Homework 6.

Amath 352
Applied Linear Algebra and Numerical Analysis

Due: 2/17/22 at 11:59pm to Gradescope

Directions:

Complete all component skills exercises and the multi-step problem as neatly as possible. Up to 2 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to some solutions of the textbook exercises and are encouraged to use them. Note that these solutions are not always correct, so double check your work just in case.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Component Skills Exercises

Exercise 1. (CS3.6)

From Olver and Shakiban, complete Exercise 5.4.1(a). Please compute the solution of this least-squares problem using **both the normal equation directly and the QR equation**. Verify that your answers are the same regardless of the approach.

5.4.1. Find the least squares solution to the linear system $A\mathbf{x} = \mathbf{b}$ when (a) $A = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Normal Equation

$A^T A\mathbf{x} = A^T\mathbf{b}$

$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$6\mathbf{x} = 3$

$\mathbf{x} = \frac{1}{2}$

QR Equation

$\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$

$Q = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} Q^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} R = \sqrt{6}$

$\sqrt{6}\mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \to \sqrt{6}\mathbf{x} = \frac{3}{\sqrt{6}} \to \mathbf{x} = \frac{1}{2}$

Exercise 2. (CS3.6)

Note: this exercise involves some programming. For this exercise, you do not need to attach your MATLAB code. You only need to provide the results of your code.

In MATLAB, create the following 20×10 matrix:

$$\mathbf{A} = \begin{pmatrix} \mathbf{1}_{10\times10} \\ 10^{-8}\mathbf{I}_{10\times10} \end{pmatrix},$$

where $\mathbf{1}_{10\times10}$ is a 10×10 matrix whose entries are all ones. This particular matrix has a very large condition number, since the columns of $\mathbf{A}$ are nearly linearly dependent.

Consider the least-squares problem

$$\mathbf{Ax} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

a. Solve this least-squares problem in MATLAB using the normal equation. That is, form the matrix $\mathbf{A}^T\mathbf{A}$ and solve $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$. You may use MATLAB's backslash command to solve this system, or you may compute the inverse of $\mathbf{A}^T\mathbf{A}$ to solve for $\mathbf{x}$ directly. Are you able to get a solution? What (if any) warnings does MATLAB give you?

Yes I was able to get a solution. I often encountered the matrix sizes are incompatible to multiple but I found a work around.

$$\mathbf{x} = \begin{pmatrix} 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \end{pmatrix}$$

b. Solve this least-squares problem in MATLAB by the QR equation. That is, obtain the reduced QR factorization of $\mathbf{A}$ and solve $\tilde{\mathbf{R}}\mathbf{x} = \tilde{\mathbf{Q}}^T\mathbf{b}$ by MATLAB's backslash command or by computing the inverse of $\tilde{\mathbf{R}}$.

Note: Please use MATLAB's built-in QR factorization function as follows: $[\mathbf{Q}, \mathbf{R}] = qr(\mathbf{A})$. Remember that MATLAB returns a full QR factorization. To get the reduced QR factorization, take only the first ten columns of $\mathbf{Q}$ and the first ten rows of $\mathbf{R}$ since $\mathbf{A}$ is a 20×10 matrix.

Rounded to the second decimal place, what is the solution $\mathbf{x}$ to this least-squares problem? Does this solution seem reasonable, *i.e.*, does $\mathbf{Ax} \approx \mathbf{b}$ for the least-squares solution $\mathbf{x}$?

$$\mathbf{x} = \begin{pmatrix} 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \\ 1.000000010000000 \end{pmatrix}$$

Rounded to the nearest second decimal place:

$$\mathbf{x} = \begin{pmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{pmatrix}$$

Yes this is a reasonable answer as it minimises the difference between the two matrixes. Since 10^{-8} is so much smaller than one it changes the values but incredibly slightly.

```
%% Excercise 2
I   =[
     1 0 0 0 0 0 0 0 0 0;
     0 1 0 0 0 0 0 0 0 0;
     0 0 1 0 0 0 0 0 0 0;
     0 0 0 1 0 0 0 0 0 0;
     0 0 0 0 1 0 0 0 0 0;
     0 0 0 0 0 1 0 0 0 0;
     0 0 0 0 0 0 1 0 0 0;
     0 0 0 0 0 0 0 1 0 0;
     0 0 0 0 0 0 0 0 1 0;
     0 0 0 0 0 0 0 0 0 1];
i = 10^-8 * I;
A = [I; i];

b = [1;1;1;1;1;1;1;1;1;1;1;1;1;1;1;1;1;1;1;1];

%% (a)
%A^tAx = A^tb
%x = A^tb/A^ta
x = inv((transpose(A)*A)) * transpose(A) * b

%% (b)
```

```
[Q, R] = qr(A);
%q is the first 10 columns
q = Q(:, 1:10);
%r is the first 10 rows
r = R(1:10, :);
%rx = q^t b
%x = inv(r) q^t b
x = inv(r) * transpose(q) * b
```

Exercise 3. (CS4.1)

From Olver and Shakiban, complete Exercise 7.1.3(a),(b),(e).

Remark: If a function doesn't satisfy the definition of a linear transformation for a specific linear combination of vectors, it isn't a linear transformation.

7.1.3. Which of the following functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ are linear?

(a) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$,

$F\left(c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + d\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = F\begin{pmatrix} cx_1 + dx_2 \\ cy_1 + dy_2 \end{pmatrix} = \begin{pmatrix} cx_1 + dx_2 - cy_1 - dy_2 \\ cx_1 + dx_2 + cy_1 - dy_2 \end{pmatrix}$

$F\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} + F\begin{pmatrix} dx_2 \\ dy_2 \end{pmatrix} = \begin{pmatrix} cx_1 - cy_1 \\ cx_1 + cy_1 \end{pmatrix} + \begin{pmatrix} dx_2 - dy_2 \\ dx_2 - dy_2 \end{pmatrix} = \begin{pmatrix} cx_1 - cy_1 + dx_2 - dy_2 \\ cx_1 + cy_1 + dx_2 - dy_2 \end{pmatrix}$

Linear Combination preserved for all arbitrary x and y the function is linear.

(b) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + 1 \\ x - y - 1 \end{pmatrix}$,

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$F\begin{pmatrix} 1 \\ 1 \end{pmatrix} + F\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - 1 + 1 \\ 1 + 1 - 1 \end{pmatrix} + \begin{pmatrix} -1 + 1 + 1 \\ -1 - 1 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

Linear combination if preserved would be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Does not preserve linear combinations, not a linear transformation.

(e) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$

$$\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$F\begin{pmatrix}1\\1\end{pmatrix}+F\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}1^2+1^2\\1^2-1^2\end{pmatrix}+\begin{pmatrix}(-1)^2+(-1)^2\\(-1)^2-(-1)^2\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}2\\0\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix}$$

Linear combination if preserved would be $\begin{pmatrix}0\\0\end{pmatrix}$

Does not preserve linear combinations, not a linear transformation.

Exercise 4. (CS4.1)

Suppose we have a linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}, \quad f\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix}, \quad f\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\-2\end{pmatrix}.$$

What is $f\left(\begin{pmatrix}2\\4\\3\end{pmatrix}\right)$?

f is a 2 x 3 matrix $\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}$

$$\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\-2\end{pmatrix}$$

$(0)a + (0)b + (1)c = 0$
$(0)d + (0)e + (1)f = -2$
$c = 0$ $f = -2$

$$\begin{pmatrix}a & b & 0\\ d & e & -2\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$$

$(1)a + (0)b + (1)(0) = 2$
$(1)d + (0)e + (1)(-2) = 2$
$a = 2$ $d = 4$

$$\begin{pmatrix}2 & b & 0\\ 4 & e & -2\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$(1)(2) + (2)b + (3)(0) = 1$
$(1)(4) + (2)e + (3)(-2) = 2$
$b = -\frac{1}{2}$ $e = 2$

$$f = \begin{pmatrix}2 & -\frac{1}{2} & 0\\ 4 & 2 & -2\end{pmatrix}$$

$$\begin{pmatrix}2 & -\frac{1}{2} & 0\\ 4 & 2 & -2\end{pmatrix}\begin{pmatrix}2\\4\\3\end{pmatrix} = \begin{pmatrix}(2)(2) + (4)(-\frac{1}{2}) + (3)(0)\\ (2)(4) + (4)(2) + (3)(-2)\end{pmatrix} = \begin{pmatrix}4 - 2 + 0\\ 8 + 8 - 6\end{pmatrix} = \begin{pmatrix}2\\10\end{pmatrix}$$

$$f\left(\begin{pmatrix}2\\4\\3\end{pmatrix}\right) = \begin{pmatrix}2\\10\end{pmatrix}$$

Exercise 5. (CS4.1)

Answer the following true-false questions. If the answer is true, prove it or cite results in the course notes or textbook. If the answer is false, provide a counterexample.

a. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the zero vector in $\mathbb{R}^n$ always gets mapped to the zero vector in $\mathbb{R}^m$.

> True, Textbook Equation 7.2. A linear transform always multiplies components of the vector. Since 0 multiplied by anything is zero it cannot change value.

b. The two-norm of a vector in $\mathbb{R}^n$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.

> False
> For example $F\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2 + y^2}$ Then $F\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sqrt{2}$ While $F\begin{pmatrix} 1 \\ 0 \end{pmatrix} + F\begin{pmatrix} 0 \\ -1 \end{pmatrix} = 1 + 1 = 2 \neq \sqrt{2}$
> Linear Transforomation is not preserved.

c. Let $\mathbf{v}$ be some fixed vector in $\mathbb{R}^n$. The function $f(\mathbf{x}) = \mathbf{v}^T\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$.

> True this is the dot product formula. Also since $\mathbf{v}^T$ is a linear transformation with columns n x 1 it will be a linear transformation from n $\to$ 1. Source what you said in class.

d. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ has a $m \times n$ canonical matrix representation.

> True.Textbook Theorem 7.5 proves this.

Exercise 6. (CS4.2)

For each of the following subexercises, construct a 2×2 canonical matrix representation of the given linear transformation.

a. Rotate a vector in $\mathbb{R}^2$ by $\pi/3$ radians clockwise.

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} = \begin{pmatrix} cos(\frac{5\pi}{3}) & -sin(\frac{5\pi}{3}) \\ sin(\frac{5\pi}{3}) & cos(\frac{5\pi}{3}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

b. Reflect a vector in $\mathbb{R}^2$ across the line $y = -x$.

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

c. Stretch the x component of a vector in $\mathbb{R}^2$ by two and contract its y component by $\frac{1}{2}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Multi-Step Problem

Note: this exercise involves some programming. For this exercise, you do not need to attach your MATLAB code. You only need to provide the results of your code.

SpaceX launches a rocket vertically upward at velocity v_0 meters per second at an altitude z_0 meters above sea level. To its rocket, SpaceX attaches a sensor that records the altitude of the rocket (in meters) relative to sea level at 5 second intervals. Here are the data from that sensor:

t	$z(t)$
0	108.5
5	1482.5
10	2651.7
15	3476.1
20	4165.7
25	4562.9
30	4678.0
35	4604.3
40	4217.6
45	3632.9
50	2853.1
55	1805.4
60	503.4

In a perfect world—one where the acceleration of Earth's gravity is perfectly constant, the rocket experiences no air drag or other external forces, and the sensor is 100% accurate, Newtonian physics tells us that $z(t)$ should be quadratic in t. In particular,

$$z(t) = -\frac{1}{2}gt^2 + v_0t + z_0,$$

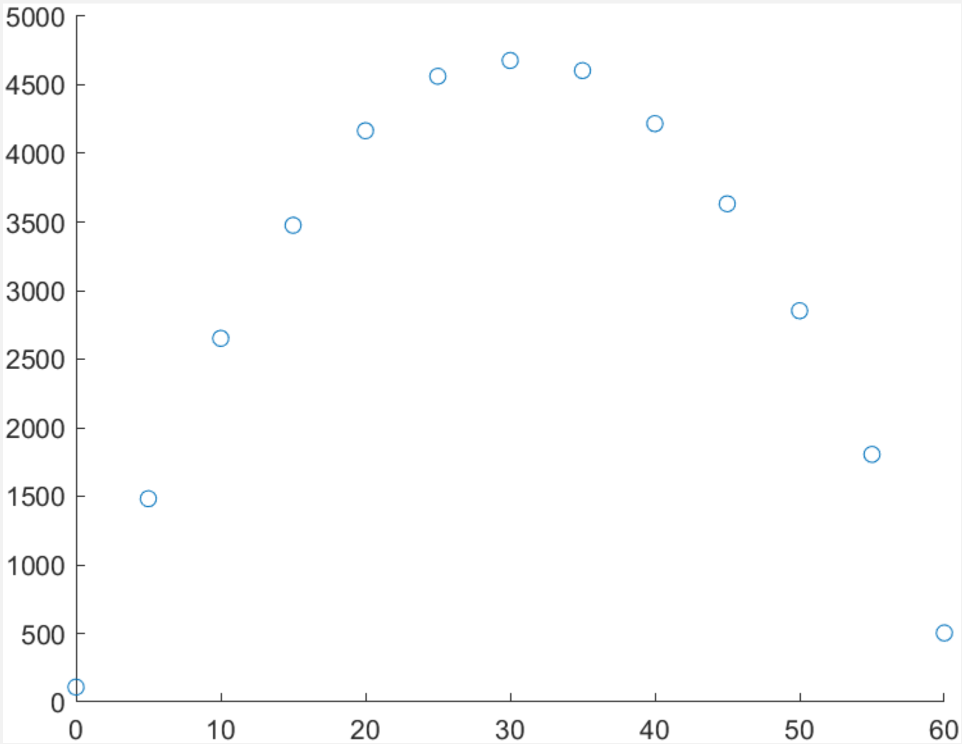
where g is the magnitude of the acceleration due to Earth's gravity (in meters per second per second).

Alas, the world is an imperfect place, and as a consequence of this imperfection, the data do not perfectly obey this quadratic equation. How might we, as applied mathematicians, smooth out this noisy data? Let us opt to fit this data with a least-squares quadratic.

a. In MATLAB, create a scatter plot of the data as follows:

```
t = 0:5:60; % Time intervals over which data has been collected
z = [108.5 1482.5 2651.7 3476.1 4165.7 4562.9 4678.0 4604.3 4217.6
     3632.9 2853.1 1805.4 503.4]; % Altitude of rocket at time intervals
figure(1);
scatter(t,z);
```

To spice up your scatter plot, you may want to check out lec14_curve_fitting.m in Lecture 14.



b. Set up the coefficient matrix of the least-squares problem as follows:

```
t = 0:5:60; % Time intervals over which data has been collected
V = vander(t); % length(t) by length(t) Vandermonde matrix:  column j of this
matrix is t^(j-length(t)) evaluated at each time interval t
A = V(:,end-2:end); % Only want columns corresponding to quadratic, linear, and
constant powers in t, i.e., the last three columns of V
```

This matrix will be 13×3 and full rank. The corresponding 13×1 vector of knowns is

```
b = transpose(z);
```

Solve this least-squares problem using the normal equation or the QR equation (your choice). To one decimal place, what are the coefficients of the quadratic polynomial that best fits the data above (at least in the least-squares sense)?

$z(t) = -4.9t^2 + 297.7t + 127.2$

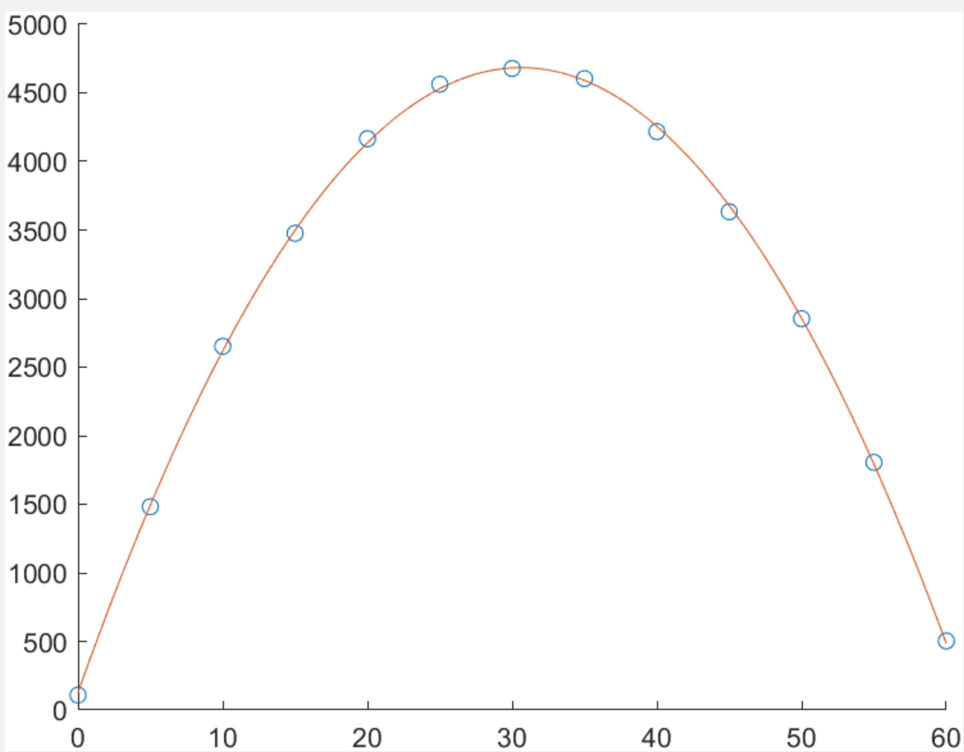
c. According to your least-squares solution in (b), what was the initial velocity of the rocket to one decimal place? What was the altitude above sea level at which the rocket was launched to one decimal place? What was the magnitude of the acceleration due to Earth's gravity to one decimal place? **Include units in your answer.**

Using the kinematics equation $z(t) = \frac{1}{2}gt^2 + v_0 t + z_0$
Initial velocity would be 297.7 meters per second.
The altitude above sea level at which the rocket was launched was 127.2 meters
The magnitude of acceleration was 9.8 meters per second per second.

d. Generate a new figure that superimposes the scatter plot data with a sketch of the least-squares quadratic that fits the data as follows:

```
s = 0:0.001:60; % Create a ‘‘continuum’’ of time intervals between 0 and 60 in
steps of 1e-3.
p = polyval(x,s); % Evaluates quadratic with coefficients given by least-squares
solution x at times s
figure(2);
scatter(t,z); % Plots the data
hold on; % Allows other plots to be superimposed atop the scatter plot
plot(s,p); % Plots the best fit quadratic
```

To spice up your plot, you may want to check out lec14_curve_fitting.m in Lecture 14.



e. According to your least-squares quadratic fit to the data, at what time does the rocket reach the pinnacle of its trajectory? Keep your answer to one decimal place and **include units**!

Vertex of a quadratic is $\frac{-b}{2a} = \frac{297.7}{(2)(-4.9)} = 30.6$ seconds

```
%% MULTISTEP PROBLEM
%Part A
t = 0:5:60; % Time intervals over which data has been collected
z = [108.5 1482.5 2651.7 3476.1 4165.7 4562.9 4678.0 4604.3
   4217.6 3632.9 2853.1 1805.4 503.4]; % Altitude of rocket at
   time intervals

figure(1);
scatter(t,z);

%% PART B Curve Fitting
t = 0:5:60;
V = vander(t);
A = V(:,end-2:end);
b = transpose(z);
% Normal Equation
%A^tAx = A^tb
%x = A^tb/A^ta
x = inv(transpose(A)*A) * transpose(A)*b

%% PART D
s = 0:0.001:60; % Create a "continuum of time intervals
   between 0 and 60 in steps of 1e-3.
p = polyval(x,s); % Evaluates quadratic with coefficients given
   by least-squares solution x at times s
figure(2);
scatter(t,z); % Plots the data
hold on; % Allows other plots to be superimposed atop the
   scatter plot
plot(s,p); % Plots the best fit quadratic
```