Homework 2

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Problem 1

For mass function

$$f_{T_n}(t) = (8n - 8n^2 t)\, I_{(1/2n,1/n)}(t)$$

the distribution function over the same domain is obtained by integration

$$F_{T_n}(t) = (8nt - 4n^2 t^2 + C)\, I_{(1/2n,1/n)}(t).$$

To be a valud distrubition function, F must equal 0 at the lower bound of the domain, $t = 1/2n$, and equal 1 at the upper bound $t = 1/n$. At theses bounds, F evaluates to

$$F_{T_n}(1/2n) = \frac{8n}{2n} - \frac{4n^2}{(2n)^2} + C \quad = 3 + C$$
$$F_{T_n}(1/n) = \frac{8n}{n} - \frac{4n^2}{n^2} + C \quad = 4 + C.$$

Therefore $C = -3$, $F(t) = 0$ for $t < 1/2n$ and $F(t) = 1$ for $t > 1/n$, and

$$F_{T_n}(t) = (8nt - 4n^2 t^2 - 3)\, I_{(1/2n,1/n)}(t)$$

meets the requirements of a distrubution function for all n. As n approaches infinity, the range of $I_{(1/2n,1/n)}(t)$ approaches the point 0. We have already defined the $F(t) = 1$ for $t > 1/n$, so we can say that

$$lim_{n\to\infty} F_{T_n}(t) = 1 \quad \forall t > 0.$$

Thus, the limiting distribution of $F_{T_n}(t)$ approaches $F(t)$, where $F(t)$ is the distribution function of $T = 0$. The probability function of the limiting distribution for the sequence $T_1, T_2, \ldots$ is

$$f(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 2

For

$$f_{X_n}(x) = \begin{cases} \dfrac{1}{n}, & x = n^2, \\ \dfrac{n-1}{n}, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

the moment-generating function is calculated by

$$M_X(t) = \sum_{x\in A} e^{tx} p(x) = e^{t\cdot 0}\left(\frac{n-1}{n}\right) + e^{tn^2}\left(\frac{1}{n}\right) = \frac{1}{n}\left(e^{tn^2} + n - 1\right).$$

To calculate the $E(X_n)$ we need the first t derivative of M_X,

$$M_X'(t) = \frac{d}{dt}\frac{1}{n}\left(e^{tn^2} + n - 1\right) = ne^{tn^2},$$

so that

$$E(X_n) = M_X'(0) = ne^{0 \cdot n^2} = n.$$

Thus,

$$\begin{aligned} T_n &= X_n - E(X_n) \\ &= X_n - n \end{aligned}$$

The probility distribution of T_n can then be defined as

$$f_{T_n}(x) = \begin{cases} \frac{1}{n}, & x = n^2 - n \\ \frac{n-1}{n}, & x = -n \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function is

$$F_{T_n}(x) = \begin{cases} 0, & x < -n \\ \frac{n-1}{n}, & -n \leq x < n^2 - n \\ 1, & n \geq n^2 - n. \end{cases}$$

As $n \to \infty$, the term $(n-1)/n$ approaches 1, so

$$\lim_{n\to\infty} F_{T_n} = \begin{cases} 0, & x < -n \\ 1, & x \geq -n. \end{cases}$$

However, $-n$ iteself goes to $-\infty$, so F_{T_n} does not converge. Since the distribution function of T_n does not converge, the a limiting distrubtion for the sequence $T_1, T_2, ...$ does not exist.

Problem 3

For the continuious probability function $f_X(x) = 3(1-x)^2\, I_{(0,1)}(x)$, the cumulative distribution function is obtained by integration

$$F_X(x) = \int 3(1-x)^2\, I_{(0,1)}(x)\, dx = x^3 - 3x^2 + 3x + C.$$

For the bounds of the probability function 0 and 1, the values of F_X are

$$\begin{aligned} x^3 - 3x^2 + 3x + C\,\Big|_0 &= C \\ x^3 - 3x^2 + 3x + C\,\Big|_1 &= 1 - 3 + 3 + C = 1 + C. \end{aligned}$$

Set $C = 0$, $F_X = 0$ for $x < 0$ and $F_X = 1$ for $x > 1$. Since the integral

$$F_X(x) = \int_0^1 3(1-x)^2\, I_{(0,1)}(x)\, dx = 1,$$

$F_X = x(x^2 - 3x - 3)\, I_{(0,1)}(x)$ is a valid distribution function. We can now use Theorem 5.4.4 from the textbook to find an expression for the probability function of $\max\{X_1, ..., X_n\}$, the n-th order statistic of X:

$$\begin{aligned} f_{X_{(n)}}(x) &= \frac{n!}{(n-1)!(n-n)!} f_X(x)\, [F_X(x)]^{n-1}\, [1 - F_X(x)]^{n-n} \\ &= n\, f_X(x)\, [F_X(x)]^{n-1} \end{aligned}$$

On the domain $0 < x < 1$. We can find an expression for

$$T_n = n^{1/3}\left(1 - n\, f_X(x)\, [F_X(x)]^{n-1}\,.\right)$$

While considering the limit with respect to n of this function, it is important to note that $F_X \leq 1$ for all x in the domain since it is a distribution function; therefore, $[F_X(x)]^{n-1}$ tends to zero as n tends to infinity.

Problem 4

The moment generating function of the sum of intependent random variables in the sequence $X_1, X_2, ...$ is

$$M_{X_1}(t)M_{X_2}(t)....$$

The moment generating function of Poisson distribution is $exp(\lambda(e^t - 1))$. Thus the moment generating function for $T_n = X_1 + ... + X_n$ is

$$\begin{aligned} M_{T_n}(t) &= \prod_{k=1}^{n} \exp\left(k^{-2}\left(e^t - 1\right)\right) \\ &= \exp\left(\left(e^t - 1\right)\sum_{k=1}^{n}\frac{1}{k^2}\right). \end{aligned}$$

This is itself the moment generating function of a Poisson distrubution with

$$\lambda = \sum_{k=1}^{n}\frac{1}{k^2}.$$

As n goes to infinity, the sum of this infinite series is $\pi^2/6$. Therefore, as n goes to infinity, the sequence $T_1, T_2, ...$ converges to a limiting distribution of Poiss($\pi^2/6$), the probability mass function of which is

$$\frac{\pi^{2j}e^{-\pi^2/6}}{6^j j!}$$

Problem 5

The joint distrubution of multiple Bernoulli random variables is the distribution of binomial random variable. For Bernoulli random varialbes X_i with mean $p = 1/2$,

$$\sum_{i=1}^{n} X_i = nE(X_i) = n^2 p = \frac{1}{2}n^2.$$

Since X_i is a Bernoulli variable, the only possible outcomes are 0 and 1. Since $0^2 = 0$ and $1^2 = 1$,

$$\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i = \frac{1}{2}n^2.$$

Therefore,

$$T_n = \sqrt{n}\left(\frac{4\sum_{i=1}^{2} X_i - 2n}{\sum_{i=1}^{n} X_i^2}\right) = \sqrt{n}\left(\frac{2n^2 - 2n}{1/2n^2}\right) = \frac{4}{\sqrt{n}}(n-1)$$

Problem 6

Part a

X is a random variable with $E(X) = \theta$. Let $g(X) = 1/\sqrt{X}$ be an estimator for $1/\sqrt{\theta}$; $g'(X) = -1/2\,x^{-3/2}$. The first order Taylor approximation to the mean of $1/\sqrt{X_n}$ is

$$\begin{aligned} g(X) &= g(\theta) + g'(\theta)(X - \theta) \\ \frac{1}{\sqrt{X_n}} &= \frac{1}{\sqrt{\theta}} - \frac{1}{2}\theta^{-\frac{3}{2}}(X_n - \theta) \\ E\left[\frac{1}{\sqrt{X_n}}\right] &= \frac{1}{\sqrt{\theta}} + -\frac{1}{2}\theta^{-\frac{3}{2}}\,E\left[X_n - \theta\right] \end{aligned}$$

Since θ is the mean of X, $E[X - \theta] = 0$, and so

$$E\left[\frac{1}{\sqrt{X}}\right] = \frac{1}{\sqrt{\theta}}$$

Part b

Since $g''(x) = 3/4x^{-5/2}$; the second order approximation to the mean of $1/\sqrt{X_n}$ is

$$g(X) = g(\theta) + g'(\theta)(X-\theta) + \frac{1}{2}g''(\theta)(X_n-\theta)^2$$
$$E\left[\frac{1}{\sqrt{X}}\right] = \frac{1}{\sqrt{\theta}} + \frac{3}{4}\theta^{-\frac{5}{2}}E\left[(X_n-\theta)^2\right]$$

Now θ is the mean of X, $E\left[(X-\theta)^2\right]$ is the definition of the variance of X, which is θ^3. Therefore

$$E\left[\frac{1}{\sqrt{X_n}}\right] = \frac{1}{\sqrt{\theta}} + \frac{3}{4}\sqrt{\theta}$$

or

$$E\left[\frac{1}{\sqrt{X_n}}\right] = \frac{1+\frac{3}{4}\theta}{\sqrt{\theta}}$$

Part c

The first order approximation to the variance of $1/\sqrt{X_n}$ is

$$\begin{aligned}\operatorname{Var}\left[\frac{1}{\sqrt{X_n}}\right] &= \left(-\frac{1}{2}\theta^{-\frac{3}{2}}\right)^2 \operatorname{Var}[X_n] \\ &= \frac{1}{4}\frac{1}{\theta^3}\theta^3 = \frac{1}{4}\end{aligned}$$

Problem 7

To generate a sufficient statistic, we must factorize the joint probability density function $f_X(\mathbf{x}|\theta)$ into two terms $g(T(\mathbf{x})|\theta)$ and $h(\mathbf{x})$ as

$$\begin{aligned}f_X(x) &= \prod_{k=1}^{n}(\theta+1)^k x_k^\theta I_{(0,1)}(x) \\ &= (\theta+1)^n \exp\left(\theta\sum_{k=1}^{n}\log x_k\right) I_{(0,1)}(x).\end{aligned}$$

Therefore

$$h(\mathbf{x}) = I_{(0,1)}(x)$$

and

$$g(T(\mathbf{x})|\theta) = (\theta+1)^n \exp\left(\theta\sum_{k=1}^{n}\log x_k\right),$$

so

$$T(\mathbf{x}) = \sum_{k=1}^{n}\log x_k$$

is a sufficiency statistic for θ for every x in the sample space.