AMATH 353 HW # 8

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1. (Knobel, 13.1) Show that the principle of superposition does not apply to solutions of the boundary value problem

$$u_{tt} = c^2 u_{xx}, 0 < x < L, t > 0 \tag{1}$$
$$u(0,t) = 1, u(L,t) = 0. \tag{2}$$

> Principle of Superposition is that if $u(x,t)$ and $v(x,t)$ satisfy the equation, then $w(x,t) = a * u(x,t) + b * v(x,t)$ also satisfies the equation, where a and b are arbitrary constants.
>
> However, this does not satisfy the boundary condition $w(0,t) = au(0,t)+bv(0,t) = a*1+b*1 = 1$ as a and b should be arbitrary constants but in order for this to work $a+b = 1$. Which does not match the principle of superposition.

2. (Knobel, 13.3) Find the solutions of

$$u_{tt} = u_{xx}, 0 < x < 1, t > 0, \tag{3}$$
$$u(0,t) = 0, u(1,t) = 0 \tag{4}$$

(a) $u(x,0) = 10\sin(\pi x) + 3\sin(4\pi x), u_t(x,0) = 0$

> $c = 1, L = 1$
> $u(x,t) = \sum_{n=1}^{N}(A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t))\sin(\frac{n\pi}{L}x)$
> $u(x,t) = \sum_{n=1}^{N}(A_n \cos(n\pi t) + B_n \sin(n\pi t))\sin(n\pi x)$
> $u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = 10\sin(\pi x) + 3\sin(4\pi x)$
> Letting $n = 1$ and $n = 4$ Matching the Coefficients.
> $A_1 = 10$
> $A_2 = 0, A_3 = 0$
> $A_4 = 3$
>
> $u_t(x,t) = (-10\pi\sin(\pi t)+B_1\pi\cos(\pi t))\sin(\pi x)+(-12\pi\sin(4\pi t)+4\pi B_4\cos(4\pi t))\sin(4\pi x)$
> $u_t(x,0) = \pi B_1 \sin(\pi x) + 4\pi B_4 \sin(4\pi x) = 0$
> $B_1 = B_4 = 0$
>
> $$\boxed{u(x,t) = 10\cos(\pi t)\sin(\pi x) + \frac{3}{4\pi}\cos(4\pi t)\sin(4\pi x)}$$

(b) $u(x,0) = \sin(2\pi x), u_t(x,0) = -3\sin(2\pi x)$

$c = 1, L = 1$
$u(x,t) = \sum_{n=1}^{N}(A_n \cos(\frac{n\pi c}{L}t) + B_n \sin(\frac{n\pi c}{L}t))\sin(\frac{n\pi}{L}x)$
$u(x,t) = \sum_{n=1}^{N}(A_n \cos(n\pi t) + B_n \sin(n\pi t))\sin(n\pi x)$
$u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = \sin(2\pi x)$
Matching Coefficients and Frequency $n = 2$
$A_2 = 1$

$u(x,t) = ((1)\cos(2\pi t) + B_2 \sin(2\pi t))\sin(2\pi x)$
$u_t(x,0) = (2\pi B_2 \cos(2\pi 0))\sin(2\pi x) = -3\sin(2\pi x)$
$u_t(x,0) = 2\pi B_2 \sin(2\pi x) = -3\sin(2\pi x)$
$2\pi B_2 = -3 \rightarrow B_2 = -\frac{3}{2\pi}$

$$\boxed{u(x,t) = \left(\cos(2\pi t) - \frac{3}{2\pi}\sin(2\pi t)\right)\sin(2\pi x)}$$

3. (from Knobel, 14.2) Find the Fourier sine expansion for the function

$$f(x) = x(1-x), \tag{5}$$

on the interval [0, 1]. Be sure to show your work for full credit.

$\sum_{n=1}^{\infty} B_n \sin(nx)$
$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$
$f(x) = -x^2 + x$, $L = 1$, $c = 1$
$B_n = 2\int_0^1 (-x^2 + x)\sin(n\pi x)dx$
Integration!
$u = -x^2 + x$, $du = -2x + 1$
$dv = \sin(n\pi x)$, $v = -\frac{1}{n\pi}\cos(n\pi x)$

$= -\frac{2(-x^2+x)\cos(n\pi x)}{n\pi}\Big|_0^1 + \int_0^1 \frac{2}{n\pi}(-2x+1)\cos(n\pi x)dx$
$u = -2x + 1$, $du = -2$
$dv = \cos(n\pi x)$, $v = \frac{1}{n\pi}\sin(n\pi x)$
$= -\frac{2(-x^2+x)\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{2(-2x+1)\sin(n\pi x)}{\pi^2 n^2}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \frac{2}{n\pi}\sin(n\pi x)dx$
$= -\frac{2(-x^2+x)\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{(-4x+2)\sin(n\pi x)}{\pi^2 n^2}\Big|_0^1 - \frac{4\cos(n\pi x)}{n^3\pi^3}\Big|_0^1$
$B_n = \frac{1}{\pi^2 n^2}(-2\sin(\pi n)) + \frac{1}{\pi^3 n^3}(-4\cos(\pi n) + 4)$
$B_n = \frac{-2\pi n\sin(\pi n) - 4\cos(\pi n) + 4}{\pi^3 n^3}$
Since $n = 0, 1, 2 \ldots .\ \sin(\pi n) = 0$

$$\boxed{-x^2 + x = \sum_{n=1}^{\infty} \frac{-4\cos(\pi n) + 4}{\pi^3 n^3}\sin(n\pi x)}$$

4. (Knobel, 14.6) Consider the displacement of a vibrating string with fixed ends given by

$$u_{tt} = u_{xx}, 0 < x < 1, t > 0, \tag{6}$$

$$u(0,t) = 0, u(1,t) = 0. \tag{7}$$

Use the Fourier sine series found in exercise 3 (above) to write down the solution of the vibrating string with the following initial conditions

(a) $u(x,0) = x(1-x), u_t(x,0) = 0,$

$c = 1, L = 1$
$u(x,t) = \sum_{n=1}^{N}(A_n \cos(n\pi t) + B_n \sin(n\pi t))\sin(n\pi x)$
$u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = x(1-x)$
Plug in sine series expansion of $f(x)$ from exercise 3.
$A_n = \frac{-4\cos(\pi n)+4}{\pi^3 n^3}$
$u_t(x,0) = \sum_{n=1}^{N} \frac{1}{n\pi} B_n \sin(n\pi x) = 0$
$B_n = 0$

$$\boxed{u(x,t) = \sum_{n=1}^{N} \frac{-4\cos(\pi n)+4}{\pi^3 n^3} \cos(n\pi t)\sin(n\pi x)}$$

(b) $u(x,0) = 0, u_t(x,0) = x(1-x).$

$c = 1, L = 1$
$u(x,t) = \sum_{n=1}^{N}(A_n \cos(n\pi t) + B_n \sin(n\pi t))\sin(n\pi x)$
$u(x,0) = \sum_{n=1}^{N} A_n \sin(n\pi x) = 0$
$A_n = 0$
$u_t(x,0) = \sum_{n=1}^{N} \frac{1}{n\pi} B_n \sin(n\pi x) = x(1-x)$
Plug in sine series expansion of $g(x)$ from exercise 3.
$B_n = \frac{-4\cos(\pi n)+4}{\pi^3 n^3}$

$$\boxed{u(x,t) = \sum_{n=1}^{N} \frac{-4\cos(\pi n)+4}{\pi^3 n^3} \sin(n\pi t)\sin(n\pi x)}$$