AMATH 353 HW # 5

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1. Find the d'Alembert solution of the initial-value problem
   $u_{tt} = u_{xx}, -\infty < x < \infty, t > 0,$
   $u(x,0) = \sin x, u_t(x,0) = xe^{-x^2}$
   Be sure to evaluate any integrals.

   $u(x,0) = f(x) = \sin x$
   $u_t(x,0) = g(x) = xe^{-x^2}$
   D'Alemberts: $c = \pm 1$
   $\frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$
   Solve Integral:
   $\frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2}ds$
   $u = -s^2, du = -2sds$
   $\frac{1}{2c}\int_{x-ct}^{x+ct} se^{-s^2}ds \to \frac{1}{2}(-\frac{1}{2})\int e^u du \to -\frac{1}{4c}e^u\Big| \to -\frac{1}{4c}e^{-s^2}\Big|_{x-ct}^{x+ct} \to -\frac{1}{4c}[e^{-(x+ct)^2} - e^{-(x-ct)^2}]$
   $\frac{1}{2}\int_{x-ct}^{x+ct} se^{-s^2}ds = -\frac{1}{4c}[e^{-(x+ct)^2} - e^{-(x-ct)^2}]$ $c = \pm 1$, If $c = 1$ then plug in 1 as normal. If $c = -1$ then the integral bounds flip to get the same value as if $c = 1$.

   $$\boxed{\frac{1}{2}[\sin(x+t) + \sin(x-t)] - \frac{1}{4}[e^{-(x+t)^2} - e^{-(x-t)^2}]}$$

2. For the wave equation
   $u_{tt} = u_{xx}$,
   write the following solutions in d'Alembert form:

   (a) $\cos x \cos t$

   $u(x,0) = f(x) = \cos(x)\cos(0) = \cos(x)$
   $u_t(x,0) = g(x) = \cos(x)(-\sin(t)) = \cos(x)(-\sin(0)) = 0$
   Since $c = \pm 1$
   $$\boxed{\frac{1}{2}[\cos(x+t) + \cos(x-t)]}$$

   (b) $t^3 + 3tx^2$

   $u(x,0) = f(x) = (0)^3 + 3(0)x^2 = 0$
   $u_t(x,0) = g(x) = 3t^2 + 3x^2 = 3x^2$
   $\frac{1}{2}\int_{x-ct}^{x+ct} 3s^2 ds \to \frac{1}{2} * s^3\Big|_{x-ct}^{x+ct} \to \frac{1}{2}[(x+ct)^3 - (x-ct)^3]$
   $$\boxed{\frac{1}{2}[(x+t)^3 - (x-t)^3]}$$