AMATH 353 HW # 4

Julien Goldstick

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1. (Knobel, Exercise 6.1) Show that the general Sine-Gordon equation

$$Au_{tt} - Ku_{xx} + T\sin(u) = 0 \tag{1}$$

can be simplified by a change of independent variables. In particular, let ζ and τ be new independent variables formed by the scalings $\zeta = ax$ and $\tau = bt$ and let $U(\zeta, \tau) = u(x, t)$. Find scaling constants a and b such that the above PDE reduces to

$$U_{\tau\tau} - U_{\zeta\zeta} + \sin(U) = 0. \tag{2}$$

$\tau = bt$ $\zeta = ax$
$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = bU_\tau$
$u_{tt} = \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \tau^2}\frac{\partial^2 \tau}{\partial t^2} = b^2 U_{\tau\tau}$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} = aU_\zeta$
$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \zeta^2}\frac{\partial^2 \zeta}{\partial x^2} = a^2 U_{\zeta\zeta}$

$Au_{tt} - Ku_{xx} + T\sin(u) = 0$

$Ab^2U_{\tau\tau} - Ka^2U_{\zeta\zeta} + T\sin(U) = 0$
Divide each term by T given $T \neq 0$
$\frac{Ab^2}{T}U_{\tau\tau} - \frac{Ka^2}{T}U_{\zeta\zeta} + sin(U) = 0$
$Ab^2 = T$
$\boxed{b = \pm\sqrt{\frac{T}{A}}}$
$Ka^2 = T$
$\boxed{a = \pm\sqrt{\frac{T}{K}}}$
Plug in to get $\frac{A(\pm\sqrt{\frac{T}{A}})^2}{T}U_{\tau\tau} - \frac{K(\pm\sqrt{\frac{T}{K}})^2}{T}U_{\zeta\zeta} + sin(U) = 0$
Simplifies to $U_{\tau\tau} - U_{\zeta\zeta} + \sin(U) = 0$

2. (Knobel, Exercise 6.3) Locate a travelling wave solution, $u(x,t) = f(z) = f(x-ct)$, of the sine-Gordon equation, $u_{tt} - u_{xx} + \sin(u) = 0$, where $f(z) \to \pi$ and $f'(z) \to 0$ as $z \to \infty$. In terms of the torsion-pendula problem, what is the physical interpretation of this solution.

$u_{tt} - u_{xx} + \sin(u) = 0$
Terms of f"
$c^2 f'' - f'' + \sin(f) = 0$
Multiply by f'
$c^2 f'' f' - f'' f' + \sin(f) f' = 0$
Integrate
$(c^2-1) * \frac{1}{2}(f')^2 - \cos(f) = C_1$
Take the limit as z goes to infinity
$f'(z) \to 0 \quad f(z) \to \pi \; (c^2-1) * \frac{1}{2}(0)^2 - \cos(\pi) = C_1$
$C_1 = -(-1) = 1$

Solve for f'
$(c^2-1) * \frac{1}{2}(f')^2 - \cos(f) = 1$
$(f')^2 = (\frac{2}{c^2-1})(1+\cos(f))$

Half Angle Formula: $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

$(f')^2 = (\frac{4}{c^2-1})\cos^2(\frac{f}{2})$
$f' = \frac{df}{dz} = \sqrt{(\frac{4}{c^2-1})}\cos(\frac{f}{2})$

Seperable ODE
$\frac{1}{\cos(\frac{f}{2})} df = \pm \frac{2}{\sqrt{c^2-1}} dz$
Plus since the wave is going towards +pi from -pi.
Integral arbitrary constant C
$2\ln|\tan(\frac{f(z)}{4} + \frac{\pi}{4})| = \frac{2}{\sqrt{c^2-1}} z + C$
Solve for f
$\tan(\frac{f(z)}{4} + \frac{\pi}{4}) = e^{\frac{z}{\sqrt{c^2-1}} + C}$
$\frac{f(z)}{4} + \frac{\pi}{4} = \arctan\left(e^{\frac{z}{\sqrt{c^2-1}} + C}\right)$
$f(z) = 4\arctan\left(Ce^{\frac{z}{\sqrt{c^2-1}}}\right) - \pi$ Where C is an arbitrary constant
$$\boxed{u(x,t) = f(x-ct) = 4\arctan\left(Ce^{\frac{(x-ct)}{\sqrt{c^2-1}}}\right) - \pi}$$

3. (Knobel, Exercise 5.1) Find a traveling-wave solution for the modified KdV equation

$$u_t + u^2 u_x + u_{xxx} = 0 \tag{3}$$

that takes the form of a pulse. Be sure to simplify your solution as much as possible. The above equation appears in electric circuit theory and in the study of multicomponent plasmas.

Hint: $\int \frac{1}{f\sqrt{6c-f^2}}df = \frac{-1}{\sqrt{6c}} \ln\left(\frac{\sqrt{6c}+\sqrt{6c-f^2}}{f}\right)$

$u(x,t) = f(x-ct)$, $u_x = f'$, $u_{xx} = f''$, $f_{xxx} = f'''$, $u_t = -cf'$

$f''' + f^2 f' - cf = 0$ Integrate $\to f'' + \frac{1}{3}f^3 - cf = a = 0$

Multiply by f' and Integrate

$f''f' + \frac{1}{3}f^3 f' - cff' = 0 \to \frac{1}{2}(f')^2 + \frac{1}{12}f^4 - \frac{1}{2}cf^2 = b = 0$

Solve for f'

$(f')^2 = cf^2 - \frac{1}{6}f^4 \to f' = f\sqrt{6c-f^2} = \frac{df}{dz} \to \int \frac{df}{f\sqrt{6c-f^2}} = \pm \int 1dz$

Solved Integral from Hint, where G_1 is an arbitrary constant, Positive Wave train Solution chosen.

$\frac{-1}{\sqrt{6c}} \ln\left(\frac{\sqrt{6c}+\sqrt{6c-f^2}}{f}\right) = z + G_1$

Solve for f

$\frac{\sqrt{6c}+\sqrt{6c-f^2}}{f} = e^{-\sqrt{6c}z+G_1} \to \sqrt{6c-f^2} = fG_1 e^{-\sqrt{6c}z} - \sqrt{6c}$

Square both Sides

$6c - f^2 = (fG_1 e^{(-\sqrt{6c}z})^2 + 2fG_1 e^{-\sqrt{6c}z} * \sqrt{6c} + 6c \to f^2(1 + G_2 e^{-2\sqrt{6c}z}) + f(2\sqrt{6c}G_1 e^{-\sqrt{6c}z}) = 0$

Divide by f where $f \neq 0$ and is not the trivial case, $G_2 = (G_1)^2$

$f(1 + G_2 e^{-2\sqrt{6c}z}) + (2\sqrt{6c}G_1 e^{-\sqrt{6c}z}) = 0 \to f(z) = \frac{-2\sqrt{6c}G_1 e^{-\sqrt{6c}z}}{1+G_2 e^{-2\sqrt{6c}z}}$

Absorb constants into arbitrary constants

$f(z) = \frac{G_3\sqrt{c}e^{-\sqrt{6c}z}}{1+G_2 e^{-2\sqrt{6c}z}}$

Traveling Wave Solution:

$$\boxed{u(x,t) = f(x-ct) = \frac{G_3\sqrt{c}e^{-\sqrt{6c}(x-ct)}}{1 + G_2 e^{-2\sqrt{6c}(x-ct)}}}$$

Multiply Numerator and Denominator by $e^{\sqrt{6c}(x-ct)}$ $f(x-ct) = \frac{G_3\sqrt{c}}{e^{\sqrt{6c}(x-ct)} + G_2 e^{-\sqrt{6c}(x-ct)}}$

When $G_2 = 1$, $f(x-ct) = \frac{G_3\sqrt{c}}{e^{\sqrt{6c}(x-ct)} + e^{-\sqrt{6c}(x-ct)}}$ This can be expressed as Hyperbolic Secant

$$\boxed{u(x,t) = f(x-ct) = G_4\sqrt{c}\ \mathrm{sech}\left(\sqrt{6c}(x-ct)\right)}$$