Homework # 6

1. Consider a system of nonlinear biochemical reactions, known as reversible Schnakenberg model, which consists of four species and three reactions:

$$A \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} C, \quad B \underset{k_{-2}}{\overset{k_{+2}}{\rightleftharpoons}} D, \quad 2C + D \underset{k_{-3}}{\overset{k_{+3}}{\rightleftharpoons}} 3C.$$

Denote the concentrations of A, B, C, and D at time t as $c_A(t)$, $c_B(t)$, $c_C(t)$, and $c_D(t)$.

(a) Write down the system of nonlinear differential equations for the chemical kinetics according to the *law of mass action*.

(b) Assuming the total initial concentration, at $t = 0$, for A, B, C and D all together is c_0. Find the steady state concentrations for all four chemical species $(c_A^*, c_B^*, c_C^*, c_D^*)$.

(c) Show that the following function of the c's:

$$L(\vec{c}) = \sum_{X=A,B,C,D} c_X \ln\left(\frac{c_X}{c_X^*}\right), \quad \text{where } \vec{c} = \big(c_A, c_B, c_C, c_D\big),$$

is a Lyapunov function of the dynamical system. That is:

(i) $L(\vec{c}) \geq 0$ and $L(\vec{c}) = 0$ if and only if $\vec{c} = \vec{c}^*$;

(ii) $L(\vec{c})$ is convex;

(iii)

$$\frac{\mathrm{d}}{\mathrm{d}t} L[\,\vec{c}(t)\,] \leq 0.$$

(d) Is the fixed point $\vec{c}^*$ stable? Is it unique?

2. Consider the FitzHugh-Nagumo equation

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= f(v) - w + I_a, \\
\frac{\mathrm{d}w}{\mathrm{d}t} &= bv - \gamma w, \\
f(v) &= v(a - v)(v - 1),
\end{aligned}$$

where $I_a = 0$, $a = 0.25$, $b = \gamma = 2 \times 10^{-3}$.

(a) Draw the null clines in the vw plane. Draw the directions of the vector field in different regions. One of the null clines has a minumum and a maximum. Analytically determine the coordinates (v, w) for the minimum and the maxium.

(b) With increasing $I_a > 0$, the fixed point (i.e., steady state) changes its stability at I_1 and I_2. What are the values for I_1 and I_2 with the above given values for a, b and γ?

(c) Plot a solution to the FitzHugh-Nagumo equation with some I_a, inside and outside the interval $[I_1, I_2]$. Discribe your finding. You need use MATLAB. If do not know how to use MATLAB, then try to find online resources such as the useful ODE solver that gives 2-dimensional phase plane at

`https://aeb019.hosted.uark.edu/pplane.html`

3. Consider a stochastic population model for SIR epidemic,

$$S + I \xrightarrow{r} 2I, \;\; I \xrightarrow{a} R.$$

In the very early time of the epidemic, the population of the infectious individuals $I(t) \ll S(t)$, the population of the susceptible individuals. So it is reasonable to assume that $S(t) \equiv S_0$ is a constant.

(a) Denote the probability of having k number of infectious individuals at time t as $\Pr\{I(t) = k\} = p_k(t)$. Show that

$$\frac{\mathrm{d}p_k(t)}{\mathrm{d}t} = r(k-1)S_0 p_{k-1} - rkS_0 p_k.$$

(b) Assuming that $I(0) = n$, show that the solution to the system of ODEs in (a) is

$$p_k(t) = \binom{k-1}{k-n} e^{-rS_0nt}\left(1 - e^{-rS_0t}\right)^{k-n},$$

where $k \geq n$. This is known as a negative binomial distribution. Furthermore,

$$p_k(0) = \left[\binom{k-1}{k-n} e^{-rS_0nt}\left(1 - e^{-rS_0t}\right)^{k-n}\right]_{t=0} = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

(c) Show that the expected value $\mathbb{E}[I(t)]$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[I(t)] = rS_0\,\mathbb{E}[I(t)].$$

Additional Problems for AMATH 523

4. Consider again the stochastic SIR model in Problem 3. We still assume that $S \equiv S_0$ to be constant.

(a) If at $t = 0$ there are $I(0) = n$ number of infectious individuals, what is the mean time for the next infection?

(b) Continue the calculation for mean time τ_m when there are m number of new infections after $I(0) = n$.

(c) According to the naive deterministic logic, the population at time τ_m is $n + m$. Therefore one expects to have

$$(n + m) = ne^{rS_0\tau_m},$$

in which rS_0 is analogous to the "per capita" birth rate. How do you reconcile this result with that you found in (b)?