AMATH 423 HW #3 | Julien Goldstick | February 1, 2025

1. Problem 1 on P. 115 of the J. D. Murray's Chapter 3.
   In the competition model for two species with populations N_1 and N_2

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right), \tag{1}$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2},\right), \tag{2}$$

   where only one species, N_1, has limited carrying capacity. Nondimensionalise the system and determine the steady states. Investigate their stability and sketch the phase plane trajectories. Show that irrespective of the size of the parameters the principle of competitive exclusion holds. Briefly describe under what ecological circumstances the species N_2 becomes extinct

1. Nondimensionalise the system
$u_1 = \frac{N_1}{K_1}, u_2 = \frac{N_2}{K_2}, a_{12} = b_{12}\frac{K_2}{K_1}, a_{21} = b_{21}\frac{K_1}{K_2}, \rho = \frac{r_2}{r_1}, \tau = r_1 t, K_1 du_1 = dN_1, K_2 du_2 = dN_2$

$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2)$
$\frac{du_2}{d\tau} = \rho u_2(1 - a_{21}u_1)$

2. Determine the steady states.
Steady states when $\frac{du_1}{d\tau} = \frac{du_2}{d\tau} = 0$

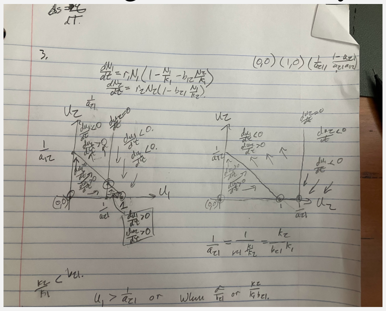
$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2) = 0$
If $u_1 = 0$ then $\frac{du_1}{d\tau} = 0$. Find steady state for $\frac{du_2}{d\tau}$
$\frac{du_2}{d\tau} = \rho u_2(1 - a_{21}u_1) = 0 \longrightarrow \rho u_2(1 - a_{21}(0)) = 0$
$\rho u_2 = 0 \longrightarrow u_2 = 0$ Steady state at (0,0)

$\frac{du_2}{d\tau} = \rho u_2(1 - a_{21}u_1) = 0$ If $u_2 = 0$ then $\frac{du_2}{d\tau} = 0$. Find steady state for $\frac{du_1}{d\tau}$
$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2) = 0 \longrightarrow u_1(1 - u_1) = 0$
$u_1 = 1, u_1 = 0$
Steady state at (1,0)

$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2) = 0$
If $(1 - u_1 - a_{12}u_2) = 0$ then $\frac{du_1}{d\tau} = 0$. so
$\frac{du_2}{d\tau} = \rho u_2(1 - a_{21}u_1) = 0 \longrightarrow \rho u_2(1 - a_{21}(0)) = 0$

$u_1 = 1 - a_{12}u_2$
(0,0), (1,0), $(\frac{1}{a_{21}}, \frac{a_{21}-1}{a_{21}a_{12}})$

3. Investigate their stability and sketch the phase plane trajectories



4. Show that irrespective of the size of the parameters the principle of competitive exclusion holds.

”The competitive exclusion principle whereby 2 species competing for the same limited resource cannot in general coexist.”

So in the example is where $\frac{1}{a_{21}} > 1$ given any inital conditions the stronger compeititor will outcompete the other population and cause it to be extinct. Meaning that the competitive exclusion speed is
So in the example where $\frac{1}{a_{21}} < 1$ if they start at the right initial conditions they can reach an equilibrium value.

5. Briefly describe under what ecological circumstances the species N_2 becomes extinct.
According to the phase plane analysis N_2 becomes extinct when $u_2 = 0$. There are two equilibirum points where this occurs (0,0) and (1,0).
The phase plan diagram shows that in the long term the population will go towards (1,0) when either $u_1 > \frac{1}{a_{21}}$ or when $\frac{K_2}{K_1 b_{21}}$

2. Write the predator-prey type equations for a system of two predators and two prey in which the prey do not compete with each other but both predator species eat both prey species. Assuming both the prey have finite carrying capacities but predators do not.

Prey Population Equations

$\frac{dx_1}{dt} = x_1 r_1(1 - \frac{x_1}{k_1}) - a_1 x_1 y_1 - b_1 x_1 y_2$

$\frac{dx_2}{dt} = x_2 r_2(1 - \frac{x_2}{k_2}) - a_2 x_2 y_1 - b_2 x_2 y_2$

Predator Population Equations

$\frac{dy_1}{dt} = f_1 a_1 x_1 y_1 + f_2 a_2 x_2 y_1 - d_1 y_2$

$\frac{dy_2}{dt} = g_1 b_1 x_1 y_2 + g_2 b_2 x_2 y_2 - d_2 y_2$

x_i is prey population

y_i is predator population

r_i the prey growth rate

d_i the predator death rate

k_i carrying capacity of population i

a_1 is the Rate at which predator 1 consumes prey 1

a_2 is the Rate at which predator 1 consumes prey 2

b_1 is the Rate at which predator 2 consumes prey 1

b_2 is the Rate at which predator 2 consumes prey 2

f_1 Efficiency of predator 1 converting prey 1 into its own population growth.

f_2 Efficiency of predator 1 converting prey 2 into its own population growth

g_1 Efficiency of predator 2 converting prey 1 into its own population growth.

g_2 Efficiency of predator 2 converting prey 2 into its own population growth

3. Problem 5 on P. 116 of the J. D. Murray's Chapter 3.
The interaction between two populations with densities N_1 and N_2 is modelled by

$$\frac{dN_1}{dt} = rN_1(1 - \frac{N_1}{K}) - aN_1N_2(1 - \exp^{[-bN_1]}), \tag{3}$$
$$\frac{dN_2}{dt} = -dN_2 + N_2e(1 - \exp^{[-bN_1]}), \tag{4}$$

where a, b, d, e, r and K are positive constants. What type of interaction exists between N_1 and N_2 ? What do the various terms imply ecologically?

1. What type of interaction exists between N_1 and N_2?
The type of interaction between N_1 and N_2 is a predator prey relationship. Where N_2 preys on N_1, N_1 is constrained by a carrying capacity, and N_2 requires N_1 to be present in order to survive.

2. What do the various terms imply ecologically?

rN_1 = per capita growth rate time the population.
$(1 - \frac{N_1}{K})$ = Term that describes the carrying capacity of the population.
$aN_1N_2(1 - \exp^{-bN_1})$ = This term describes the effect of N_2 on the N_1 population. where a is the proportionality constant and $1 - \exp^{-bN_1}$ is for when N_1 is large the populations stabilize sigmoidally.

$-dN_2$ = is the term that describes the death rate of predation
$N_2e(1 - \exp^{-bN_1})$ = is the term that describes when N_1 is large and the populations stabilize sigmoidally

Nondimensionalise the system by writing

$$u = \frac{N_1}{K}, v = \frac{aN_2}{r}, \tau = rt, \alpha = \frac{e}{r}, \delta = \frac{d}{r}, \beta = bK \tag{5}$$

Nondimensionalization:

$Kdu = dN_1$, $d\tau = rdt$ so

$\frac{Kdu}{d\tau} = \frac{dN_2}{rdt}$

$\frac{du}{d\tau} = \frac{1}{rK}\frac{dN_2}{dt}$

$\frac{du}{dt}K = rKu(1-u) - uvrK(1-\exp^{-\beta u})$

$$\boxed{\frac{du}{d\tau} = u(1-u) - uv(1-\exp^{-\beta u})}$$

$\frac{r}{a}dv = dN_2$, $d\tau = rdt$ so

$\frac{\frac{r}{a}dv}{d\tau} = \frac{dN_2}{rdt} \longrightarrow \frac{dv}{d\tau} = \frac{1}{ar^2}\frac{dN_2}{dt}$

$\frac{dv}{dt} = -\frac{dr}{a}v + \frac{er}{a}v(1-\exp^{-\beta u})$

$\frac{dv}{dt} = -\frac{dr}{ar^2}v + \frac{er}{ar^2}v(1-\exp^{-\beta u})$

$\frac{dv}{dt} = -\frac{d}{r}v + \frac{e}{r}v(1-\exp^{-\beta u})$

$$\boxed{\frac{dv}{d\tau} = -\delta v + \alpha v(1-\exp^{-\beta u})}$$

Determine the nonnegative equilibria and note any parameter restrictions. Discuss the linear stability of the equilibria. Show that a nonzero N_2-population can exist if $\beta > \beta_c = -\ln(1-\delta/\alpha)$. Briefly describe the bifurcation behaviour as β increases with $0 < \delta/\alpha < 1$

2. Determine the nonnegative equilibria and note any parameter restrictions.
$\frac{du}{d\tau} = u(1-u) - uv(1-\exp^{-\beta u})$
$\frac{dv}{d\tau} = -\delta v + \alpha v(1-\exp^{-\beta u})$

Fixed Point 1.
$\boxed{u = 0, v = 0}$ obvious

Fixed Point 2.
Let $u = 1$
$\frac{du}{d\tau} = (1)(1-(1)) - (1)v(1-\exp^{-\beta(1)}) = 0$
$\frac{du}{d\tau} = (0) - v(1-\exp^{-\beta}) = 0$
$\frac{du}{d\tau} = v(1-\exp^{-\beta}) = 0$
So $v = 0$ check if $\frac{dv}{d\tau} = 0$
$\frac{dv}{d\tau} = -\delta v + \alpha v(1-\exp^{-\beta u}) = -\delta(0) + \alpha(0)(1-\exp^{-\beta u}) = 0$

$\boxed{u = 1, v = 0}$

Fixed point 3.
$\frac{dv}{d\tau} = -\delta v + \alpha v(1-\exp^{-\beta u}) = 0$
$-\delta + \alpha(1-\exp^{-\beta u}) = 0$
$1 - \exp^{-\beta u} = \frac{\delta}{\alpha}$
$-\exp^{-\beta u} = \frac{\delta}{\alpha} - 1$
$\exp^{-\beta u} = -\frac{\delta}{\alpha} + 1$
$-\beta u = \ln(-\frac{\delta}{\alpha} + 1)$
$u = (-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))$

$\frac{du}{d\tau} = u(1-u) - uv(1-\exp^{-\beta u}) = 0$
$\frac{du}{d\tau} = (-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))(1-(-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))) - (-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))v(1-\exp^{-\beta(-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))}) = 0$

$\frac{du}{d\tau} = (1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha})) - v(1-(1-\frac{\delta}{\alpha})) = 0$

$\frac{du}{d\tau} = (1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha})) - v(\frac{\delta}{\alpha}) = 0$

$-v(\frac{\delta}{\alpha}) = -(1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))$

$v(\frac{\delta}{\alpha}) = (1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))$

$v = \frac{\alpha}{\delta}(1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}))$

$$\boxed{u = -\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}), v = \frac{1+\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha})}{\frac{\delta}{\alpha}}}$$

For $\frac{\delta}{\alpha} < 1$ because $\ln(x)$ can only handle nonzero positive values. And for $u, v \geq 0$ because these are real populations.

3. Discuss the linear stability of the equilibria.
Point (0,0) is unstable. If a single population is present the system will go towards a different stable equilibrium.
(1,0) is stable. If the population reaches 1,0 it will not diverge from that equilibrium in the long term as only one population is present N_1 and N_2 is dead. Meaning there is no way for N_2 to grow back.
For $(-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha}), \frac{1-\frac{1}{\beta}\ln(1-\frac{\delta}{\alpha})}{\frac{\delta}{\alpha}})$. Then if $\frac{\delta}{\alpha} < 1$ there exists an equilibrium point where both populations can live in coexistence. That would be a stable equilibrium point.

Jacobian Matrix
$$\begin{pmatrix} \frac{\partial}{\partial u}\frac{\partial u}{\partial \tau} & \frac{\partial}{\partial v}\frac{\partial u}{\partial \tau} \\ \frac{\partial}{\partial u}\frac{\partial v}{\partial \tau} & \frac{\partial}{\partial v}\frac{\partial v}{\partial \tau} \end{pmatrix}$$
$$\begin{pmatrix} 1-2u-v+v\exp(-\beta u)-\beta uv\exp(-\beta u) & -u+u\exp(-\beta u) \\ \alpha\beta v\exp(-u\beta) & -\delta+\alpha-\alpha\exp(-\beta u) \end{pmatrix}$$
At $u = 0, v = 0$
$$\begin{pmatrix} 1 & 0 \\ 0 & -\delta \end{pmatrix}$$
det = $-\delta < 0$ this is a saddle node and thus a saddle point.
At $u = 1, v = 0$
$$\begin{pmatrix} 1 & -1+\exp(-\beta) \\ 0 & -\delta+\alpha-\alpha\exp(-\beta) \end{pmatrix}$$
det = $-\delta+\alpha-\alpha\exp(-\beta)$
Stability depends on sign of the determinant. For $\alpha > \delta$ this is a saddle point. While for $\alpha < \delta$ it is a stable node.

4. Show that a nonzero N_2-population can exist if $\beta > \beta_c = -\ln(1 - \delta/\alpha)$

Biologically, this resembles an environment with abundant u, promoting the growth of the v population. The system stabilizes at a new fixed point where both U and v are positive, allowing for harmonious coexistence. The stable fixed point represents a large v-population supported by the strong influence of u.
If $\beta > \beta_c$ then
If $\beta = -\ln(1 - \frac{\delta}{\alpha})$ then when $u = -\frac{1}{\beta_c}\ln(1 - \frac{\delta}{\alpha}) = -\frac{1}{-\ln(1-\frac{\delta}{\alpha})}\ln(1 - \frac{\delta}{\alpha}) = 1$ then
$\frac{dv}{d\tau} = -\delta v + \alpha v(1 - \exp^{-\beta u})$

$\frac{dv}{d\tau} = -\delta v + \alpha v(1 - \exp^{-(-\ln(1-\delta/\alpha))(1)})$

$\frac{dv}{d\tau} = -\delta v + \alpha v(1 - \exp^{-(-\ln(1-\delta/\alpha))(1)})$

$\frac{dv}{d\tau} = -\delta v + \alpha v(1 - \exp^{\ln(1-\delta/\alpha))})$

$\frac{dv}{d\tau} = -\delta v + \alpha v(1 - (1 - \frac{\delta}{\alpha}))$

$\frac{dv}{d\tau} = -\delta v + \alpha v\frac{\delta}{\alpha}$

$\frac{dv}{d\tau} = -\delta v + v\delta$

$\frac{dv}{d\tau} = -\delta v + v\delta = 0$

Thus β must be greater than that value for a v population to be nonzero. If β is less the derivative is negative and will go to zero. While if the β is great the derivative will be positive or zero making it larger.

5. Briefly describe the bifurcation behaviour as β increases with $0 < \delta/\alpha < 1$

When β is small, $1 - \exp[-\beta u]$ is also small, indicating weak interaction between the populations. The growth term for v,$\alpha v(1 - e - \beta u)$, is insufficient to overcome the decay rate δv, leading to $u \to 0, v \to 0$. So Only the u-population persists, corresponding to a stable trivial fixed point for $u = 0 v = 0$
At $\beta = \beta_c = -\ln(1 - \frac{\delta}{\alpha})$, a saddle-node bifurcation occurs. A non-trivial fixed point so both u and v populations can coexist.
As β increases further, $1 - \exp[-\beta u] \to 1$, implying a strong interaction effect.
The bifurcation reflects how increasing β strengthens the coupling between u and v, transitioning the system from a single-species equilibrium to a stable coexistence state.