AMATH 423 HW # 1

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1. An "exponential waiting/sojourn/resident time distribution" follows $f_T(t) = re^{-rt}$

   (a) What is the *expected value* of random variable T?

$$\mathbb{E}[T] = \int_0^\infty t f_T(t)dt?$$

$\mathbb{E}[T] = \int_0^\infty t f_T(t)dt? \rightarrow \int_0^\infty tre^{-rt}dt \rightarrow r\int_0^\infty te^{-rt}dt \rightarrow$
Integration by parts:
$u = t, v = \frac{1}{r}e^{-rt}$
$du = 1, dv = e^{-rt}$
$uv - \int vdu \rightarrow (r)(-\frac{1}{r}te^{-rt} - \int -\frac{1}{r}e^{-rt}dt) \rightarrow (r)(-\frac{1}{r}te^{-rt} - [-\frac{1}{r^2}e^{-rt}]) \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty + \frac{1}{r^2}e^{-rt}|_0^\infty) \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty - \int_0^\infty -\frac{1}{r}e^{-rt}dt) \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty + \frac{1}{r}\int_0^\infty e^{-rt}dt) \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty + \frac{1}{r}\left[-\frac{1}{r}e^{-rt}\right]_0^\infty) \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty + \frac{1}{r^2}e^{-rt}|_0^\infty \rightarrow (r)(-\frac{1}{r}te^{-rt}|_0^\infty + \frac{1}{r^2}(0-1)) \rightarrow (r)(0 + \frac{1}{r^2}) \rightarrow \boxed{\mathbb{E}[T] = \frac{1}{r}}$.

   (b) Discuss the parameter r.

The parameter r represents the rate on average the event occurs. r is inversely proportional to the time between events T. As r increases the time between events should decrease.

   (c) What are the several possible mechanistic explanations for the exponentially distributed random time, as a phenomenon? Try to outline them using mathematics.

There are several possible mechanistic explanations.
Example 1. Radioactive decay. Every atom decays at a fixed rate r per unit time. The decay of the atom is independent of other atoms and independent of time. This results in the exponential distribution equation. e^{-rt}
Example 2. Waiting time between customer arrivals between customer arrivals at a store. The exponential distribution works because the time you spend waiting for the next person is the same regardless of waiting and independent on if a customer is there. To a certain extent.

   (d) What are the most important features of an exponentially distributed random time?

1) Memorylessness
The probability of an event occurring in the future is independent of how much time has elapsed
2) The events are independent.
Events do not depend on other events to occur.
3) The rate parameter r determines the expected value and standard deviation
Meaning $\mathbb{E}[T] = \frac{1}{r}$, stdev = $\frac{1}{r}$

2. Two independent random events with waiting time T_1 and T_2 and corresponding cumulative distribution functions (CDF) $F_{T1}(x) = 1 - e^{-r_1 x}$ and $F_{T2}(x) = e^{-r_2 x}$.

   (a) What is the probability distribution for the random time of the "first of the two" occurs?

   $T = min(T_1, T_2)$
   $P(T_1 > x, T_2 > x) = P(T_1 > x) * P(T_2 > x)$
   $= e^{-r_1 t} * e^{-r_2 t} = \boxed{e^{(-r_1 - r_2)t}}$

   (b) What is the probability distribution for the random time of both occur?

   $T = max(T_1, T_2)$
   $P(T \le x) = P(T_1 \le x, T_2 \le x) = P(T_1 \le x) * P(T_2 \le x)$
   $= (1 - e^{-r_1 t})(1 - e^{-r_2 t}) = \boxed{1 + e^{(-r_1 - r_2)t} - e^{-r_1 t} - e^{-r_2 t}}$

3. In probabilistic terms, the Poisson process

$$Pr\{N(t) = n\} = \frac{(rt)^n}{n!} e^{-rt}$$

   is a fundamental for random events that occurs one by one in continuous time with a rate $r > 0$

   (a) Discuss how this mathematical formulation resolves both objections of the differential equation formulation,

$$\frac{dN(t)}{dt} = r, \text{a constant} \tag{1}$$

$$\tag{2}$$

   based on "discreteness" and "randomness

discreteness.
N(T) represents an integer value of the number of events so it is inherently discrete. Meaning differentiating it does not make sense as it is not continous. However the formulation given in the Poisson process $\frac{(rt)^n}{n!}e^{-rt}$ is continuous and thus can be differentiated.

randomness.
The poisson equation fixes the randomness problem by giving an expected value. We can calculate the expected value but not the actual value. While the $\frac{dN(t)}{dt} = r$ is completely deterministic.

(b) When, if possible, are these two mathematical formulations being the same, and/or being approximately the same?

Lets take the Poisson process expected value.
$\mathbb{E}[N(t)] = \sum_{n=0}^{\infty} n\frac{(rt)^n}{n!}e^{-rt}$
$\mathbb{E}[N(t)] = \sum_{n=0}^{\infty} \frac{(rt)^n}{(n-1)!}e^{-rt}$
$\mathbb{E}[N(t)] = rte^{-rt}\sum_{n=0}^{\infty} \frac{(rt)^{n-1}}{(n-1)!}$
*identity: $e^x \sum \frac{x^m}{m!}$
$\mathbb{E}[N(t)] = rte^{-rt}e^{rt}$
$\mathbb{E}[N(t)] = rt$
Then taking the derivative
$\boxed{\frac{d\mathbb{E}[N(t)]}{dt} = r}$

Thus the poisson process expected value is equivalent to the differential equation when the expected value can be taken.
At sufficient time t to take the expected value. so when the expected value and standard deviation can be used reasonably. The equations are the same when the standard deviations are equivalent. Meaning when $\frac{\sqrt{Var}}{\mathbb{E}[T]} = \frac{\sqrt{rt}}{rt}$
The equations are equivalent when rt is sufficiently large. For example when $rt = 10000$, they only differ by 1%. So as $t \to \infty$ or as $r \to \infty$.

4. A nonlinear ordinary differential equation (ODE)

$$\frac{dX(t)}{dt} = f(X)$$

is called *autonomous* if $f(X)$ is not an explicit function of time t. Assuming that the right-hand-side $f(X)$ is sufficiently smooth, according to the ODE and with a given value of $X(0) = x_0$, the value of $X(t)$ is determined for time $t > 0$. If the solution to the ODE $X(t)$ represents the population size of a biological species in an ecosystem, or the concentration of a biochemical "species" in a test tube, at time t, then the ODE represents a single-specie population dynamics.

(a) Explain when the ODE is a *linear* ODE, and when it is not? Is there anything special about the solution(s) to a linear ODE?

The ODE is a linear ODE when f(X) is linear function of the derivatives of X. For example if $f(X)$ is $3X + 2$ or 3, But not $\sin(X)$.

The special thing about solutions to linear ODEs is the superposition principle. Any linear combination of solutions to a homogenous linear ODE is a solution. Also for any given initial condition there is one unique solution.

(b) For general smooth function $F(X)$, let x^* be a root to the algebraic equation $f(X) = 0$. Note there could be many roots. Please discuss the significance of these root in terms of the dynamics.

The roots of $f(X)$ correspond to points where the slope of $F(X)$ is zero. In a mechanics view point the roots are an equilibrium point with no change unless perturbed by an outside force. For example this would be when the velocity is zero. This is either a local max or a local min point.

(c) Show that for a single-specie population dynamics described by an autonomous ODE, it is not possible to have oscillatory, nor periodic, solution X(t).

Say $x(t)$ is oscillatory then
$x(t_1) = x(t_2)$
$x'(t_1) > 0$
$x'(t_2) < 0$
$\frac{dx}{dt} = f(x)$
So then it is not autonomous. As autonomous differential equations cannot go from a negative to a positive slope.

(d) Now, some people might say that $X(t) = 2 + sin(t)$, which is oscillatory and periodic, is a solution to the differential equation $(dX/dt)2 = 1 - (X - 2)2$. Try to explain why this is not a legitimate *counter-example* to the statement in (c).

Not a solution because it is not an autonomous equation.
The example can be reduced to this equation here.

$$\frac{dx}{dt} = \begin{cases} \sqrt{1-(X-2)^2} & [-\frac{\pi}{2}, \frac{\pi}{2}] + 2k\pi \\ -\sqrt{1-(X-2)^2} & [-\frac{3\pi}{2}, \frac{3\pi}{2}] + 2k\pi \end{cases}$$

$k = 0, 1, 2 \ldots$
The example is discontinous and thus is not autonomous. The square root of a linear differential equation is not the same as the linear function.

5. Consider the dynamics of a population that consists of n sub-populations, with their respective sizes at time t represented by $X_1(t), X_2(t), \ldots, X_n(t)$. Let us assume that for each sub-population,

$$\frac{dX_1(t)}{dt} = r_1 X_1, \frac{dX_2(t)}{dt} = r_2 X_2, \ldots, \frac{dX_n(t)}{dt} = r_n X_n. \tag{3}$$

So the dynamics of these n sub-poluations are completely independent from each other, and each is growing or decaying "exponentially" depending on the sign of its r. Now introducing the per capita growth rate (PCGR) of the total population at time t

$$\bar{r} = \frac{1}{X_{tot}(t)} \frac{dX_{tot}(t)}{dt}, \text{ where } X_{tot}(t) = \sum_{i=1}^{n} X_i(t) \tag{4}$$

(a) The PCGR of the total population is not a constant over time. Show that it is the "average" of the PCGRs of the sub-populations weighted by the population

$$\bar{r} = \frac{\sum_{i=1}^{n} X_i r_i}{\sum_{i=1}^{n} X_i} \tag{5}$$

$\bar{r} = \frac{1}{X_{tot}(t)} \frac{dX_{tot}(t)}{dt}$
$X_{tot}(t) = \sum_{i=1}^{n} X_i(t)$
$\frac{dX_{tot}(t)}{dt} = \sum_{i=1}^{n} \frac{dX_i(t)}{dt} = \sum_{i=1}^{n} r_i X_i$
thus
$\bar{r} = \frac{\sum_{i=1}^{n} X_i r_i}{\sum_{i=1}^{n} X_i}$

(b) More interestingly, show that

$$\frac{d}{dt}\bar{r} = \frac{\sum_{i=1}^{n} X_i (r_i - \bar{r}(t))^2}{\sum_{i=1}^{n} X_i} \geq 0. \tag{6}$$

$\frac{d}{dt}\bar{r} = \frac{d}{dt}\frac{\sum_{i=1}^{n} X_i r_i}{\sum_{i=1}^{n} X_i}$
Quotient rule and linear derivatives.
$\frac{d}{dt}(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$
$u = \sum_{i=1}^{n} r_i X_i, u' = \sum_{i=1}^{n} r_i^2 X_i$
$v = \sum_{i=1}^{n} X_i, v' = \sum_{i=1}^{n} r_i X_i$

$\frac{(\sum_{i=1}^{n} X_i r_i^2)(\sum_{i=1}^{n} X_i) - (\sum_{i=1}^{n} X_i r_i)(\sum_{i=1}^{n} X_i r_i)}{(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i)}$

$\frac{(\sum_{i=1}^{n} X_i r_i^2)(\sum_{i=1}^{n} X_i)}{(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i)} - \frac{(\sum_{i=1}^{n} X_i r_i)(\sum_{i=1}^{n} X_i r_i)}{(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i)}$

$\frac{(\sum_{i=1}^{n} X_i r_i^2)}{(\sum_{i=1}^{n} X_i)} - \frac{(\sum_{i=1}^{n} X_i r_i)\bar{r}(t)}{(\sum_{i=1}^{n} X_i)}$

$\frac{\sum_{i=1}^{n} X_i r_i^2 - (\sum_{i=1}^{n} X_i r_i)\bar{r}(t)}{(\sum_{i=1}^{n} X_i)}$

$\frac{(\sum_{i=1}^{n} X_i)(r_i^2 - r_i\bar{r}(t))}{(\sum_{i=1}^{n} X_i)}$

$\frac{\sum_{i=1}^{n}(X_i)\left((r_i - \bar{r}(t))^2 - \bar{r}(t)^2 + \bar{r}(t)r_i\right)}{(\sum_{i=1}^{n} X_i)}$

$\frac{\sum_{i=1}^{n} X_i\left((r_i - \bar{r}(t))^2)\right)}{\sum_{i=1}^{n} X_i} - \frac{\sum_{i=1}^{n} X_i(\bar{r}(t)^2 + \bar{r}(t)r_i}{\sum_{i=1}^{n} X_i}$

The right term is equivalent to zero.
$\frac{\sum_{i=1}^{n} X_i\bar{r}(t)^2 + X_i r_i\bar{r}(t)}{\sum_{i=1}^{n} X_i}$

$\frac{\sum_{i=1}^{n} X_i\bar{r}(t)^2 + \sum_{i=1}^{n} X_i r_i\bar{r}(t)}{\sum_{i=1}^{n} X_i}$

$\frac{\sum_{i=1}^{n} X_i\bar{r}(t)^2}{\sum_{i=1}^{n} X_i} + \frac{\sum_{i=1}^{n} X_i r_i\bar{r}(t)}{\sum_{i=1}^{n} X_i}$

$\bar{r}(t)^2 - \bar{r}(t)\bar{r}(t) = 0$
Thus
$\frac{d}{dt}\bar{r} = \frac{\sum_{i=1}^{n} X_i\left((r_i - \bar{r}(t))^2)\right)}{\sum_{i=1}^{n} X_i}$

(c) For n = 2, show that
$\frac{d}{dt}\left(\frac{X_1}{X_1+X_2}\right)\frac{(r_1 - r_2)X_1X_2}{(X_1+X_2)^2}$.

Quotient rule.
$\frac{d}{dt}(\frac{u}{v}) = \frac{vu' - uv'}{v^2}$
$u = X_1, u' = r_1X_1$
$v = X_1 + X_2, v' = r_1X_1 + r_2X_2$
$= \frac{(X_1+X_2)(r_1X_1) - (X_1)(r_1X_1 + r_2X_2)}{(X_1+X_2)^2}$
$= \frac{r_1X_1^2 + r_1X_1X_2 - r_1X_1^2 - r_2X_2X_1}{(X_1+X_2)^2}$
$= \frac{(r_1 - r_2)X_1X_2}{(X_1+X_2)^2}$

Based on this mathematical expression, discuss what happens to the two sub-populations if $r_1 > r_2$, and if $r_2 > r_1$? What general conclusions can you reach for the total population $X_1(t) + X_2(t)$?

If $r_1 > r_2$, then $\frac{d}{dt}\frac{X_1}{X_1+X_2} > 0$, means the X_1 population growth rate is greater than the X_2 population growth rate
If $r_2 > r_1$, then $\frac{d}{dt}\frac{X_1}{X_1+X_2} < 0$, a mean the X_1 population growth rate is less than the X_2 population growth rate
At long enough t the population will consist entirely of the subpopulation with the higher growth rate, $\max(r_1, r_2)$.

(d) In the biological context, the per capita growth rate is often used as the *fitness* of a population. Discuss the mathematical result in (b) with respect to the statement that "The fitness of a population always increases when there are variations within the population."

"The fitness of a population always increases when there is variation within the population." Means that the more r_i varies from $\bar{r}(t)$ the more $\bar{r}(t)$ increases as well.
This makes sense because the derivative of $\bar{r}(t)$ increases as the top part $(r_i - \bar{r}(t))^2$ increases which is essentially equivalent to the variation in the population. The equation supports the quote.