Homework # 1

(Due Jan. 17, Friday 11:59pm)

Students in 523 are required to complete additional problems on the third page

1. An "exponential waiting/sojourn/resident time distribution" follows $f_T(t) = re^{-rt}$.

(a) What is the *expected value* of random variable T

$$\mathbb{E}[T] = \int_0^\infty t f_T(t)\mathrm{d}t?$$

(b) Discuss the parameter r.

(c) What are the several possible mechanistic explanations for the exponentially distributed random time, as a phenomenon? Try to outline them using mathematics.

(d) What are the important features of an exponentially distributed random time?

2. Two independent random events with waiting time T_1 and T_2 and corresponding cumulative distribution functions (CDF) $F_{T_1}(x) = e^{-r_1 x}$ and $F_{T_2}(x) = e^{-r_2 x}$.

(a) What is the probability distribution for the random time of the "first of the two" occurs?

(b) What is the probability distribution for the random time of both occur?

3. In probabilistic terms, the *Poisson process*

$$\Pr\left\{N(t) = n\right\} = \frac{(rt)^n}{n!}e^{-rt}$$

is a fundamental for random events that occurs one by one in continuous time with a rate $r > 0$.

(a) Discuss how this mathematical formulation resolves both objections of the differential equation formulation,

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = r, \text{ a constant,}$$

based on "discreteness" and "randomness".

(b) When, if possible, are these two mathematical formulations being the same, and/or being approximately the same?

4. A nonlinear ordinary differential equation (ODE)

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = f(X),$$

is called *autonomous* if $f(X)$ is not an explicit function of time t. Assuming that the right-hand-side $f(X)$ is sufficiently smooth, according to the ODE and with a given value of $X(0) = x_0$, the value of $X(t)$ is determined for time $t > 0$. If the solution to the ODE $X(t)$ represents the population size of a biological species in an ecosystem, or the concentration of a biochemical "species" in a test tube, at time t, then the ODE represents a single-specie population dynamics.

(a) Explain when the ODE is a *linear* ODE, and when it is not? Is there anything special about the solution(s) to a linear ODE?

(b) For general smooth function $f(X)$, let x^* be a root to the algebraic equation $f(X) = 0$. Note there could be many roots. Please discuss the significance of these root in terms of the dynamics.

(c) Show that for a single-specie population dynamics described by an autonomous ODE, it is not possible to have oscillatory, nor periodic, solution $X(t)$.

(d) Now, some people might say that $X(t) = 2 + \sin(t)$, which is oscillatory and periodic, is a solution to the differential equation $(\mathrm{d}X/\mathrm{d}t)^2 = 1 - (X-2)^2$. Try to explain why this is not a legitimate *counter-example* to the statement in (c).

5. Consider the dynamics of a population that consists of n sub-populations, with their respective sizes at time t represented by $X_1(t)$, $X_2(t)$, $\cdots$, $X_n(t)$. Let us assume that for each sub-population,

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = r_1 X_1,\ \frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = r_2 X_2,\ \cdots,\ \frac{\mathrm{d}X_n(t)}{\mathrm{d}t} = r_n X_n.$$

So the dynamics of these n sub-poluations are completely independent from each other, and each is growing or decaying "exponentially" depending on the sign of its r. Now introducing the *per capita growth rate* (PCGR) of the total population at time t:

$$\overline{r}(t) = \frac{1}{X_{tot}(t)} \frac{\mathrm{d}X_{tot}(t)}{\mathrm{d}t}, \text{ where } X_{tot}(t) = \sum_{i=1}^{n} X_i(t).$$

(a) The PCGR of the total population is not a constant over time. Show that it is the "average" of the PCGRs of the sub-populations weighted by the population:

$$\overline{r} = \frac{\sum_{i=1}^{n} X_i r_i}{\sum_{i=1}^{n} X_i.}$$

(b) More interestingly, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{r}(t) = \frac{\sum_{i=1}^{n} X_i \Big(r_i - \overline{r}(t) \Big)^2}{\sum_{i=1}^{n} X_i(t)} \geq 0.$$

(c) For $n = 2$, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{X_1}{X_1 + X_2} \right) = \frac{(r_1 - r_2) X_1 X_2}{(X_1 + X_2)^2}.$$

Based on this mathematical expression, discuss what happens to the two sub-populations if $r_1 > r_2$, and if $r_2 > r_1$? What general conclusions can you reach for the total population $X_1(t) + X_2(t)$?

(d) In the biological context, the per capita growth rate is often used as the *fitness* of a population. Discuss the mathematical result in (b) with respect to the statement that "The fitness of a population always increases when there are variations within the population."

Additional Problems for AMATH 523

6. This is a continuation of Problem 3.

(a) A Poisson process is a mathematical theory about a sequence of **random events** that occur in random time one after another. There are two different representations:

(i) A counting process $N(t)$, where N takes non-negative integer values and $t \geq 0$, representing the number of events that have occured in time duration $[0, t)$;

(ii) A point process T_n, where T takes non-negative real value and $n = 1, 2, \cdots$, representing the random time for the the nth event to occur.

Using

$$P_n(t) = \Pr\left\{N(t) = n\right\} = \frac{(rt)^n}{n!} e^{-rt}$$

for the counting process and denoting

$$F_n(t) = \Pr\left\{\, T_n \leq t \,\right\}$$

for the point process, show that the probability density function $f_n(t) = \mathrm{d}F_n(t)/\mathrm{d}t$ is

$$f_n(t) = \frac{r^n\, t^{n-1}}{(n-1)!} e^{-rt}.$$

(b) Consider n independent and identially distributed random time $T^{(1)}$, $T^{(2)}$, $\cdots$, $T^{(n)}$ all have the same exponential distribution with rate r. Find the cumulative distribution function (CDF) and probability density function(PDF) for

$$T^{(1)} + T^{(2)} + \cdots + T^{(n)}.$$

(c) Discuss your results in (a) and (b).

7. This problem should be studied in compare and contrast with the earlier Problem 5 in which the growths of two the populations are independent:

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = r_1 X_1 \text{ and } \frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = r_2 X_2.$$

Now consider a biological population following sexual reproduction. A simple population dynamic model for the two sexes X_1 and X_2 is

$$\begin{aligned}\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} &= r_1 X_1 X_2, \\ \frac{\mathrm{d}X_2(t)}{\mathrm{d}t} &= r_2 X_1 X_2,\end{aligned}$$

in which r's can be either positive or negative.

(a) Based on your biological knowledge, why are r_1 and r_2 not the same? How to explain the "possibility" of a negative r?

(b) Derive the expression for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{X_1}{X_1+X_2}\right).$$

Based on the mathematical expression, discuss what happens to the two sub-populations. What general conclusions can you reach for the total population? Discuss your result in connection to the result in 5(c).

(c) For both cases of 5(c) and 7(b), what is the steady population ***ratio*** between X_1 and X_2, if exists, in the long-time limit?

(d) Checking out the concept of *Hardy–Weinberg principle*, and discuss the possible biological significance of the result in (c).