

Homework 2

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Problem 1

For mass function

$$f_{T_n}(t) = (8n - 8n^2t) I_{(1/2n, 1/n)}(t)$$

the distribution function over the same domain is obtained by integration

$$F_{T_n}(t) = (8nt - 4n^2t^2 + C) I_{(1/2n, 1/n)}(t).$$

To be a valid distribution function, F must equal 0 at the lower bound of the domain, $t = 1/2n$, and equal 1 at the upper bound $t = 1/n$. At these bounds, F evaluates to

$$F_{T_n}(1/2n) = \frac{8n}{2n} - \frac{4n^2}{(2n)^2} + C = 3 + C$$

$$F_{T_n}(1/n) = \frac{8n}{n} - \frac{4n^2}{n^2} + C = 4 + C.$$

Therefore $C = -3$, $F(t) = 0$ for $t < 1/2n$ and $F(t) = 1$ for $t > 1/n$, and

$$F_{T_n}(t) = (8nt - 4n^2t^2 - 3) I_{(1/2n, 1/n)}(t)$$

meets the requirements of a distribution function for all n . As n approaches infinity, the range of $I_{(1/2n, 1/n)}(t)$ approaches the point 0. We have already defined the $F(t) = 1$ for $t > 1/n$, so we can say that

$$\lim_{n \rightarrow \infty} F_{T_n}(t) = 1 \quad \forall t > 0.$$

Thus, the limiting distribution of $F_{T_n}(t)$ approaches $F(t)$, where $F(t)$ is the distribution function of $T = 0$. The probability function of the limiting distribution for the sequence T_1, T_2, \dots is

$$f(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 2

For

$$f_{X_n}(x) = \begin{cases} \frac{1}{n}, & x = n^2, \\ \frac{n-1}{n}, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

the moment-generating function is calculated by

$$M_X(t) = \sum_{x \in A} e^{tx} p(x) = e^{t \cdot 0} \left(\frac{n-1}{n} \right) + e^{tn^2} \left(\frac{1}{n} \right) = \frac{1}{n} (e^{tn^2} + n - 1).$$

To calculate the $E(X_n)$ we need the first t derivative of M_X ,

$$M'_X(t) = \frac{d}{dt} \frac{1}{n} (e^{tn^2} + n - 1) = ne^{tn^2},$$

so that

$$E(X_n) = M'_X(0) = ne^{0 \cdot n^2} = n.$$

Thus,

$$\begin{aligned} T_n &= X_n - E(X_n) \\ &= X_n - n \end{aligned}$$

The probability distribution of T_n can then be defined as

$$f_{T_n}(x) = \begin{cases} \frac{1}{n}, & x = n^2 - n \\ \frac{n-1}{n}, & x = -n \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function is

$$F_{T_n}(x) = \begin{cases} 0, & x < -n \\ \frac{n-1}{n}, & -n \leq x < n^2 - n \\ 1, & n \geq n^2 - n. \end{cases}$$

As $n \rightarrow \infty$, the term $(n-1)/n$ approaches 1, so

$$\lim_{n \rightarrow \infty} F_{T_n} = \begin{cases} 0, & x < -n \\ 1, & x \geq -n. \end{cases}$$

However, $-n$ itself goes to $-\infty$, so F_{T_n} does not converge. Since the distribution function of T_n does not converge, the a limiting distribution for the sequence T_1, T_2, \dots does not exist.

Problem 3

For the continuous probability function $f_X(x) = 3(1-x)^2 I_{(0,1)}(x)$, the cumulative distribution function is obtained by integration

$$F_X(x) = \int 3(1-x)^2 I_{(0,1)}(x) dx = (x-1)^3 + C.$$

For the bounds of the probability function 0 and 1, the values of F_X are

$$\begin{aligned} (x-1)^3 + C \Big|_0 &= -1 + C \\ (x-1)^3 + C \Big|_1 &= C. \end{aligned}$$

Set $C = 1$, $F_X = 0$ for $x < 0$ and $F_X = 1$ for $x > 1$, and then

$$F_X(x) = (x-1)^3 + 1 I_{(0,1)}(x)$$

is a valid distribution function. As n increases, we expect $\max\{X_1, \dots, X_n\}$ to approach the maximum value, which is $x = 1$. Using the distribution function, we can say that for all X_i with $0 < i < n$,

$$P(X_i \leq x) = (x-1)^3 + 1.$$

Since the X_i are independent of each other, the probability that all X_i in the series X_1, X_2, \dots, X_n are less than x is

$$P(\max\{X_1, \dots, X_n\} < x) = \left((x-1)^3 + 1\right)^n.$$

This is the definition of a distribution function

$$F_{\max}(x) = \begin{cases} 0, & x < 0 \\ \left((x-1)^3 + 1\right)^n, & 0 \leq x \leq 1 \\ 1, & x > 1. \end{cases}$$

The limit of this distribution in n is

$$F_{\max}(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1. \end{cases}$$

Problem 4

The moment generating function of the sum of independent random variables in the sequence X_1, X_2, \dots is

$$M_{X_1}(t)M_{X_2}(t)\dots$$

The moment generating function of Poisson distribution is $\exp(\lambda(e^t - 1))$. Thus the moment generating function for $T_n = X_1 + \dots + X_n$ is

$$\begin{aligned} M_{T_n}(t) &= \prod_{k=1}^n \exp(k^{-2}(e^t - 1)) \\ &= \exp\left((e^t - 1) \sum_{k=1}^n \frac{1}{k^2}\right). \end{aligned}$$

This is itself the moment generating function of a Poisson distribution with

$$\lambda = \sum_{k=1}^n \frac{1}{k^2}.$$

As n goes to infinity, the sum of this infinite series is $\pi^2/6$. Therefore, as n goes to infinity, the sequence T_1, T_2, \dots converges to a limiting distribution of $\text{Pois}(\pi^2/6)$, the probability mass function of which is

$$\frac{\pi^{2j} e^{-\pi^2/6}}{6^j j!}$$

Problem 5

The sum of the distributions of multiple Bernoulli random variables is the binomial distribution. For Bernoulli random variables X_i with mean $p = 1/2$, as n goes to infinity,

$$\sum_{i=1}^n X_i = nE(X_i) = n^2 p = \frac{1}{2}n^2$$

due to the Law of Large Numbers. Since X_i is a Bernoulli variable, the only possible outcomes are 0 and 1. Since $0^2 = 0$ and $1^2 = 1$, $X_i^2 = X_i$, and

$$\sum_{i=1}^n X_i^2 = \sum_{i=1}^n X_i = \frac{1}{2}n^2.$$

Therefore, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} T_n = \lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{4 \sum_{i=1}^n X_i - 2n}{\sum_{i=1}^n X_i^2} \right) = \lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{2n^2 - 2n}{1/2 n^2} \right) = \lim_{n \rightarrow \infty} 4 \left(\sqrt{n} - \frac{1}{\sqrt{n}} \right).$$

This limit goes to infinity. Therefore, as n goes to infinity, the sequence T_1, T_2, \dots does not converge.

Problem 6

Part a

X is a random variable with $E(X) = \theta$. Let $g(X) = 1/\sqrt{X}$ be an estimator for $1/\sqrt{\theta}$; $g'(X) = -1/2 X^{-3/2}$. The first order Taylor approximation to the mean of $1/\sqrt{X_n}$ is

$$\begin{aligned} g(X) &= g(\theta) + g'(\theta)(X - \theta) \\ \frac{1}{\sqrt{X_n}} &= \frac{1}{\sqrt{\theta}} - \frac{1}{2} \theta^{-\frac{3}{2}} (X_n - \theta) \\ E \left[\frac{1}{\sqrt{X_n}} \right] &= \frac{1}{\sqrt{\theta}} + -\frac{1}{2} \theta^{-\frac{3}{2}} E[X_n - \theta] \end{aligned}$$

Since θ is the mean of X , $E[X - \theta] = 0$, and so

$$E \left[\frac{1}{\sqrt{X}} \right] = \frac{1}{\sqrt{\theta}}$$

Part b

Since $g''(x) = 3/4x^{-5/2}$; the second order approximation to the mean of $1/\sqrt{X_n}$ is

$$g(X) = g(\theta) + g'(\theta)(X - \theta) + \frac{1}{2}g''(\theta)(X_n - \theta)^2$$

$$E\left[\frac{1}{\sqrt{X}}\right] = \frac{1}{\sqrt{\theta}} + \frac{3}{4}\theta^{-\frac{5}{2}}E[(X_n - \theta)^2]$$

Now θ is the mean of X , $E[(X - \theta)^2]$ is the definition of the variance of X , which is θ^3 . Therefore

$$E\left[\frac{1}{\sqrt{X_n}}\right] = \frac{1}{\sqrt{\theta}} + \frac{3}{4}\sqrt{\theta}$$

or

$$E\left[\frac{1}{\sqrt{X_n}}\right] = \frac{1 + \frac{3}{4}\theta}{\sqrt{\theta}}$$

Part c

The first order approximation to the variance of $1/\sqrt{X_n}$ is

$$\text{Var}\left[\frac{1}{\sqrt{X_n}}\right] = \left(-\frac{1}{2}\theta^{-\frac{3}{2}}\right)^2 \text{Var}[X_n]$$

$$= \frac{1}{4} \frac{1}{\theta^3} \theta^3 = \frac{1}{4}$$

Problem 7

To generate a sufficient statistic, we must factorize the joint probability density function $f_X(\mathbf{x}|\theta)$ into two terms $g(T(\mathbf{x})|\theta)$ and $h(\mathbf{x})$ as

$$f_X(x) = \prod_{k=1}^n (\theta + 1)^k x_k^\theta I_{(0,1)}(x)$$

$$= (\theta + 1)^n \exp\left(\theta \sum_{k=1}^n \log x_k\right) I_{(0,1)}(x).$$

Therefore

$$h(\mathbf{x}) = I_{(0,1)}(x)$$

and

$$g(T(\mathbf{x})|\theta) = (\theta + 1)^n \exp\left(\theta \sum_{k=1}^n \log x_k\right),$$

so

$$T(\mathbf{x}) = \sum_{k=1}^n \log x_k$$

is a sufficient statistic for θ for every x in the sample space.