AMATH 353
Homework #4

Show your work to earn credit ! Due on Wednesday, April 26, 2023

1. (Knobel, Exercise 6.1)

   Show that the general sine-Gordon equation

   $$A\,u_{tt} - K\,u_{xx} + T\sin u \;=\; 0$$

   can be simplified by a change of independent variables. In particular, let ξ and τ be new independent variables formed by the scalings $\xi = ax$ and $\tau = bt$ and let $U(\xi,\tau) = u(x,t)$. Find scaling constants a and b such that the above PDE reduces to

   $$U_{\tau\tau} - U_{\xi\xi} + \sin U \;=\; 0\,.$$

2. (Knobel, Exercise 6.3)

   Locate a travelling wave solution, $u(x,t) = f(z) = f(x-ct)$, of the sine-Gordon equation, $u_{tt} - u_{xx} + \sin u = 0$, where $f(z) \to \pi$ and $f'(z) \to 0$ as $z \to \infty$. In terms of the torsion-pendula problem, what is the physical interpretation of this solution.

3. (Knobel, Exercise 5.1)

   Find a traveling-wave solution for the modified KdV equation

   $$u_t + u^2\,u_x + u_{xxx} \;=\; 0$$

   that takes the form of a pulse. Be sure to simplify your solution as much as possible. The above equation appears in electric circuit theory and in the study of multicomponent plasmas.

   Hint:

   $$\int \frac{1}{f\sqrt{6c - f^2}}\,df \;=\; \frac{-1}{\sqrt{6c}}\ln\left(\frac{\sqrt{6c} + \sqrt{6c - f^2}}{f}\right)\,.$$