Assignment 0 Solution

Question A:

1. $2a - b = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$
2. $\hat{a} = \dfrac{a}{\|a\|} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
3. $\|a\| = \sqrt{14}$ and the angle of a relative to the positive x axis is: $\arccos(\frac{1}{\sqrt{14}}) \approx 74.499^\circ$
4. The direction of cosines of a is: $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$
5. The angle between a and b is: $\arccos(\frac{a \cdot b}{\|a\|\|b\|}) = \arccos(\frac{32}{\sqrt{14 \cdot 77}}) \approx 12.933^\circ$
6. $a \cdot b = b \cdot a = 4 + 10 + 18 = 32$. The dot product is commutative: $a \cdot b = b \cdot a$
7. $a \cdot b$ using the angle between a and b is: $a \cdot b = \|a\|\|b\| \cos(\theta) = 32$
8. Scalar projection b onto $\hat{a}$: $b \cdot \frac{\hat{a}}{\|\hat{a}\|} = \frac{32}{\sqrt{14}} \approx 8.5524$
9. $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \quad \Rightarrow$ One of the solutions can be $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
10. $a \times b = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$ and $b \times a = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$. The cross product is anticommutative: $a \times b = -(b \times a)$
11. The cross product of two vectors is another vector that is perpendicular to both:
    $a \times b = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$
12. $xa + yb + zc = 0 \xRightarrow{x=3, y=-1, z=-1} 3a - b - c = 0$
13. $a^T b = a \cdot b = 32$ (dot product) and $ab^T = a \otimes b = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$ (outer product)

Question B:

1. $2A - B = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$
2. $AB = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$ and $BA = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$
3. $(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$ and $(AB)^T = B^T A^T$
4. $|A| = 55$. Because of ***A-12*** (matrix is linearly dependent), $|C| = 0$
5. The row vectors of B form an orthogonal set, because BB^T is a diagonal matrix.
6. $A^{-1} = \dfrac{1}{|A|} \mathrm{adj}(A) = \dfrac{1}{55} \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$.

Because of ***B-5***, we have $BB^T = D^2$ where $D = diag([\ \|B_{1,:}\|, \|B_{2,:}\|, \|B_{3,:}\|\ ]^T)$. Thus, B^{-1} is cheap to compute:

$$B^{-1} = B^T D^{-2} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{21} & 0 \\ 0 & 0 & \frac{1}{14} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} & \frac{3}{14} \\ \frac{2}{6} & \frac{1}{21} & \frac{-2}{14} \\ \frac{1}{6} & \frac{-4}{21} & \frac{1}{14} \end{bmatrix}$$

B is not an orthogonal matrix, because the rows and columns are not **orthonormal** vectors.

7. Because of ***B-4***, C^{-1} does not exist.
8. $Ad = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$
9. The scalar projection of the rows of A onto the vector d (with normalizing d)

   scalar projection: $a_i = A_{i,:} \cdot \frac{d}{\|d\|}$. If d is normalized, then scalar projection is dot product.

   $a_1 = \frac{14}{\sqrt{14}}$, $a_2 = \frac{9}{\sqrt{14}}$, and $a_3 = \frac{7}{\sqrt{14}}$
10. The vector projection of the rows of A onto the vector d (with normalizing d)

    vector projection: $a_i = (A_{i,:} \cdot \frac{d}{\|d\|})\frac{d}{\|d\|}$

    $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} \frac{9}{14} \\ \frac{18}{14} \\ \frac{27}{14} \end{bmatrix}$, and $a_3 = \begin{bmatrix} \frac{7}{14} \\ \frac{14}{14} \\ \frac{21}{14} \end{bmatrix}$
11. The linear combination of the columns of A using the elements of d (right-multiplication):

    $$1 \times \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + 2 \times \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} + 3 \times \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$
12. $x = B^{-1}d = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} & \frac{3}{14} \\ \frac{2}{6} & \frac{1}{21} & \frac{-2}{14} \\ \frac{1}{6} & \frac{-4}{21} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
13. No solution, because C is not invertible (***B-7***).

Question C:

1. - Eigenvalues: $|D - \lambda I| = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0 \Rightarrow \lambda_1 = 4 \text{ and } \lambda_2 = -1$
   - Eigenvectors (nonzero vector):

     $(D - \lambda_1 I)e_1 = 0 \Rightarrow e_1 = \begin{bmatrix} a \\ \frac{3}{2}a \end{bmatrix}$. One solution can be $e_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, if assuming $a = 2$

     $(D - \lambda_2 I)e_2 = 0 \Rightarrow e_2 = \begin{bmatrix} b \\ -b \end{bmatrix}$. One solution can be $e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, if assuming $b = 1$

     Eigenvectors are often normalized so that $e_i^T e_i = \|e_i\|^2 = 1$. So, e_1 and e_2 can be $\begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.
2. $e_1 \cdot e_2 = -\frac{1}{2}ab$. Solutions can be -1 if using $a = 2$ and $b = 1$, or $-\frac{1}{\sqrt{26}} \approx -0.1961$ if using normalized eigenvectors (same as Numpy result).
3. Eigenvalues: $|E| = \lambda_1\lambda_2 = 6$ and $\mathrm{tr}(E) = \lambda_1 + \lambda_2 = 7 \Rightarrow \lambda_1 = 6$ and $\lambda_2 = 1$

Eigenvectors: $(E - \lambda_1 I)e_1 = 0 \Rightarrow e_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$ and $(E - \lambda_2 I)e_2 = 0 \Rightarrow e_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

Therefore, $e_1 \cdot e_2 = 0$

4.
   - Eigenvectors belonging to distinct eigenvalues of a real symmetric matrix are orthogonal.
   - Let A be a real symmetric matrix with eigenvalues (λ_1, λ_2) and eigenvectors (e_1, e_2)

     Eigenvalues and eigenvectors of A satisfy: $Ae_1 = \lambda_1 e_1$ and $Ae_2 = \lambda_2 e_2$

$$\begin{aligned} & Ae_1 = \lambda_1 e_1 \\ \text{Pre-multiply } e_2^T \text{ on both sides:} & \\ & e_2^T(Ae_1) = e_2^T(\lambda_1 e_1) = \lambda_1 e_2^T e_1 \\ & (A^T e_2)^T e_1 = \lambda_1 e_2^T e_1 \\ & (Ae_2)^T e_1 = \lambda_1 e_2^T e_1 \\ \text{Substitute } Ae_2 \text{ by } Ae_2 = \lambda_2 e_2\text{:} & \\ & \lambda_2 e_2^T e_1 = \lambda_1 e_2^T e_1 \\ & (\lambda_2 - \lambda_1) e_2^T e_1 = 0 \Rightarrow \boxed{(\lambda_2 - \lambda_1)(e_2 \cdot e_1) = 0} \end{aligned}$$

     Therefore, $\lambda_1 \neq \lambda_2$ implies eigenvectors e_1 and e_2 are orthogonal.

5. Trivial solution: $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
6. $Fx = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -2t \\ t \end{bmatrix}$. Two non-trivial solutions ($t \neq 0$) can be: $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
7. Only has trivial solution: $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, because the columns of D are linearly independent.

   The homogeneous system, $Ax = 0$, will either have trivial solution as its only solution if $|A| \neq 0$ (**C-7**), or it will have an infinite number of solutions if $|A| = 0$ (**C-6**).

Question D:

1. $f'(x) = 2x$ and $f''(x) = 2$
2. $\frac{\partial q}{\partial x} = 2x, \quad \frac{\partial q}{\partial y} = 2y$
3. $\nabla q(x, y) = \begin{bmatrix} \frac{\partial q}{\partial x} \\ \frac{\partial q}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$
4. $f(x) = x^2 + 3$, $g(x) = x^2$, and $f(g(x)) = g(x)^2 + 3 = x^4 + 3$
   - With chain rule: $\frac{d}{dx} f(g(x)) = \frac{d}{dg(x)} f(g(x)) \cdot \frac{d}{dx} g(x)$ $\quad \boxed{(f \circ g)(x) \equiv f(g(x))}$

     $\frac{d}{dg(x)} f(g(x)) = \frac{d}{dg(x)} (g(x)^2 + 3) = 2g(x)$, and $\frac{d}{dx} g(x) = 2x$

     $\Rightarrow \frac{d}{dx} f(g(x)) = 2g(x) \cdot 2x = 4x^3$
   - Without chain rule: $\frac{d}{dx} f(g(x)) = \frac{d}{dx} (x^4 + 3) = 4x^3$