Homework 7.

Amath 352
Applied Linear Algebra and Numerical Analysis

Due: 2/24/23 at 11:59pm to Gradescope

Directions:

Complete all component skills exercises and the multi-step problem as neatly as possible. Up to 2 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to some solutions of the textbook exercises and are encouraged to use them. Note that these solutions are not always correct, so double check your work just in case.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Component Skills Exercises

Exercise 1. (CS4.1)

In the subexercises that follow, $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}, \quad f\left(\begin{pmatrix}-1\\2\\0\end{pmatrix}\right) = \begin{pmatrix}-2\\2\end{pmatrix}, \quad f\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}.$$

What is the canonical matrix representation of f?

Hint: You need to determine $f(\mathbf{e}_1)$, $f(\mathbf{e}_2)$, and $f(\mathbf{e}_3)$, where $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are the canonical basis vectors in $\mathbb{R}^3$.

$1a + 0b - 1 = 1 \to a - c = 1, \to a = c + 1$
$1d + 0e - 1f = 1 \to d - f = 1 \to d = f + 1$
$-1a + 2b + 0c = -2 \to -1a = -2 - 2b \to a = 2 + 2b$
$-1d + 2e + 0f = 2 \to -d = 2 - 2e \to d = -2 + 2e$
$0a + 1b + 1c = 0 \to b + c = 0 \to b = -c$
$0d + 1e + 1f = 0 \to e + f = 0 \to e = -f$

$2 + 2b = a = c + 1 \quad (b = -c)$
$2 + 2(-c) = c + 1$
$c = \frac{1}{3}$
$b = -c \to b = \frac{1}{3}$
$a = c + 1$
$a = (\frac{1}{3}) + 1 \to a = \frac{4}{3}$
$f + 1 = d = -2 + 2e \quad (f = -e)$
$(-e) + 1 = -2 + 2e$
$e = 1$
$f = -e \to f = -1$
$d = f + 1$
$d = (-1) + 1 \to d = 0$
$f = \begin{pmatrix}\frac{4}{3} & -\frac{1}{3} & \frac{1}{3}\\ 0 & 1 & -1\end{pmatrix}$

Exercise 2. (CS 4.2-4.3)

Consider the following 2×2 matrices:

$$\mathbf{A} = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}.$$

In the following subexercises, interpret these matrices as the canonical representation of a linear transformation in $\mathbb{R}^2$.

a. Describe in words how each of these matrices transforms the unit square in $\mathbb{R}^2$. A picture is helpful to include, but I am specifically looking for a few words that describe geometrically what has happened to the unit square, similar to the course notes.

> **A** Stretches the y plane by two.
> **B** Reflects vectors about the line y = 0
> **C** Rotates vectors by $\frac{\pi}{4}$ radians counterclockwise
> **D** Projects vectors on to the x-axis

b. Suppose I want a linear transformation that first reflects a vector about the y axis and then stretches its y coordinate by two. Using the matrices above, what is the canonical matrix representation of this linear transformation?

> **B** then **A** Canonical representation is $\mathbf{A} * \mathbf{B}$
> $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} * \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

c. Suppose I want a linear transformation that first rotates a vector $\pi/4$ radians counter-clockwise and then reflects a vector about the y axis. Using the matrices above, what is the canonical matrix representation of this linear transformation?

> **C** then **B** Canonical representation is $\mathbf{B} * \mathbf{C}$
> $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

d. Suppose I want a linear transformation that first projects a vector onto the x axis and then rotates that vector $\pi/4$ radians counter-clockwise. Using the matrices above, what is the canonical matrix representation of this linear transformation?

> **D** then **C** Canonical representation is $\mathbf{C} * \mathbf{D}$
> $\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{pmatrix}$

e. One of these matrices above describes a linear transformation that is not invertible (meaning the linear transformation does not have an inverse to "undo" the transformation that has taken place). Which of these matrices is the canonical matrix representation of that linear transformation?

> The uninvertible matrix is matrix D. As it is the only matrix that is not full rank and therefore does not have an inverse.

Exercise 3. (CS4.4)

The *rank* of a $m \times n$ matrix $\mathbf{A}$ is the dimension of the Col($\mathbf{A}$). Similarly, the *nullity* of $\mathbf{A}$ is the dimension of the Null($\mathbf{A}$). Recall from lecture that these two dimensions are related by the rank-nullity theorem:

$$\dim(\mathrm{Null}(\mathbf{A})) + \dim(\mathrm{Col}(\mathbf{A})) = n.$$

As an example, consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a. Determine a basis for Col($\mathbf{A}$). What is the rank of $\mathbf{A}$?

b. Determine a basis for Null($\mathbf{A}$). What is the nullity of $\mathbf{A}$?

c. Verify that the rank-nullity theorem is true for $\mathbf{A}$.

(a)
Guassian Elimination
$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$ $R_2 = R_2 - 2R_1$ $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Basis for Column A $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Rank of A = 1

(b)
$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
$x_1 - x_2 + 2x_3 = 0$
$x_2 = free$
$x_3 = free$

Basis for null($\mathbf{A}$) $= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Dimension of null space aka nullity = 2

(c)
$\dim(\mathrm{Null}(\mathbf{A})) + \dim(\mathrm{Col}(\mathbf{A})) = n.$
$1 + 2 = 3$

Exercise 4. (CS4.5)

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

a. Compute $\det(\mathbf{A})$ by Laplace expansion about the third row. When computing the determinant of your 3×3 matrix, expand about the third row.

Third Row $\begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}$
$\det(\mathbf{A}) =$

$$-1 * \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} - 0 * \det \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} + 0 * \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} - 0 * \det \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Third Row $\begin{pmatrix} 2 & 0 & 0 \end{pmatrix}$

$$= -1 * (2 * \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + 0\det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + 0\det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}) + 0 + 0 + 0$$

$$= -1 * 2 * 4 = -8$$

b. Compute $\det(\mathbf{A})$ by Laplace expansion about the first column. When computing the determinant of your 3×3 matrix, expand about the second column.

First column $\begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix}^T$
$\det(\mathbf{A}) =$

$$0 * \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} + 0 * \det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} + -1 * \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} + 0 * \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Second Column $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}^T$

$$0 + 0 + -1 * (2 * \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + 0 * \det \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} + 0 * \det \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + 0)$$

$$= 0 + 0 + -1 * (2 * (4 + 0 + 0)) + 0 = -8$$

c. Compute $\det(\mathbf{A})$ by Laplace expansion about the third column. When computing the determinant of your 3×3 matrix, expand about the second row.

Third column $\begin{pmatrix}2 & 0 & 0 & 0\end{pmatrix}^T$
$\det(\mathbf{A}) =$
$2 * \det\begin{pmatrix}0 & 1 & 2\\ -1 & 0 & 0\\ 0 & 2 & 0\end{pmatrix} + 0 * \det\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 2 & 0\end{pmatrix} + 0 * \det\begin{pmatrix}0 & 1 & 0\\ 0 & 1 & 2\\ 0 & 2 & 0\end{pmatrix} + 0 * \det\begin{pmatrix}0 & -1 & 0\\ 0 & 1 & 2\\ -1 & 0 & 0\end{pmatrix}$
Second Row $\begin{pmatrix}-1 & 0 & 0\end{pmatrix}$
$2 * (+0 * \det\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} - (-1) * \det\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix} + 0 * \det\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix})0 + 0 + 0$
$= 2 * (1 * (-4 + 0 + 0)) + 0 = -8$

d. Compute $\det(\mathbf{A})$ by reducing $\mathbf{A}$ to an upper-triangular matrix.

$\mathbf{A} = \begin{pmatrix}0 & 1 & 2 & 0\\ 0 & 1 & 0 & 2\\ -1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\end{pmatrix}.$
Swap Column 1 and Column 3
$\mathbf{A} = \begin{pmatrix}2 & 1 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 0 & -1 & 0\\ 0 & 2 & 0 & 0\end{pmatrix}.$
Swap Column 2 and Column 4
$\mathbf{A} = \begin{pmatrix}2 & 0 & 0 & 1\\ 0 & 2 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}.$
A is now upper triangular
$\det(\mathbf{A}) = 2 * 2 * -1 * 2 * (-1)^2 = -8$

Exercise 5. (CS4.5)

a. Prove that $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$, provided $\mathbf{A}^{-1}$ exists.

Hint: Use the definition of $\mathbf{A}^{-1}$.

$\mathbf{A} * \mathbf{A}^{-1} = \mathbf{I}_n$
$\det(\mathbf{A}) * \det(\mathbf{A}^{-1}) = \det(\mathbf{I}_n)$
$\det(\mathbf{A}) * \det(\mathbf{A}^{-1}) = 1$
$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$

b. Prove that $\det(\mathbf{A}^T) = \det(\mathbf{A})$.

Hint: Consider the PLU decomposition of $\mathbf{A}$. Note that the $\mathbf{P}$ matrix is orthogonal.

$\mathbf{A} = \mathbf{PLU}$
$(\mathbf{A})^T = (\mathbf{PLU})^T$
$\mathbf{A}^T = \mathbf{U}^T\mathbf{L}^T\mathbf{P}^T$
$\det(\mathbf{A}^T) = \det(\mathbf{U}^T)\det(\mathbf{L}^T)\det(\mathbf{P}^T)$
Matrices $\mathbf{U}$ and $\mathbf{P}$ are still upper triangular after transposition, meaning their determinants do not change, and thus can be simplified to
$\det(\mathbf{A}^T) = \det(\mathbf{U})\det(\mathbf{L})\det(\mathbf{P}^T)$
Since the Matrix $\mathbf{P}$ is orthogonal, then $\mathbf{P}^T$ it is now a rotated version of the original $\mathbf{P}$ matrix. Rotation does not change the value of the determinant so $\mathbf{P}^T$ has the same determinant of the $\mathbf{P}$ matrix.
$\det(\mathbf{A}^T) = \det(\mathbf{U})(1)(-1)^k$
The right side is the same as the determinant of $\mathbf{A}$
$\det(\mathbf{A}^T) = \det(\mathbf{A})$

c. Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are called *similar* if there exists an $n \times n$ invertible matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{SAS}^{-1}$. Prove that $\det(\mathbf{A}) = \det(\mathbf{B})$ when $\mathbf{A}$ and $\mathbf{B}$ are similar.

$\mathbf{B} = \mathbf{SAS}^{-1}$.
$\det(\mathbf{B}) = \det(\mathbf{S})\det(\mathbf{A})\det(\mathbf{S}^{-1})$.
$\det(\mathbf{B}) = \det(\mathbf{S})\det(\mathbf{A})\frac{1}{\det(\mathbf{S})}$.
Determinants Cancel
$\det(\mathbf{B}) = \det(\mathbf{A})$

d. Prove that $\det(c\mathbf{A}) = c^n\det(\mathbf{A})$, where $c \in \mathbb{R}$ and $\mathbf{A}$ is $n \times n$.
Hint: Write $c\mathbf{A}$ as $\mathbf{CA}$, where $\mathbf{C}$ is an appropriately chosen diagonal matrix.

$c * \mathbf{I}_n = \mathbf{C}$
$\det(\mathbf{CA}) = \det(\mathbf{C}) * \det(\mathbf{A})$
n columns of $\mathbf{C}$, which is a diagonal matrix so the determinant is c^n
$\det(\mathbf{CA}) = c^n \det(\mathbf{A})$

Multi-Step Problem

In this Multi-Step Problem, we restrict to linear transformations that map $\mathbb{R}^n$ to $\mathbb{R}^n$. Typically we describe these transformations by their canonical matrix representations. In so doing we implicitly assume that vectors in $\mathbb{R}^n$ are expressed in terms of the canonical basis vectors, but you and I both know that there are many other bases we could choose for $\mathbb{R}^n$. If we choose to express our vectors with respect to a different basis, then the entries of the matrix representation of the linear transformation change[1].

As you will see in this Multi-Step Problem, if a linear transformation is described by the canonical matrix representation $\mathbf{A}$, then that same linear transformation can be equally well-described with respect to a new basis $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathbb{R}^n$ by the matrix

$$\tilde{\mathbf{A}} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B},$$

where $\mathbf{B} = (\mathbf{b}_1 | \cdots | \mathbf{b}_n)$. This formula is called the **change-of-base formula**.

To see why the change-of-base formula is true, you will watch the 3Blue1Brown YouTube video "Change of basis | Chapter 13, Essence of linear algebra," which can be found at

https://www.youtube.com/watch?v=P2LTAUO1TdA. (Click to access video.)

As some of you may know, 3Blue1Brown is one of the most insightful and charismatic mathematicians on YouTube: you are in for a treat. As you watch his video, answer the following.

a. When associating the vector $\begin{pmatrix} 3 & 2 \end{pmatrix}^T$ to an arrow in $\mathbb{R}^2$, what have we implicitly assumed?

That the first coordinate indicates to the rightward motion, the second coordinate indicates upwards motion, and the unit of distance.

b. What is a coordinate system?

Anyway to translate between vectors and a sets of numbers.

c. What vector would Jennifer's coordinate system associate to the vector $\begin{pmatrix} 3 & 2 \end{pmatrix}^T$?

$\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \end{pmatrix}$

[1] In fact, the whole reason we are studying eigenvalues and eigenvectors in this unit is to find a basis for which this matrix representation becomes a diagonal matrix.

d. In Jennifer's coordinate system, what is the meaning of the first component of her vector, the second component?

The first component is how far the $\mathbf{b_1}$ unit vector should be stretched and the second component is how far the $\mathbf{b_2}$ unit vector should be stretched.

e. Expressed in our canonical coordinate system, what vectors does Jennifer use to define her basis? In Jennifer's coordinate system, how are these basis vectors expressed?

In our system $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ In her system $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

f. Space has no grid, true or false?

True, space has no intrinsic grid.

g. The vector $\begin{pmatrix} -1 & 2 \end{pmatrix}^T$ expressed in Jennifer's coordinates is what vector expressed in our canonical coordinates?

$$-1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

h. What is the matrix that transforms a vector expressed in Jennifer's coordinates to that same vector expressed in our canonical coordinates?

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

i. What is the matrix that transforms a vector expressed in our canonical coordinates to that same vector expressed in Jennifer's coordinates?

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

j. The canonical representation of a 90° counter-clockwise rotation of $\mathbb{R}^2$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

How does Jennifer represent this linear transformation in her coordinate system?

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{5}{3} & -\frac{1}{3} \end{pmatrix}$$