— QUIZ 2 —

AMATH 352: Applied Linear Algebra & Numerical Analysis

Due to Gradescope **February 6, 2023** at 11:59pm PST

This quiz should take $\sim$ 1 hour to complete, but you have all day to turn it in. **Follow all directions carefully**. Credit will not be given for work that does not follow directions. You are **not** allowed to work with other students or use your notes, homework assignments, MATLAB, or external sources. Please submit your work to Quiz 2 on Gradescope. Only .pdf files will be accepted.

Good luck!

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1. (5 points total) Answer the following true-false questions.

For (a)-(b), consider the following matrices:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

(a) (1 point) $E_1E_2E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$

True ☒ False ☐

(b) (1 point) $(E_1E_2E_3)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$

That's not how inverses work.

True ☐ False ☒

For (c)-(e), consider the following system:

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) (1 point) This system has a unique solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & -1 & 0 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right) \begin{array}{l} \\ +2R_1 \\ +R_1 \end{array}$$

True ☐ False ☒

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right)$$

$2x_1 + x_2 = 0$

$3x_2 + 6x_3 = 0$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ Linearly Dependendant.

(d) (1 point) The coefficient matrix of this system has a LU factorization.

True ☒ False ☐

$$U = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \\ -1 & 1 & 3 \end{pmatrix} \begin{array}{l} \\ +2R_1 \\ +R_1 \end{array} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{pmatrix} \begin{array}{l} \\ \\ -R_2 \end{array} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$ ✓

(e) (1 point) The dimension of Span $\left\{ \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \right\}$ is 3.

True ☐ False ☒

Linearly Dependent so these 3 vectors so has a dimension less than the number of vectors which is 3.

2. (5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (4 points) Obtain the LU factorization of $\mathbf{A}$.

$$u = \begin{pmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ +R_1 \\ -2R_1 \\ -R_1 \end{matrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & -4 & -1 \end{pmatrix} \begin{matrix} \\ \\ \\ -R_3 \end{matrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

(b) (1 point) Based on this LU factorization, is $\mathbf{A}$ a full rank matrix? Why or why not? **This should be a one sentence answer.**

Yes, A is full rank because In row echelon form, there are four pivots, and A has 4 columns.

2. (5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (4 points) Obtain the LU factorization of $\mathbf{A}$.

$$U = \begin{pmatrix} 1 & 0 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ +R_1 \\ -2R_1 \\ -R_1 \end{matrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & -4 & -1 \end{pmatrix} \begin{matrix} \\ \\ \\ -R_3 \end{matrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & -8 & -1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}}$$

(b) (1 point) Based on this LU factorization, is $\mathbf{A}$ a full rank matrix? Why or why not? **This should be a one sentence answer.**

Yes, A is full rank because In row echelon form, there are four pivots, and A has 4 columns.

3. (5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 2 & 4 & 4 \\ 0 & 1 & -7 & 2 & 3 \\ 1 & 4 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 & 2 \end{pmatrix}.$$

(a) (4 points) Obtain the *PLU* factorization of **A**. Use partial pivoting to determine whether to exchange rows.

$R_1 \leftrightarrow R_3$ To get:

$$U = \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

R_2 good ✓ $R_4 = R_4 - \frac{1}{2} R_3$

$R_5 = R_5 - \frac{1}{2} R_3$

$$U = \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \quad P = //$$

$R_5 = R_5 + 2R_4$

$$U = \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -2 & 1 \end{pmatrix} \quad //$$

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -7 & 2 & 3 \\ 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (1 point) Based on this *PLU* factorization, what is the dimension of the span of **A**'s columns?

4 Dimensions. Since the Upper Matrix has a rank of 4.

4. (5 points total) Consider the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) (1 point) Reduce $\mathbf{A}$ to an upper-triangular matrix. Is $\mathbf{A}$ invertible?

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$R_2 = R_2 + R_1$

$R_3 = R_3 - R_1$

A is invertible because in row-echelon form it has a pivot in every column, so its rank is 3!

(b) (2 points) Compute $\mathbf{A}^{-1}$ by Gauss-Jordan elimination. Show each elimination step!

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ +R_1 \\ -R_1 \end{array} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right) \begin{array}{c} +\frac{1}{2}R_3 \\ +\frac{1}{2}R_3 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right) \rightarrow \left(\right.$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right) \quad A^{-1}$$

$$\boxed{\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}$$

(c) (1 point) Use your inverse from (b) to solve the linear system.

$Ax = b \qquad x = A^{-1}b$

$$x = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cdot 1 + 0\cdot 1 + \frac{1}{2}\cdot 1 \\ \frac{1}{2}\cdot 1 + 1\cdot 1 + \frac{1}{2}\cdot 1 \\ -\frac{1}{2}\cdot 1 + 0\cdot 1 + \frac{1}{2}\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$$

(d) (1 point) For large $n \times n$ matrices, Gauss-Jordan elimination is more expensive than computing the *LU* decomposition.

True ☒ False ☐

5. (10 points total) Consider the following set:

$$\begin{array}{cccc} & \vec{v_1} & \vec{v_2} & \vec{v_3} \\ \mathcal{S} = & \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \right. & \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}, & \left. \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \right\}. \end{array}$$

(a) (5 points) Is $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \in \text{Span}\{\mathcal{S}\}$? If so, what linear combination of the vectors in $\mathcal{S}$ results in $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$?

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -1 & -3 & 5 & 4 \\ 1 & -1 & 3 & 4 \end{array}\right) \begin{matrix} \\ +\frac{1}{2}R_1 \\ -\frac{1}{2}R_1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 0 & -2 & 4 & 4 \\ 0 & -2 & 4 & 4 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{+\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{matrix} x_1 + x_3 = 2 \\ -x_2 + 2x_3 = 2 \\ x_3 = \text{free} \end{matrix} \longrightarrow \begin{matrix} x_1 = 2 - x_3 \\ x_2 = -2 + 2x_3 \\ x_3 = \text{free} \end{matrix} \quad x_3 = 0 \rightarrow \begin{matrix} x_1 = 2 \\ x_2 = -2 \end{matrix}$$

$\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ is in the span of the vectors: $\quad 2\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + -2\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

(b) (4 points) Are the vectors in $\mathcal{S}$ linearly independent? If not, which vector(s) are linearly dependent and which are linearly independent according to row-echelon form?

No. $\vec{v_3}$ is linearly Dependent on $\vec{v_1}$ and $\vec{v_2}$

$\vec{v_3} = 1 \cdot \vec{v_1} + -2 \cdot \vec{v_2}$

In row echelon form:

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ \vec{v_1} \;\; \vec{v_2} \;\; \vec{v_3} \end{array} \qquad \vec{v_3} = -2\vec{v_2} + \vec{v_1}$$

(c) (1 point) Which of the following describes $\text{Span}\{\mathcal{S}\}$ geometrically?

- $\text{Span}\{\mathcal{S}\}$ is $\mathbb{R}^3$.
- $\text{Span}\{\mathcal{S}\}$ is a plane through the origin in $\mathbb{R}^3$.
- $\text{Span}\{\mathcal{S}\}$ is a line through the origin in $\mathbb{R}^3$.