Homework 6.

Amath 383
Introduction to Continuous Mathematical Modeling

Due: 11/17/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 2.4)

Recall from class the Lorenz equations

$$\frac{dx}{dt} = f(x, y, z) = \sigma(y - x), \tag{1a}$$
$$\frac{dy}{dt} = g(x, y, z) = rx - y - xz, \tag{1b}$$
$$\frac{dz}{dt} = h(x, y, z) = xy - bz, \tag{1c}$$

where σ, r, and b are positive constants.

a. Let $\mathbf{F}(x, y, z) = \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix}$. Show that $\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$ equals a negative constant.

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x}(\sigma(y - x)) = -\sigma$
$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} g(x, y, z) = \frac{\partial}{\partial y}(rx - y - xz) = -1$
$\frac{\partial h}{\partial z} = \frac{\partial}{\partial z} g(x, y, z) = \frac{\partial}{\partial y}(xy - bz) = -b$

$\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$
$\nabla \cdot \mathbf{F} = -\sigma - 1 - b$

b. Imagine a small 3D region of initial conditions in phase space. Call this region D_0 and its volume V_0. If we turn on the dynamics of the Lorenz system, then trajectories emerge from each of the initial conditions in D_0, consequently distorting this region and its volume. Let $D(t)$ represent our region of initial conditions after they have evolved according to the Lorenz system for t units of time, and let $V(t)$ represent the volume of $D(t)$.

A standard identity in multivariable calculus tells us that

$$\frac{dV}{dt} = \iiint_{D(t)} \nabla \cdot \mathbf{F}\ dx\ dy\ dz. \tag{2}$$

Substitute your result from (a) into (2) and use the fact that

$$V(t) = \iiint_{D(t)} 1\ dx\ dy\ dz$$

to get a differential equation for $V(t)$. Solve this differential equation subject to the initial condition $V(0) = V_0$. What happens to the volume of our initial conditions as $t \to \infty$?

Remark: This result tells us that trajectories of the Lorenz system in phase space eventually settle down on a surface in 3D space. (Of course, in the chaotic regime, this surface consists of the butterfly wings of the strange attractor.)

$\frac{dV}{dt} = \iiint_{D(t)} \nabla \cdot \mathbf{F} \; dx \; dy \; dz$
$\iiint_{D(t)} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \; dy \; dz.$
$\iiint_{D(t)} (-\sigma - 1 - b) \, dx \; dy \; dz.$
$(-\sigma - 1 - b) \iiint_{D(t)} (1) \, dx \; dy \; dz.$
$(-\sigma - 1 - b) V(t)$
$\boxed{\frac{dV}{dt} = (-\sigma - 1 - b) V(t)}$

c. Our course textbook uses the results obtained in (b) to imply that there are no periodic solutions of the Lorenz equations, see pages 223 and 228. The textbook argument goes as follows:

> If there are periodic solutions of the Lorenz equations, then start with a region D_0 in phase space whose initial conditions correspond to the periodic solutions. Let the volume of these initial conditions be V_0. The trajectories emenating from the initial conditions in D_0 are periodic, which implies both $D(t)$ and $V(t)$ are periodic. If $V(t)$ is periodic, there must be a time t_* for which $V'(t_*) = 0$. (This is by Rolle's theorem.) However, this contradicts your result in (b). Thus, there are no periodic solutions of the Lorentz equations.

The argument is wonderful, except the last sentence. It's not fair to say that the Lorenz equations don't have periodic solutions. Instead, all we conclude from the argument above is that the Lorentz equations can't have a 3D "clump" of trajectories where each trajectory in the clump is periodic. It says nothing about the possibility of an isolated periodic trajectory. (This is a trajectory that is periodic, but its nearest neighbors are not.)

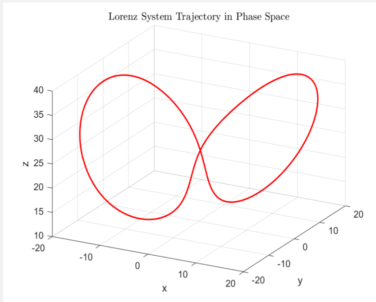
As you might imagine, finding these isolated periodic trajectories is quite challenging, but they have been found[1]. For example, take $\sigma = 10$, $b = 8/3$, and $r = 28$. Consider initial conditions

$$x(0) = -13.763610682134201,$$
$$y(0) = -19.578751942451796,$$
$$z(0) = 27.$$

Solve the Lorenz equations numerically over the time interval $0 \leq t \leq 1.558652210716175$ using ode45 (or an equivalent Runge-Kutta fourth-order method). Use $N = 1000$ time steps. Plot the trajectory of the solution in phase space.

Hint: Use my code from lecture as a guideline.

[1]See "The Fractal Property of the Lorenz Attractor" by D. Viswanath for more.

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%% AMATH 383 HW 6 Question 1(c)

%Define variables
sigma = 10;
b = 8/3;
r = 28;
tspan = linspace(0, 1.558652210716175, 1000);

%%Lorenz system.
lorenz = @(t, X) [sigma * (X(2) - X(1));
                  r*X(1) - X(2) - X(1)*X(3);
                  X(1) * X(2) - b * X(3)];
%%\sigma (y - x)
%%rx - y - xz
%%xy - bz

%Initial conditions
X0 = [-13.763610682134201, -19.578751942451796, 27];

%Solve using ode45
[t, X] = ode45(lorenz, tspan, X0);

%Plot
figure;
plot3(X(:,1), X(:,2), X(:,3), 'r', 'LineWidth', 1.5);
xlabel('x');
ylabel('y');
zlabel('z');
title('Lorenz System Trajectory in Phase Space','Interpreter','latex');
grid on;
view(3);
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Exercise 2. (Unit 3.1)

Consider a robot driving towards a wall. We want the robot to stop when its distance to the wall is equal to zero. The robot is programmed to take a discrete number of steps such that, on its nth step, the robot moves with a velocity proportional to its current distance to the wall. We can model the position of the robot by the following difference equation

$$d_n = d_{n-1} + hv_{n-1},$$

where d_n is the robot's distance to the wall after n steps have been taken, d_{n-1} is the robot's distance to the wall after $n-1$ steps have been taken, h is the amount of time the robot takes to change its position per step, and v_{n-1} is the velocity at which the robot moves during this step. According to our assumption,

$$v_{n-1} = \gamma d_{n-1},$$

where $\gamma \in \mathbb{R}$ is a constant of proportionality. Thus, our difference equation becomes

$$d_n = d_{n-1} + \gamma h d_{n-1}. \tag{3}$$

a. Suppose the robot's initial distance to the wall is $d_0 > 0$. What is the robot's position for any step $n \geq 0$?

$d_n = d_{n-1} + \gamma h d_{n-1} \to d_n = (1+\gamma h)d_{n-1} \to d_n = (1+\gamma h)(1+\gamma h)d_{n-2} \to$
$d_n = (1+\gamma h)(1+\gamma h)(1+\gamma h)d_{n-3} \to d_n = (1+\gamma h)(1+\gamma h)(1+\gamma h)\ldots(1+\gamma h)d_0 \to$
$\boxed{d_n = (1+\gamma h)^n d_0}$

b. According to your solution in (a), which values of the constant γ ensure that the robot does not bang into the wall?

Distance from wall must be greater than zero. $d_n > 0$
$0 < d_n = (1+\gamma h)^n d_0 \to 0 < (1+\gamma h)^n \to 0 < 1+\gamma h \to -1 < \gamma h \to$
$\boxed{-\frac{1}{h} < \gamma}$

c. Suppose the robot's program terminates if the distance to the wall becomes smaller than 10^{-3}. If $d_0 = 10$, $h = 0.1$, and $\gamma = -1$, roughly how many steps are needed to terminate the program? How long does the program run?

Robot stops when $10^{-3} > d_n$.
$10^{-3} > d_n = (1+\gamma h)^n d_0 \to (10^{-3}) > (1+(-1)(0.1))^n(10) \to (10^{-4}) > (0.9)^n \to \log(10^{-4}) > \log(0.9)^n \to -4\log(10) > n\log(0.9) \to -4 > n\log(0.9) \to n > -\frac{4}{\log(0.9)} = 87.41738 \to$
Discrete amount of steps so
$\boxed{n = 88 \text{ steps}}$
88 steps $\times 0.1 \frac{\text{units of time}}{\text{steps}} = 8.8$ units of time
$\boxed{8.8 \text{ units of time}}$

d. Let's now consider what's called the "continuum limit" of the difference equation (3). Let $d_n = d(t)$ and $d_{n+1} = d(t+h)$. Here, $d(t)$ is a continuous function that well-approximates the behavior of the robot if the time step h is taken sufficiently small. Show that, in the limit $h \to 0$, the difference equation (3) becomes

$$d'(t) = \gamma d(t).$$

(This should take all of two lines.) Solve this differential equation subject to the initial condition $d(0) = d_0 > 0$. For which values of γ will this solution converge to the wall? Are these values consistent with what you obtained in (b)?

$d_n = d(t), d_{n+1} = d(t+h)$
Assuming linear relationship. where the function value changes by the slope γ and distance h.
$d(t+h) = d(t) + h\gamma d(t) \to d(t+h) - d(t) = h\gamma d(t) \to \frac{d(t+h)-d(t)}{h} = \gamma d(t)$
Taking the limit as $h \to 0$ to get. $\lim_{h\to 0} \frac{d(t+h)-d(t)}{h} = \lim_{h\to 0} \gamma d(t)$
This is the definition of a derivative simplify to get. $d'(t) = \gamma d(t)$

e. Suppose $d_0 = 10$ and $\gamma = 1$. According to the solution obtained in (d), roughly how long will it take the robot's program to terminate (again, assuming it does so when the distance to the wall is first less than 10^{-3})? How does this value compare to what you computed in (c)?

$d'(t) = \gamma d(t) \to \frac{d(d(t))}{dt} = \gamma d(t)$
Separable ODE.
$\frac{d(d(t))}{dt} = \gamma d(t) \to \frac{d(d(t))}{d(t)} = \gamma dt \to \int \frac{d(d(t))}{d(t)} = \int \gamma dt \to \ln(d(t)) = \gamma t + C_1 \to d(t) = e^{\gamma t + C_1} \to d(t) = d_0 e^{\gamma t}$
Plug in variables. $d_0 = 10, \gamma = -1$ typo on assignment
$10^{-3} > d(t) = d_0 e^{(-\gamma t)} \to 10^{-3} > d(t) = 10e^{(-1)t} \to 10^{-4} > e^{-t} \to \ln(10^{-4}) > \ln(e^{-t}) \to -4\ln(10) > -t\ln(e) \to -4\ln(10) > -t \to 4\ln(10) < t \to t > 4\ln(10) = 9.2103$
$\boxed{t > 9.2103}$
Compared to what was computed in (c) this time value is higher. In this part we figured out $t > 9.21$ units of time while in part (c) $t > 8.8$ units of time. However, both values are relatively close, meaning both models can be useful. Of course in both models there are simplifying assumptions. For example, the robot cannot have an instantaneous velocity because cannot just go from 0 to 100 immediately and that time is not discrete.