— QUIZ 4 —

AMATH 352:
Applied Linear Algebra & Numerical Analysis

Due to Gradescope **March 6, 2023** at 11:59pm PST

This quiz should take ∼ 1 hour to complete, but you have all day to turn it in.
Follow all directions carefully. Credit will not be given for work that does not follow directions. You are **not** allowed to work with other students or use your notes, homework assignments, MATLAB, or external sources. Please submit your work to Quiz 4 on Gradescope. Only .pdf files will be accepted.

Sally hopes you enjoy this quiz!

Name: ________________________________

Student ID: ________________________________

Problem	Points	Score
1	10	
2	5	
3	5	
4	5	
5	5	

Total	

1. (10 points total) Answer the following true-false questions.

(a) (1 point) Consider the shift-up operator:

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{pmatrix}.$$

The shift-up operator is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.

True ☐ False ☐

(b) (1 point) The canonical matrix representation for a linear transformation in the plane that first projects onto the line $y = x$ and then rotates 90° clockwise is $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$.

True ☐ False ☐

(c) (1 point) Recall the canonical matrix representation for the linear transformation that rotates the plane by θ radians counter-clockwise:

$$\mathbf{A} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This matrix never has real eigenvalues for $0 < \theta < 2\pi$.

True ☐ False ☐

(d) (1 point) Let $\mathbf{A} \in \mathbb{R}^{5\times 3}$. If $\text{Rank}(\mathbf{A}) = 3$, then $\text{Nullity}(\mathbf{A}) = 2$.

True ☐ False ☐

(e) (1 point) If $\mathbf{A} \in \mathbb{R}^{n\times n}$, then the only vector in the intersection of $\text{Null}(\mathbf{A})$ and $\text{Range}(\mathbf{A})$ is the zero vector.

True ☐ False ☐

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(f) (1 point) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ and Nullity($\mathbf{A}$) ≥ 1, then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

True ☐ False ☐

(g) (1 point) The area of the parallelogram with sides $\mathbf{u} = (2,1)^T$ and $\mathbf{v} = (-1,4)^T$ is 9.

True ☐ False ☐

(h) (1 point) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is diagonalizable with eigenvalues λ_j for $1 \leq j \leq n$, then $\det(\mathbf{A}) = \lambda_1 \cdots \lambda_n$.

True ☐ False ☐

(i) (1 point) If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{A}$, then $(\lambda + 1, \mathbf{v})$ is an eigenpair of $\mathbf{A} + \mathbf{I}$.

True ☐ False ☐

(j) (1 point) There exists an eigenvalue λ of $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that its geometric multiplicity is greater than its algebraic multiplicity.

True ☐ False ☐

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2. (5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & 3 & 1 \\ 4 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 \end{pmatrix}.$$

(a) (4 points) Compute $\det(\mathbf{A})$ either by Laplace expansion or Gaussian elimination (your choice).

$$\boxed{\det(\mathbf{A}) = \square}$$

(b) (1 point) Prove the following statement or provide a counterexample:

$$\text{For any } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n},\ \det\left(\mathbf{A}^2 - \mathbf{B}^2\right) = \det\left(\mathbf{A} - \mathbf{B}\right)\det\left(\mathbf{A} + \mathbf{B}\right).$$

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3. (5 points total) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 4 & 0 \\ 3 & -2 & 8 \end{pmatrix}.$$

(a) (4 points) Find a basis $\mathcal{B}_{\mathrm{N}}$ for the nullspace of $\mathbf{A}$ and a basis $\mathcal{B}_{\mathrm{R}}$ for the range of $\mathbf{A}$.

$$\boxed{\mathcal{B}_{\mathrm{N}} = \left\{ \qquad\qquad \right\} \qquad \mathcal{B}_{\mathrm{R}} = \left\{ \qquad\qquad \right\}}$$

(b) (1 point) What is Rank($\mathbf{A}$) and Nullity($\mathbf{A}$)?

$$\boxed{\mathrm{Rank}(\mathbf{A}) = \square \qquad \mathrm{Nullity}(\mathbf{A}) = \square}$$

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4 (5 points total) Consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1/2 & 3/2 & 0 \\ 3/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) (1.5 points) Using the characteristic polynomial, determine the eigenvalues of $\mathbf{A}$. Arrange your eigenvalues so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$ to ensure we all arrive at the same conclusions in the boxes below.

$$\lambda_1 = \square \qquad \lambda_2 = \square \qquad \lambda_3 = \square$$

(b) (1.5 points) Obtain the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$.

$$\mathbf{x}_1 = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

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(c) (2 points) Find a spectral decomposition of $\mathbf{A}$ of the form $\mathbf{Q\Lambda Q}^T$, where $\mathbf{Q}$ is an orthogonal matrix.

$$\mathbf{Q} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \quad \mathbf{\Lambda} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

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5. (5 points total) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -2 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) (3 points) Obtain a spectral decomposition for $\mathbf{A}$ of the form $\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$.

$$\mathbf{X} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \qquad \mathbf{\Lambda} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

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(b) (2 points) For $x \in \mathbb{R}$, the hyperbolic sine is defined as

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

Use your spectral decomposition from (a) to compute $\sinh(\mathbf{A})$.

$$\boxed{\sinh(\mathbf{A}) = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}}$$

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