Homework 4.

Amath 383
Introduction to Continuous Mathematical Modeling

Due: 11/1/22 at 11:59pm to Gradescope

Directions:

Complete all exercises as neatly as possible. Up to 3 points may be deducted for homework that is illegible and/or poorly organized. You are encouraged to type homework solutions, and a half bonus point will be awarded to students who use LaTeX. (Check out my LaTeX beginner document and overleaf.com if you are new to LaTeX.) If you prefer not to type homeworks, I ask that **homeworks be scanned.** (I will not accept physical copies.) In addition, **homeworks must be in .pdf format.**

Pro-Tips:

- You have access to the textbook, which inspired some of these exercises. You may find that the textbook offers alternative explanations that can help you solve these exercises.
- If a result was already derived in class, no need to rederive it here. Just make sure you cite where in the lecture notes the result your using can be found.
- Teamwork makes the dream work, but please indicate at the top of your assignment who your collaborators are.
- Don't wait until the last minute. ☺

Exercise 1. (Unit 2.1)

Lidocaine is a local anesthetic that prevents pain signals from reaching the brain without causing you to lose consciousness. The drug is typically injected into a muscle that needs to be numbed during an operation. A model for the dynamics of lidocaine in the body takes the form

$$\frac{dP}{dt} = -\alpha P + \beta Q,$$
$$\frac{dQ}{dt} = \gamma P - \beta Q,$$

where P is the concentration of lidocaine in the bloodstream, Q is the concentration of lidocaine in the target muscle, $\alpha > 0$ is a constant that represents the overall decay rate of lidocaine in the bloodstream, $\beta > 0$ is a constant that represents the loss of lidocaine from the target muscle to the bloodstream, and $\gamma > 0$ is a constant that represents the reabsorption of lidocaine into the target muscle from the bloodstream.

a. Obtain the eigenvalues of the coefficient matrix for this system. Show that these eigenvalues are always real-valued and distinct for all $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

   Hint: Recall that $x^2 - 2xy + y^2 = (x - y)^2$.

b. Show that both eigenvalues are negative precisely when $\alpha > \gamma$. In this parameter regime, do solutions grow or decay in time?

   Hint: Consider the larger of your two eigenvalues, require this eigenvalue to be negative, and follow the chain of inequalities.

c. Suppose $\alpha = 5$, $\beta = 3$, and $\gamma = 1$. Find the general solution of the linear system above. Is your result consistent with (b)?

d. Suppose lidocaine of concentration Q_0 is initially injected into the target muscle of a patient. (Assume no lidocaine is present in the patient's bloodstream prior to injection.) Using your general solution from (c), determine P and Q for all $t > 0$. Your final answer should depend only on one parameter, which is Q_0.

e. Using software of your choice, plot the concentration of lidocaine in the bloodstream and target muscle over the time interval $0 \le t \le 2$ for the patient described in (d). Use $Q_0 = 1$ for concreteness. *Be sure to include labels on your axes!* (Don't worry about units.)

f. Show that the maximum concentration of lidocaine in the bloodstream for the patient described in (d) is

$$P_{\max} = \frac{Q_0}{2\sqrt{3}},$$

which occurs at time

$$t_{\max} = \frac{\ln(3)}{4}.$$

Hint: Consider when the first derivative of $P(t)$ is zero. You *don't* have to check the second derivative to confirm this is a maximum.

g. Too much lidocaine in the bloodstream can be fatal. Suppose the patient in (d) is at risk for lidocaine poisoning if at any point in time the concentration of lidocaine in the bloodstream exceeds 30% of the initial injection concentration. Is the patient at risk for lidocaine poisoning, or did their anesthesiologist do their calculations correctly?

Exercise 2. (Unit 2.1)

Suppose Jeff Bezos causes an oil spill in South Lake Union to spite the city of Seattle. Some of this oil will naturally decay in the lake, but a great deal of this oil will flow into Lake Washington through the Montlake Cut, inadvertently affecting Bezos' cushy beachfront property in Medina. In this exercise, we will model how the concentration of oil in Lake Union and Lake Washington changes as a function of time. To do so, we introduce the following variables:

- Let $P(t)$ represent the concentration of oil in Lake Union, and let $Q(t)$ represent the concentration of oil in Lake Washington.
- Let $f > 0$ be a constant that represents the flow rate of the polluted water from Lake Union into Lake Washington.
- Let $r_1 > 0$ represent the natural rate of decay of oil in Lake Union, and let $r_2 > 0$ represent the natural rate of decay of oil in Lake Washington.

A model for the change in concentration of the oil in both lakes is

$$\frac{dP}{dt} = -fP - r_1 P, \tag{1a}$$

$$\frac{dQ}{dt} = fP - r_2 Q. \tag{1b}$$

Unless otherwise stated, assume $r_2 \neq f + r_1$ in what follows.

a. Find the general solution of the system (1a)-(1b).

b. Suppose the initial concentration of oil in Lake Union is P_0. If there is initially no oil in Lake Washington, what is the concentration of oil in both lakes for all $t > 0$? (Note that your answers will depend on t as well as f, r_1, r_2, and P_0.)

c. Based on your answer to (b), will Lake Union and Lake Washington eventually recover from the spill?

d. Suppose $f = r_1 = r_2 = 1$ in your answer to (b). Bezos' beachfront property will be ruined if the concentration of oil in Lake Washington reaches $P_0/4$ at any moment in time. Did Bezos inadvertently destroy his own beachfront property?

e. Find the general solution of the system (1a)-(1b) assuming $r_2 = f + r_1$. What is the solution subject to the initial conditions $P(0) = P_0$ and $Q(0) = 0$?

Exercise 3. (Units 2.1-2.2)

Consider the constant-coefficient, homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \mathrm{A}\mathbf{x},$$

where $\mathbf{x}(t) = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}$ and $\mathrm{A} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$.

a. What is the general solution of this linear system?

b. What is the solution of this linear system subject to the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

c. Plot the vector field of this system in the Q *vs.* P phase plane over the rectangular domain $[-5,5] \times [-5,5]$ using MATLAB (or a programming language of your choice). For ease of readability, normalize your vectors so that they have unit length. Also superimpose the trajectory $\mathbf{x}(t)$ obtained in (b) to your vector field plot.

   Hint: Use my code from class for inspiration.

d. Based on your answers to (a) and (c), is the fixed point at the origin stable or unstable? What type of fixed point is at the origin? (Is it a stable/unstable node, saddle point, stable/unstable spiral, center, *etc.*?)

e. How does the behavior at the origin change if $\mathrm{A} = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$?

Exercise 4. (Unit 2.2)

Read the article "Differential Equations and Love Affairs" by S. Strogatz provided on the Homework

4 Canvas page. It should be a short, entertaining read[1]. Once you've read the article, have a go at the problems below.

a. Suppose in our tale of Romeo and Juliet that Romeo is fickle while Juliet is receptive. According to Strogatz, their love is modeled according to

$$\frac{dr}{dt} = -aj, \tag{2a}$$
$$\frac{dj}{dt} = br, \tag{2b}$$

where $r(t)$ represents Romeo's attraction to Juliet, $j(t)$ represents Juliet's attraction to Romeo, and a and b are positive constants. What is the behavior nearby the origin of the phase plane for this system?

b. Suppose initially that Romeo is interested in Juliet but Juliet is unaware of Romeo, meaning $r(0) = r_0 > 0$ while $j(0) = 0$. If the dynamics of Romeo and Juliet's attraction for each other is governed by (2a)-(2b), what is Romeo's attraction for Juliet as a function of time? What is Juliet's attraction for Romeo as a function of time? (Note that your answers will depend not only on time, but also on the constants a, b, and r_0.)

c. Strogatz claims the situation in (b) devolves into a neverending cycle of love and hate. Is that what your equations predict, or do your equations predict that Romeo and Juliet's attraction for each other settles to some fixed state?

d. Let's suppose now that Romeo is an eager-beaver while Juliet is more cautious, meaning

$$\frac{dr}{dt} = a_{11}r + a_{12}j, \tag{3a}$$
$$\frac{dj}{dt} = a_{21}r + a_{22}j, \tag{3b}$$

where a_{11}, a_{12}, and a_{21} are positive constants and a_{22} is a negative constant. Let

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Show that the behavior near the origin is always a saddle node in this case.

e. Suppose $a_{11} = 1$, $a_{12} = 2$, $a_{21} = 4$, and $a_{22} = -1$. What are the eigenvalues of A? Is this consistent with (d)?

f. What are the eigenvectors corresponding to the eigenvalues in (e)?

[1]Strogatz is as much a talented writer as he is a mathematician. For more of his amusing mathematical prose, check out his textbook on nonlinear dynamics and chaos. It's a standard in the field.

g. Roughly sketch the behavior near the origin of the phase plane for the system (3a)-(3b) with coefficients given in (e). Include in your sketch the lines through the origin that are parallel to your eigenvectors, the direction of motion along these lines, and some sample trajectories in between these lines.

h. We say that Romeo and Juliet live happily ever after if both $r(t)$ and $j(t)$ head to positive infinity as $t \to \infty$. (That is, the trajectory in phase space heads off to infinity in the first quadrant.) Similarly, we say that Romeo and Juliet are caught in a bad romance[2] if both $r(t)$ and $j(t)$ head to negative infinity as $t \to \infty$. (That is, the trajectory in phase space heads off to infinity in the third quadrant.)

Based on your sketch in (g), what region in the phase plane corresponds to initial conditions that lead to a happily ever after? To a bad romance?

[2]Yes, I did shamelessly steal Lady Gaga lyrics for a math homework.