Homework # 2

1. (i) Obtain the general solution to the pair of linear, homogeneous, ordinary differential equations (ODEs) with constant coefficients:

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t} &= -X-3Y,\\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= -2X-2Y.\end{aligned} \tag{1}$$

(ii) Find the particular solution to the above equation with initial values $X(0) = Y(0) = 3$.

(iii) Plot the two lines in a (X, Y) graph:

$$-X-3Y=0,$$

$$-2X-2Y=0;$$

what is the meaning of these two lines; and what is the meanng of the intersection point, for the pair of ODEs?

(iv) What is meant by a *trajectory* of the above system? Describe the behaviour of the trajectories of the system as $t \to +\infty$ and $t \to -\infty$.

(v) Suppose $x(t)$ and $y(t)$ are population sizes of two competing biological species which have an equilibrium point at (x_c, y_c) where x_c, y_c are both positive. If one denotes $X = x - x_c$ and $Y = y - y_c$, and they satisfy the above pair of equations, roughly sketch the phase portrait in the first quadrant of the (x, y) plane. Discuss, in detail, the fate of populations of x and y.

2. Let us again consider a pair of ODEs:

$$\begin{aligned}\frac{dX(t)}{dt} &= aX+bY,\\ \frac{dY(t)}{dt} &= cX+dY,\end{aligned}$$

where a, b, c, d are four arbitrary constants. This time try to eliminate the unknown function Y and obtain a 2nd order ODE for $X(t)$ in the form of

$$\frac{d^2X}{dt^2}+\alpha\frac{dX}{dt}+\beta X=0. \tag{2}$$

Find out α and β in terms of a, b, c, d. Discuss the significance of your finding.

3. Consider a biological population with size $S(t)$ at time t on an island. Assume that the per capita growth rate for the population in the absence of immigration is

$$\text{per capita growth rate } = -r,$$

where $r > 0$. Furthermore, the rate of population immigration from mainland into the island is expressed by

$$\text{rate of immigration} = I_0 - \frac{I_0}{T} S,$$

where I_0 is the rate of immigration to the island when it is empty. The immigration rate decreases with increasing S, and the T represents the carrying capacity of the island.

(i) Show that the equilibrium population size on the island is given by

$$\hat{S} = \frac{T I_0}{rT + I_0}.$$

Explain this result by graphs.

(ii) Suppose that I_0 is linearly related to the distance between the island and the mainland. Show that in this case

$$\hat{S} = \frac{T(D^* - D)}{D^*(1+\mu) - D},$$

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where D is the distance from the mainland and D^* is the critical distance beyond which no species can immigrate. What is the meaning of the dimensionless parameter μ?

4. Consider a population consisting of identical and independent individual organisms, each with an exponentially distributed time for giving "birth", with rate λ, and going "death", with rate μ.

(i) Now when the population has exactly n individuals, what is the probability distribution for the waiting time to the next birth? What is the probability distribution for the waiting time to the next death? What is the probability distribution for the waiting time to the next birth or death event?

(ii) Let $p_n(t)$ be the probability of having exactly n individuals in the population at time t. The $p_n(t)$ satisfies a system of (infinite number!) differential equations:

$$\begin{aligned}
\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} &= \mu p_1(t), \\
\frac{\mathrm{d}p_n(t)}{\mathrm{d}t} &= (n-1)\lambda p_{n-1}(t) - n(\lambda+\mu)p_n(t) + (n+1)\mu p_{n+1}(t), \\
& \quad n = 1, 2, \cdots
\end{aligned}$$

Explain why this set of equations are true.

(iii) Assuming that the order of derivative and infinit summation below can exchange, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} p_n(t) = \sum_{n=0}^{\infty} \frac{\mathrm{d}p_n(t)}{\mathrm{d}t},$$

show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty} p_n(t) = 0.$$

(iv) The mean population at time t is defined as

$$u(t) := \sum_{n=0}^{\infty} np_n(t).$$

Based on the system of differential equations in (ii), show that

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \big(\lambda - \mu\big)u(t).$$

Additional Problems for AMATH 523

5. A chemostat is a continuous culture device used for growing and studying bacteria. Nutrient is added at a constant rate, say $S_0\lambda$ (Mass/Volume)(1/Time), to the growth chamber where living cells are stirred in the enriched media. The growth chamber is continually adjusted to keep a constant volume by removing bacteria-containing fluid at the flow rate λ. Let $S(t)$ denote the concentration of nutrient in the growth chamber at time t, and let $B(t)$ denote the concentration of bacteria. (concentration = number/volume)

One mathematical model of the chemstat accounts for the addition of nutrient and the washing out of bacterial cells:

$$\begin{aligned} \frac{dS}{dt} &= \lambda(S_0 - S) - \frac{VSB}{K+S} \\ \frac{dB}{dt} &= \frac{VSB}{K+S}Y - \lambda B \end{aligned}$$

where λ is the flow rate, V is the maximum uptake rate of the nutrient by per bacterial cell, K is the saturation constant of nutrient uptake and Y is the yield of cells per unit nutrient taken up.

(a) Show that if λ is too large, then the cell population washes out. Determine the critical λ.

(b) Determine the optimal steady flow for maximal bacteria output while keeping the cell population steady.