb{R}^3$

2.4.2 (d) $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$?

Can be rewritten as:
$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ 0 & 2 & -1 & 2 \\ -2 & -1 & 0 & 1 \end{array}\right)$$
Gaussian Elimination!
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$$
Can be expressed as linear combination of other vectors.
$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -2 & -1 & 0 \end{array}\right)$$
Gaussian Elimination!
$$\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -1 \end{array}\right)$$
$0 \neq -1$ So three of the vectors are linearly independent with one linearly dependent. Thus these vectors have a dimension of three and are thus a bases for $\mathbb{R}^3$.

Exercise 7. (CS3.2)

From Olver and Shakiban, complete Exercise 2.4.5(a),(c). What is the dimension of each of these planar subspaces?

2.4.5 Find a basis for (a) the plane given by the equation $z - 2y = 0$ in $\mathbb{R}^3$

Plane Which is two dimensions. My bases vectors would be
$\mathbf{v}_1 = x * \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = y * \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

2.4.5 Find a basis for (c) the hyperplane $x + 2y + z - w = 0$ in $\mathbb{R}^4$

Hyperplane plane Which is Three dimensions. My bases vectors would be
$\mathbf{v}_1 = x * \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\mathbf{v}_2 = y * \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v}_3 = z * \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

NO MULTI-STEP PROBLEM OR CODING THIS WEEK! ☺