Homework 1.

Amath 352
Applied Linear Algebra and Numerical Analysis

Due: 1/13/22 at 11:59pm to Gradescope

Directions:

Complete all component skills exercises and the multi-step problem as neatly as possible. Up to 2 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to some solutions of the textbook exercises and are encouraged to use them. Note that these solutions are not always correct, so double check your work just in case.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Component Skills Exercises

Exercise 1. (CS1.1)

A chemist must determine the coefficients x_j (for $1 \leq j \leq 7$) that balance the reaction

$$x_1\mathrm{PCH_4} + x_2\mathrm{KMnO_4} + x_3\mathrm{H_2SO_4} \to x_4\mathrm{PCH_2O_2} + x_5\mathrm{K_2SO_4} + x_6\mathrm{MnSO_4} + x_7\mathrm{H_2O}.$$

In order to balance this reaction, there must be an equal number of each element on both sides of the reaction. Use this information to derive a linear system satisfied by the unknown coefficients x_j. What are the dimensions of this system?

Remark: Please derive an equation for each element. Note that some of your equations may be redundant. For sake of consistency, keep these redundant equations in your system.

Phosphorous $x_1\mathrm{PCH_4} \to x_4\mathrm{PCH_20_2}$
$x_1 = x_4$
Carbon $x_1\mathrm{PCH_4} \to x_4\mathrm{PCH_20_2}$
$x_1 = x_4$
Hydrogen $x_1\mathrm{PCH_4} + x_3\mathrm{H_2SO_4} \to x_4\mathrm{PCH_2O_2} + x_7\mathrm{H_2O}.$
$4x_1 + 2x_3 = 2x_4 + 2x_7$
Potassium $x_2\mathrm{KMnO_4} \to x_5\mathrm{K_2SO_4}$
$x_2 = 2x_5$
Manganese $x_2\mathrm{KMnO_4}+ \to x_6\mathrm{MnSO_4}$
$x_2 = x_6$
Oxygen $x_2\mathrm{KMnO_4} + x_3\mathrm{H_2SO_4} \to x_4\mathrm{PCH_2O_2} + x_5\mathrm{K_2SO_4} + x_6\mathrm{MnSO_4} + x_7\mathrm{H_2O}.$
$4x_2 + 4x_3 = 2x_4 + 4x_5 + 4x_6 + x_7$
Sulfur $x_3\mathrm{H_2SO_4} \to x_5\mathrm{K_2SO_4} + x_6\mathrm{MnSO_4}$
$x_3 = x_5 + x_6$

System of Linear Equations:
$x_1 - x_4 = 0$
$x_1 - x_4 = 0$
$4x_1 + 2x_3 - 2x_4 - 2x_7 = 0$
$x_2 - 2x_5 = 0$
$x_2 - x_6 = 0$
$4x_2 + 4x_3 - 2x_4 - 4x_5 - 4x_6 - x_7 = 0$
$x_3 - x_5 - x_6 = 0$
Dimensions are 7 x 8 (7 rows by 8 columns)

Exercise 2. (CS1.2)

Consider the following triangular system:

$$\begin{cases} x + y + z & = 2 \\ a_{22}y + 2z & = b_2 \\ 3z & = 9 \end{cases}$$

a. Suppose $a_{22} = 1$ and $b_2 = 6$. What is the solution of this system?

$$\begin{cases} \text{x + y + z} & = 2 \\ a_{22}y + 2z & = b_2 \\ 3z & = 9 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ a_{22}y & = b_2 - 6 \\ z & = 3 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ (1)y & = (6) - 6 \\ z & = 3 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ y & = 0 \\ z & = 3 \end{cases}$$

Solution: x = -1 y = 0 z = 3

b. Suppose $a_{22} = 0$ and $b_2 = 6$. What is the solution of this system?

$$\begin{cases} x + y & = -1 \\ a_{22}y & = b_2 - 6 \\ 3z & = 9 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ (0)y & = (6) - 6 \\ z & = 3 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ 0 & = 0 \\ z & = 3 \end{cases}$$

Infinitely Many Solutions!

c. Suppose $a_{22} = 0$ and $b_2 = 0$. What is the solution of this system?

$$\begin{cases} x + y & = -1 \\ a_{22}y & = b_2 - 6 \\ 3z & = 9 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ (0)y & = (6) - 6 \\ z & = 3 \end{cases} \rightarrow \begin{cases} x + y & = -1 \\ 0 & = -6 \\ z & = 3 \end{cases}$$

$0 \neq 6$ No solutions!

Exercise 3. (CS1.2)

Consider the following triangular system:

$$\begin{cases} x + y + z + w & = 2 \\ -6z + 5w & = b_2 \\ 3z - w & = -1 \\ 2w & = -4 \end{cases},$$

where b_2 is a real number.

a. For what value(s) of b_2, if any at all, does this system have a unique solution? Briefly justify your answer with a short explanation or short calculation.

$$\begin{cases} x+y+z+w &= 2 \\ -6z+5w &= b_2 \\ 3z-w &= -1 \\ w &= -2 \end{cases} \rightarrow \begin{cases} x+y+z &= 4 \\ -6z &= b_2+10 \\ 3z &= -3 \\ w &= -2 \end{cases} \rightarrow \begin{cases} x+y &= 5 \\ 0 &= b_2+4 \\ z &= -1 \\ w &= -2 \end{cases}$$

For no value of b_2 does the system have a Unique Solutions since x and y cannot be solved for

b. For what value(s) of b_2, if any at all, does this system have infinitely many solutions? Briefly justify your answer with a short explanation or short calculation.

$$\begin{cases} x+y &= 5 \\ 0 &= b_2+4 \\ z &= -1 \\ w &= -2 \end{cases}$$

For the value of $b_2 = -4$
Otherwise if $b_2 \neq 4$ then $0 \neq b_2 + 4$ making the system have no solutions. The system has infinitely many solutions since the solution does exist but the variables x and y have infinite combination of solutions

c. For what value(s) of b_2, if any at all, does this system have no solution? There's no need to justify your answer to (c) as it will be the complement of your answers to (a) and (b).

For the value of $b_2 \neq -4$

Exercise 4. (CS1.2)

Consider the following systems of square linear systems:

a. $\begin{cases} x+3y-z &= 4 \\ 4x-y+2z &= 8 \\ 2x-7y+4z &= -3 \end{cases}$.

b. $\begin{cases} x+3y-z &= 4 \\ 4x-y+2z &= 8. \\ 2x-7y+4z &= 0 \end{cases}$

c. $\begin{cases} x+3y-z &= 4 \\ 4x-y+2z &= 8. \\ 2x-7y+6z &= 0 \end{cases}$

Which of these systems has no solution? (There is only one.) What are the solutions of the remaining two systems?

Solution of a:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 4 & -1 & 2 & 8 \\ 2 & -7 & 4 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & -13 & 6 & -11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

a has no solution since $0 \neq 3$

Solution of b

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 4 & -1 & 2 & 8 \\ 2 & -7 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & -13 & 6 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -\frac{6}{13} & -\frac{8}{13} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{13} & \frac{28}{13} \\ 0 & 1 & -\frac{6}{13} & -\frac{8}{13} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely Many solutions!

$x = \frac{28}{13} - \frac{5}{13}z$
$y = \frac{8}{13} + \frac{6}{13}z$
$z =$ free

Solution of c

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 4 & -1 & 2 & 8 \\ 2 & -7 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & -13 & 8 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & -13 & 0 & -8 \\ 0 & 0 & 2 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{28}{13} \\ 0 & 1 & 0 & \frac{8}{13} \\ 0 & 0 & 2 & 0 \end{array}\right]$$ Solution: $x = \frac{28}{13}, y = \frac{8}{13}, z = 0$

Exercise 5. (CS1.3)

Express the linear system in **Exercise 1** in matrix-vector form and as an augmented matrix.

System of Linear Equations:
$x_1 - x_4 = 0$
$x_1 - x_4 = 0$
$4x_1 + 2x_3 - 2x_4 - 2x_7 = 0$
$x_2 - 2x_5 = 0$
$x_2 - x_6 = 0$
$4x_2 + 4x_3 - 2x_4 - 4x_5 - 4x_6 - x_7 = 0$
$x_3 - x_5 - x_6 = 0$

Augmented Matrix Form:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & -2 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 4 & 4 & -2 & -4 & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right]$$

Matrix Vector Form:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & 2 & -2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 4 & 4 & -2 & -4 & -4 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_P \\ x_C \\ x_H \\ x_K \\ x_{Mn} \\ x_O \\ x_S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise 6. (CS1.4)

From Olver and Shakiban, complete Exercises 1.2.7(b), (c), (f), and (h). For (h), I is the appropriately sized identity matrix.

Remark: Unless very special circumstances are at play, remember $AB \neq BA$.

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & -2 \\ 3 & 0 & 6 \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix} \quad \mathrm{C} = \begin{bmatrix} 2 & 3 \\ -3 & -4 \\ 1 & 2 \end{bmatrix}$$

1.2.7 (b) compute AB

No answer since A is a 3x3 matrix and B is a 2x3 matrix so the columns of A do not match the rows of B meaning they cannot be multiplied.

1.2.7 (c) compute BA

$$\mathrm{BA} = \begin{bmatrix} -6*1+0*-1+3*3 & -6*-1+0*4+3*0 & -6*3+0*-2+3*6 \\ 4*1+2*-1+-1*3 & 4*-1+2*4+-1*0 & 4*3+2*-2+-1*6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 0 \\ -1 & 4 & 2 \end{bmatrix}$$

1.2.7 (f) compute A + 2CB

$$\mathrm{CB} = \begin{bmatrix} 2*-6+3*4 & 2*0+3*2 & 2*3+3*-1 \\ -3*-6+-4*4 & -3*0+-4*2 & -3*3+-4*-1 \\ 1*-6+2*4 & 1*0+2*2 & 1*3+2*-1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3 \\ 2 & -8 & -5 \\ 2 & 4 & 1 \end{bmatrix}$$

$$2\mathrm{CB} = \begin{bmatrix} 0 & 12 & 6 \\ 4 & -16 & -10 \\ 4 & 8 & 2 \end{bmatrix} \quad \rightarrow A + 2CB = \begin{bmatrix} 0+1 & 12-1 & 6+3 \\ 4-1 & -16+4 & -10-2 \\ 4+3 & 8+0 & 2+6 \end{bmatrix} = \begin{bmatrix} 1 & 11 & 9 \\ 3 & -12 & -12 \\ 7 & 8 & 8 \end{bmatrix}$$

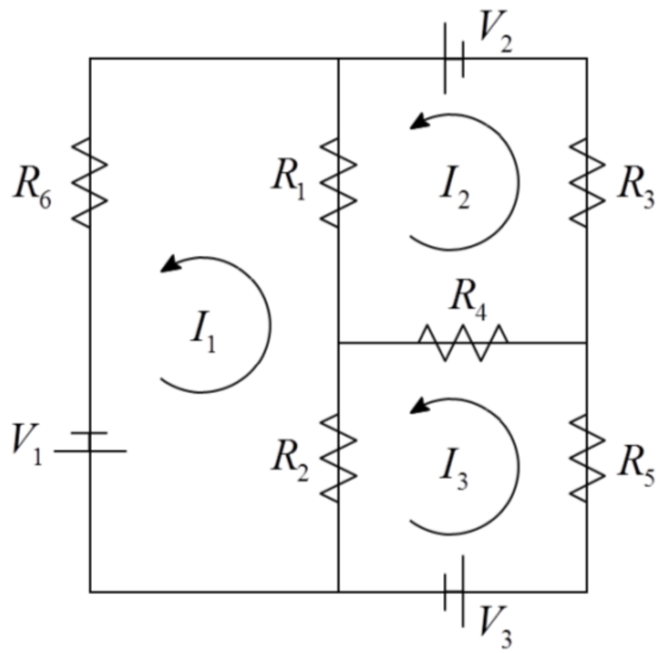
1.2.7 (h) compute$\mathrm{A}^2 - 3A + I$

$$A^2 = \begin{bmatrix} 11 & -5 & 23 \\ -11 & 17 & -23 \\ 21 & -3 & 45 \end{bmatrix} \quad 3A = \begin{bmatrix} 3 & -3 & 9 \\ -3 & 12 & -6 \\ 9 & 0 & 18 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 - 3A = \begin{bmatrix} 8 & -2 & 14 \\ -8 & 5 & -17 \\ 12 & -3 & 27 \end{bmatrix} \rightarrow + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 14 \\ -8 & 6 & -17 \\ 12 & -3 & 28 \end{bmatrix}$$

Multi-Step Problem

Consider the circuit diagram below:



Here, each V represents a change in voltage (measured in volts) at a battery, each R represents a resistance (measured in ohms) at a resistor, and each I represents a current (measured in amps) through a wire. These quantities obey two simple laws:

- **Ohm's Law:** The voltage drop across a resistor is $V = IR$,
- **Kirchhoff's Second Law:** The sum of all the voltage drops in a closed loop is zero.

Using these two laws, we can construct the following system of equations:

$$\begin{aligned} V_1 &= I_1 R_6 + (I_1 - I_2) R_1 + (I_1 - I_3) R_2, \\ V_2 &= I_2 R_3 + (I_2 - I_3) R_4 + (I_2 - I_1) R_1, \\ V_3 &= I_3 R_5 + (I_3 - I_2) R_4 + (I_3 - I_1) R_2. \end{aligned}$$

Given the values of the resistances R_j and voltage drops V_j, we want to calculate the currents I_j.

(**a**) Express the system above as a matrix equation $\mathbf{Ax} = \mathbf{b}$. What are the dimensions of $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$?

$I_1 R_6 + I_1 R_1 - I_2 R_1 + I_1 R_2 - I_3 R_2 = V_1$
$I_2 R_3 + I_2 R_4 - I_3 R_4 + I_2 R_1 - I_1 R_1 = V_2$
$I_3 R_5 + I_3 R_4 - I_2 R_4 + I_3 R_2 - I_1 R_2 = V_3$

$(R_6+R_1+R_2)I_1+(-R_1)I_2+(-R_2)I_3=V_1$
$(-R_1)I_1+(R_3+R_4+R_1)I_2+(-R_4)I_3=V_2$
$(-R_2)I_1+(-R_4)I_2+(R_5+R_4+R_2)I_3=V_3$

$\begin{pmatrix} R_6+R_1+R_2 & -R_1 & -R_2 \\ -R_1 & R_3+R_4+R_1 & -R_4 \\ -R_2 & -R_4 & R_5+R_4+R_2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$ A is 3x3, x is 3x1, and b is 3x1

(**b**) Let

$$R_1 = 20, \quad R_2 = 10, \quad R_3 = 5, \quad R_4 = 25, \quad R_5 = 20, \quad R_6 = 10.$$

Suppose we have no idea what the voltages of our batteries are. What must these voltages be if a 100 amp current flows uniformly across all three loops of the circuit above[1]?

Plug in values
$\begin{pmatrix} (10)+(20)+(10) & -(20) & -(10) \\ -(20) & (5)+(25)+(20) & -(25) \\ -(10) & (-25) & (20)+(25)+(10) \end{pmatrix} \begin{pmatrix} (100) \\ (100) \\ (100) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$
$\begin{pmatrix} 40 & -20 & -10 \\ -20 & 50 & -25 \\ -10 & -25 & 55 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$
$\begin{pmatrix} 1000 \\ 500 \\ 2000 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$ $V_1 = 1000, V_2 = 500, V_3 = 2000$

(**c**) Let

$$V_1 = 10, \quad V_2 = 385, \quad V_3 = 35,$$

and the resistances be as in the previous subexercise. Using MATLAB's backslash command, determine the currents I_1, I_2, and I_3 from your matrix equation $\mathbf{Ax} = \mathbf{b}$. Supposing any part of the circuit cannot handle currents beyond 100 amps, does this circuit break down?

» A = [40 -20 -10; -20 50 -25; -10 -25 55]
» V = [10; 385; 35]
» A\V =
12
18
11
$I_1 = 12$ $I_2 = 18$ $I_3 = 11$
No current is more than 100 amps, the circuit does not break down

[1]This means $I_1 = I_2 = I_3 = 100$ amps.