Homework 5.

Amath 383
Introduction to Continuous Mathematical Modeling

Due: 11/10/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 2.2)

Remark: This is definitely an exercise for the pure mathematician's at heart, but applied mathematicians can make use of these results, too!

Consider a generic homogeneous 2×2 constant-coefficient linear system

$$\frac{d\mathbf{x}}{dt} = \mathrm{A}\mathbf{x},$$

where $\mathbf{x}(t) = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}$, $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $a, b, c, d \in \mathbb{R}$.

a. The *trace* of a matrix, denoted $\mathrm{tr}(\mathrm{A})$, is defined to be the sum of the diagonal entries of A:

$$\mathrm{tr}(\mathrm{A}) = a + d.$$

The *determinant* of a matrix $\det(\mathrm{A})$ is (as you already know) defined to be

$$\det(\mathrm{A}) = ad - bc.$$

Obtain an expression for the eigenvalues of A as a function of $\mathrm{tr}(\mathrm{A})$ and $\det(\mathrm{A})$.

$\det(\mathrm{A} - \lambda I) = 0$
$(a - \lambda)(d - \lambda) - (b)(c) = 0 \rightarrow \lambda^2 - a\lambda - d\lambda + ad - bc = 0 \rightarrow \lambda^2 - (a + d)\lambda + (ad - bc) = 0$

Quadratic formula
$\lambda_i = \frac{1}{2}\left(-(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)(1)}\right)$
$$\boxed{\lambda_i = \frac{1}{2}\left(-\mathrm{tr}(\mathrm{A}) \pm \sqrt{\mathrm{tr}(\mathrm{A})^2 - 4\det(\mathrm{A})}\right)}$$

b. Show that the conditions

- $\mathrm{tr}(\mathrm{A})^2 - 4\det(\mathrm{A}) > 0$,
- $\mathrm{tr}(\mathrm{A}) > 0$, and
- $\det(\mathrm{A}) > 0$,

imply that the origin $(P = 0, Q = 0)$ is an *unstable node* of the linear system.

Hint: First, you need the eigenvalues to be real-valued and distinct. This gives you the first condition. Then, you need to require that the smaller of your eigenvalues is positive. This gives you the remaining conditions above after some work.

$\mathrm{tr}(A) = \lambda_1 + \lambda_2 > 0$ At least one eigenvalue must be positive.
$\det(A) = \lambda_1\lambda_2 > 0$ Means eigenvalues must share signs so means both eigenvalues must be positive.
thus both eigenvalues are positive, and thus unstable node of the linear system.

c. Show that the condition

- $\det(A) < 0$

implies that the origin $(P = 0, Q = 0)$ is a *saddle node* of the linear system.

Hint: First, you need the eigenvalues to be real-valued and distinct. This gives you the first condition. Next, you need to require that these eigenvalues are distinct. This will modify your first condition slightly. Then, you need to require that the product of your eigenvalues is negative. (This implies that one eigenvalue is positive while the other is negative.) This will give you a second condition. Argue why this second condition makes your first condition obsolete.

$\det(A) = \lambda_1\lambda_2 < 0$
Means one eigenvalue must be positive one must be negative.
Which is the basis of a saddle node at the origin.

d. Show that the conditions

- $\mathrm{tr}(A)^2 - 4\det(A) > 0$,
- $\mathrm{tr}(A) < 0$, and
- $\det(A) > 0$,

imply that the origin $(P = 0, Q = 0)$ is a *stable node* of the linear system.

Hint: First, you need the eigenvalues to be real-valued and distinct. This gives you the first condition. Then, you need to require that the larger of your eigenvalues is negative. This gives you the two remaining conditions. *Be careful with your inequalities in this case, particularly when deriving the third condition!*

$\det(A) = \lambda_1\lambda_2 > 0$
Either both negative or both positive eigenvalues.
$tr(A) = \lambda_1 + \lambda_2 < 0$
Both negative or one positive one negative eigenvalue.
Meaning that both eigenvalues are negative which is the definition of a stable node of the linear system.

e. Show that the conditions

- $\mathrm{tr}(A)^2 - 4\det(A) < 0$ and
- $\mathrm{tr}(A) > 0$

imply that the origin $(P = 0, Q = 0)$ is an *unstable spiral* of the linear system.

Hint: First, you need the eigenvalues to be complex-valued. This gives you the first condition. Then, you need to require that the real part of your eigenvalues is positive. This gives you the second condition.

> $\mathrm{tr}(A) = \lambda_1 + \lambda_2$
> Discriminant is negative. Thus there are two complex distinct λ. Then since $\mathrm{tr}(A) > 0$ one eigenvalues will be negative and one eigenvalue will be positive. which is an unstable spiral of the linear system.

f. Show that the conditions

- $\mathrm{tr(A)}^2 - 4\mathrm{det(A)} < 0$ and
- $\mathrm{tr(A)} < 0$

imply that the origin $(P = 0, Q = 0)$ is a *stable spiral* of the linear system.

Hint: First, you need the eigenvalues to be complex-valued. This gives you the first condition. Then, you need to require that the real part of your eigenvalues is negative. This gives you the second condition.

> $\lambda_i = \frac{1}{2}\left(-\mathrm{tr(A)} \pm \sqrt{\mathrm{tr(A)}^2 - 4\det(\mathrm{A})}\right)$
> Discriminant is negative. Thus there are two complex distinct λ. Then since $\mathrm{tr}(A) < 0$ both eigenvalues will be negative. which is a stable spiral of the linear system.

g. Show that the conditions

- $\mathrm{tr(A)} = 0$ and
- $\det(\mathrm{A}) > 0$

imply that the origin $(P = 0, Q = 0)$ is a *center* of the linear system.

Hint: First, you need the eigenvalues to be complex-valued. This gives you a temporary condition. Then, you need to require that the real part of your eigenvalues is zero. This gives you the first condition. If you combine your temporary condition and the first condition, you will obtain the second condition above.

> $\lambda_i = \frac{1}{2}\left(-\mathrm{tr(A)} \pm \sqrt{\mathrm{tr(A)}^2 - 4\det(\mathrm{A})}\right)$ $\lambda_i = \frac{1}{2}\left(\pm\sqrt{-4\det(\mathrm{A})}\right)$
> Therefore the two eigenvalues are complexed valued. This is a *center* of the linear system.

Remark: Going forward, you are encouraged to use the conditions above to determine the behavior nearby fixed points in the phase plane instead of relying on eigenvalues.

Exercise 2. (Unit 2.2)

The Battle of Gettysburg (July 1-3, 1863) is considered by most historians to be the turning point of the American Civil War between the northern and southern states[1]. It was also one of the bloodiest battles in American history, with over 50,000 soldiers dead, wounded, or missing over the course of a few days. A great deal of this carnage came on the final day of battle during an infamous military maneuver called *Pickett's Charge*, where Confederate (southern) soldiers attempted to break the main defense line of Union (northern) soldiers by brute force. The maneuver was a massive failure for the Confederates and ultimately cost them the battle (and possibly even the war).

For this exercise, I would like you to read an excerpt from the paper "Refighting Pickett's Charge: Mathematical Modeling of the Civil War Battlefield," which can be found on Canvas. You need only read the subsection "Main Equations," beginning on page three. Once you've read that subsection, answer the following questions.

a. Write down the model equations used in the study. Be sure to describe what each variable represents in these model equations.

> $\frac{dU}{dt} = -aC_t$
> $\frac{dC}{dt} = -bU_t$
>
> $\frac{dU}{dt}$ is the change in the number of Union infantrymen per instant of time
> $\frac{dC}{dt}$ is the change in the number of Confederate infantrymen per instant of time.
> t is the time, where $t = 0$ is the start of the skirmish between the two infantry forces
> U_t is the number of Union infantrymen still fighting at time t.
> C_t is the number of Confederate infantrymen still fighting at time t.
> b is the rate at which each Union soldier incapacitates Confederate soldiers
> a is the rate at which each Confederate soldier incapacitates Union soldier

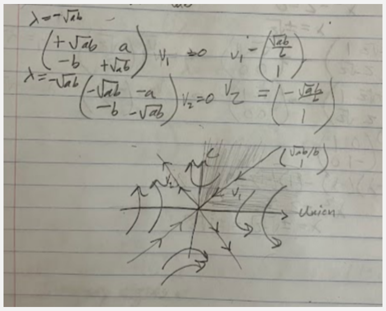
b. Find the fixed point of the model equations. What is the stability of this fixed point? (Assume that $a, b > 0$.)

> $\frac{dU}{dt} = -aC_t = 0$
> $\frac{dC}{dt} = -bU_t = 0$
>
> Fixed point is when $C_t = U_t = 0$.
> $\begin{pmatrix} 0 & -a \\ -b & 0 \end{pmatrix} \Rightarrow (-\lambda)(-\lambda) - (-a)(-b) = 0 \Rightarrow \lambda^2 - ab = 0 \Rightarrow \lambda^2 = ab \Rightarrow \lambda = \pm\sqrt{ab}, \quad a > 0, b > 0.$
> One positive eigenvalue, one negative eigenvalue this is a saddle node of the linear system.

c. Roughly sketch the trajectories of solutions to this model equation near the fixed point. Include arrows on your trajectories to indicate the flow of the solutions. If your fixed point is

[1] Fun fact: your professor lived about 45 minutes away from the battlefield. It's definitely a must-see for anyone interested in military history.

a saddle node, compute the eigenvectors to get an accurate representation for the stable and unstable trajectories.



d. Suppose there are initially U_0 Union soldiers and C_0 Confederate soldiers. We say that the Union wins the battle if the trajectory starting at (U_0, C_0) in the phase plane eventually intersects the Union axis. (This is because the number of Confederate soldiers at the time of intersection is zero.) Similarly, the Confederacy wins the battle if the trajectory starting at (U_0, C_0) in the phase plane eventually intersects the Confederate axis. Based on your sketch from (c), for which initial conditions does the Union win the battle? For the Confederacy? Are there any initial conditions that lead to a stalemate where neither side wins?

For initial conditions when above the eigenvector the confederacy wins. For initial conditions below the eigenvector the Union wins
The Union wins if $C_0 < \frac{\sqrt{ab}}{a} U_0$
The Confederacy wins if $C_0 > \frac{\sqrt{ab}}{a} U_0$
There is a stalemate if $C_0 = \frac{\sqrt{ab}}{a} U_0$

e. Pickett's Charge began with 8,000 Union soldiers and 12,500 Confederate soldiers. What was the smallest possible ratio of Union to Confederate kill rates to guarantee victory for the Union?

Hint: Use your result from (d).

$C_0 < \frac{\sqrt{ab}}{a} U_0 \Rightarrow \frac{C_0}{U_0} < \frac{\sqrt{ab}}{a} \Rightarrow \left(\frac{C_0}{U_0}\right)^2 < \left(\frac{\sqrt{ab}}{a}\right)^2 \Rightarrow \frac{C_0^2}{U_0^2} < \frac{ab}{a^2} \Rightarrow \frac{C_0^2}{U_0^2} < \frac{b}{a} \Rightarrow \frac{(12{,}500)^2}{(8{,}000)^2} < \frac{b}{a} \Rightarrow$ $2.4414 < \frac{b}{a}$
The smallest possible ratio of Union to Confederate kill rates is slightly above $\frac{b}{a} \approx 2.4414$.

Exercise 3. (Unit 2.3)

Hooke's law states that the restoring force of a stretched spring is proportional to the length the string has been stretched. If a mass is attached to the end of a stretched spring governed by Hooke's law, the position of the mass for all time $t > 0$ satisfies

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0.$$

Here, x is the position of the mass relative to its location when the string is unstretched, and ω_0 is the angular frequency at which the mass oscillates as the spring stretches and compresses.

We can improve this model by adding friction in the string and by modifying Hooke's law to make it more realistic. This leads to a new differential equation

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \omega_0^2 x + \beta x^3 = 0,$$

where $\delta > 0$ is a damping rate due to frictional forces in the spring and βx^3 is a correction to Hooke's law that accounts for the nonlinear dependence of the restoring force on the length the string is stretched. If $\beta > 0$, the string is "hard", meaning the restoring force of the string is larger than what the unmodified Hooke's law predicts. If $\beta < 0$, the string is "soft", meaning the restoring force of the string is smaller than what the unmodified Hooke's law predicts.

By rescaling the t variable and redefining the parameters δ and β appropriately, we can eliminate the parameter ω_0 from the equation, leaving us with

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + x + \beta x^3 = 0.$$

This is called the *Duffing equation*, a universal model for any damped, weakly nonlinear oscillator. In this problem, we use linear stability analysis to study solutions of this equation.

a. Let $y = \frac{dx}{dt}$. Show that the Duffing equation can then be rewritten as the planar system

$$\frac{dx}{dt} = y, \tag{1a}$$
$$\frac{dy}{dt} = -\left(\delta y + x\left(1 + \beta x^2\right)\right). \tag{1b}$$

> Duffing equation: $\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + x + \beta x^3 = 0.$
> Setting $\frac{dx}{dt} = y$, $\frac{d^2x}{dt^2} = \frac{dy}{dt}$,
> $\frac{d^2x}{dt^2} = -\delta\frac{dx}{dt} - x - \beta x^3 \rightarrow \frac{dy}{dt} = -\delta\frac{dx}{dt} - x - \beta x^3 \rightarrow \frac{dy}{dt} = -\delta y - x - \beta x^3$
> Rearrange.
> $\frac{dy}{dt} = -\left(\delta y + x\left(1 + \beta x^2\right)\right).$

b. Determine the fixed points of the planar system (1a)-(1b) when (i) $\beta > 0$, (ii) $\beta = 0$, and (iii) $\beta < 0$.

Fixed points when $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0,$
$\frac{dx}{dt} = y = 0 \rightarrow y = 0$
$\frac{dy}{dt} = -(\delta y + x(1+\beta x^2)) = 0 \rightarrow \delta y + x(1+\beta x^2) = 0 \rightarrow *y = 0 \rightarrow x(1+\beta x^2) = 0 \rightarrow (0)(1+\beta(0)^2) = 0 \rightarrow x = 0$
$\frac{dy}{dt} = -(\delta y + x(1+\beta x^2)) = 0 \rightarrow \delta y + x(1+\beta x^2) = 0 \rightarrow *y = 0 \rightarrow x(1+\beta x^2) = 0 \rightarrow 1+\beta x^2 = 0 \rightarrow \beta x^2 = -1 \rightarrow x^2 = -\frac{1}{\beta} \rightarrow x = \pm\sqrt{-\frac{1}{\beta}}$
Fixed points at: $\left(+\sqrt{-\frac{1}{\beta}}, 0\right), \left(-\sqrt{-\frac{1}{\beta}}, 0\right), (0,0)$
(i) $\beta > 0$ then two imaginary fixed points. One real fixed point at (0,0)
(ii) $\beta = 0$ then divide by zero error. One real fixed point at (0,0)
(iii) $\beta < 0$ then three real fixed points. $\left(+\sqrt{-\frac{1}{\beta}}, 0\right), \left(-\sqrt{-\frac{1}{\beta}}, 0\right), (0,0)$

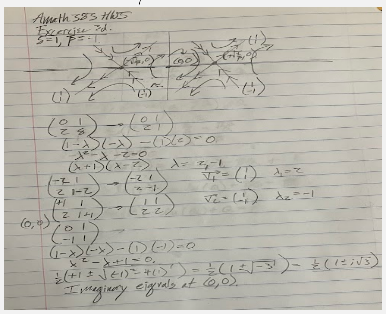
c. Determine the stability of the fixed points found in (b) when (i) $\beta > 0$, (ii) $\beta = 0$, and (iii) $\beta < 0$.

(i) $\beta > 0$ then two imaginary fixed points. One real fixed point at (0,0)
(ii) $\beta = 0$ then divide by zero error. One real fixed point at (0,0)
(iii) $\beta < 0$ then three real fixed points. Linear stability analysis using a jacobian:
$$\begin{pmatrix} 0 & 1 \\ -3\beta x^2 - 1 & \delta \end{pmatrix}_{\left(\pm\sqrt{-\frac{1}{\beta}},0\right)} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta\left(\pm\sqrt{-\frac{1}{\beta}}\right)^2 - 1 & \delta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta\left(-\frac{1}{\beta}\right) - 1 & \delta \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 0 & 1 \\ -3(-1) - 1 & \delta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 2 & \delta \end{pmatrix}$$
Eigenspace comse off of the node.
$(-\lambda)(\delta - \lambda) - (1)(2) = 0 \rightarrow \lambda^2 - \delta\lambda - 2 = 0$
Quadratic Formula. $\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4(1)(-2)}\right)$ $\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 + 8}\right)$
Always one positive and one negative eigenvalue. Thus always a saddle node.
For (0,0)
$$\begin{pmatrix} 0 & 1 \\ -3\beta x^2 - 1 & \delta \end{pmatrix}_{(0,0)} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta(0)^2 - 1 & \delta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$$
$(-\lambda)(\delta - \lambda) - (1)(-1) = 0 \rightarrow \lambda^2 - \delta\lambda + 1 = 0$

$\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4(1)(1)}\right)$ $\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4}\right)$
If $\delta > 2, \delta < -2$ two real eigenvalues. One positive one negative. A stable node.
If $\delta = 2, \delta = -2$ two real eigenvalues. 0 and δ.
If $-2 < \delta < 2$ two imaginary eigenvalues. Either unstable or stable sprial of the linear system. Depending on if $\delta < 0$ or $\delta > 0$.

d. Roughly sketch trajectories of solutions in the phase plane when $\delta = 1$ and $\beta = -1$. Include in your sketch the location of the fixed points and arrows indicating the flow along trajectories. If any of your fixed points are saddle nodes, be sure that you compute your eigenvectors to get the correct directions of the stable and unstable trajectories.

$\delta = 1$ and $\beta = -1$



e. Assuming $\delta = 0$, meaning our spring has no friction, what are our fixed points and their stability for (i) $\beta < 0$, (ii) $\beta = 0$, and (iii) $\beta > 0$?

Fixed points when $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$,
$\frac{dx}{dt} = y = 0 \rightarrow y = 0$
$\frac{dy}{dt} = -\left(\delta y + x\left(1 + \beta x^2\right)\right) = 0 \rightarrow \delta y + x\left(1+\beta x^2\right) = 0 \rightarrow *y = 0 \rightarrow x\left(1+\beta x^2\right) = 0 \rightarrow$
$1 + \beta x^2 = 0 \rightarrow \beta x^2 = -1 \rightarrow x^2 = -\frac{1}{\beta} \rightarrow x = \pm\sqrt{-\frac{1}{\beta}}$
Fixed points at: $\left(+\sqrt{-\frac{1}{\beta}}, 0\right), \left(-\sqrt{-\frac{1}{\beta}}, 0\right)$
(i) $\beta > 0$ then two imaginary fixed points.
(ii) $\beta = 0$ then divide by zero error. No fixed points.
(iii) $\beta < 0$ then two real fixed points.
Linear stability analysis using a jacobian:

$$\begin{pmatrix} 0 & 1 \\ -3\beta x^2 - 1 & \delta \end{pmatrix}_{\left(\pm\sqrt{-\frac{1}{\beta}},0\right)} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta\left(\pm\sqrt{-\frac{1}{\beta}}\right)^2 - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta\left(-\frac{1}{\beta}\right) - 1 & 0 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 0 & 1 \\ -3(-1) - 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$(-\lambda)(-\lambda) - (1)(2) = 0 \rightarrow \lambda^2 - 2 = 0 \rightarrow \lambda_{1,2} = \pm\sqrt{2}$
Always one positive and one negative eigenvalue. Thus always a saddle node.
For (0,0)

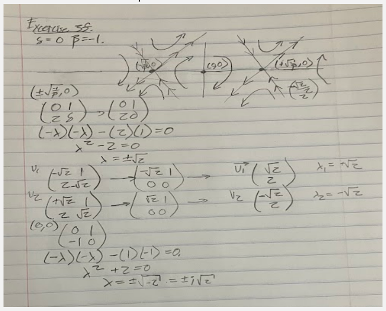
$$\begin{pmatrix} 0 & 1 \\ -3\beta x^2 - 1 & \delta \end{pmatrix}_{(0,0)} \rightarrow \begin{pmatrix} 0 & 1 \\ -3\beta(0)^2 - 1 & \delta \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}$$

$(-\lambda)(\delta - \lambda) - (1)(-1) = 0 \rightarrow \lambda^2 - \delta\lambda + 1 = 0$

$\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4(1)(1)}\right)$ $\lambda_i = \frac{1}{2}\left(\delta \pm \sqrt{\delta^2 - 4}\right) \rightarrow \lambda_i = \frac{1}{2}\left((0) \pm \sqrt{(0)^2 - 4}\right)$
$\lambda_i = \frac{1}{2}\left(\pm\sqrt{-4}\right) \rightarrow \lambda_i = \frac{1}{2}\left(\pm i\sqrt{2}\right) \rightarrow \lambda_{1,2} = \pm i\frac{\sqrt{2}}{2}$ at (0,0)

f. Roughly sketch trajectories of solutions in the phase plane when $\delta = 0$ and $\beta = -1$. Include in your sketch the location of the fixed points and arrows indicating the flow along trajectories. If any of your fixed points are saddle nodes, be sure that you compute your eigenvectors to get the correct directions of the stable and unstable trajectories.

$\delta = 0$ and $\beta = -1$



g. Suppose $\delta = 0$. Dividing (1b) by (1a) gives us

$$\frac{dy}{dx} = -\frac{x(1+\beta x^2)}{y}. \tag{2}$$

Show that the solution of this differential equation is

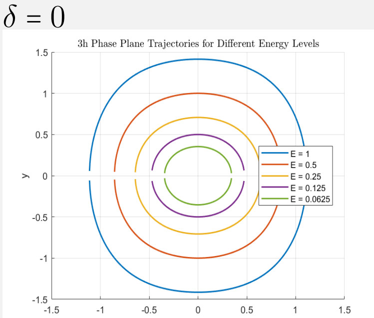
$$\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\beta}{4}x^4 = E, \tag{3}$$

where E is an arbitrary constant.

$\frac{dy}{dx} = -\frac{x(1+\beta x^2)}{y}$ $ydy = -x(1+\beta x^2)dx \rightarrow ydy = (-x-\beta x^3)dx \rightarrow \int ydy = \int(-x-\beta x^3)dx \rightarrow$
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 - \frac{1}{4}\beta x^4 + E \rightarrow \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}\beta x^4 = E$

h. Assuming β is known ahead of time, (3) gives us a one-parameter family of curves in the phase plane corresponding to the trajectories of solutions to the Duffing equation with $\delta = 0$. Using graphical software of your choice, plot these curves for $E = 1, 1/2, 1/4, 1/8$, and $1/16$ assuming $\beta = 1$. How do these curves compare to your sketch in (f)?

$\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}\beta x^4 = E$
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 - \frac{1}{4}\beta x^4 + E \rightarrow y^2 = -x^2 - \frac{1}{2}\beta x^4 + 2E \rightarrow y = \sqrt{-x^2 - \frac{1}{2}\beta x^4 + 2E}$
$\delta = 0$



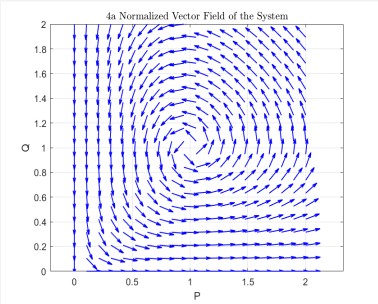
Exercise 4. (Unit 2.3)

Recall from class the Lotka-Volterra equations

$$\frac{dP}{dt} = P(\alpha - \beta Q), \tag{4a}$$
$$\frac{dQ}{dt} = Q(\delta P - \gamma). \tag{4b}$$

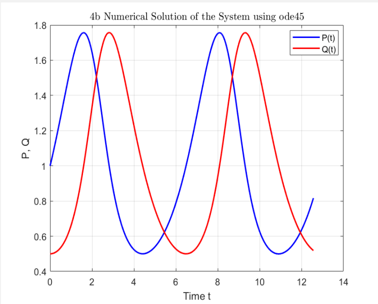
Here, P is the population of prey, Q is the population of predators, and α, β, γ, and δ are positive constants.

a. Using a programming language of your choice, plot the vector field of (4a)-(4b) in the phase plane when $\alpha = \beta = \gamma = \delta = 1$. Take the domain of your vector field to be $[0,2] \times [0,2]$. For ease of readability, normalize your vectors.



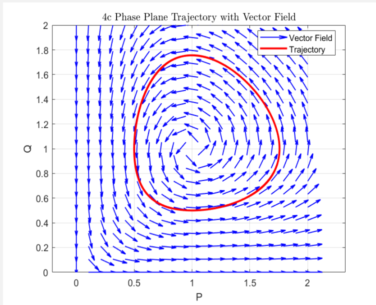
b. Solve (4a)-(4b) numerically using ode45 in MATLAB (or an equivalent Runge-Kutta fourth-order method). Take $\alpha = \beta = \gamma = \delta = 1$ and the initial conditions to be $P(0) = 1$ and $Q(0) = 1/2$. Solve the equations over the time interval $0 \le t \le 4\pi$ with $N = 1000$ time steps. Plot P and Q as a function of time on the same graph.

Hint: Use my code from lecture as a guideline.



c. Plot the trajectory in the phase plane corresponding to the solution obtained in (b). Superimpose onto this plot the vector field obtained in (a).

Hint: Use my code from lecture as a guideline.



d. If we divide (4a) by (4b), we arrive at the differential equation

$$\frac{dP}{dQ} = \frac{P(\alpha - \beta Q)}{Q(\delta P - \gamma)}. \tag{5}$$

Show that the solutions of (5) can be written implicitly as

$$\delta P - \gamma \ln(P) + \beta Q - \alpha \ln(Q) = E, \tag{6}$$

where E is an arbitrary constant.

$\frac{dP}{dQ} = \frac{P(\alpha-\beta Q)}{Q(\delta P-\gamma)} \to Q(\delta P - \gamma)dP = P(\alpha - \beta Q)dQ \to \frac{(\delta P-\gamma)}{P}dP = \frac{(\alpha-\beta Q)}{Q}dQ \to \frac{\delta P}{P}dP - \frac{\gamma}{P}dP = \frac{\alpha}{Q}dQ - \frac{\beta Q}{Q}dQ \to \int \frac{\delta P}{P}dP - \int \frac{\gamma}{P}dP = \int \frac{\alpha}{Q}dQ - \int \frac{\beta Q}{Q}dQ \to \int \delta dP - \int \frac{\gamma}{P}dP = \int \frac{\alpha}{Q}dQ - \int \beta dQ \to \delta P - \gamma \ln(P) = \alpha \ln(Q) - \beta Q + E \to \delta P - \gamma \ln(P) - \alpha \ln(Q) + \beta Q = E$

e. Upon choosing constants α, β, γ, and δ, (6) represents a one parameter family that describes the trajectories of solutions to (4a)-(4b) in the phase plane. Let $\alpha = \beta = \gamma = \delta = 1$. Plot (6) when $E = 2.01, 2.05, 2.1, 2.15, 2.2, 2.25$, and 2.3. Superimpose on your plot the vector field computed in (a).

Hint: Use my code from lecture as a guideline.