Chapter 1

Sets

1.1 Sets

Definition 1.1.1: Set

A *set* is a collection of objects, called *elements*.

Anything you can think of can be a set. This is known as the *Axiom of Abstraction*.

However, there are problems with the Axiom of Abstraction. Paradoxes can arise, the most famous example being *Russell's Paradox*.

Hence, some sources define a set as a "well-defined" collection of objects.

Definition 1.1.2: Set-builder Notation

A set can be specified by means of such a propositional function:

$$S = \{x : P(x)\}$$

which means **S is the set of all objects that satisfy the property P**.
Symbolically,

$$\forall x : (x \in S \iff P(x))$$

Example 1.1.3
Let S be the set defined as:

$$S := \{n^2 : n \in \mathbb{Z}^+\}$$

Then S is the set of all squares of strictly positive integers.

$$S = \{1, 4, 9, 16, \ldots\}$$

Definition 1.1.4: Subset

Let A and B be sets.
A is a *subset* of B if and only if all of the elements of A are also elements of B.
Symbolically,

$$A \subseteq B \iff \forall x : (x \in A \Rightarrow x \in B)$$

If the elements of A are not all also elements of B, then A is not a subset of B:

$$A \nsubseteq B \iff \neg(A \subseteq B)$$

Example 1.1.5

$$\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$$
$$\{1, 2, 3\} \nsubseteq \{1, 2, 4\}$$

Theorem 1.1.6 Empty Set is Subset of All Sets

The empty set $\varnothing$ is a subset of every set, including itself.
Symbolically,

$$\forall S : \varnothing \in S$$

Proof: By the definition of subset,

$$\varnothing \in S \Rightarrow (\forall x : (x \in \varnothing \Rightarrow x \in S))$$

And by the definition of the empty set,

$$\forall x : \neg(x \in \varnothing)$$

Thus $\varnothing \in S$ is vacuously true. ■

Theorem 1.1.7 Set is a Subset of Itself

Every set is a subset of itself.
In other words,

$$\forall S : S \subseteq S$$

Proof: Left as an exercise for the reader. ■

Definition 1.1.8: Proper Subset

Let A and B be sets such that A is a subset of B.
Let $A \neq B$.
Then, A is a *proper subset* of B, which is annotated as:

$$A \subset B$$

or

$$A \subsetneq B$$

1.2 Set Operations

Definition 1.2.1: Union of Sets

Let A and B be sets.
The *union* of A and B is the set $A \cup B$, which consists of all the elements which are contained in either or both of A and B:

$$A \cup B := \{x : x \in A \vee x \in B\}$$

Definition 1.2.2: Intersection of Sets

Let A and B be sets.
The *intersection* of A and B is the set $A \cup B$, which consists of all the elements which are contained in both of A and B:

$$A \cap B := \{x : x \in A \wedge x \in B\}$$

Definition 1.2.3: Finite Union

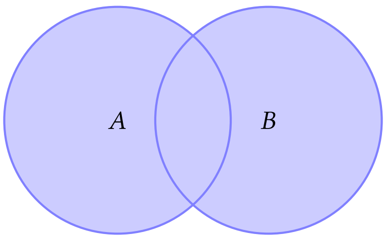
Let $A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$.
Then,

$$A = \bigcup_{i=1}^{n} A_i.$$

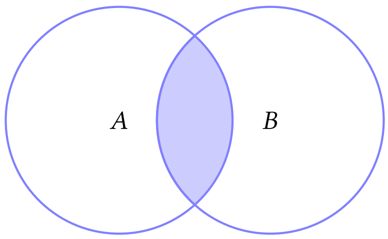
Definition 1.2.4: Finite Intersection

Let $A = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$.
Then,

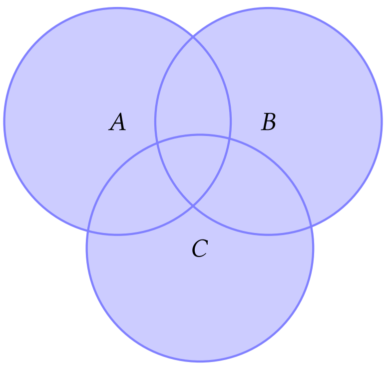
$$A = \bigcap_{i=1}^{n} A_i.$$



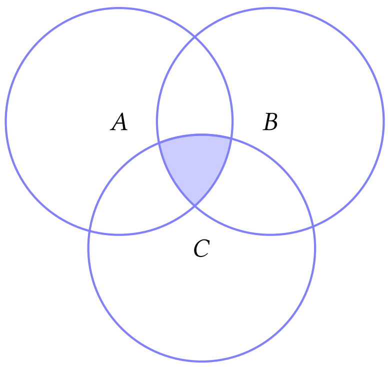
$A \cup B$



$A \cap B$



$A \cup B \cup C$



$A \cap B \cap C$

Definition 1.2.5: Set Difference

The *difference* between two sets A and B, written $A \setminus B$ is the set that contains elements of A which are not elements of B.

$$A \setminus B := \{x \in A : x \notin B\}$$

Definition 1.2.6: Universe

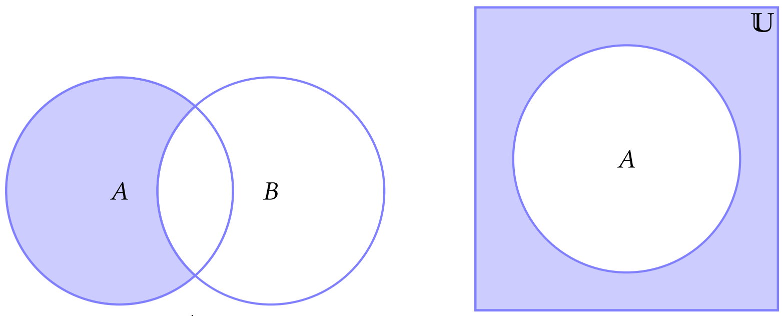
A *universal set* is a set where all sets are subsets of.
What the universe exactly is depends on the context.
However, note that, due to the Russell's Paradox, there exists no absolutely universal set, so this universe cannot be everything that there is.
The universal set is usually written as $\mathbb{U}$, however this is not universal (pun not intended) - some uses $\mathfrak{U}, X$ or even $\mathbb{X}$.

Definition 1.2.7: Set Complement

A *set complement* of a set A in a universe $\mathbb{U}$ is defined as:

$$A^c = \mathbb{U} \setminus A$$

There is no standard notation for a set complement. Common ones include $C(A), c(A), \mathrm{C}(A), CA, \mathbf{C}(A), A^c$, $A', A*, -A, \overline{A}$ and $\sim A$.



$A \setminus B$ A^c

Definition 1.2.8: Power Set

The *power set* of a set A is the set defined as:

$$\mathcal{P}(A) := \{B : B \subseteq S\}$$

Example 1.2.9

$$\mathcal{P}(\{1,2,3\}) = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{\varnothing\}\}$$

Theorem 1.2.10 Empty Set is Element of Power Set

$$\forall A : \varnothing \in \mathcal{P}(A)$$

Proof: Obvious. ■

Definition 1.2.11: Cardinality of a Finite Set

Let A be a finite set.
The *cardinality* $|A|$ is the number of elements in A.
The formal definition is beyond the scope of this course.

Definition 1.2.12: Cardinality of an Infinite Set

Let B be an infinite set.
The *cardinality* $|B|$ is

$$|B| = \infty$$

It should be noted that this just means that the cardinality of B cannot be assigned a number $n \in \mathbb{N}$. In other words, $|B|$ is at least $\aleph_0$. However, this is beyond the scope of this course.

Example 1.2.13

$$\begin{aligned} |\{1,2,3\}| &= 3 \\ |\{\{1,2\},\{a,b,c\}\}| &= 2 \\ |\mathbb{N}| &= \infty \end{aligned}$$

Definition 1.2.14: Cartesian Product

Let A and B be sets.
The *cartesian product* of A and B is defined as:

$$A \times B := \{(x,y) : x \in A \land y \in B\}$$

In other words,

$$\forall p : (p \in A \times B \iff \exists x : \exists y : x \in A \land y \in B \land p = (x,y))$$

Note that the elements of $A \times B$ are ordered pairs, not sets.

Example 1.2.15

$$\{1,2,3\} \times \{a,b\} = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

1.3 Results of Sets

Lemma 1.3.1

The subset relation is transitive.
In other words,

$$(A \subseteq B) \land (B \subseteq C) \Rightarrow A \subseteq C$$

Proof:

$$\begin{aligned} &(A \subseteq B) \land (B \subseteq C) \\ \Rightarrow\ &(x \in A \Rightarrow x \in B) \land (x \in B \Rightarrow x \in C) \\ \Rightarrow\ &(x \in A \Rightarrow x \in C) \\ \Rightarrow\ &A \subseteq C \end{aligned}$$

■

Theorem 1.3.2 Power Set of Subset

Let A and B be sets.
Let $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Then,

$$A \subseteq B$$

Proof: Let $x \in A$.

By the definition of the subset,

$$\{x\} \subseteq A$$

By the definition of the power set,

$$\{x\} \subseteq \mathcal{P}(A)$$

By Lemma 1.3.1

$$\begin{aligned} \{x\} &\subseteq \mathcal{P}(B) \\ \{x\} &\subseteq B \\ x &\in B \end{aligned}$$

We have shown that $x \in A$ implies $x \in B$, hence $A \subseteq B$ ∎

Theorem 1.3.3 De Morgan's Laws

Let A and B be sets.
Then,

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c \\ (A \cap B)^c &= A^c \cup B^c \end{aligned}$$

In other words, if everything is, then there exists nothing that is not. If everything is not, then there exists nothing that is. If not everything is, then there exists something that is not. If not everything is not, then there exists something that is.

Proof: Let A and B be in the universe $\mathbb{U}$.

$$\begin{aligned} A^c \cup B^c &= \{x \in \mathbb{U} : x \in A^c \vee x \in B^c\} \\ &= \{x \in \mathbb{U} : x \notin A \vee x \notin B\} \\ &= \{x \in \mathbb{U} : x \notin (A \cap B)\} \\ &= (A \cap B)^c \end{aligned}$$

$$\begin{aligned} A^c \cap B^c &= \{x \in \mathbb{U} : x \in A^c \wedge x \in B^c\} \\ &= \{x \in \mathbb{U} : x \notin A \wedge x \notin B\} \\ &= \{x \in \mathbb{U} : x \notin (A \cup B)\} \\ &= (A \cup B)^c \end{aligned}$$

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