Homework 1.

Amath 383
Introduction to Continuous Mathematical Modeling

Due: 10/7/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 1.1)

Solve the following initial value problems:

a. $\frac{dy}{dt} = \frac{ty^3}{\sqrt{1+t^2}}$, subject to $y(0) = -1$

$\frac{dy}{dt} = \frac{ty^3}{\sqrt{1+t^2}}$
$\frac{dy}{y^3} = \frac{tdt}{\sqrt{1+t^2}}$
$\int \frac{dy}{y^3} = \int \frac{tdt}{\sqrt{1+t^2}}$
$\frac{1}{-2y^2} = \sqrt{1+t^2} + C$
$y(t) = \sqrt{\frac{1}{-2\sqrt{1+t^2}+C}}$
$y(0) = -1 = \sqrt{\frac{1}{-2\sqrt{1+(0)^2}+C}} = \sqrt{\frac{1}{-2+C}}$
$C = 3$
$$\boxed{y(t) = \sqrt{\frac{1}{-2\sqrt{1+t^2}+3}}}$$

b. $t\frac{dy}{dt} - y = t^2$, subject to $y(1) = 1$

$\frac{dy}{dt} - \frac{1}{t}y = t$
First order linear integrating factor generic solution: $p(t) = e^{\int -(1/t)dt} = e^{-ln(t)}$ $y(t) = \frac{1}{e^{-\ln(t)}} \int e^{-\ln(t)} * t \rightarrow y(t) = t * \int (\frac{1}{t} * t)dt \rightarrow y(t) = t(t+C) \rightarrow y(t) = t^2 + Ct$
$y(1) = 1 = (1)^2 + C(1) \rightarrow C = 0$
$$\boxed{y(t) = t^2}$$

c. $\frac{dy}{dt} = -yt - y + 3t + 3$, subject to $y(0) = 0$

$\frac{dy}{dt} + y(t+1) = 3t + 3$
First order linear integrating factor generic solution:
$p(t) = e^{\int (t+1)dt} = e^{\frac{1}{2}t^2+t} \rightarrow y(t) = \frac{1}{e^{\frac{1}{2}t^2+t}} \int e^{\frac{1}{2}t^2+t} * (3t+3)dt$
U substitution:
$u = \frac{1}{2}t^2 + t$, $du = t + 1dt \rightarrow \int 3e^u du \rightarrow 3e^u + C \rightarrow 3e^{\frac{1}{2}t^2+t}$
$y(t) = \frac{1}{e^{\frac{1}{2}t^2+t}}(3e^{\frac{1}{2}t^2+t} + C) \rightarrow y(t) = 3 + \frac{C}{e^{\frac{1}{2}t^2+t}}$

$y(0) = 0 = 3 + \frac{C}{e^{\frac{1}{2}(0)^2+(0)}} \rightarrow 0 = 3 + \frac{C}{e^0} \rightarrow 0 = 3 + C \rightarrow C = -3$

$y(t) = -\frac{3}{e^{\frac{1}{2}t^2+t}} + 3$ $\boxed{y(t) = -3e^{-\frac{1}{2}t^2-t} + 3}$

Exercise 2. (Unit 1.2)

The isotope technetium-99m is used by doctors to image the skeleton and heart muscles. Following injection, the concentration of this isotope decays exponentially in the body.

a. Let $N(t)$ represent the number of Technetium-99m isotopes present in a person's body t hours after injection. Propose a differential equation that can be used to model the decay of Technetium-99m in the body. Your model should include an unknown constant that controls the exponential growth rate. *This constant should be positive.*

$$\boxed{\frac{dN}{dt} = -\lambda N(t)}$$

b. Solve your differential equation from (a) with the initial condition $N(0) = N_0$, where N_0 represents the initial amount of Technetium-99m in the body.

$\frac{dN}{dt} = -\lambda N(t) \to \frac{dN}{N(t)} = \frac{-\lambda}{dt} \to \ln(N(t)) = -\lambda t + C \to N(t) = e^{-\lambda t + C} = Ce^{-\lambda t} \to N(0) = N_0 = Ce^{-\lambda t} \to \boxed{N(t) = N_0 e^{-\lambda t}}$

c. Suppose a third of the initial technetium-99m remains in the body after 10 hours have past. What is the half-life of technetium-99m? *Don't forget units in your answer!*

$t_{1/2} = \frac{\ln(2)}{\lambda}$

$\frac{1}{3}N_0 = N_0 e^{-\lambda t_{1/3}} \to \frac{1}{3} = e^{-\lambda t_{1/3}} \to \ln\left(\frac{1}{3}\right) = -\lambda t_{1/3} \to \ln(3) = \lambda t_{1/3} \to \frac{\ln(3)}{t_{1/3}} = \lambda$

$\lambda = \frac{\ln(3)}{10hr} = 0.10986$

$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{\ln(2)}{0.10986}$

$\boxed{t_{1/2} = 6.30929 \text{ hours}}$

d. What percent of technetium-99m remains in the body 1 day after injection?

$N_f = N_0 e^{-0.10986 * t} \to N_f = N_0 e^{-0.10986 * 24hr} = e^{-0.10986 * 24hr} = 0.071601$

$\boxed{7.16\% \text{ in the body 1 day after injection.}}$

Exercise 3. (Unit 1.2)

In very old stars, heavy metallic elements like uranium (U), osmium (Os), and iridium (Ir) are hard to come by. (Usually, these elements are created during supernovae, when stars die in an epic explosion.) In 2001, there was a star detected in our Milky Way that had trace (but still detectable) amounts of these elements in it, suggesting the star was very old and may have only been formed by one previous supernovae in the history of our universe. The following ratios for the amounts of U, Os, and Ir in the star were measured at the time of the study:

$$\log_{10}\left(\frac{^{238}\mathrm{U}}{\mathrm{Os}}\right) = -2.19, \tag{1a}$$

$$\log_{10}\left(\frac{^{238}\mathrm{U}}{\mathrm{Ir}}\right) = -2.10. \tag{1b}$$

Theoretical arguments from nuclearsynthesis models (whose complexity is far beyond this class) suggest that the initial ratio of these amounts at the time the star formed were

$$\log_{10}\left(\frac{^{238}\mathrm{U}}{\mathrm{Os}}\right) = -1.27, \tag{2a}$$

$$\log_{10}\left(\frac{^{238}\mathrm{U}}{\mathrm{Ir}}\right) = -1.30. \tag{2b}$$

a. The amount of isotope $^{238}\mathrm{U}$ in the star exponentially decays in time with a half-life of 4.47Gyr. (That's 4.47×10^9 years.) Suppose the initial amount of $^{238}\mathrm{U}$ in the star is $^{238}\mathrm{U}_0$. Find a formula for the amount of $^{238}\mathrm{U}$ in the star for all time $t > 0$ measured in giga-years.

   Note: your answer should only have one unknown constant in it, which is $^{238}\mathrm{U}_0$.

   $^{238}U(t) =^{238} U_0 e^{-\lambda t}$
   $t_{1/2} = \frac{\ln(2)}{\lambda}$
   $\lambda = \frac{\ln(2)}{t_{1/2}} = \frac{\ln(2)}{4.47 gigayears} = 0.1550664$
   $\boxed{^{238}U(t) =^{238} U_0 e^{-0.1550664t}}$

b. The elements Os and Ir are stable elements, meaning the amount of these elements in the star does not change in time. Using (1a) to estimate the present amount of $^{238}\mathrm{U}$ in the star and (2a) to estimate the initial amount of $^{238}\mathrm{U}$ in the star, how old is the star? *Don't forget units in your answer!*

   Rerranging the equations we get.
   $\frac{U_f}{10^{-2.19}} = Os$ $\frac{U_i}{10^{-1.27}} = Os$
   Set them equal to each other.
   $\frac{U_i}{10^{-1.27}} = \frac{U_f}{10^{-2.19}}$

   $\frac{U_f}{U_i} = 10^{1.27} * 10^{-2.19} \rightarrow \frac{U_f}{U_i} = 0.12$
   $0.12 = e^{-0.1551t} \rightarrow \ln(0.12) = -0.1551t \rightarrow t = \frac{\ln(0.12)}{-0.1551}$
   $\boxed{t = 13.6610 \text{ gigayears}}$

c. If instead (1b) is used to estimate the present amount of $^{238}\mathrm{U}$ in the star and (2b) is used to estimate the initial amount of $^{238}\mathrm{U}$ in the star, how old is the star? *Don't forget units in your* *answer!*

Rerranging the equations we get.
$\frac{U_f}{10^{-2.10}} = Os$ $\frac{U_i}{10^{-1.30}} = Os$
Set them equal to each other.
$\frac{U_i}{10^{-1.30}} = \frac{U_f}{10^{-2.10}}$

$\frac{U_f}{U_i} = 10^{1.30} * 10^{-2.10} \to \frac{U_f}{U_i} = 0.1585$
$0.16 = e^{-0.1551t} \to \ln(0.16) = -0.1551t \to t = \frac{\ln(0.16)}{-0.1551}$
$\boxed{t = 11.8792 \text{ gigayears}}$

d. Taking an average of your answers from (b) and (c), what is a reasonable estimate for the age of our universe? *Don't forget units in your answer!*

Average. $\frac{13.6610+11.8792}{2} = 12.7701$ gigayears
A reasonable estimate for the age of our univerise with standard deviation is:
$\boxed{12.7701 \text{ gigayears} \pm 0.8909 \text{ gigayears}}$

Exercise 4. (Unit 1.2)

For this exercise, you'll need to skim the article "HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time" by Perelson *et al.*, which can be found on Canvas under Homework 1. You don't have to understand everything you read in this article (or even read the full article), but you should feel comfortable answering the following questions.

a. The article introduces a mathematical model for the growth of HIV virus particles (called virions) in an infected individual's blood (or plasma). According to the abstract of this article, what is the estimated half-life $t_{\frac{1}{2}}$ of virions in plasma?

Plasma virions half life in the abstract is 0.24 days

b. According to the mathematical model presented in equation (2) of the article, the concentration of virions V in plasma is modeled as follows:

$$\frac{dV}{dt} = N\delta T^* - cV, \tag{3}$$

where V is the concentration of the virions and t is time in days. What are the physical interpretations of the variables N, δ, T^*, and c? (Your answer need only be 1-2 complete sentences.)

δ is the rate of loss of the virus producing cells.
N is the number of new virions produced per infected cell during its lifetime.
$T*$ is the concentration of infected cells.
c rate constant of virion clearance

c. If no T-cells are infected by HIV virions and the initial concentration of virions in the blood is V_0, use (3) to find a formula for V as a function of time t.

Note: your final answer should still depend on two unknown constants: V_0 and c.

$\frac{dV}{dt} = -cV$
$\frac{dV}{dt} = -cV(t) \to \frac{dV}{V(t)} = -c\,dt \to \ln(V(t)) = -ct + C \to V(t) = e^{-ct+C} = Ce^{-ct} \to V(0) = V_0 = Ce^0 = C \to \boxed{V(t) = V_0 e^{-ct}}$

d. Using the half-life of HIV virions obtained in (a), estimate the value of c in your answer to (c). *Don't forget units in your final answer!*

Half life formula to calculate growth rate. $t_{1/2} = \frac{\ln(2)}{c} \to c = \frac{\ln(2)}{0.24} = 2.8881 \to \boxed{c = 2.8881}$

e. Using your formula in (c) together with the value of c computed in (d), estimate how long it takes for 99% of the initial concentration of virions to exit the body. *Don't forget units in your final answer!*

$V(t) = e^{-ct} \to \ln(0.01) = -ct \to t = \frac{\ln(0.01)}{-2.8881} \to \boxed{t = 1.594 \text{ days}}$

Exercise 5. (Unit 1.3)

According to the SI model of disease discussed in class, the number of individuals I infected by a disease changes in time t according to the logistic equation

$$\frac{dI}{dt} = rI\left(1 - \frac{I}{N}\right),$$

where $r > 0$ is the initial growth rate of the infected population and $N > 0$ is the total population (assumed to be constant, *i.e.*, no one dies). The number of individuals susceptible to the disease S can be determined according to

$$S = N - I.$$

a. Assume the initial infected population is I_0. Determine a formula for the infected population as a function of time t.

$\frac{dI}{dt} = rI\left(1 - \frac{I}{N}\right)$
$\frac{dI}{I(1-\frac{I}{N})} = rdt$
$\int \frac{dI}{I(1-\frac{I}{N})} = \int rdt$

Solve the left integral using partial fraction decomposition

$\frac{1}{I(1-\frac{I}{N})} = \frac{A}{I} + \frac{B}{1-\frac{I}{N}} \to 1 = A(1 - \frac{I}{N}) + BI \to 1 = A + I(-\frac{A}{N} + B)$ when I = 0. A = 1
$1 = 1 + I(-\frac{1}{N} + B)$
when I = 1.
$1 = 1 + (1)(-\frac{1}{N} + B)$
$0 = (-\frac{1}{N} + B) \to 0 = -\frac{1}{N} + B \to B = \frac{1}{N}$
$\int \frac{1}{I} dI + \int \frac{\frac{1}{N}}{1-\frac{I}{N}} dI = -\int rdt$
Solved integrals:
$\ln(I) - \ln(N - I) = rt + C$
$-\ln\left(\frac{N-I}{I}\right) = rt + C$
$\ln\left(\frac{N-I}{I}\right) = -rt + C$
$\frac{N-I}{I} = e^{-rt+C}$

Solve for I using algebra
$N - I = e^{-rt+C} I \to N = e^{-rt+C} I + I \to N = I(e^{-rt+C} + 1) \to I = \frac{N}{e^{-rt+C}+1}$
$I(t) = \frac{N}{C_2 e^{-rt}+1}$

Solve for C_2 using algebra
$I(0) = I_0 = \frac{N}{C_2 e^{(0)}+1} \to C_2 + 1 = \frac{N}{I_0} \to C_2 = \frac{N}{I_0} - 1$
$C_2 = \frac{N-I_0}{I_0}$
$$\boxed{I(t) = \frac{N}{\frac{N-I_0}{I_0} e^{-rt} + 1}}$$

b. Suppose $I_0 < N/2$. At what time is the infected population equal to the susceptible population? Does this time increase or decrease if we...

t when $N - I = \frac{N}{2}$. so when $I(t) = \frac{N}{2}$

$I(t) = \frac{N}{\frac{N-I_0}{I_0} e^{-rt}+1}$
$\frac{N}{2} = \frac{N}{\frac{N-I_0}{I_0} e^{-rt}+1} \to \frac{1}{2} = \frac{1}{\frac{N-I_0}{I_0} e^{-rt}+1} \to 2 = \frac{N-I_0}{I_0} e^{-rt} + 1 \to 1 = \frac{N-I_0}{I_0} e^{-rt} \to \frac{I_0}{N-I_0} = e^{-rt} \to$
$\ln\left(\frac{I_0}{N-I_0}\right) = -rt \to -\frac{1}{r} \ln\left(\frac{I_0}{N-I_0}\right) = t \to \boxed{t = \frac{1}{r} \ln\left(\frac{N - I_0}{I_0}\right)}$

- increase the initial growth rate of disease r?

Using $t = \frac{1}{r}\ln\left(\frac{N-I_0}{I_0}\right)$, this equation is inversely proportional for r and t. As r increases, t will decrease.

- increase the population size N?

Using $t = \frac{1}{r}\ln\left(\frac{N-I_0}{I_0}\right)$, this equation is naturally log proportional for N and t. As N increases, t will increase.

- increase the initial number of infected individuals I_0?

Using $t = \frac{1}{r}\ln\left(\frac{N-I_0}{I_0}\right)$, this equation is negatively proportional for I_0 and t. As I_0 increases, t will decrease.

c. According to the SI model, is anyone safe from disease or are we all doomed? Use your answer from (a) to justify your answer.

Remark: This is one of the reasons why the SI model is incomplete/unphysical.

As $t \to \infty$ then the exponenet goes to zero. $I(t) = \frac{N}{\frac{N-I_0}{I_0}e^{-rt}+1} \to I(t) = \frac{N}{\frac{N-I_0}{I_0}(0)+1}$. It simplifies to $\frac{N}{0+1} = N$ Meaning that the infected population will equal the total population which is not representative of the real world. There is no guarantee that everyone will get infected as t becomes very large.