AMATH 423 HW #5 | Julien Goldstick | February 24, 2025

1. Consider the dimensionless activator (u)-inhibitor (v) system represented by

$$\frac{du}{dt} = a - bu + \frac{u^2}{v} = f(u, v) \tag{1a}$$

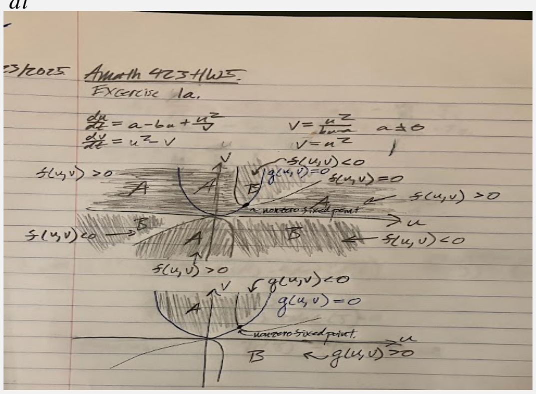
$$\frac{dv}{dt} = u^2 - v = g(u, v) \tag{1b}$$

where b is a positive parameter while a can have both signs

(a) Sketch the null clines for the system; mark the signs of f and g in the (u, v) phase plot.

$\frac{du}{dt} = a - bu + \frac{u^2}{v} = 0 \Rightarrow \frac{u^2}{v} = bu - a \Rightarrow u^2 = (bu - a)v \Rightarrow v = \frac{u^2}{bu-a}, \quad bu - a \neq 0$

$\frac{dv}{dt} = u^2 - v = 0 \Rightarrow u^2 = v$



(b) Determine the (a, b) parameter domain where the system might have periodic solutions.

$J = \begin{bmatrix} \frac{df}{du} & \frac{df}{dv} \\ \frac{dg}{du} & \frac{dg}{dv} \end{bmatrix} \Rightarrow \begin{pmatrix} -b + \frac{2u}{v} & -\frac{u^2}{v^2} \\ 2u & -1 \end{pmatrix}_{(\frac{a+1}{b},(\frac{a+1}{b})^2)} \Rightarrow \begin{pmatrix} -b + \frac{2(\frac{a+1}{b})}{(\frac{a+1}{b})^2} & -\frac{(\frac{a+1}{b})^2}{((\frac{a+1}{b})^2)^2} \\ 2(\frac{a+1}{b}) & -1 \end{pmatrix} \Rightarrow$
$\begin{pmatrix} -b + 2\frac{b}{a+1} & -\frac{b^2}{(a+1)^2} \\ 2(\frac{a+1}{b}) & -1 \end{pmatrix}$
For Hopf Bifurcation which allows for periodic solutions the two conditions must be satisfied:
$tr(A) = 0$ and $\det(A) > 0$
$tr(A) = 0 = -1 - b + 2\frac{b}{a+1} = 0 \Rightarrow -1 - b + 2\frac{b}{a+1} = 0 \Rightarrow 2\frac{b}{a+1} = b + 1 \Rightarrow 2b = (b+1)(a+1) \Rightarrow$
$2b = ba + b + a + 1 \Rightarrow b - ba = a + 1 \Rightarrow b(1-a) = 1 + a \Rightarrow \boxed{b = \frac{1+a}{1-a}, a \neq 1}$
$\det(A) = (-b + 2\frac{b}{a+1})(-1) - (-\frac{b^2}{(a+1)^2})(2(\frac{a+1}{b})) > 0$
$b - 2\frac{b}{a+1} + (\frac{b^2}{(a+1)^2})(2(\frac{a+1}{b})) > 0 \Rightarrow b - 2\frac{b}{a+1} + 2\frac{b}{a+1} > 0 \Rightarrow b > 0$

$b = \frac{1+a}{1-a} > 0 \Rightarrow 1 + a > 0 \Rightarrow a > -1$
$1 - a > 0 \Rightarrow -a > -1 \Rightarrow a < 1$
Parameter Domain with potentially periodic solutions:
$\boxed{b = \frac{1+a}{1-a} > 0, a \neq 1, -1 < a < 1}$

(c) Show that the (a, b) parameter space in which u and v may exhibit periodic behavior is bounded by the curve

$$b = \frac{1+a}{1-a} \tag{2}$$

Periodic behavior is bounded by the curve when a Hopf bifurcation exists. which is when the Trace of the matrix is zero. Solving for the trace of the matrix to equal zero results in the above equation.
$tr(A) = 0 = -1 - b + \frac{2b}{a+1} = 0 \Rightarrow -1 - b + \frac{2b}{a+1} = 0 \Rightarrow \frac{2b}{a+1} = b + 1 \Rightarrow 2b = (b+1)(a+1) \Rightarrow$
$2b = ba + b + a + 1 \Rightarrow b - ba = a + 1 \Rightarrow b(1-a) = 1 + a \Rightarrow b = \frac{1+a}{1-a}$

2. Consider the Hopfield neural network of n neurons in continuous time in terms of a system of ODEs [Hopfield, J. J. (1984) Proc. Natl. Acad. Sci. USA **81**, 3088–3092]:

$$C_i\left(\frac{du_i}{dt}\right) = \sum_{j=1}^{n} T_{ij}V_j - \frac{u_i}{R_i} + I_i, \tag{3}$$

here $T_{ij} = T_{ji}$, and $V_i = g_i(u_i)$ for $i = 1 \dots n$. The functions $g_i(u)$ are monotonically increasing thus invertible; one denotes the inverse functions

$$u_i = g_i^{-1}(V_i), V_i = g_i(u_i). \tag{4}$$

All I_i are constant.

(a) Eq. 1(a) is motivated by the Hodgkin-Huxley model. The V_i is the output electrical potential of the i^{th} neuron, and u_j is the input electrical potential to the j^{th} neuron; they can be different. Discuss the equation as well as the possible meaning of all the parameters.

> The Hopfield neural network is an example of a model used for associative memory, where neurons can update their memory states based on the weighted inputs given. The symmetry in T_{ij} ensures that the network can be described using an energy function, leading to convergence toward stable states. The model allows for memory to be stored in a dynamic and changing way not relying on a recursive neural network model.
> C_i is the capacitance term
> $T_{ij}Vj$ is the Synaptic Coupling Term which is the summation term represents the weighted influence of other neurons on neuron i. The weights T_{ij} dictate the connectivity between neurons and define how much neuron j contributes to the potential of neuron i
> $-\frac{u_i}{R_i}$ is the Leakage Term. This term accounts for the dissipation of electrical charge due to passive membrane properties
> I_i is the External Input Term. This represents a constant external input to neuron i, such as an external stimulus or a bias current.
> $V_i = g_i(u_i)$ is the Activation Function. The function g_i defines the transformation between input potential u_i and output potential V_i.

(b) Introducing a scalar function of all the V's:

$$E\left(V_1 \dots V_n\right) = -\frac{1}{2}\sum_{i,j=1}^{n} V_i T_{ij} V_j + \sum \frac{1}{R_i}\int_0^{V_i} g_i^{-1}(v)dv - \sum_{i=1}^{n} I_i V_i. \tag{5}$$

Denoting $(u_1, , u_n)(t)$ as a solution to Eq. 1, then correspondingly

$$\left(V_1(t), V_2(t), \dots, V_n(t)\right) = \left(g_1(u_1(t)), g_2(u_2(t)), \dots, g_n(u_n(t))\right). \tag{6}$$

show that

$$\frac{d}{dt}E(V_1(t), \dots, V_n(t)) \le 0 \tag{7}$$

$E\left(V_1 \ldots V_n\right) = -\frac{1}{2}\sum_{i,j=1}^{n} V_i T_{ij} V_j + \sum \frac{1}{R_i}\int_0^{V_i} g_i^{-1}(v)dv - \sum_{i=1}^{n} I_i V_i.$
$\frac{d}{dt}E\left(V_1 \ldots V_n\right) = -\frac{d}{dt}\frac{1}{2}\sum_{i,j=1}^{n} V_i T_{ij} V_j + \frac{d}{dt}\sum \frac{1}{R_i}\int_0^{V_i} g_i^{-1}(v)dv - \frac{d}{dt}\sum_{i=1}^{n} I_i V_i.$
Differentiate the Quadratic Term
$\frac{d}{dt}\left(-\frac{1}{2}\sum_{i,j=1}^{n} V_i T_{ij} V_j\right) = -\sum_{i,j=1}^{n} \frac{dV_i}{dt} T_{ij} V_j.$
Since T_{ij} is symmetric, we can rewrite this as: $-\sum_{i=1}^{n} \frac{dV_i}{dt}\sum_{j=1}^{n} T_{ij} V_j.$
Differentiate the Integral Term Using the fundamental theorem of calculus:
$\frac{d}{dt}\sum_{i=1}^{n} \frac{1}{R_i}\int_0^{V_i} g_i^{-1}(v)dv = \sum_{i=1}^{n} \frac{1}{R_i} g_i^{-1}(V_i)\frac{dV_i}{dt}.$
Differentiate the Linear Term
$\frac{d}{dt}\left(-\sum_{i=1}^{n} I_i V_i\right) = -\sum_{i=1}^{n} I_i \frac{dV_i}{dt}.$
Combine the Terms
$\frac{d}{dt}E = -\sum_{i=1}^{n} \frac{dV_i}{dt}\sum_{j=1}^{n} T_{ij} V_j + \sum_{i=1}^{n} \frac{1}{R_i} g_i^{-1}(V_i)\frac{dV_i}{dt} - \sum_{i=1}^{n} I_i \frac{dV_i}{dt}.$
Factoring out $\frac{dV_i}{dt}$:
$\frac{d}{dt}E = \sum_{i=1}^{n} \frac{dV_i}{dt}\left(\frac{1}{R_i} g_i^{-1}(V_i) - I_i - \sum_{j=1}^{n} T_{ij} V_j\right).$
$\frac{d}{dt}E = \sum_{i=1}^{n} \frac{d}{dt} g_i^{-1}(V_i)\left(\frac{1}{R_i} g_i^{-1}(V_i) - I_i - \sum_{j=1}^{n} T_{ij} V_j\right).$
$\frac{d}{dt}E = \sum_{i=1}^{n} \frac{d}{dt} g_i^{-1}(V_i)\left(\frac{1}{R_i} u_i - I_i - \sum_{j=1}^{n} T_{ij} V_j\right).$
Using the differential equation:
$C_i \frac{dV_i}{dt} = \sum_{j=1}^{n} T_{ij} V_j - \frac{1}{R_i} g_i^{-1}(V_i) + I_i, \Rightarrow -\frac{dV_i}{dt} = \frac{1}{C_i}\left(\frac{1}{R_i} g_i^{-1}(V_i) - \sum_{j=1}^{n} T_{ij} V_j - I_i\right).$
Substitute in $\frac{dV_i}{dt}$
$\frac{d}{dt}E = \sum_{i=1}^{n}\left(\frac{1}{C_i} * -\left(\frac{1}{R_i} g_i^{-1}(V_i) - \sum_{j=1}^{n} T_{ij} V_j - I_i\right)^2\right).$
The squared term and C_1 are positive. The negative sign in front makes the expression always < 0. Therfore the derivative must be less than zero. We conclude:
$\frac{d}{dt}E = \sum_{i=1}^{n} -\frac{1}{C_i}\left(-\sum_{j=1}^{n} T_{ij} V_j + \frac{1}{R_i} g_i^{-1}(V_i) - I_i\right)^2 \leq 0$

(c) Taking $n = 2$, show that if $T_{12} = T_{21}$ in Eq. 1(a), the neural network cannot oscillate

$C_i\left(\frac{du_i}{dt}\right) = \sum_{j=1}^{n} T_{ij}V_j - \frac{u_i}{R_i} + I_i,$
$n = 2$
$C_1\left(\frac{du_1}{dt}\right) = \sum_{j=1}^{2} T_{ij}V_i - \frac{u_1}{R_1} + I_1 \Rightarrow C_1\left(\frac{du_1}{dt}\right) = T_{11}V_1 + T_{12}V_2 - \frac{u_1}{R_1} + I_1$
$C_2\left(\frac{du_2}{dt}\right) = \sum_{j=1}^{2} T_{ij}V_i - \frac{u_2}{R_2} + I_2 \Rightarrow C_2\left(\frac{du_2}{dt}\right) = T_{21}V_1 + T_{22}V_2 - \frac{u_2}{R_2} + I_2$
$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \begin{pmatrix} -\frac{u_1}{R_1} + I_1 \\ -\frac{u_2}{R_2} + I_2 \end{pmatrix}$
$V_1 = g_1(u_1) = u_1, V_2 = g_2(u_2) = u_2$
$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -\frac{u_1}{R_1} + I_1 \\ -\frac{u_2}{R_2} + I_2 \end{pmatrix}$
Calculate Jacobian Matrix.
$\begin{pmatrix} \frac{T_{11}}{C_1} - \frac{1}{C_1R_1} & \frac{T_{12}}{C_1} \\ \frac{T_{21}}{C_2} & \frac{T_{22}}{C_2} - \frac{1}{C_2R_2} \end{pmatrix}$
$\det(J - \lambda I) = 0 \rightarrow \lambda^2 - Tr(J) + \det(J) = 0 \rightarrow \lambda_{1,2} = \frac{1}{2}\left(\mathrm{Tr(J)} \pm \sqrt{\mathrm{Tr(J)}^2 - 4\det(\mathrm{J})}\right)$
For there to be no oscillation there must be two real eigenvalues. Meaning the discriminant must be > 0 which is $\mathrm{Tr(J)}^2 - 4\det(\mathrm{J}) > 0$
$Tr(J) = (\frac{T_{11}}{C_1} - \frac{1}{C_1R_1}) + (\frac{T_{22}}{C_2} - \frac{1}{C_2R_2})$
$\det(J) = (\frac{T_{11}}{C_1} - \frac{1}{C_1R_1})(\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}) - (\frac{T_{12}}{C_1})(\frac{T_{21}}{C_2})$
$\mathrm{Tr(J)}^2 - 4\det(\mathrm{J}) > 0 \rightarrow$
$((\frac{T_{11}}{C_1} - \frac{1}{C_1R_1}) + (\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}))^2 + (-4)((\frac{T_{11}}{C_1} - \frac{1}{C_1R_1})(\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}) - (\frac{T_{12}}{C_1})(\frac{T_{21}}{C_2})) > 0$
Simplifies to.
$\frac{1}{C_1C_2}(-3T_{11}T_{22} + \frac{3T_{11}}{R_1} + \frac{3T_{22}}{R_2} - \frac{3}{R_1R_2} + 4T_{12}T_{21}) > 0$
$\frac{1}{C_1C_2}(-3(T_{11} - \frac{1}{R_2})(T_{22} - \frac{1}{R_1}) + 4T_{12}T_{21}) > 0$
So, when $T_{12} = T_{21}$ the discriminant is always non-negative, meaning the eigenvalues are always real. This confirms that the neural network cannot oscillate when the connectivity matrix is symmetric.

(d) Again for $n = 2$ and assume $g_1(u_1) = u_1$ and $g_2(u_2) = u_2$. Give an example in which $T_{12} \neq T_{21}$ and the system has oscillations.

$C_i\left(\frac{du_i}{dt}\right) = \sum_{j=1}^{n} T_{ij}V_j - \frac{u_i}{R_i} + I_i,$
$n = 2$
$C_1\left(\frac{du_1}{dt}\right) = \sum_{j=1}^{2} T_{ij}V_i - \frac{u_1}{R_1} + I_1 \Rightarrow C_1\left(\frac{du_1}{dt}\right) = T_{11}V_1 + T_{12}V_2 - \frac{u_1}{R_1} + I_1$
$C_2\left(\frac{du_2}{dt}\right) = \sum_{j=1}^{2} T_{ij}V_i - \frac{u_2}{R_2} + I_2 \Rightarrow C_2\left(\frac{du_2}{dt}\right) = T_{21}V_1 + T_{22}V_2 - \frac{u_2}{R_2} + I_2$
$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \begin{pmatrix} -\frac{u_1}{R_1} + I_1 \\ -\frac{u_2}{R_2} + I_2 \end{pmatrix}$
$V_1 = g_1(u_1) = u_1,\ V_2 = g_2(u_2) = u_2$
$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -\frac{u_1}{R_1} + I_1 \\ -\frac{u_2}{R_2} + I_2 \end{pmatrix}$
Calculate Jacobian Matrix.
$\begin{pmatrix} \frac{T_{11}}{C_1} - \frac{1}{C_1R_1} & \frac{T_{12}}{C_1} \\ \frac{T_{21}}{C_2} & \frac{T_{22}}{C_2} - \frac{1}{C_2R_2} \end{pmatrix}$
$\det(J - \lambda I) = 0 \to \lambda^2 - Tr(J) + \det(J) = 0 \to \lambda_{1,2} = \frac{1}{2}\left(\mathrm{Tr(J)} \pm \sqrt{\mathrm{Tr(J)}^2 - 4\det(\mathrm{J})}\right)$
For there to be oscillations there must be two imaginary eigenvalues. Meaning the discriminant must be < 0 which is $\mathrm{Tr(J)}^2 - 4\det(\mathrm{J}) < 0$
$Tr(J) = (\frac{T_{11}}{C_1} - \frac{1}{C_1R_1}) + (\frac{T_{22}}{C_2} - \frac{1}{C_2R_2})$
$\det(J) = (\frac{T_{11}}{C_1} - \frac{1}{C_1R_1})(\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}) - (\frac{T_{12}}{C_1})(\frac{T_{21}}{C_2})$
$\mathrm{Tr(J)}^2 - 4\det(\mathrm{J}) > 0 \to$
$((\frac{T_{11}}{C_1} - \frac{1}{C_1R_1}) + (\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}))^2 + (-4)((\frac{T_{11}}{C_1} - \frac{1}{C_1R_1})(\frac{T_{22}}{C_2} - \frac{1}{C_2R_2}) - (\frac{T_{12}}{C_1})(\frac{T_{21}}{C_2})) < 0$ Simplifies to. $\frac{1}{C_1C_2}(-3T_{11}T_{22} + \frac{3T_{11}}{R_1} + \frac{3T_{22}}{R_2} - \frac{3}{R_1R_2} + 4T_{12}T_{21}) < 0$
$\frac{1}{C_1C_2}(-3(T_{11} - \frac{1}{R_2})(T_{22} - \frac{1}{R_1}) + 4T_{12}T_{21}) < 0$
If $T_{12} \neq T_{21}$ then it is possible to have this expression be satisfied and thus there would be negative eigenvectors and oscillations. This ensures the discriminant is negative and the eigenvalues are complex, leading to oscillations.

3. One of the simplest mathematical models for infection epidemics is the SIR mode

$$\begin{cases} \frac{dS}{dt} = -rSI, \\ \frac{dI}{dt} = rSI - aI, \\ \frac{dR}{dt} = aI, \end{cases} \tag{8}$$

n which S represents the number of susceptible individuals, I stands for the population size of infectious individuals, and R for the number of removed, i.e. immune and/or deceased individuals. In this model, it is assumed that individuals after infection either recover with immunity from or die of the disease

(a) Design a system of chemical reactions with chemical species S, I, and R, which under the law of mass action yields the above differential equations.

$S + I \xrightarrow{r} 2I$ I converts S to I with rate r.
$I \xrightarrow{a} R$ I converts to R with rate a.

(b) At the very beginning of the spreading of the disease, one assumes that total $S(0) = S_0$, $R(0) = 0$, and $I(0) = I_0$. Then at $t = 0$ if $\frac{dI}{dt}(0) < 0$, the population of the infectious individuals decreases and there will not be an epidemic. On the other hand, if

$$\frac{dI}{dt}(0) > 0 \tag{9}$$

then $I(t)$ grows and there is an epidemic. Find the condition on S_0 and I_0, in terms of the two parameters r and a, that is indicative of the occurrence of an epidemic.

$\frac{dI}{dt} = rSI - aI \Rightarrow \frac{dI}{dt}(0) = rS_0I_0 - aI_0 > 0 \Rightarrow rS_0I_0 - aI_0 > 0 \Rightarrow rS_0I_0 > aI_0 \Rightarrow rS_0 > a \Rightarrow$ $\boxed{S_0 > \frac{a}{r}}$

If $S_0 > \frac{a}{r}$ is indicative of the occurrence of an epidemic, because then the intial rate of infected population will be increasing and $I(t)$ grows and there is an epidemic. From the equation it shows that I_0 can be anything for an epidemic to still occur.

(c) The first two equations in the above system can be transformed into

$$\frac{dI}{dS} = -\frac{rSI - aI}{rSI} = -1 + \left(\frac{a}{r}\right)\frac{1}{S} \tag{10}$$

Solve this differential equation, show that

$$I(S) = I_0 + (S_0 - S) + \frac{a}{r}\log\frac{S}{S_0} \tag{11}$$

and discuss your finding.

$\frac{dI}{dS} = -1 + \left(\frac{a}{r}\right)\frac{1}{S} \Rightarrow dI = \left(-1 + \left(\frac{a}{r}\right)\frac{1}{S}\right) dS \Rightarrow dI = (-1)\,dS + \left(\frac{a}{rS}\right) dS \Rightarrow$
$\int dI = \int (-1)\,dS + \int \left(\frac{a}{rS}\right) dS \Rightarrow \int_{I_0}^{I(S)} dI = \int_{S_0}^{S} (-1)\,dS + \int_{S_0}^{S} \left(\frac{a}{rS}\right) dS \Rightarrow$
$I(S) - I_0 = -(S - S_0) + \frac{a}{r}(\log(S) - \log(S_0)) \Rightarrow I(S) = I_0 + (S_0 - S) + \frac{a}{r}\log(\frac{S}{S_0}), r \neq 0$
This finding shows that a Linear term, a nonlinear logarithmic term and a constant term.
We care about when $I(S) = 0$ which is when there are no infected people.
Taking $a = 1, r = 4, I_0 = 1, S = 2$ $I(S)$ looks like this: