Daniel Hartig
Stat 652
Homework 4

Problem 1

a.

$$
\begin{aligned}
f(\mathbf{x}|\theta) &= \prod_{i=1}^{n}\left[\frac{1}{2\theta^2}\exp\left(\frac{-\sqrt{x_i}}{\theta}\right)I_{(0,\infty)}(x_i)\right] \\
&= \frac{1}{2^n\theta^{2n}}\exp\left(\frac{-1}{\theta}\sum_{i=1}^{n}\sqrt{x_i}\right)\prod_{i=1}^{n}I_{(0,\infty)}(x_i)
\end{aligned}
$$

This pdf is in the exponential family, and can be decomposed as follows,

$$
c(\theta) = \frac{1}{2^n\theta^{2n}}, \qquad h(\mathbf{x}) = \prod_{i=1}^{n} I_{(0,\infty)}(x_i), \qquad w_1 = \frac{-1}{\theta}, \qquad t = \sum_{i=1}^{n}\sqrt{x_i}
$$

Since

$$
\{w_1(\theta) : \theta \in \Theta\} \quad \rightarrow \quad \left\{\frac{-1}{\theta} : \theta > 0\right\} \quad \rightarrow \quad (-\infty, 0)
$$

contains an open set on $\mathbb{R}'$, $T(\mathbf{x}) = \sum_{i=0}^{n}\sqrt{x_i}$ is a complete sufficient statistic for θ. The expectation of any X_i can be calculated as

$$
\begin{aligned}
E(X) &= \int_0^\infty x\frac{1}{2\theta^2}\exp\left(\frac{-\sqrt{x}}{\theta}\right)dx \\
&= \frac{1}{2\theta^2}\exp\left(\frac{-\sqrt{x}}{\theta}\right)\left(-12\theta^4 - 12\theta^3\sqrt{x} - 6\theta^2 x - \theta x^{3/2}\right)\Big|_0^\infty && \text{solved with Wolfram Alpha} \\
&= \exp\left(\frac{-\sqrt{x}}{\theta}\right)\left(-6\theta^2 - 6\theta\sqrt{x} - 3x - \frac{1}{2\theta}x^{3/2}\right)\Big|_0^\infty \\
&= e^{-\infty}(...) - e^0\left(-6\theta^2 - 0\right) = 6\theta^2
\end{aligned}
$$

Using a similar derivation, the expectation of $\sqrt{X}$ is

$$
\begin{aligned}
E(\sqrt{X}) &= \int_0^\infty \sqrt{x}\frac{1}{2\theta^2}\exp\left(\frac{-\sqrt{x}}{\theta}\right)dx \\
&= \exp\left(\frac{-\sqrt{x}}{\theta}\right)\left(-2\theta - 2\sqrt{x} - \frac{1}{\theta}x\right)\Big|_0^\infty && \text{solved with Wolfram Alpha} \\
&= 2\theta
\end{aligned}
$$

To find an unbiased estimator, we investigate the expectation of $T(\mathbf{x})$,

$$
\begin{aligned}
E\left(T(\mathbf{x})\right) &= E\left(\sum_{i=1}^{n}\sqrt{x_i}\right) \\
&= \sum_{i=1}^{n}E\left(\sqrt{x_i}\right) \\
&= n(2\theta)
\end{aligned}
$$

Therefore, the expectation of

$$
\hat{\theta} = \frac{T(\mathbf{x})}{2n}; \qquad E(\hat{\theta}) = E\left(\frac{T}{2n}\right) = \theta
$$

shows that $\hat{\theta}$ is an unbiased estimator of θ. Since it is also a function of a complete sufficient statistic of θ, by Theorem 7.3.23, $\hat{\theta}$ is the unique best unbiased estimator of θ.

b.

The Cramer-Rao lower bound for the variance of $\hat{\theta}$ given that $\hat{\theta}$ is a function of $T(\mathbf{X})$ and $X_1, X_2, ...$ are iid variables is given by

$$\frac{1}{nI_1(\theta)} = \frac{1}{nE_\theta\left(\left[\frac{d}{d\theta}\log f(X|\theta)\right]^2\right)}.$$

We resolve the denominator using values of expectation from part a,

$$\begin{aligned}
\frac{d}{d\theta}\log f(X|\theta) &= \frac{d}{d\theta}\left(-2\log(\sqrt{2}\theta) - \frac{\sqrt{x}}{\theta}\right) = \frac{-2}{\theta} + \frac{\sqrt{x}}{\theta^2} \\
\left[\frac{d}{d\theta}\log f(X|\theta)\right]^2 &= \frac{4}{\theta^2} - \frac{4\sqrt{x}}{\theta^3} + \frac{x}{\theta^4} \\
E\left(\left[\frac{d}{d\theta}\log f(X|\theta)\right]^2\right) &= E\left(\frac{4}{\theta^2} - \frac{4\sqrt{x}}{\theta^3} + \frac{x}{\theta^4}\right) = \frac{4}{\theta^2} - \frac{4E(\sqrt{x})}{\theta^3} + \frac{E(x)}{\theta^4} \\
&= \frac{4}{\theta^2} - \frac{8\theta}{\theta^3} + \frac{6\theta^2}{\theta^4} = \frac{2}{\theta^2}
\end{aligned}$$

which gives us a Cramer-Rao lower bound of

$$\frac{1}{n \cdot \frac{2}{\theta^2}} = \frac{\theta^2}{2n}.$$

c.

$$\begin{aligned}
\operatorname{Var}(\hat{\theta}) &= \operatorname{Var}\left(\frac{T}{4n^2}\right) = \frac{1}{4n^2}\operatorname{Var}(T) = \frac{1}{4n^2}\operatorname{Var}\left(\sum_{i=1}^n \sqrt{X_i}\right) = \frac{1}{4n^2}\sum_{i=1}^n \operatorname{Var}(\sqrt{X_i}) \\
&= \frac{n}{4n^2}\left(E(X) - \left[E(\sqrt{X})\right]^2\right) \\
&= \frac{1}{4n}\left[6\theta^2 - (2\theta)^2\right] \\
&= \frac{\theta^2}{2n}
\end{aligned}$$

d.

We can reuse the complete, sufficient statistic $T(\mathbf{x})$ derived in part a. From part a we see that $E(T(\mathbf{x})) = 2n\theta$. Therefore, to find an unbiased estimator for $6\theta^2$, we investigate the expectation of T^2 using information from part c,

$$\begin{aligned}
E\left(T^2\right) &= \operatorname{Var}(T) + [E(T)]^2 \\
&= 2n\theta^2 + (2n\theta)^2 \\
&= 2n(2n+1)\theta^2
\end{aligned}$$

Thus

$$\widehat{6\theta^2} = \frac{3T^2}{n(2n+1)}; \qquad E\left(\widehat{6\theta^2}\right) = E\left(\frac{3T^2}{n(2n+1)}\right) = \frac{3E\left(T^2\right)}{n(2n+1)} = 6\theta^2$$

is an unbiased estimator for $6\theta^2$. Since it is also a function of $T(\mathbf{x})$, and we have determined that this is a complete, sufficient statistic in part a, by Theorem 7.3.23, $\widehat{6\theta^2}$ is the unique best unbiased estimator of $6\theta^2$.

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e.

Let $\tau(\theta) = 6\theta^2$, so that $\tau'(\theta) = 12\theta$. Since we have alrady calculated the Fisher information for the f in part b, the Cramer-Rao lower bound is given by

$$\frac{[\tau'(\theta)]^2}{nI_1(\theta)} = \frac{144\theta^2}{n \cdot \frac{2}{\theta^2}} = \frac{72\theta^4}{n}$$

f.

$$\begin{aligned}
\mathrm{Var}\left(\widehat{6\theta^2}\right) &= \mathrm{Var}\left(\frac{3T^2}{n(2n+1)}\right) \\
&= \frac{9}{n^2(2n+1)^2}\mathrm{Var}(T^2) \\
&= \frac{9}{n^2(2n+1)^2}\left(E(T^4) - \left[E(T^2)\right]^2\right) \\
&= \frac{9}{n^2(2n+1)^2}\left(??? - \left[2n(2n+1)\theta^2\right]^2\right)
\end{aligned}$$

g.

The MME is obtained by setting $m_1 = \bar{X}$ equal to $\mu_1' = E(X)$. Since $E(X) = 6\theta^2$, we get $\widehat{6\theta^2} = \bar{X}$ as the MME estimator of $6\theta^2$. To find the variance we must take

$$\begin{aligned}
\mathrm{Var}\left(\widehat{6\theta^2}\right) = \mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) \\
&= \frac{1}{n}\mathrm{Var}(X_i)
\end{aligned}$$

The variance of X (any of the iid X_i) can be calculated from its definition, using the values of $E(X)$ from part a, and

$$\begin{aligned}
E(X^2) &= \int_0^\infty \frac{x^2}{2\theta^2}\exp\left(\frac{-\sqrt{x}}{\theta}\right)dx \\
&= 120\theta^4 \qquad\qquad \text{solved with Wolfram Alpha}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(\bar{X}) &= \frac{1}{n}\left(E(X^2) - [E(X)]^2\right) \\
&= \frac{1}{n}\left(120\theta^4 - 36\theta^4\right) = \frac{84\theta^4}{n}
\end{aligned}$$

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Problem 2

A sum of iid Bernoulli random variables has a binomial distribution with probability θ and $n = 4$. From the notes on page 6.2.16, we see that $T(\mathbf{X}) = \sum_{i=1}^{4} X_i$ is a complete, sufficient statistic for a binomial distribution with $n = 4$. Since the sufficient statistic T itself has a binomial distribution, we know that

$$E(T) = np = 4\theta$$

$$\text{Var}(T) = np(1-p) = 4\theta - 4\theta^2$$

$$\begin{aligned}E(T^2) &= \text{Var}(T) + [E(T)]^2 \\ &= 4\theta - 4\theta^2 + 16\theta^2 \\ &= 4\theta + 12\theta^2\end{aligned}$$

To find an unbiased estimator of $(1-\theta)^2$ we work backwards from the desired result, utilizing the information above

$$\begin{aligned}(1-\theta)^2 &= (1 - 2\theta + \theta^2) \\ &= \frac{12 - 28\theta + 4\theta + 12\theta^2}{12} \\ &= \frac{1}{12}\left(12 - 7E(T) + E(T^2)\right) \\ &= E\left[\frac{1}{12}(4-T)(3-T)\right]\end{aligned}$$

Because $T(\mathbf{X})$ is a complete sufficient statistic,

$$\widehat{(1-\theta)^2} = \frac{(4-T)(3-T)}{12}$$

is the unique best unbiased estimator of $(1-\theta)^2$ by Theorem 7.3.23.

Problem 3

a.

From the class notes, page 6.2.18, $T(\mathbf{X}) = (\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2)$ is a complete sufficient statistic for a normal distribution. The sample variance is defined as

$$S^2 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}.$$

Now with $n = 7$,

$$\frac{6S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{\sigma^2} \sim \chi^2(6).$$

To calculate the expectation of $1/S^2$, let

$$U = \frac{1}{\frac{6S^2}{\sigma^2}} = \frac{\sigma^2}{6S^2}; \qquad E\left(\frac{1}{S^2}\right) = \frac{6}{\sigma^2}E(U)$$

and we use the pdf of $\chi^2(6)$,

$$\begin{aligned}\frac{6}{\sigma^2}E(U) &= \frac{6}{\sigma^2}\int_0^\infty \frac{1}{u}\frac{1}{2^3 \cdot 2}u^2 \exp\left(\frac{-u}{2}\right) \\ &= \frac{6}{8\sigma^2}\int_0^\infty u\frac{1}{2}u^0 \exp\left(\frac{-u}{2}\right)\end{aligned}$$

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The integral of this last expression is equivalent to $E(Y); Y \sim \chi^2(2)$, which evaluates to 2 since the mean of a chi-squared distribution is equal to the parameter k.

$$E\left(\frac{1}{S^2}\right) = \frac{3}{2\sigma^2}.$$

Since $\bar{X}$ and S^2 are stochastically independant,

$$E\left(\frac{\bar{X}}{S^2}\right) = E(\bar{X})E\left(\frac{1}{S^2}\right) = \mu\frac{3}{2\sigma^2}.$$

Therefore

$$\widehat{\frac{\mu}{\sigma^2}} = \frac{2\bar{X}}{3S^2}; \qquad E\left(\frac{2\bar{X}}{3S^2}\right) = \frac{2}{3}E(\bar{X})E\left(\frac{1}{S^2}\right) = \frac{\mu}{\sigma^2}.$$

Since

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$$
$$S^2 = \frac{\sum_{i=1}^n \left(X_i - \bar{X}\right)^2}{n-1},$$

$\bar{X}$ is a function of $\sum_{i=1}^n X_i$ and and S^2 is a function of $\sum_{i=1}^n X_i$, $\sum_{i=1}^n X_i^2$, and $\bar{X}$. Therefore, both $\bar{X}$ and S^2 are functions of only $T(\mathbf{X})$, and by Theorem 7.3.23, $\widehat{\mu/\sigma^2}$ is the unique best unbiased estimator of μ/σ^2.

b.

The inverse of the Fisher information matrix for the joint distribution of n iid Normal distributions is given in the class notes page 7.3.39 as

$$\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

The gradient for $\tau(\vec{\theta}) = \mu/\sigma^2$ is

$$\nabla_{\tau(\vec{\theta})} = \left(\frac{d}{d\mu}\tau(\vec{\theta}), \frac{d}{d(\sigma^2)}\tau(\vec{\theta})\right)$$
$$= \left(\frac{1}{\sigma^2}, -\frac{\mu}{(\sigma^2)^2}\right)$$

The lower bound for estimating $\tau(\vec{\theta})$ then becomes

$$\begin{bmatrix} \frac{1}{\sigma^2} & -\frac{\mu}{\sigma^4} \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} \\ -\frac{\mu}{\sigma^4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & -\frac{\mu}{\sigma^4} \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ -\frac{2\mu}{n} \end{bmatrix} = \frac{1}{n\sigma^2} + \frac{2\mu^2}{n\sigma^4}$$

c.

The variance of $\widehat{\mu/\sigma^2}$ is

$$\text{Var}\left(\frac{2\bar{X}}{3S^2}\right) = \frac{4}{9}\text{var}\left(\frac{\bar{X}}{S^2}\right) = \frac{4}{9}\left(E\left(\frac{\bar{X}^2}{S^4}\right) - \left[E\left(\frac{\bar{X}}{S^2}\right)\right]^2\right)$$

$$\left[E\left(\frac{\bar{X}}{S^2}\right)\right]^2 = \left[\frac{3\mu}{2\sigma^2}\right]^2 = \frac{9\mu^2}{4\sigma^4}$$

$$E(\bar{X}^2) = \text{Var}(\bar{X}) + \left[E(\bar{X})\right]^2$$
$$= \frac{\sigma^2}{7} + \mu^2$$

To calculate the expectation of the squared inverse of the sample variance, we use the U transformation from part a,

$$
\begin{aligned}
E\left(\frac{1}{(S^2)^2}\right) &= \frac{6^2}{(\sigma^2)^2} E(U) = \frac{36}{\sigma^4} \int_0^\infty \frac{1}{u^2} \frac{1}{2^3 \cdot 2} u^2 \exp\left(\frac{-u}{2}\right) \\
&= \frac{36}{16\sigma^4} \int_0^\infty \exp\left(\frac{-u}{2}\right) \\
&= \frac{9}{4\sigma^4} \; -2 \exp\left(\frac{-u}{2}\right)\Bigg|_0^\infty \\
&= \frac{-9}{2\sigma^4} (0 - 1) = \frac{9}{2\sigma^4}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\frac{\bar{X}^2}{S^4}\right) &= E(\bar{X}^2) E\left(\frac{1}{(S^2)^2}\right) \\
&= \left(\frac{\sigma^2 + 7\mu^2}{7}\right)\left(\frac{9}{2\sigma^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}\left(\frac{2\bar{X}}{3S^2}\right) &= \frac{4}{9}\left(E\left(\frac{\bar{X}^2}{S^4}\right) - \left[E\left(\frac{\bar{X}}{S^2}\right)\right]^2\right) \\
&= \frac{4}{9}\left[\left(\frac{\sigma^2 + 7\mu^2}{7}\right)\left(\frac{9}{2\sigma^4}\right) - \frac{9\mu^2}{4\sigma^4}\right] \\
&= \frac{2}{7\sigma^2} + \frac{\mu^2}{\sigma^4}
\end{aligned}
$$