CS512 S23

HW#0

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Answer to the Question No. A

Given,

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

A.1

$$2a - b = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

A.2

$$\hat{a} = \frac{a}{\|a\|} = \frac{1}{\sqrt{1^2+2^2+3^2}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A.3

$$\|a\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\text{Angle, } \theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.50^\circ$$

A.4 The direction cosines of a are:

$$\alpha = \frac{1}{\sqrt{14}}, \quad \beta = \frac{2}{\sqrt{14}}, \quad \gamma = \frac{3}{\sqrt{14}}$$

A.5 The angle between a and b $= \cos^{-1}\left(\frac{a \cdot b}{\|a\|\,\|b\|}\right)$

$$= \cos^{-1}\left(\frac{1\times4+2\times5+3\times6}{\sqrt{14}\times\sqrt{77}}\right)$$

$$= 12.93^\circ$$

A.6

$$a \cdot b = 1\times4 + 2\times5 + 3\times6 = 32$$

$$b \cdot a = 4\times1 + 5\times2 + 6\times3 = 32$$

A.7 $\cos\theta = \dfrac{a.b}{\|a\| \, \|b\|}$

$\therefore a.b = \|a\| \, \|b\| \cos\theta = \sqrt{14}\sqrt{77} \cos(12.93°)$

$= 32.00$

A.8 The scalar projection of b ~~a~~ onto $\hat{a} = \dfrac{b.a}{\|a\|}$

$= b. \dfrac{a}{\|a\|}$

$= b.\hat{a}$

$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= \dfrac{4 \times 1 + 5 \times 2 + 6 \times 3}{\sqrt{14}}$

$= \dfrac{32}{\sqrt{14}}$

A.9 Let $V = [V_1 \ V_2 \ V_3]$ is a vector perpendicular to a. We know that the dot product of two orthogonal vector is 0.

So, $1 \times V_1 + 2 \times V_2 + 3 \times V_3 = 0$

If $V_1 = 1$ and $V_2 = 1$, then

$1 \times 1 + 2 \times 1 + 3 \times V_3 = 0$

$\therefore V_3 = -1$

Hence, $V = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is perpendicular to a.

A.10 $a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (12-15)i - (6-12)j + (5-8)k$

$= \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$

$b \times a = \begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = (15-12)i - (12-6)j + (8-5)k$

$= \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$

11 The cross product of a and b produces a vector Q that is both perpendicular to the plane containing a and b. So,

$$Q = a \times b = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

A.12 The linear dependence of a, b and c follows the relation $ax + by + cz = 0$

$$\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Forming the row-echelon form

$$\begin{bmatrix} 1 & 4 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, a, b and c are not linearly independent.

A.13

$$a^T b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$$

$$a b^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Answer to the Question B

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$$

and $d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

B.1 $$2A - B = \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

B.2 $$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$$

B.3 $$(AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

~~$B^T A^T$~~ $$B^T A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 2 & -2 & 5 \\ 3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

B.4 $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} = 1(2-15) - 2(-4-0) + 3(20-0)$$
$$= 55$$

$$|C| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 1(15-6) - 2(12+6) + 3(4+5)$$
$$= 0$$

B.5 Two vectors are orthogonal if their inner product is

For matrix A, row 1 (dot) row 2 $= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} = 9$

row 2 (dot) row 3 $= -13$

row 1 (dot) row 3 $= 7$

for matrix B, dot product of the rows pairwise yield 0. So, the rows of the matrix B are normal to one another.

For ~~the~~ matrix C, dot product of the rows pairwise doesnot not yield zero. So, the ~~the~~ rows of the matrix C are not orthogonal to one another.

B.6

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{55}\begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$$B^{-1} = \frac{adj(B)}{|B|} = \frac{1}{42}\begin{bmatrix} 7 & 4 & 9 \\ 14 & 2 & -6 \\ 7 & -8 & 3 \end{bmatrix}$$

B.7

$$C^{-1} = \frac{adj(C)}{|C|}$$

As $|C| = 0$, the inverse of C doesn't exist.

B.8

$$Ad = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 9 \\ 7 \end{bmatrix}$$

B.9

$$\text{row 1 (dot) } d = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14$$

$$\text{row 2 (dot) } d = \begin{bmatrix} 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 9$$

$$\text{row 3 (dot) } d = \begin{bmatrix} 0 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 7$$

B.10

$$\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix} a_2 + \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} a_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & -2 & 3 & 2 \\ 0 & 5 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -10 & -9 & -2 \\ 0 & 0 & 11 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 9/10 & 1/5 \\ 0 & 0 & 1 & -4/11 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1.2 & 3/5 \\ 0 & 1 & 0.9 & 1/5 \\ 0 & 0 & 1 & -4/11 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 24/25 \\ 0 & 1 & 0 & 29/55 \\ 0 & 0 & 1 & -4/11 \end{array}\right]$$

So, $a_1 = 24/25$

$a_2 = 29/55$

$a_3 = -4/11$

B.11 $Bx = d$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 1 & -4 & 2 \\ 3 & -2 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -3 & -6 & 0 \\ 0 & -8 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So, $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

B.12 Matrix C is a non invertible matrix or singular matrix, so there is no solution.

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Answer to the Question No. C

Given,

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

C.1 To find the eigenvalues of D,

$$Dx = \lambda x$$

$$(D - \lambda I)x = 0$$

so, $|D - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda^2 + \lambda - 4\lambda - 4 = 0$$

$$\lambda(\lambda+1) - 4(\lambda+1) = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\therefore \lambda = -1, 4$$

For $\lambda = -1$,

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

So, $u_1 = \begin{bmatrix} c_1 \\ -c_1 \end{bmatrix}$

let $c_1 = 1$

so, $u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 4$,

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$$

so, $u_2 = \begin{bmatrix} \frac{2}{3}c_2 \\ c_2 \end{bmatrix}$

let $c_2 = 3$

so, $u_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

So, $x = [u_1 \quad u_2] = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

(Ans)

C.2 $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -1$

C.3 $\begin{bmatrix} -0.89442719 & -0.4472136 \end{bmatrix} \begin{bmatrix} 0.4472136 \\ -0.89442719 \end{bmatrix} = 0$

C.4 Since the dot product of the eigenvectors of E ~~a~~ is 0, they are orthogonal.

C.5
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented matrix,
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The trivial solution is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

C.6 The two non trivial solutions are

$$u_1 = \begin{bmatrix} -2c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} c_2 \\ -\frac{1}{2}c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So, $x = [u_1 \ \ u_2] = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$

C.7 The solution is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As the determinant of D is non-zero, the homogeneous system of linear equations must have a unique (trivial) solution.

Answer to the Question No. D

Given, $f(x) = x^2 + 3$, $g(x) = x^2$, $v(x,y) = x^2 + y^2$

D.1

$$f'(x) = 2x$$

$$f''(x) = 2$$

D.2

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = 2y$$

D.3

$$\nabla v(x,y) = \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

D.4

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= f'(g(x)) \times g'(x) \\ &= f'(x^2) \times 2x \\ &= \frac{d}{dx}\left(f(x^2)\right) \times 2x \\ &= \frac{d}{dx}(x^4 + 3) \times 2x \\ &= 4x^3 \times 2x \\ &= 8x^4 \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= \frac{d}{dx} f(x^2) \\ &= \frac{d}{dx}(x^4 + 3) \\ &= 4x^3 \end{aligned}$$