Homework 2.

Amath 383
Introduction to Continuous Mathematical Modeling

Due: 10/18/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 1.4)

Perform a phase plane analysis for each of the following model equations:

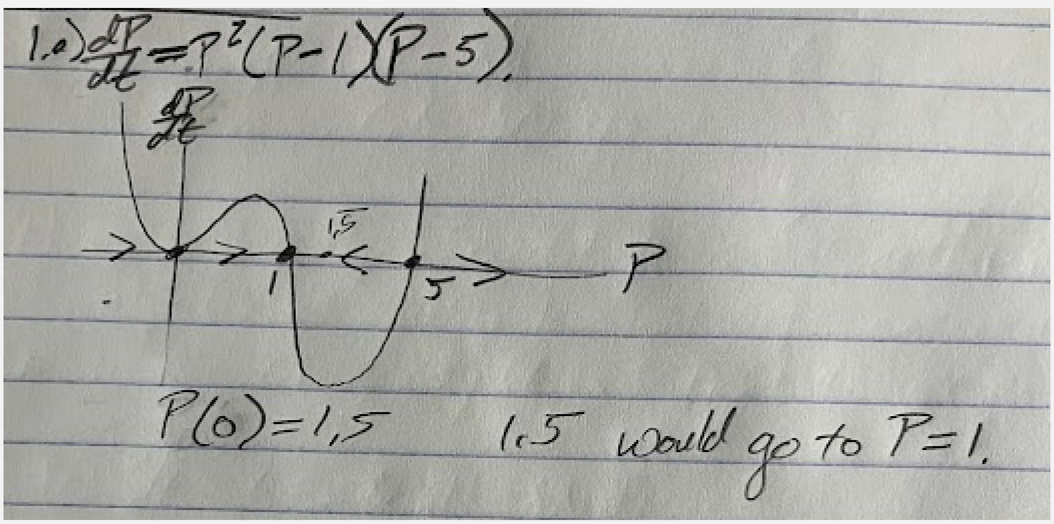
a. $\frac{dP}{dt} = P^4 - 6P^3 + 5P^2$

b. $\frac{dP}{dt} = P^4 - 6P^3 + 13P^2 - 12P + 4$ **Hint**: Use the rational root theorem.

Include in your analysis a sketch of the relevant curve in the phase plane as well as arrows to indicate the stability of fixed points. For each model equation above, also determine the long-time behavior of the solution with initial condition $P(0) = 1.5$.

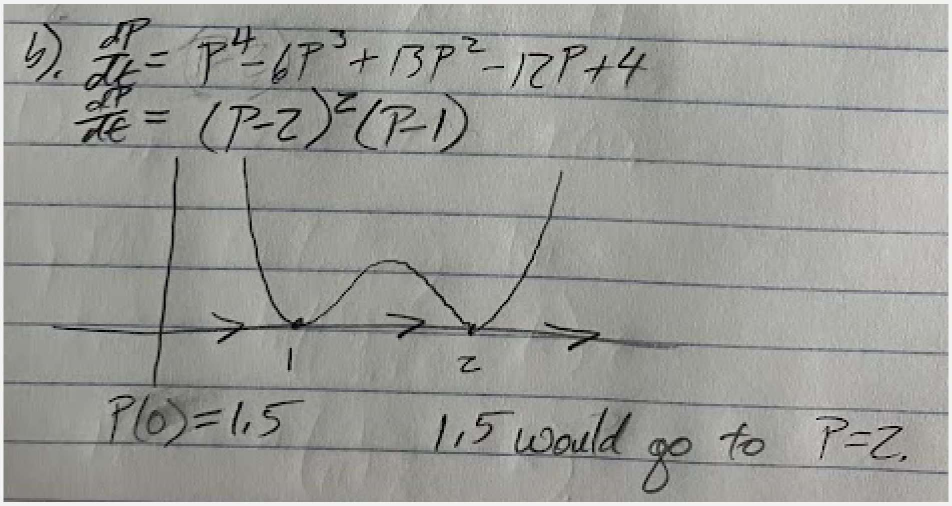
$\frac{dP}{dt} = P^4 - 6P^3 + 5P^2$
$\frac{dP}{dt} = P^2(P^2 - 6P + 5)$

roots at $P = 0, 1, 5$
Stability of the roots is semistable, stable, unstable.
With initial conditions $P(0) = 1.5$ at long enough time it would go to $P = 1$.



$\frac{dP}{dt} = P^4 - 6P^3 + 13P^2 - 12P + 4$
$\frac{dP}{dt} = (P-2)(P^3 - 4P^2 + 5P - 2)$
$\frac{dP}{dt} = (P-2)(P-1)(P^2 - 3P + 2)$
$\frac{dP}{dt} = (P-2)(P-1)(P-2)(P-1)$

roots at $P = 1, 2$
With initial conditions $P(0) = 1.5$ at long enough time it would go to $P = 2$.



Exercise 2. (Unit 1.4)

D'Arcy Wentworth Thompson, a noted scientist of natural history, wrote in his book *On Growth and Form* (1917), "But why, in the general run of shells, all the world over, in the past and in the present, one direction of twist is so overwhelmingly commoner than the other, no man knows." Thomson was referring to the spirals observed in snail shells when he wrote this passage. Most snail species are dextral (right-handed) in their shell pattern. Sinistral (left-handed) snails are exceedingly rare.

A plausible model for the evolution in the "handedness" of a spiral shell population is the following:

$$\frac{dP}{dt} = \alpha P(1-P)\left(P - \frac{1}{2}\right),$$

where $P(t)$ represents the proportion of dextral snail shells to the total snail population and $\alpha > 0$ is a constant of proportionality. If $P = 1$, all snails are right-handed. If instead $P = 0$, all snails are left-handed. The model equation above is proposed so that there is no bias towards the growth of either the left- or right-handed spiral shells.

a. Identify the fixed points of this model.

$\frac{dP}{dt} = \alpha P(1-P)(P-\frac{1}{2}) = 0$
Looking at the factored form of the equation the fix points are at
$\boxed{P = 0, 1, \frac{1}{2}}$

b. Suppose the population of snails is initially evenly divided between left- and right-handed shells. Suppose at some later time a right-handed snail dies by accident/natural selection. What happens to the snail population over time? Suppose instead at some later time a left-handed snail dies by accident/natural selection. What happens to the snail population over time?

Divided evenly initially means the population starts at the 1/2 equilibrium. After one right handed snail dies the ratio goes to below 1/2. Since now is it the spot between 0 and 1/2 it will go down below the spot. According to our model at long enough time the population will go towards 0.
If instead a left handed snail died. The population ratio would shift to above 1/2. Now it is in the spot above 1/2 and 1. At long enough time according to our model the population will go towards 1.

c. In order for the population of snails today to be overwhelmingly right-handed, what must've happened in the distant past according to your analysis in (b)?

At some time in history the proportion right handed to left handed snails was greater than equilibrium value.

d. Based on your answer to (c), did the current proportion of right-handed snails occur by accident or by choice?

Most likely by accident. Any event that made the population equilibrium slightly above 1/2 would make it go all the way towards the right-handed snails.

Exercise 3. (Units 1.4-1.5)

Consider a population model of the form

$$\frac{dP}{dt} = -P(P^2 + r(1-P)),$$

where $r > 0$ is a free parameter.

a. For what values of r does this model have 3 distinct *real* fixed points? What are these fixed points?

$\frac{dP}{dt} = -P(P^2 + r(1-P))$
$\frac{dP}{dt} = -P(P^2 - rP + r)) = 0$
$P^2 - rP + r = 0$
$\frac{-(-r) \pm \sqrt{(-r)^2 - 4(1)(r)}}{2(1)}$
$\frac{r \pm \sqrt{r^2 - 4r}}{2}$
Then 3 real points
3 distinct real roots as long as discriminate $r^2 - 4r > 0$
So when
$r^2 > 4r$
$\boxed{r > 4}$

The fixed points are at $\boxed{0, \frac{r - \sqrt{r^2 - 4r}}{2}, \frac{r + \sqrt{r^2 - 4r}}{2}}$

b. Show that the three distinct fixed points found in (a) are all non-negative (as they should be, since we are talking about a model for a non-negative population).

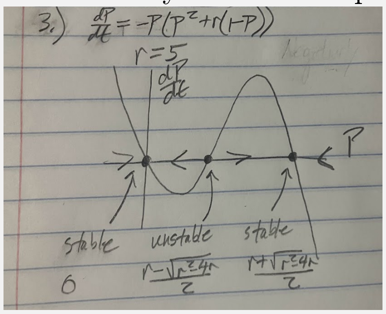
Hint: Look at what happens when you require your smaller nontrivial root from (a) to be greater than or equal to zero. Do you get a consistent chain of inequalities?

Roots at $P = 0, \frac{r+\sqrt{r^2-4r}}{2}, \frac{r-\sqrt{r^2-4r}}{2}$
Proof:
$r > 0$ $0 > -r$
$0 > -4r$
$r^2 > r^2 - 4r$
$r > \sqrt{r^2 - 4r}$
$r - \sqrt{r^2 - 4r} > 0$ could potentially be negative.
However since $r > 4$ this is always positive.
$r + \sqrt{r^2 - 4r} > 0$ is always positive and true.
$r \pm \sqrt{r^2 - 4r} > 0$
$r \pm \sqrt{r^2 - 4r} > 0$
$\frac{r \pm \sqrt{r^2 - 4r}}{2} > 0$

So all three fixed points are positive as long as r is true. Which is sufficiently true because there are only three points when $r > 4$.

c. Assess the stability of the fixed points obtained in (a).

The fixed points are at 0, $\frac{r-\sqrt{r^2-4r}}{2}$, $\frac{r+\sqrt{r^2-4r}}{2}$
The stability of the fixed points is stable, unstable, stable.



d. What happens to the number of fixed points and their stability when $r = 4$? When $0 < r < 4$? What type of bifurcation is this?

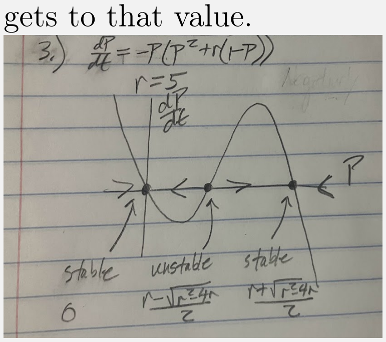
$r = 4$
$\frac{r \pm \sqrt{r^2-4r}}{2}$
$\frac{4 \pm \sqrt{(4)^2-4(4)}}{2}$
$\frac{4 \pm \sqrt{0}}{2}$
$\frac{4}{2}$
Two fixed points, $P = 0, 2$

$0 < r < 4$
So then $r^2 - 4r < 0$ which means there are no real solutions.
One fixed point at $P = 0$

$\boxed{\text{The type of bifurcation is saddle node}}$ because the two fixed points appear out of nowhere at the same time.

e. Assuming $r > 4$, what is the maximum possible population that can be achieved in the long term with this model?

$r > 4$
then there are three fixed points 0, $\frac{r-\sqrt{r^2-4r}}{2}$, $\frac{r+\sqrt{r^2-4r}}{2}$
According to the phase plot diagram in the long term the maximum value be at the fixed point $\frac{r+\sqrt{r^2-4r}}{2}$.. Because if it goes any greater than that it will decrease until it gets to that value.



Exercise 4. (Unit 1.5)

The following model of a laser was introduced in the book *Lasers* (1988) by Milonni and Eberly. Let N be the number density of excited atoms in a material absorbing and emitting laser light, and let n be the number density of photons of light passing through the material. (The larger the value of n, the stronger the laser.) When a photon meets an excited atom, it can trigger the excited atom to relax back to its ground state, emitting more photons in the process. This is called stimulated emission. A model of stimulated emission is the following:

$$\frac{dn}{dt} = GnN - kn, \tag{1a}$$
$$\frac{dN}{dt} = -GnN - fN + p, \tag{1b}$$

where $G > 0$ is a constant that measures the efficiency of increasing photons in the laser through the stimulated emission process, $k > 0$ is constant that measures the decay rate of the photons in the laser due to other processes (such as reflecting across mirrors), $f > 0$ is a constant that measures the natural relaxation rate of excited atoms in the laser, and $p > 0$ is a constant that measures the rate of production of excited atoms in the laser due to external sources.

a. Suppose the number density of excited atoms N in the laser does not change significantly in time. Using (1b) for inspiration, what must N be as a function of n, G, f, and p?

$\frac{dN}{dt} = -GnN - fN + p = 0$
$-GnN - fN = -p$
$(-Gn - f)N = -p$
$(Gn + f)N = p$
$\boxed{N = \frac{p}{Gn + f}}$

b. Using your answer to (a), show that, if N does not change significantly in time, then

$$\frac{dn}{dt} = F(n), \qquad \text{where} \qquad F(n) = n\left(\frac{p}{n + \frac{f}{G}} - k\right). \tag{2}$$

$\frac{dn}{dt} = GnN - kn = F(n)$
$F(n) = GnN - kn$
$N = \frac{p}{Gn+f}$ $F(n) = Gn(\frac{p}{Gn+f}) - kn$
$F(n) = Gn(\frac{p}{Gn+f} - k)$

$F(n) = n(\frac{Gp}{Gn+f} - k)$
$\boxed{F(n) = n\left(\frac{p}{n + \frac{f}{G}} - k\right)}$

c. Suppose G, k, and f are fixed constants, but p is allowed to vary. Show that there is a critical value of p, denoted p_c, for which the $n = 0$ fixed point of (2) is stable if $0 < p < p_c$ and unstable if $p > p_c$.

Hint: You will not be able to graph $F(n)$ to assess the change in stability of $n = 0$, as you aren't given G, k, and f. However, if $n = 0$ is stable, $F(n)$ must decrease through $n = 0$, meaning $F'(0) < 0$. Similarly, if $n = 0$ is unstable, $F(n)$ must increase through $n = 0$, meaning $F'(0) > 0$. If $n = 0$ is transitioning from stable to unstable, $F(n)$ can neither increase nor decrease through $n = 0$, meaning $F'(0) = 0$. This should give you an equation for p_c.

p_c is the critical point

$F(n) = n\left(\frac{p_c}{n+\frac{f}{G}} - k\right) = 0$

$n\left(\frac{p_c}{n+\frac{f}{G}} - k\right) = 0$

$\left(\frac{p_c}{n+\frac{f}{G}} - k\right) = 0$

$\frac{p_c}{n+\frac{f}{G}} = k$

$p_c = k(n + \frac{f}{G})$

$\frac{p_c}{k} - \frac{f}{G} = n$

We know that if $n = 0$ is transitioning between a stable and unstable fixed point, then the first derivative of that location must be zero.

$F'(n) = \frac{d}{dn}\left(n\left(\frac{p}{n+\frac{f}{G}} - k\right)\right)$

$F'(n) = \left(\frac{p}{n+\frac{f}{G}} - k\right) + \left(\frac{-p}{(n+\frac{f}{G})^2}\right)$

$F'(n = 0) = \frac{p_c}{\frac{f}{G}} - k = 0$

$$\boxed{p_c = \frac{kf}{G}}$$

d. Following a similar argument as in (c), show that the other fixed point of (2) is unstable when $0 < p < p_c$ and stable when $p > p_c$.

Find roots of polynomial.

$p = kn + \frac{kf}{G}$

$kn + \frac{kf}{G} < p_c = \frac{kf}{G}$

$kn + \frac{kf}{G} < \frac{kf}{G}$

$kn < 0$

$n < 0$ since $k > 0$

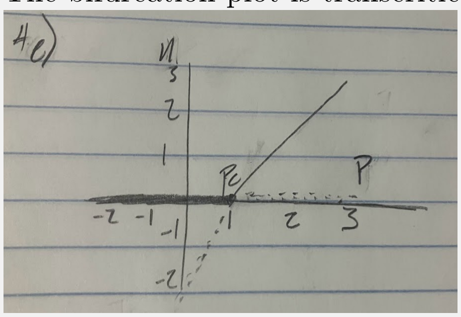
Which means the fixed point is to the left of $n = 0$. $n = 0$ fixed point is stable. Each root has a multiplicity of 1. Thus the fixed point must have an opposite stability to the $n = 0$ fixed point.

e. Plot a bifurcation diagram of the fixed points of (2) as a function of p. For concreteness, let

$G = k = f = 1$. Identify p_c as well as the bifurcation point. What type of bifurcation happens at p_c?

Remark: The value p_c is sometimes called the critical pumping parameter. Only when atoms are excited enough by a pumping factor $p > p_c$ does the laser emit a nonzero number density of photons in the long term.

$G = k = f = 1$
$p_c = \frac{kf}{g} = \frac{(1)(1)}{(1)} = 1$
$p_c = 1$
Bifurcation point is (1,0).
The bifurcation plot is transcritical.



Exercise 5. (Unit 1.5)

Two metal sheets connected by a torsion spring are pressed together to create a mousetrap. A model for the evolution of the angle θ between these two metal sheets is

$$\frac{d\theta}{dt} = \mu \sin(\theta) - \theta,$$

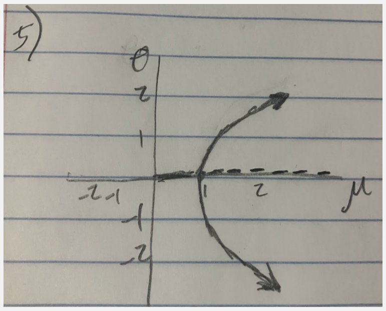
where $\mu > 0$ is a constant that represents the ratio of compressive forces keeping the plates together and repulsive forces acting in opposition to the compressive forces.

Remark: This simple model of a mousetrap was studied by Euler in 1757 and demonstrates one of the first successes of bifurcation theory to explain the buckling of elastic plates under stress.

a. Show that this model has a pitchfork bifurcation at $\mu_* = 1$ and $\theta_* = 0$.

This has a pitchfork bifurcation at $\mu_* = 1$ and $\theta_* = 0$ because it satisfies these six inequalities.
$F(0;1) = \mu \sin(\theta) - \theta = 0,$
$\frac{\partial F(0;1)}{\partial P} = \mu \cos(\theta) - 1 = 1 - 1 = 0$
$\frac{\partial F(0;1)}{\partial r} = \sin(\theta) = 0$
$\frac{\partial^2 F}{\partial P^2}(0;1) = -\mu \sin(\theta) = 1 * 0 = 0,$
$\frac{\partial^2 F}{\partial P \partial r}(0;1) = \cos(\theta) = 1 \neq 0,$
$\frac{\partial^3 F}{\partial P^3}(0;1) = -\mu \cos(\theta) = -1 \neq 0.$

b. Sketch the bifurcation diagram.



c. Is this pitchfork bifurcation supercritical or subcritical?

supercritical. because the stable fixed point splits into one unstable fixed point and two new stable fixed points and