Homework # 5

1. Consider the dimensionless activator (u)-inhibitor (v) system represented by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a - bu + \frac{u^2}{v} = f(u,v),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = u^2 - v = g(u,v),$$

where b is a positive parameter while a can have both signs.

(a) Sketch the null clines for the system; mark the signs of f and g in the (u,v) phase plot.

(b) Determine the (a,b) parameter domain where the system might have periodic solutions.

(c) Show that the (a,b) parameter space in which u and v may exhibit periodic behavior is bounded by the curve

$$b = \frac{1+a}{1-a}.$$

2. Consider the Hopfield neural network of n neurons in continuous time in terms of a system of ODEs [Hopfield, J. J. (1984) *Proc. Natl. Acad. Sci. USA* **81**, 3088–3092]:

$$C_i\left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right) = \sum_{j=1}^{n} T_{ij}V_j - \frac{u_i}{R_i} + I_i, \tag{1a}$$

where $T_{ij} = T_{ji}$, and $V_i = g_i(u_i)$ for $i = 1, \cdots, n$. The functions $g_i(u)$ are monotonically increasing thus invertible; one denotes the inverse functions

$$u_i = g_i^{-1}(V_i), \;\; V_i = g_i(u_i). \tag{1b}$$

All I_i are constant.

(a) Eq. 1(a) is motivated by the Hodgkin-Huxley model. The V_i is the output electrical potential of the i^{th} neuron, and u_j is the input electrical potential to the j^{th} neuron; they can be different. Discuss the equation as well as the possible meaning of all the parameters.

(b) Introducing a scalar function of all the V's:

$$E\big(V_1, \cdots, V_n\big) = -\frac{1}{2}\sum_{i,j=1}^{n} V_iT_{ij}V_j + \sum_{i=1}^{n}\frac{1}{R_i}\int_0^{V_i} g_i^{-1}(v)\mathrm{d}v - \sum_{i=1}^{n} I_iV_i.$$

Denoting $\big(u_1, \cdots, u_n\big)(t)$ as a solution to Eq. 1, then correspondingly

$$\Big(V_1(t), V_2(t), \cdots, V_n(t)\Big) = \Big(g_1\big(u_1(t)\big), g_2\big(u_2(t)\big), \cdots, g_n\big(u_n(t)\big)\Big).$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}E\big(V_1(t), \cdots, V_n(t)\big) \le 0.$$

(c) Taking $n = 2$, show that if $T_{12} = T_{21}$ in Eq. 1(a), the neural network cannot oscillate.

(d) Again for $n = 2$ and assume $g_1(u_1) = u_1$ and $g_2(u_2) = u_2$. Give an example in which $T_{12} \neq T_{21}$ and the system has oscillations.

3. One of the simplest mathematical models for infection epidemics is the SIR model,

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = -rSI, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = rSI - aI, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = aI, \end{cases}$$

in which S represents the number of susceptible individuals, I stands for the population size of infectious individuals, and R for the number of removed, *i.e.* immune and/or deceased individuals. In this model, it is assumed that individuals after infection either recover with immunity from or die of the disease.

(a) Design a system of chemical reactions with chemical species S, I, and R, which under the law of mass action yields the above differential equations.

(b) At the very beginning of the spreading of the disease, one assumes that total $S(0) = S_0$, $R(0) = 0$, and $I(0) = I_0$. Then at $t = 0$ if $\frac{\mathrm{d}I}{\mathrm{d}t}(0) < 0$, the population of the infectious individuals decreases and there will not be an epidemic. On the other hand, if

$$\frac{\mathrm{d}I}{\mathrm{d}t}(0) > 0,$$

then $I(t)$ grows and there is an epidemic. Find the condition on S_0 and I_0, in terms of the two parameters r and a, that is indicative of the occurence of an epidemic.

(c) The first two equations in the above system can be transformed into

$$\frac{\mathrm{d}I}{\mathrm{d}S} = -\frac{rSI - aI}{rSI} = -1 + \left(\frac{a}{r}\right)\frac{1}{S}.$$

Solve this differential equation, show that

$$I(S) = I_0 + \big(S_0 - S\big) + \frac{a}{r}\log\frac{S}{S_0},$$

and discuss your finding.

Additional Problems for AMATH 523

4. If the time duration of an epidemic process prolongs, the population birth and death that are not related to the epidemic become relevant. Then one has

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu(S+I+R) - \gamma S - rSI, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = -\gamma I + rSI - aI, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = -\gamma R + aI, \end{cases}$$

where μ and γ are the per capita birth and death rates.

(a) Explain the additional terms in the equations.

(b) Denoting the total population $N(t) = S(t) + I(t) + R(t)$, what is the differential equation for $N(t)$:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = ?$$

(c) Introducing

$$\hat{S}(t) = \frac{S(t)}{N(t)}, \quad \hat{I}(t) = \frac{I(t)}{N(t)}, \quad \hat{R}(t) = \frac{R(t)}{N(t)},$$

as the fractions of the populations within the total $N(t)$. Show that

$$\begin{aligned} \frac{\mathrm{d}\hat{S}}{\mathrm{d}t} &= \mu\big(\hat{I} + \hat{R}\big) - rN_0 e^{(\mu-\gamma)t}\hat{S}\hat{I}, \\ \frac{\mathrm{d}\hat{I}}{\mathrm{d}t} &= rN_0 e^{(\mu-\gamma)t}\hat{S}\hat{I} - \big(a+\mu\big)\hat{I}, \\ \frac{\mathrm{d}\hat{R}}{\mathrm{d}t} &= \big(a-\mu\big)\hat{R}. \end{aligned}$$