Homework 3.

Amath 383
Introduction to Continuous Mathematical Modeling

© Ryan Creedon, University of Washington

Due: 10/25/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 1.5)

Consider the model equation

$$\frac{dP}{dt} = 1 - rP + P^2. \tag{1}$$

a. For what value(s) of r does (1) have two fixed points? How about one fixed point? No fixed points?

b. Show, using the appropriate conditions, that this model equation has two saddle-node bifurcations. What are the bifurcation points?

c. Using software of your choice, plot a bifurcation diagram for $-5 \le r \le 5$. Be sure to identify the stable and unstable branches in your plot.

Now consider the model equation

$$\frac{dP}{dt} = P(r - e^P). \tag{2}$$

d. For what value(s) of r does (2) have two fixed points? How about one fixed point? No fixed points?

e. Show, using the appropriate conditions, that this model equation has a transcritical bifurcation. What is the bifurcation point?

f. Using software of your choice, plot a bifurcation diagram for $-1.5 \le r \le 3$. Be sure to identify the stable and unstable branches in your plot.

Exercise 2. (Units 1.5-1.6)

The FitzHugh-Nagumo equations were proposed independently by Richard FitzHugh in 1961 and Jin-ichi Nagumo in 1962 as a model for the firing of individual nerve cells, such as neurons in the brain. The (dimensionless) model equations are as follows:

$$\frac{dV}{dt} = V - \frac{1}{3}V^3 - W + I, \tag{3a}$$

$$\frac{dW}{dt} = \frac{1}{T_R}\left(V - \frac{W}{T_F}\right), \tag{3b}$$

where V represents the voltage potential across the membrane of the cell, I represents the net current due to ions flowing into and out of the cell, W is an artificial relaxation variable that controls how V relaxes after the cell fires, $T_R > 0$ is a constant that controls the time scale of the relaxation period between cell firings, and $T_F > 0$ is a constant that controls the time scale for the decay of the relaxation variable, allowing the cell to unrelax and refire.

a. Suppose $T_R \gg 1$, meaning $\frac{dW}{dt}$ is effectively zero. In this case, what is W as a function of V and T_F?

b. Show that, when $T_R \gg 1$ and there is no net current flowing across the cell membrane, we have

$$\frac{dV}{dt} = V\left(1 - T_F - \frac{1}{3}V^2\right). \tag{4}$$

c. Show, using the appropriate conditions, that (4) has a Pitchfork bifurcation with respect to the parameter T_F. Sketch the bifurcation diagram. Is this Pitchfork bifurcation sub- or super-critical?

d. Suppose initially $V(0) = 1$. Solve (4) with $T_F = 0.5$ by the following numerical methods:

- Euler's method,
- The midpoint method,
- The Runge-Kutta method,
- MATLAB's ode45 method.

Use $N = 20$ time steps over the interval $0 \le t \le 10$. The code for these methods can be found on the Homework 3 Canvas page. Superimpose the graphs of each numerical solution onto one graph and attach this graph to your homework assignment. *Include a legend on your graph to distinguish each of the numerical solutions.*

e. Repeat the analysis in (d) but with $T_F = 1.5$. How are your plots qualitatively different than those in (d)? How can your analysis in (c) account for this difference?

Remark: As you have seen, T_F must be sufficiently small in order for the cell to unrelax and build up voltage potential. (Otherwise, the cell continues to relax.) This makes sense, since a small T_F means that W decays quickly, allowing V to increase more rapidly.

Exercise 3. (Unit 1.6)

Consider the logistic equation with initial growth rate $r = 1$ and carrying capacity $K = 1$:

$$\frac{dP}{dt} = P(1 - P).$$

a. Find the exact solution of this equation satisfying the initial condition $P(0) = 1/2$.

b. Find the solution of this equation satisfying the initial condition $P(0) = 1/2$ by the following numerical methods:

- Euler's method,
- The midpoint method,
- The Runge-Kutta method.

Use $N = 20$ time steps over the interval $0 \le t \le 10$. Superimpose the graph of the exact solution and the graphs of each of the numerical solutions onto one graph and attach your graph to your homework assignment. *Include a legend on your graph to distinguish each of the solutions.*

Define the error of each method as the maximum difference in absolute value between the exact solution and the numerical solution over $0 \le t \le 10$:

$$\mathcal{E} = \max_{0 \le t \le 10} \left| P_{\text{exact}}(t) - P_{\text{numerical}}(t) \right|.$$

What is the error of each numerical method?

Hint: You may find MATLAB's built-in max function useful.

c. Obtain the errors for each numerical method using $N = 200$ time steps. Have the errors improved?

d. According to theory, the error $\mathcal{E}$ of a finite-difference method is approximately given by

$$\mathcal{E} = Ch^p,$$

where h is the step size used for the method, p is the order of the method, and C is a constant of proportionality. Given two errors $\mathcal{E}_1$ and $\mathcal{E}_2$ for two different step sizes h_1 and h_2, show that

$$p = \frac{\ln\left(\frac{\mathcal{E}_2}{\mathcal{E}_1}\right)}{\ln\left(\frac{h_2}{h_1}\right)}.$$

e. Using the errors computed in (b) and (c) and the formula derived in (d), estimate the order of Euler's method, the midpoint method, and the Runge-Kutta method. Do these orders match what you expect?