

Problem 1

a.

$$\begin{aligned} f(\mathbf{x}|\theta) &= \prod_{i=0}^n \left[\frac{1}{2\theta} \exp\left(\frac{-\sqrt{x_i}}{\theta}\right) I_{(0,\infty)}(x_i) \right] \\ &= \frac{1}{2^n \theta^{2n}} \exp\left(\frac{-1}{\theta} \sum_{i=0}^n \sqrt{x_i}\right) \prod_{i=0}^n I_{(0,\infty)}(x_i) \end{aligned}$$

Let

$$c(\theta) = \frac{1}{2^n \theta^{2n}}, \quad h(x) = \prod_{i=0}^n I_{(0,\infty)}(x_i), \quad w_1 = \frac{-1}{\theta}, \quad t = \sum_{i=0}^n \sqrt{x_i}$$

Since

$$\{w_1(\theta) : \theta \in \Theta\} \rightarrow \left\{ \frac{-1}{\theta} : \theta > 0 \right\} \rightarrow (-\infty, 0)$$

contains an open set on \mathbb{R}' , $T(\mathbf{x}) = \sum_{i=0}^n \sqrt{x_i}$ is a complete sufficient statistic. The expectation of any X can be calculated as

$$\begin{aligned} E(X) &= \int_0^\infty x \frac{1}{2\theta^2} \exp\left(\frac{-\sqrt{x}}{\theta}\right) dx \\ &= \frac{1}{2\theta^2} \exp\left(\frac{-\sqrt{x}}{\theta}\right) \left(-12\theta^4 - 12\theta^3 \sqrt{x} - 6\theta^2 x - \theta x^{3/2} \right) \Big|_0^\infty \quad \text{solved with Wolfram Alpha} \\ &= \exp\left(\frac{-\sqrt{x}}{\theta}\right) \left(-6\theta^2 - 6\theta \sqrt{x} - 3x - \frac{1}{2\theta} x^{3/2} \right) \Big|_0^\infty \\ &= e^{-\infty}(\dots) - e^0(-6\theta^2 - 0) = 6\theta^2 \end{aligned}$$

Using a similar derivation, the expectation of \sqrt{X} is

$$\begin{aligned} E(\sqrt{X}) &= \int_0^\infty \sqrt{x} \frac{1}{2\theta^2} \exp\left(\frac{-\sqrt{x}}{\theta}\right) dx \\ &= \exp\left(\frac{-\sqrt{x}}{\theta}\right) \left(-2\theta - 2\sqrt{x} - \frac{1}{\theta} x \right) \Big|_0^\infty \quad \text{solved with Wolfram Alpha} \\ &= 2\theta \end{aligned}$$

To find an unbiased estimator, we investigate the expectation of $T(\mathbf{x})$,

$$\begin{aligned} E(T(\mathbf{x})) &= E\left(\sum_{i=0}^n \sqrt{x_i}\right) \\ &= \sum_{i=0}^n E(\sqrt{x_i}) \\ &= n(2\theta) \end{aligned}$$

Therefore, the expectation of $\hat{\theta} = T/2n$,

$$E(\hat{\theta}) = E\left(\frac{T}{2n}\right) = \theta$$

shows that $\hat{\theta}$ is an unbiased estimator of θ . Since it is also a function of a complete sufficient statistic of θ , by Theorem 7.3.23, $\hat{\theta}$ is the unique best unbiased estimator of θ .

b.

The Carmer-Rao lower bound for the variance of $\hat{\theta}$ given that $\hat{\theta}$ is a function of $T(\mathbf{X})$ and X_1, X_2, \dots are iid variables is given by

$$\frac{1}{nE_\theta\left(\left[\frac{d}{d\theta} \log f(X|\theta)\right]^2\right)}.$$

We resolve the denominator using values of expectation from part a,

$$\begin{aligned}\frac{d}{d\theta} \log f(X|\theta) &= \frac{d}{d\theta} \left(-2 \log(\sqrt{2}\theta) - \frac{\sqrt{x}}{\theta} \right) = \frac{-2}{\theta} + \frac{\sqrt{x}}{\theta^2} \\ \left(\frac{d}{d\theta} \log f(X|\theta) \right)^2 &= \frac{4}{\theta^2} - \frac{4\sqrt{x}}{\theta^3} + \frac{x}{\theta^4} \\ E \left[\left(\frac{d}{d\theta} \log f(X|\theta) \right)^2 \right] &= E \left[\frac{4}{\theta^2} - \frac{4\sqrt{x}}{\theta^3} + \frac{x}{\theta^4} \right] = \frac{4}{\theta^2} - \frac{4E(\sqrt{x})}{\theta^3} + \frac{E(x)}{\theta^4} \\ &= \frac{4}{\theta^2} - \frac{8\theta}{\theta^3} + \frac{6\theta^2}{\theta^4} = \frac{2}{\theta^2}\end{aligned}$$

which gives us a Cramer-Rao lower bound of

$$\frac{\theta^2}{2n}$$

c.

$$\begin{aligned}\text{Var}(\hat{\theta}) &= \frac{1}{4n^2} \text{Var}(T) = \frac{1}{4n^2} \text{Var} \left(\sum_{i=0}^n \sqrt{X_i} \right) = \frac{1}{4n^2} \sum_{i=0}^n \text{Var}(\sqrt{X_i}) \\ &= \frac{n}{4n^2} \left(E(X) - [E(\sqrt{X})]^2 \right) \\ &= \frac{1}{4n} [6\theta^2 - (2\theta)^2] \\ &= \frac{\theta^2}{2n}\end{aligned}$$

d.

From part a we see that

$$E(T(\mathbf{x})) = 2n\theta.$$

Therefore, to find an unbiased estimator for $6\theta^2$, we investigate T^2 using information from part c,

$$\begin{aligned}E(T^2) &= \text{Var}(T) + [E(T)]^2 \\ &= 2n\theta^2 + (2n\theta)^2 \\ &= 2n(n+1)\theta^2\end{aligned}$$

Thus

$$\widehat{6\theta^2} = \frac{3T^2}{n(n+1)}; \quad E(\widehat{6\theta^2}) = E\left(\frac{3T^2}{n(n+1)}\right) = \frac{3E(T^2)}{n(n+1)} = 6\theta^2$$

is an unbiased estimator for $6\theta^2$. Since it is also a function of $T(\mathbf{x})$, and we have determined that this is a complete, sufficient statistic in part a, by Theorem 7.3.23, $\widehat{6\theta^2}$ is the unique best unbiased estimator of $6\theta^2$.