Assignment_3 Solution

1. Line detection

a. The range of slope and y-intercept are infinity.

b. Line appears as sinusoid curve.

c. $x\cos(\theta) + y\sin(\theta) - d = 0 \Rightarrow d = 1\cos(0) + 1\sin(0) = 1$. Therefore, it casts a vote at location $(0, 1)$ in Hough space.

d. Each edge point is transformed to a line in the Hough space, and the areas where most lines intersect in the Hough space is interpreted as true lines in the edge map.

e. Bin size trade-off:

- Big bin: fewer votes, faster, lower accuracy
- Small bin: more votes, slower, higher accuracy

f. When the normal θ_n is known, we can narrow down the searching space from $\theta \in [0, 180]$ to $\theta \in [\theta_n - \Delta\theta, \theta_n + \Delta\theta]$.

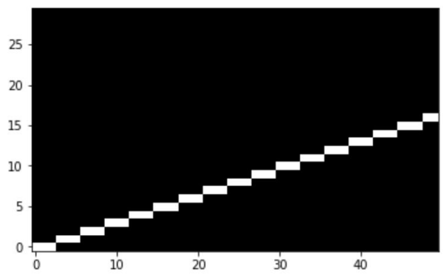
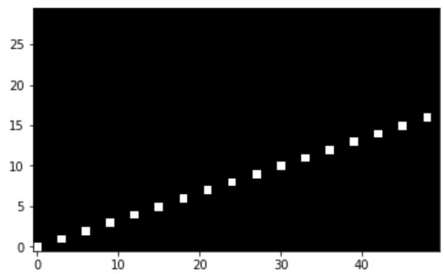
g. Three dimensions: circle center (a, b) and radius r

h. Given the implicit line equation: $x\cos(\theta) + y\sin(\theta) - d = 0$.

- For the nearly horizontal line ($\theta \in [45^\circ, 135^\circ]$),

$$\begin{aligned}
&x\cos(\theta) + y\sin(\theta) - d = 0\\
&y\sin(\theta) = -x\cos(\theta) + d\\
&\text{Since } \theta \in [45^\circ, 135^\circ],\ \ \sin(\theta) \neq 0\\
&y = -\frac{\cos(\theta)}{\sin(\theta)}x + \frac{d}{\sin(\theta)}
\end{aligned}$$

- The reason why we need to scan x and compute y for the nearly horizontal line:
  - It can avoid zero denominator. If scanning y and computing x, we may encounter $\cos(90^\circ) = 0$ on the denominator.
  - For the nearly horizontal line, the absolute slope of the line is smaller than 1. If we scan y (increment for the y-values is one), the x value of the line will change by more than one which leads to the gaps in the representation of the line in raster graphics.
  - Example: $y = \frac{1}{3}x$ (nearly horizontal line).

Scan x and compute y	Scan y and compute x
	

i. Similar to **1.h**

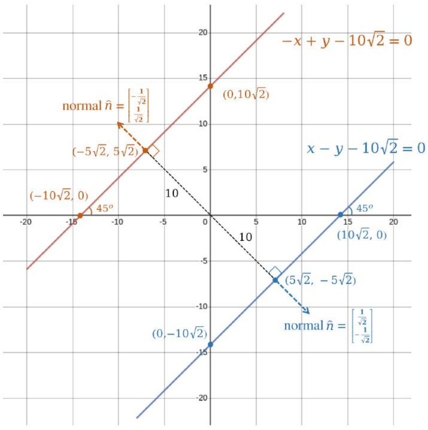
2. Model fitting

a.

- With $y = ax + b$, we cannot model vertical or near vertical lines because the slope would have to be infinite. This shows that we need to be careful when choosing the model so that it can describe all possible (and not only a subset) of observations.
- Vertical and near vertical lines.

b. Lines meet the requirements: $-x + y - 10\sqrt{2} = 0$ and $x - y - 10\sqrt{2} = 0$

- Parameters
  - Line 1: $a = -1, b = 1$, and $c = -10\sqrt{2}$
  - Line 2: $a = 1, b = -1$, and $c = -10\sqrt{2}$
- Draw and points on the line:

draw	points on line
	Line 1: $-x + y - 10\sqrt{2} = 0$ Points: $(-10\sqrt{2}, 0)$ and $(0, 10\sqrt{2})$ Line 2: $x - y - 10\sqrt{2} = 0$ Points: $(10\sqrt{2}, 0)$ and $(0, -10\sqrt{2})$

c.

- Implicit equation: $-x + y = 0$ or $x - y = 0$
- Normalized normal vector: $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^\top$ or $[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^\top$

d. $l = [\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -2]^\top$

e. -2.5

f.

- To fit a line using the implicit line equation: $l^\top p_i = 0$
  1. Build the correlation matrix $S = \sum_i p_i p_i^\top$.
  2. Find the eigenvector of S belonging to zero eigenvalue.
- The equation needs to solve:

$$\begin{cases} E(l) = l^\top S l \\ l^* = \arg\min_{l} E(l) \end{cases}, \quad \text{where } S = \begin{bmatrix} \sum x_i^2 & \Sigma x_i y_i & \Sigma x_i \\ \sum x_i y_i & \Sigma y_i^2 & \Sigma y_i \\ \sum x_i & \sum y_i & n \end{bmatrix}$$

Let $\nabla E(l) = 0$, so the equation that has to be solved is $Sl = 0$. The solution l^* is the eigenvector of S belonging to zero eigenvalue.

g. $S = \sum_{i=1}^{n} p_i p_i^\top = D^\top D$, where $D = \begin{bmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix}$. Therefore, $S = \begin{bmatrix} 5 & 15 & 3 \\ 15 & 46 & 10 \\ 3 & 10 & 3 \end{bmatrix}$.

h. When a point p is off the curve f,

- the algebraic distance is: $d(p, f) = |f(p)|$.
- the geometric distance is: $d(p, f) = |p - x^*| = \frac{|f(p)|}{|\nabla f(x^*)|}$, where x^* is the closest point on the curve f.

i.

- The geometric distance:
  - Exact way: $d(p, f) = |p - x^*| = \frac{|f(p)|}{|\nabla f(x^*)|}$
  - Approximated: $d(p, f) \approx \frac{|f(p)|}{|\nabla f(p)|}$
- The reason for approximating is we cannot compute the exact value of x^*.

j. Algebraic distance: $d(p, f) = |f(p)| = 1$

k. Approximated geometric distance: $d(p, f) \approx \frac{|f(p)|}{|\nabla f(p)|} = \frac{1}{2}$

l.

- The objective function for active contours:

$$E(\phi(s)) = \int_{\phi(s)} (\underbrace{\alpha(s)E_{\text{cont}} + \beta(s)E_{\text{curv}}}_{\text{internal energy}} + \underbrace{\gamma(s)E_{\text{img}}}_{\text{external energy}})ds,$$

  where $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are coefficients of the different energy terms.
- Components:
  - Continuity energy describes the contour behavior regarding elasticity or smoothness: $E_{\text{cont}} = |\frac{d\phi}{ds}|^2$
  - Curvature energy describes the contour behavior regarding curvature: $E_{\text{curv}} = |\frac{d^2\phi}{ds^2}|^2$
  - Image energy describes how the deformable curve will match with objects of the image: $E_{\text{img}} = -|\nabla I|^2$.

m. $p_1 = (1, 2)$, $p_2 = (2, 3)$, and $p_3 = (3, 4)$. At point p_2,

- $E_{\text{cont}} = |p_3 - p_2|^2 = |(1, 1)|^2 = (\sqrt{1^2 + 1^2})^2 = 2$
- $E_{\text{curv}} = |(p_3 - p_2) - (p_2 - p_1)|^2 = |(1, 1) - (1, 1)|^2 = 0$

n. We can set β to zero at corners to allow discontinuity.

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3. Robust estimation

a.

- Outliers are the observation points that are distant from other observations.
- The fundamental problem is that the model fitting can be mismatched by the influence of outliers.

b.

- $E(\theta) = \sum_{i=1}^{n} \xi_\sigma(d(x_i;\theta))$
- In robust estimation, the function gives low weight for high-value outlier, however, the least squares objective function gives higher weight for high-value outlier.

c.

- Geman-McClure function: $\xi_\sigma(x) = \frac{x^2}{x^2+\sigma^2}$
- Advantages:
  - We can control the loss function.
  - The weight of outliers is up to 1 (has upper bound).
- Start with large σ and decrease as converging: $\sigma^{(n)} = 1.5 \times \text{median}\{d(x_i;\theta^{(n-1)})\}$

d. Geman-McClure function: $\xi_\sigma(x=1,\sigma=1) = \frac{x^2}{x^2+\sigma^2} = \frac{1}{2}$

e.

- RANSAC algorithm estimates the model parameters by repeated random sampling of observed data (the observed data contains both inliers and outliers). RANSAC uses the consensus scheme to find the best result. The principle of RANSAC algorithm:
  - Perform multiple experiments
  - Choose the best result
  - Use small sets in hope that at least one set will not have outliers
- The number of points drawn at each attempt should be small.
- A small number of points drawn can avoid including more outliers.

f.

- Parameters of RANSAC algorithm:
  - n : the number of points at each evaluation.
  - d : the minimum number of points needed.
  - t : the threshold to identify inliers.
  - k : the number of trials.
- The number of trials: $k = \frac{\log(1-p)}{\log(1-w^n)}$, where w is the probability that a data point is an inlier and p is at least one of the draws is free from outliers.

g. Number of experiments: $k = \frac{\log(0.01)}{\log(1-0.9^n)}$

n	k	n	k	n	k
2	2.7729774493739905	3	3.5271458279290657	4	4.314363283401872
5	5.15815564748312	6	6.074675084464547	7	7.077727323270934
8	8.18058587666166	9	9.396838847904254	10	10.740876345462974