Daniel Hartig
Stat 652
Homework 3

Problem 1

a.

$$m_1 = \bar{X} = \mu_1', \qquad \mu_1' = \mathbb{E}\left[\frac{\theta x^{\theta-1}}{3^\theta} I_{(0,3)}(x)\right]$$
$$= \int_0^3 x \frac{\theta x^{\theta-1}}{3^\theta} dx = \frac{\theta}{3^\theta} \int_0^3 x^\theta dx$$
$$= \frac{\theta}{3^\theta} \frac{x^{\theta+1}}{\theta+1}\bigg|_0^3 = \frac{3\theta}{\theta+1}$$

$$\theta = \frac{-\mu_1'}{\mu_1' - 3} = \frac{\bar{X}}{3 - \bar{X}}$$

This answer is intuitive because X_i has the range $(0, 3)$; therefore $\bar{X}$ must be in this range as well. The denominator must therefore always be a positive number, and the range of θ becomes $(0, \infty)$, as given in the problem statement.

b.

$$L(\theta|x) = \prod_{i=1}^{n} \frac{\theta x_i^{\theta-1}}{3^\theta} I_{(0,3)}(x)$$
$$\mathcal{L}(\theta|x) = \sum_{i=1}^{n} \log\theta + (\theta - 1)\log x_i\, I_{(0,3)}(x) - \theta \log 3$$
$$\frac{d\mathcal{L}(\theta|x)}{d\theta} = \sum_{i=1}^{n} \frac{1}{\theta} + \log x_i\, I_{(0,3)}(x) - \log 3$$

Solve for a maximum by setting the derivative of the log-likelihood function equal to zero:

$$\sum_{i=1}^{n} \frac{1}{\theta} = -\sum_{i=1}^{n} \log\frac{x}{3}$$
$$\frac{n}{\theta} = -\sum_{i=1}^{n} \log\frac{x}{3}$$
$$\theta = \frac{-n}{\sum_{i=1}^{n} \log\frac{x}{3}}$$

Check the second derivative to ensure that this is a global maximum:

$$\frac{d^2\mathcal{L}(\theta|x)}{d\theta^2} = \sum_{i=1}^{n} -\frac{1}{\theta^2} = -\frac{n}{\theta^2}$$

The second derivative is always negative, so the point in question must be a global maximum. This answer is intuitive, because $x < 3$; therefore $x/3 < 1$; therefore $\log x/3 < 0$. The sum of negative numbers is negative in the denominator, while $-n$ is negative in the numerator. Therefore, θ will be in the range $(0, \infty)$, as given in the problem statement.

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Problem 2

a.

$$m_1 = \bar{X} = \mu_1', \qquad \mu_1' = \mathbb{E}\left[\frac{3x_i^2}{\theta^3} I_{(0,\theta]}(x)\right]$$
$$= \int_0^\theta x \frac{3x^2}{\theta^3} dx = \frac{3}{\theta^3} \int_0^\theta x^3 dx$$
$$= \frac{3}{\theta^3} \left.\frac{x^4}{4}\right|_0^\theta = \frac{3\theta}{4}$$
$$\theta = \frac{4\bar{X}}{3}$$

b.

$$L(\theta|x) = \prod_{i=1}^{n} \frac{3x^2}{\theta^3} I_{(0,\theta]}(x)$$
$$= \begin{cases} \frac{3^n}{\theta^{3n}} \prod_{i=1}^n x_i^2 & \theta \geq x_{(n)} \\ 0 & \theta < x_{(n)} \end{cases}$$
$$\frac{dL(\theta|x)}{d\theta} = \frac{-n3^{n+1}}{\theta^{3n+1}} \prod_{i=0}^{n} x_i^2$$

This derivative is always negative, but never reaches zero. Since the derivative is negative, L is always decreasing on $(x_{(n)}, \infty)$. Therefore, the maximum value of the likelihood function is attained when $\theta = x_{(n)}$.

c.

$$F(x|\theta) = \int \frac{3x^2}{\theta^3} dx$$
$$= \frac{x^3}{\theta^3} = P(X < x)$$

The probablility that all of n X_i are less than x is equal to this expression raised to the n power. Let $Y = X_{(n)}$, then

$$P(Y < y) = \frac{y^{3n}}{\theta^{3n}} = F(y|\theta)$$
$$f(y|\theta) = \frac{d}{dx} \frac{y^{3n}}{\theta^{3n}} = \frac{3ny^{3n-1}}{\theta^{3n}}$$
$$E\left[f(y|\theta)\right] = \int_0^\theta y \frac{3ny^{3n-1}}{\theta^{3n}} dx$$
$$= \frac{3n}{3n+1} \left.\frac{y^{3n+1}}{\theta^{3n}}\right|_0^\theta = \frac{3n}{3n+1}\theta$$

As n approaches infinity, $E(X_{(n)}) \to \theta$.

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Problem 3

0.0.1 a.

$$\begin{aligned}
p_2 \text{ observed} &= \frac{5}{25} = \theta(1-\theta) \\
\theta^2 - \theta + 1/5 &= 0 \\
\theta &= \frac{\sqrt{5} \pm 1}{2\sqrt{5}}
\end{aligned}$$

Since $\theta \in (0, 1/2)$, only the lower of the two roots is applicable, so

$$\theta = 0.276.$$

b.

$$\begin{aligned}
L(\theta | n_1 = 11, n_2 = 5, n_3 = 9) &= \frac{25!}{11!\,5!\,9!} (\theta)^1 \, 1 \, (\theta(1-\theta))^5 \left((1-\theta)^2\right)^9 \\
&= C\theta^{16} (1-\theta)^{23}
\end{aligned}$$

where $C = 25!/(11!\,5!\,9!) = 8923714800$.

$$\begin{aligned}
\mathcal{L}(\theta) &= \log C + 16 \log \theta + 23 \log (1-\theta) \\
\frac{d\mathcal{L}(\theta)}{d\theta} &= \frac{16}{\theta} + \frac{23}{\theta - 1}
\end{aligned}$$

Set this derivative equal to zero to solve for maximum:

$$\begin{aligned}
\frac{16}{\theta} &= \frac{23}{1-\theta} \\
\theta &= \frac{16}{39} = 0.410
\end{aligned}$$

Check the second derivative to ensure that this is a global maximum:

$$\frac{d^2\mathcal{L}(\theta)}{d\theta^2} = -\frac{16}{\theta^2} - \frac{23}{(\theta-1)^2}$$

Since the second derivative is always negative, $\theta = 0.410$ is a global maximum of the likelihood function.

Problem 4

a.

$$\begin{aligned}
m_1 = \bar{X}, \qquad \mu_1' &= \frac{1 + \theta x}{2} I_{[-1,1]}(x) \\
&= \frac{1}{2} \int_{-1}^{1} x\,(1 + \theta x)\, dx \\
&= \frac{1}{2} \left(\frac{x^2}{2} + \frac{\theta}{3} x^3 \right)\Bigg|_{-1}^{1} = \frac{1}{2} \left(\frac{2\theta}{3} \right) \\
&= \frac{\theta}{3} \\
\theta &= 3\bar{X}
\end{aligned}$$

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b.

$$L(\theta|x) = \frac{1 + \theta x_1}{2}$$

The range of both x and θ are $[-1, 1]$. If $x > 0$, then likelihood increases with increasing θ, so maximum likelihood is when $\theta = 1$. If $x < 0$, likelihood decreases with increasing θ, so maximum likelihood is when $\theta = -1$. If $x = 0$, likelihood is a constant $1/2$ for all θ. Therefore the maximum likelihood expression is

$$\theta = \begin{cases} 1 & x > 0 \\ [-1, 1] & x = 0 \\ -1 & x < 0 \end{cases}$$

c.

Problem 5

$$\begin{aligned} m_1 &= \bar{X} = 0.516\bar{6} \\ m_2 &= \frac{1}{n}\sum_{i=1}^{n} x_i^2 = 1.3113 \end{aligned}$$

$$\begin{aligned} \mu_i' &= \mathbb{E}\left[\frac{1}{2\delta} I_{[\gamma-\delta,\gamma+\delta]}(u)\right] \\ &= \int_{\gamma-\delta}^{\gamma+\delta} \frac{1}{2\delta} u du \\ &= \frac{1}{4\delta}\, u^2\Big|_{\gamma-\delta}^{\gamma+\delta} = \frac{1}{4\delta}\,(4\gamma\delta) \\ &= \gamma \end{aligned}$$

$$\begin{aligned} \mu_2' &= \mathbb{E}\left[\left(\frac{1}{2\delta}\right)^2 I_{[\gamma-\delta,\gamma+\delta]}(u)\right] \\ &= \int_{\gamma-\delta}^{\gamma+\delta} \frac{1}{4\delta^2} u du \\ &= \frac{1}{8\delta^2}\,(4\gamma\delta) \\ &= \frac{\gamma}{2\delta} \end{aligned}$$

$$\begin{aligned} \gamma &= m_1 = 0.517 \\ \delta &= \frac{\gamma}{2m_2} = \frac{0.516\bar{6}}{2 \cdot 1.3113} = 0.197 \end{aligned}$$