AMATH 353 HW # 9

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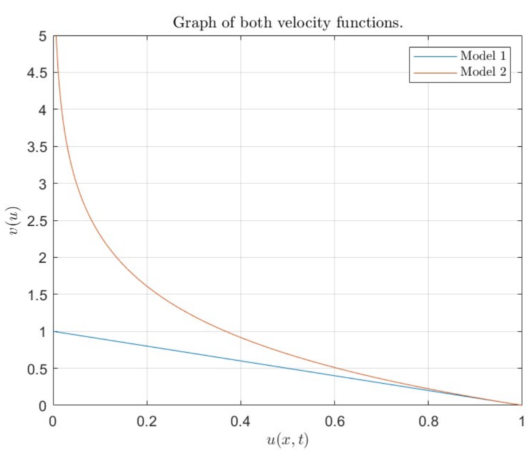
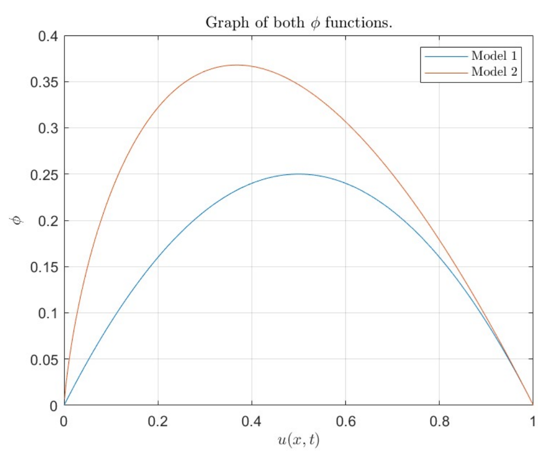
1. In class, we considered a model for traffic velocity (as a function of traffic density u) that took the form $v(u) = v_1[1 - (\frac{u}{u_1})]$. A second, different model of traffic velocity is $v(u) = k\ln(\frac{u_1}{u})$, where k is a positive constant.

   (a) Draw a graph of both functions and describe how the driver-behavior described by the second model differs from the driver-behavior of the first (linear) model.

   Model 1
   $v(u) = v_1(1 - \frac{u}{u_1}) \quad \phi = v_1(1 - \frac{u}{u_1})u$
   Model 2
   $v(u) = k\ln\left(\frac{u_1}{u}\right) \quad \phi = k\ln\left(\frac{u_1}{u}\right)u$

   Setting $u_1 = 1, v_1 = 1, k = 1$
   We can see that in model 1 an increase in density leads to a decrease in velocity, with total gridlock at $u = u_1$ and max speed at $u = 0$. While in the second model the same trends continue with an increase in density leading to a decrease in velocity. However, the velocity is asymptotic where the drivers tend to infinite speed as density goes to zero. While also leading to gridlock at the density value of $u = u_1$





   (b) For the second velocity function, what is the flux ϕ and the conservation law for traffic density $u(x,t)$? (For the conservation law, assume that there are no sources or sinks, i.e., $f = 0$.)

   $\boxed{\text{Conservation law: } u_t + \phi_x = 0}$ $\phi = k\ln(\frac{u_1}{u})u$

   $\phi_x = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} = k\ln(\frac{u_1}{u})u_x + ku(-\frac{1}{u})u_x \rightarrow \boxed{\phi_x = \left(k\ln\left(\frac{u_1}{u}\right) - k\right)u_x}$

   $u_t + \left(k\ln\left(\frac{u_1}{u}\right) - k\right)u_x = 0$

2. Consider the initial-value problem

$u_t + txu_x = 0, -\infty < x < \infty, t > 0,$
$u(x,0) = \frac{1}{1+x^2}$

(a) Use the method of characteristics to rewrite your PDE as a system of three first-order ordinary differential equations for t, x, and u. What are the initial conditions for these equations ?

$$\frac{du}{ds} = \frac{dt}{ds}\frac{du}{dt} + \frac{dx}{du}\frac{du}{dx}$$
$$\frac{dt}{dx} = 1 \quad \frac{dx}{ds} = tx \quad \frac{du}{ds} = 0$$

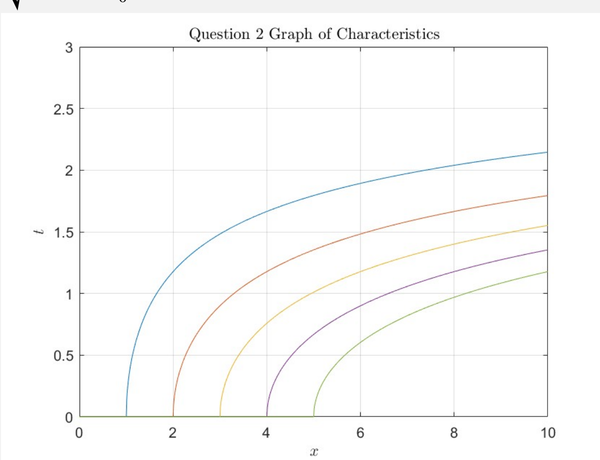
(b) Solve this system of ODEs to get an exact solution of your PDE.

$\frac{dt}{dx} = 1$ Integrate $t = s + 0$
$$\frac{dx}{ds} = tx$$
$$\frac{dx}{x} = tds \longrightarrow \frac{dx}{x} = sds$$
$$\ln(x) = \frac{1}{2}s^2 + C \longrightarrow x = x_0 e^{\frac{1}{2}s^2} \longrightarrow x_0 = xe^{-\frac{1}{2}s^2}$$
$$x_0 = xe^{-\frac{1}{2}t^2}$$
$$\frac{du}{ds} = 0$$
$$u = u_0(x_0) = \frac{1}{1+x_0^2} \longrightarrow u(x,t) = \frac{1}{1+\left(xe^{-\frac{1}{2}t^2}\right)^2}$$

$$\boxed{u(x,t) = \frac{1}{1+x^2e^{-t^2}}}$$

(c) What are the characteristics for this problem ? Draw the characteristics in the (x, t) plane.

$$x_0 = xe^{-\frac{1}{2}t^2} \longrightarrow x = x_0 e^{\frac{1}{2}t^2}$$
$$\sqrt{2 \cdot \ln(\frac{x}{x_0})} = t$$



3. Consider the initial-value problem

$u_t + xu_x + u = 0, -\infty < x < \infty, t > 0,$
$u(x, 0) = x.$

(a) Use the method of characteristics to rewrite your PDE as a system of three first-order ordinary differential equations for t, x, and u. What are the intial conditions for these equations?

$\frac{du}{ds} = \frac{dt}{ds}\frac{du}{dt} + \frac{dx}{du}\frac{du}{dx}$
$\frac{dt}{dx} = 1 \quad \frac{dx}{ds} = x \quad \frac{du}{ds} + u = 0$

(b) Solve this system of ODEs to get an exact solution of your PDE.

$\frac{dt}{dx} = 1 \quad t = s + 0$
$\frac{dx}{ds} = x \quad \frac{dx}{x} = ds \ \ln(x) = s + g(x_0) \longrightarrow x = x_o e^s \longrightarrow x_0 = xe^{-s} \longrightarrow x_0 = xe^{-t}$
$\frac{du}{ds} + u = 0$
$\frac{du}{ds} = -u \longrightarrow \frac{du}{u} = -ds \longrightarrow \ln(u) = -s + u_0(x_0) \longrightarrow u = u(0)e^{-s} \longrightarrow u = u(x_0)e^{-s}$
$u = u(x_0)e^{-t} \longrightarrow u = (xe^{-t})e^{-t}$
$\boxed{u(x, t) = xe^{-2t}}$

(c) What are the characteristics for this problem ? Draw the characteristics in the (x, t) plane.

$x_0 = xe^{-t} \longrightarrow x = x_0 e^t$
$\ln(\frac{x}{x_0}) = t$