AMATH 423 HW # 2

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1. Question 1

$$\frac{dX}{dT} = -X - 3Y \tag{1}$$
$$\frac{dY}{dT} = -2X - 2Y \tag{2}$$

(a) Obtain the general solution to the pair of linear, homogeneous, ordinary differential equations (ODEs) with constant coefficients

Rewrite in the form of matrices
$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$
Using the characteristic polynomial solve for eigenvalues
$\det(A - \lambda I) = 0$
$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$
$\lambda^2 - (-1 + -2)\lambda + ((-1)(-2) - (-2)(-3)) = 0$
$\lambda^2 + 3\lambda - 4 = 0$
$(\lambda + 4)(\lambda - 1) = 0$
$\lambda = -4, 1$

Solve for eigen vectors:
$(A - \lambda) * v_1 = 0$
$\begin{pmatrix} -1+4 & -3 \\ -2 & -2+4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$
$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Sovle for eigen vectors:
$(A - \lambda) * v_1 = 0$
$\begin{pmatrix} -1-1 & -3 \\ -2 & -2-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} -2 & -3 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$
$\begin{pmatrix} \frac{3}{2} & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

Generic solution
$Z(t) = C_1 e^{\lambda_1 t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + C_2 e^{\lambda_2 t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\boxed{Z(t) = C_1 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}}$

(b) Find the particular solution to the above equation with initial values $X(0) = Y(0) = 3$

$Z(t) = C_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$
$C_1 + -\frac{3}{2} C_2 = 3$
$C_1 + C_2 = 3$
$C_1 + C_2 = C_1 + -\frac{3}{2} C_2$
$C_2 = -\frac{3}{2} C_2$
$C_2 = 0$
$C_1 + (0) = 3$
$C_1 = 3$

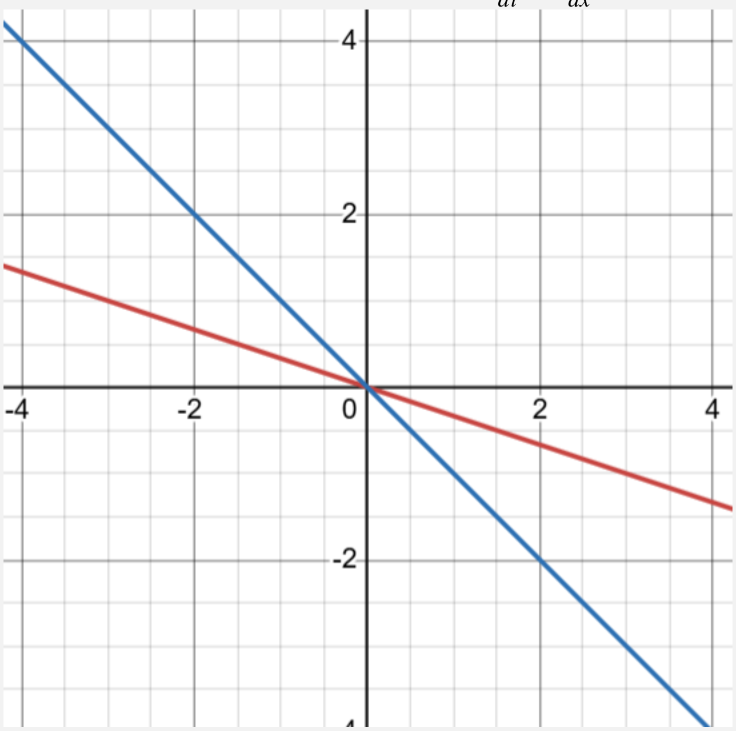
$\boxed{Z(t) = 3e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

(c) Plot the two lines in a (X, Y) graph:

$$-X - 3Y = 0 \tag{3}$$
$$-2X - 2Y = 0 \tag{4}$$

what is the meaning of these two lines; and what is the meanng of the intersection point, for the pair of ODEs?

The two equations signify
$\frac{dX}{dt} = -X - 3Y = 0$ $\frac{dX}{dt} - 2X - 2Y = 0$ Meaning this is when there is no change in the derivative of X or Y.
The intersection of lines is when the derivatives are equal to each other. When both derivatives are zero the system is in equilibrium and there is no change. In this case the intersection is at 0 when $X = Y = 0 = \frac{dX}{dt} = \frac{dY}{dx}$

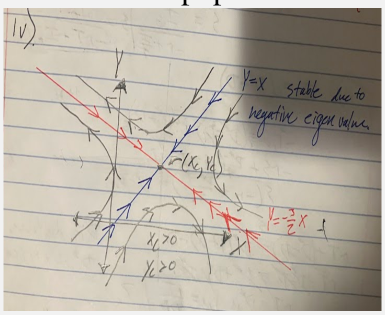


(d) What is meant by a trajectory of the above system? Describe the behavior of the trajectories of the system as $t \longrightarrow \infty$ and $t \longrightarrow -\infty$

The trajectory of the above system means the path that represents the solution of $X(t), Y(t)$ in the phase plane as it changes over time. The time trajectory shows how the system is evolving in time.
$t = \infty, \rightarrow X(t) = C_1 e^{-4(\infty)} - \frac{3}{2} C_2 e^{\infty} = 0 + -\frac{3}{2} C_2 * \infty = -\frac{3}{2} C_2 * \infty$
$t = \infty, \rightarrow Y(t) = C_1 e^{-4(\infty)} + C_2 e^{\infty} = C_1 * 0 + C_2 * \infty = C_2 * \infty$
$t = -\infty, \rightarrow X(t) = C_1 e^{-4(-\infty)} - \frac{3}{2} C_2 e^{-\infty} = C_1 * \infty + 0 = C_1 * \infty$
$t = -\infty, \rightarrow Y(t) = C_1 e^{-4(-\infty)} + C_2 e^{-\infty} = C_1 * \infty + C_2 * 0 = C_1 * \infty$
we can see that as $t \longrightarrow \infty$ X(t) and Y(t) seperate in opposite directions because the resulting values differ by a negative sign. When $t \longrightarrow -\infty$ then X(t) and Y (t) go in the same direction.

(e) Suppose x(t) and y(t) are population sizes of two competing biological species which have an equilibrium point at (x_c, y_c) where x_c, y_c are both positive. If one denotes X $= x - x_c$ and $Y = y - y_c$, and they satisfy the above pair of equations, roughly sketch the phase portrait in the first quadrant of the (x, y) plane. Discuss, in detail, the fate of populations of x and y.

Blue line has eigenvalue -4 and the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, since the eigenvalue is negative it is stable so solutions converge to equilibrium (x_c, y_c) along this vector.
Red line has eigenvalue 1 and the eigenvector $\begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$, since the eigenvalue is positive it is unstable so solutions will diverge and not go to equilibrium (x_c, y_c) along this vector.
According to the plot we can see how the initial conditions will either approach the equilibrium or if the populations will separate along the unstable direction.



2. Let us again consider a pair of ODEs:

$$\frac{dX(t)}{dt} = aX + bY \tag{5}$$
$$\frac{dY(t)}{dt} = cX + dY \tag{6}$$

where *a, b, c, d* are four arbitrary constants. This time try to eliminate the unknown function Y and obtain a 2nd order ODE for $X(t)$ in the form of

$$\frac{d^2X(t)}{dt^2} + \alpha\frac{dX(t)}{dt} + \beta X = 0 \tag{7}$$
$$\tag{8}$$

Find out α and β in terms of *a,b,c,d* Discuss the significance of your finding.

Take the derivative of the first function
$\frac{dX(t)}{dt} = aX + bY$
$\frac{dX(t)}{dt} - aX = bY$
$\frac{d^2X(t)}{dt^2} - a\frac{dX(t)}{dt} = b\frac{dY(t)}{dt}$
$\frac{d^2X(t)}{dt^2} - a\frac{dX(t)}{dt} = b(cX + dY)$
$\frac{d^2X(t)}{dt^2} - a\frac{dX(t)}{dt} = b(cX + dY)$
$\frac{d^2X(t)}{dt^2} - a\frac{dX(t)}{dt} = b(cX + d(\frac{\frac{dX}{dt} - aX}{b})$
$\frac{d^2X(t)}{dt^2} - a\frac{dX(t)}{dt} = bcX + d\frac{dX}{dt} - adX$
$\frac{d^2X(t)}{dt^2} - (a + d)\frac{dX(t)}{dt} + (ad - bc)X = 0$
$\alpha = (-a - d), \beta = (ad - bc)$
This the characteristic polynomial of the equation. The result shows that you can rewrite a linear, homogeneous system of 2 first order ODE's as a linear homogeneous 2nd order ODE

3. Consider a biological population with size $S(t)$ at time t on an island. Assume that the per capita growth rate for the population in the absence of immigration is

$$\text{per capita growth rate} = -r, \tag{9}$$
$$\tag{10}$$

where $r > 0$. Furthermore, the rate of population immigration from mainland into the island is expressed by

$$\text{rate of immigration} = I_0 - \frac{I_0}{T}S(t) \tag{11}$$
$$\tag{12}$$

where I_0 is the rate of immigration to the island when it is empty. The immigration rate decreases with increasing S, and the T represents the carrying capacity of the island

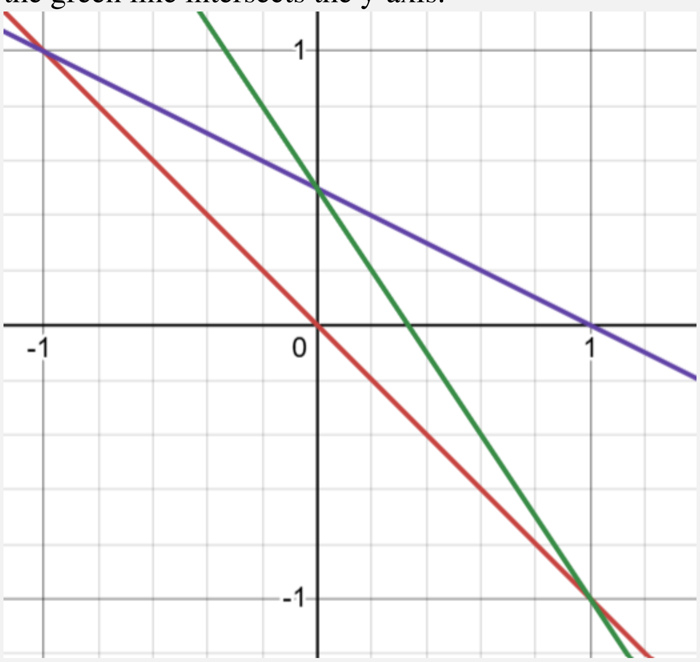
(a) Show that the equilibrium population size on the island is given by

$$\hat{S} = \frac{TI_0}{rT + I_0} \tag{13}$$

Explain this result by graphs.

Equilibrium is when the negative growth rate is equal to the immigration population.
$r\hat{S}(t) = I_0 - \frac{I_0}{T}\hat{S}(t)$
$rS(t) + \frac{I_0}{T}S(t) = I_0$
$S(t)(r + \frac{I_0}{T}) = I_0$
$S(t) = \frac{I_0}{r+\frac{I_0}{T}}$
$\hat{S}(t) = \frac{I_0 T}{rT+I_0}$

Red line is $-rS(t)$, Blue line is rate of immigration. Green line is both combined. Setting $I_0 = \frac{1}{2}, T = 1, r = 1$ we get the following graph. The population equilibrium occurs when the green line intersects the y-axis.



(b) Suppose that I_0 is linearly related to the distance between the island and the mainland. Show that in this case

$$\hat{S} = \frac{T(D^* - D)}{D^*(1+\mu) - D} \tag{14}$$

where D is the distance from the mainland and D^* is the critical distance beyond which no species can immigrate. What is the meaning of the dimensionless parameter μ?

$I_0 = c(D^* - D)$
c is the linear scaling constant.
Plugging $I_0 = c(D^* - D)$ into the equation we get:
$\hat{S}(t) = \frac{c(D^*-D)T}{rT+c(D^*-D)}$
Letting $\mu = \frac{rT}{cD^*}$, $rT = cD^*\mu$
$\hat{S}(t) = \frac{c(D^*-D)T}{cD^*\mu+c(D^*-D)}$
$\hat{S}(t) = \frac{(D^*-D)T}{D^*\mu+(D^*-D)}$
$\hat{S}(t) = \frac{(D^*-D)T}{D^*(1+\mu)-D)}$
Assuming $D^* \neq 0, c \neq 0$
μ is the negative growth rate r divided by the max inital immigration $D^* - 0$ scaled by the islands carry capacity. A large μ signifies a decrease in population due to more negative growth than immigration. Vice verse for a small μ

4. Consider a population consisting of identical and independent individual organisms, each with an exponentially distributed time for giving "birth", with rate λ, and going "death", with rate μ.

   (a) Now when the population has exactly n individuals, what is the probability distribution for the waiting time to the next birth? What is the probability distribution for the waiting time to the next death? What is the probability distribution for the waiting time to the next birth or death event?

   Population has n individuals.
   $P_{birth}(T \leq t) = n\lambda e^{-n\lambda t}$
   $P_{death}(T \leq t) = n\mu e^{-n\mu t}$

   $P_{nextevent}(T \leq t) = n(\lambda + \mu)e^{-n(\lambda+\mu)t}$

   (b) Let $p_n(t)$ be the probability of having exactly n individuals in the population at time t. The $p_n(t)$ satisfies a system of (infinite number!) differential equations:

$$\frac{dp_0(t)}{dt} = \mu p_1(t) \tag{15}$$

$$\frac{dp_n(t)}{dt} = (n-1)\lambda p_{n-1}(t) - n(\lambda + \mu)p_n(t) + (n+1)\mu p_{n+1}(t) \tag{16}$$

$$n = 1, 2, \ldots \tag{17}$$

   Explain why this set of equations are true.

The first equation makes sense, because the rate of change at $p = 0$ requires the population to have come from 1 or -1. -1 is not a real possibility, so the popualtion must have come from 1. Thus in one time span the change is the death rate to go from population of 1 to 0. The second equation makes sense, because the rate of change at the population of p depends on the instantaneous increase from n-1 or n+1 to n and the instantaneous decrease from n to n-1 or n+1.
The instantaneous increase from n-1 to n is given by $(n-1)\lambda p_{n-1}(t)$
The instantaneous increase from n+1 to n is given by $(n+1)\mu p_{n+1}(t)$
The instantaneous decrease from n to either n+1 or n-1 is given by $n(\lambda+\mu)p_n(t)$
Combining all these results in the second equation for this problem to get
$\frac{dp_n(t)}{dt} = (n-1)\lambda p_{n-1}(t) - n(\lambda+\mu)p_n(t) + (n+1)\mu p_{n+1}(t)$
$n = 1, 2, \ldots$
The infinite system of equation shows how the evolution of the probability distribution of the population size n over time.

(c) Assuming that the order of derivative and infinite summation below can exchange, i.e.,

$$\frac{d}{dt}\sum_{n=0}^{\infty} p_n(t) = \sum_{n=0}^{\infty} \frac{dp_n(t)}{dt} \tag{18}$$

Show that

$$\frac{d}{dt}\sum_{n=0}^{\infty} p_n(t) = 0 \tag{19}$$

$\frac{d}{dt}\sum_{n=0}^{\infty} p_n(t) = \sum_{n=0}^{\infty} \frac{dp_n(t)}{dt}$

$\sum_{n=0}^{\infty} \frac{dp_n(t)}{dt} = \mu p_1(t) + \sum_{n=1}^{\infty}\left((n-1)\lambda p_{n-1}(t) - n(\lambda+\mu)p_n(t) + (n+1)\mu p_{n+1}(t)\right)$

$= \mu p_1(t) + \sum_{n=1}^{\infty}(n-1)\lambda p_{n-1}(t) - \sum_{n=1}^{\infty} n\lambda p_n(t) - \sum_{n=1}^{\infty} n\mu p_n(t) + \sum_{n=1}^{\infty}(n+1)\mu p_{n+1}(t)$
Indexes shift.
$= \mu p_1(t) + \sum_{n=0}^{\infty}(n)\lambda p_n(t) - \sum_{n=1}^{\infty} n\lambda p_n(t) - \sum_{n=0}^{\infty}(n+1)\mu p_{n+1}(t) + \sum_{n=1}^{\infty}(n+1)\mu p_{n+1}(t)$
$= \mu p_1(t) + \sum_{n=0}^{\infty}(n)\lambda p_n(t) - \sum_{n=1}^{\infty} n\lambda p_n(t) - \mu p_1 - \sum_{n=1}^{\infty}(n+1)\mu p_{n+1}(t) + \sum_{n=1}^{\infty}(n+1)\mu p_{n+1}(t)$
Now all the terms cancel thus
$\frac{d}{dt}\sum_{n=0}^{\infty} p_n(t) = 0$

(d) The mean population at time t is defined as

$$u(t) := \sum_{n=0}^{\infty} np_n(t). \tag{20}$$

Based on the system of differential equations in (ii), show that

$$\frac{du(t)}{dt} = (\lambda - \mu)u(t) \tag{21}$$

$\frac{du(t)}{dt} = \frac{d}{dt} \sum_{n=0}^{\infty} np_n(t) = \sum_{n=0}^{\infty} \frac{dnp_n(t)}{dt}$

$\sum_{n=0}^{\infty} \frac{dp_n(t)}{dt} = n\mu p_1(t) + \sum_{n=1}^{\infty} (n(n-1)\lambda p_{n-1}(t) - n(\lambda+\mu)p_n(t) + n(n+1)\mu p_{n+1}(t))$

$= n\mu p_1(t) + \sum_{n=1}^{\infty} n(n-1)\lambda p_{n-1}(t) - \sum_{n=1}^{\infty} n^2\lambda p_n(t) - \sum_{n=1}^{\infty} n^2\mu p_n(t) + \sum_{n=1}^{\infty} n(n+1)\mu p_{n+1}(t)$

Shift Indexes

$= n\mu p_1(t) + \sum_{n=1}^{\infty} n(n-1)\lambda p_{n-1}(t) - \sum_{n=1}^{\infty} n^2\lambda p_n(t) - \sum_{n=1}^{\infty} n^2\mu p_n(t) + \sum_{n=1}^{\infty} n(n+1)\mu p_{n+1}(t)$

$= \sum_{n=0}^{\infty} n\lambda p_n(t) - \mu p_1(t) - \sum_{n=2}^{\infty} n\mu p_n(t).$

$= \lambda p_1(t) \sum_{n=2}^{\infty} n\lambda p_n(t) - \mu p_1(t) - \sum_{n=2}^{\infty} n\mu p_n(t).$

$= (\lambda - \mu) \sum_{n=1}^{\infty} np_n(t).$

$= (\lambda - \mu)u(t)$

QED.