Daniel Hartig
Stat 652
Homework 3

Problem 1

a.

$$m_1 = \bar{X} = \mu_1', \qquad \mu_1' = E\left[\frac{\theta x^{\theta-1}}{3^\theta} I_{(0,3)}(x)\right]$$
$$= \int_0^3 x \frac{\theta x^{\theta-1}}{3^\theta} dx = \frac{\theta}{3^\theta} \int_0^3 x^\theta dx$$
$$= \frac{\theta}{3^\theta} \frac{x^{\theta+1}}{\theta+1}\Big|_0^3 = \frac{3\theta}{\theta+1}$$

$$\theta = \frac{-\mu_1'}{\mu_1' - 3}, \qquad \hat{\theta}_{MME} = \frac{\bar{X}}{3 - \bar{X}}$$

This answer is intuitive because X_i has the range $(0, 3)$; therefore $\bar{X}$ must be in this range as well. The denominator must therefore always be a positive number, and the range of θ becomes $(0, \infty)$, as given in the problem statement.

b.

$$L(\theta|x) = \prod_{i=1}^{n} \frac{\theta x_i^{\theta-1}}{3^\theta} I_{(0,3)}(x)$$
$$\mathcal{L}(\theta|x) = \sum_{i=1}^{n} \log\theta + (\theta - 1)\log x_i \, I_{(0,3)}(x) - \theta \log 3$$
$$\frac{d\mathcal{L}(\theta|\mathbf{x})}{d\theta} = \sum_{i=1}^{n} \frac{1}{\theta} + \log x_i \, I_{(0,3)}(x) - \log 3$$

Solve for a maximum by setting the derivative of the log-likelihood function equal to zero:

$$\sum_{i=1}^{n} \frac{1}{\theta} = -\sum_{i=1}^{n} \log \frac{x_i}{3}$$
$$\frac{n}{\theta} = -\sum_{i=1}^{n} \log \frac{x_i}{3}$$
$$\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \log \frac{X_i}{3}}$$

Check the second derivative to ensure that this is a global maximum:

$$\frac{d^2\mathcal{L}(\theta|x)}{d\theta^2} = \sum_{i=1}^{n} -\frac{1}{\theta^2} = -\frac{n}{\theta^2}$$

The second derivative is always negative, so the point in question must be a global maximum. This answer is intuitive, because $x_i < 3$; therefore $x_i/3 < 1$; therefore $\log x_i/3 < 0$. The sum of negative numbers is negative in the denominator, while $-n$ is negative in the numerator. Therefore, θ will be in the range $(0, \infty)$, as given in the problem statement.

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Problem 2

a.

$$m_1 = \bar{X} = \mu_1', \qquad \mu_1' = E\left[\frac{3x_i^2}{\theta^3} I_{(0,\theta]}(x)\right]$$
$$= \int_0^\theta x\frac{3x^2}{\theta^3}dx = \frac{3}{\theta^3}\int_0^\theta x^3 dx$$
$$= \frac{3}{\theta^3}\left.\frac{x^4}{4}\right|_0^\theta = \frac{3\theta}{4}$$
$$\hat{\theta}_{MME} = \frac{4\bar{X}}{3}$$

b.

$$L(\theta|x) = \prod_{i=1}^n \frac{3x^2}{\theta^3} I_{(0,\theta]}(x)$$
$$= \begin{cases} \frac{3^n}{\theta^{3n}} \prod_{i=1}^n x_i^2 & \theta \geq x_{(n)} \\ 0 & \theta < x_{(n)} \end{cases}$$
$$\frac{dL(\theta|x)}{d\theta} = \frac{-n3^{n+1}}{\theta^{3n+1}} \prod_{i=0}^n x_i^2$$

This derivative is always negative, but never reaches zero. Since the derivative is negative, L is always decreasing on $(x_{(n)}, \infty)$. Therefore, the maximum value of the likelihood function is attained when $\hat{\theta}_{MLE} = X_{(n)}$.

c.

$$\int f(x|\theta) = \int \frac{3x^2}{\theta^3}dx = \frac{x^3}{\theta^3}$$
$$F(X|\theta) = P(X < x) = \begin{cases} \frac{x^3}{\theta^3} & x \in (0,\theta] \\ 0 & x \notin (0,\theta] \end{cases}$$

The probablility that all of n X_i's are less than x is equal to this expression raised to the n power. Let $Y = X_{(n)}$, then

$$P(Y < y) = \frac{y^{3n}}{\theta^{3n}} = F(y|\theta)$$
$$f(y|\theta) = \frac{d}{dx}\frac{y^{3n}}{\theta^{3n}} = \frac{3ny^{3n-1}}{\theta^{3n}}$$
$$E\left[f(y|\theta)\right] = \int_0^\theta y\frac{3ny^{3n-1}}{\theta^{3n}}dx$$
$$= \frac{3n}{3n+1}\left.\frac{y^{3n+1}}{\theta^{3n}}\right|_0^\theta = \frac{3n}{3n+1}\theta$$

As n approaches infinity, $E(\hat{\theta}) = E(X_{(n)}) \to \theta$.

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Problem 3

0.0.1 a.

$$
\begin{aligned}
p_2 \text{ observed} &= \frac{5}{25} = \theta(1-\theta) \\
\theta^2 - \theta + 1/5 &= 0 \\
\theta &= \frac{\sqrt{5} \pm 1}{2\sqrt{5}}
\end{aligned}
$$

Since $\theta \in (0, 1/2)$, only the lower of the two roots is applicable, so

$$\hat{\theta} = 0.276.$$

b.

$$
\begin{aligned}
L(\theta | n_1 = 11, n_2 = 5, n_3 = 9) &= \frac{25!}{11!\,5!\,9!} (\theta)^{11} (\theta(1-\theta))^5 \left((1-\theta)^2\right)^9 \\
&= C\theta^{16} (1-\theta)^{23}
\end{aligned}
$$

where $C = 25!/(11!\,5!\,9!) = 8923714800$.

$$
\begin{aligned}
\mathcal{L}(\theta) &= \log C + 16 \log \theta + 23 \log (1-\theta) \\
\frac{d\mathcal{L}(\theta)}{d\theta} &= \frac{16}{\theta} + \frac{23}{\theta - 1}
\end{aligned}
$$

Set this derivative equal to zero to solve for maximum:

$$
\begin{aligned}
\frac{16}{\theta} &= \frac{23}{1-\theta} \\
\hat{\theta}_{MLE} &= \frac{16}{39} = 0.410
\end{aligned}
$$

Check the second derivative to ensure that this is a global maximum:

$$\frac{d^2\mathcal{L}(\theta)}{d\theta^2} = -\frac{16}{\theta^2} - \frac{23}{(\theta-1)^2}$$

Since the second derivative is always negative, $\hat{\theta}_{MLE} = 0.410$ is a global maximum of the likelihood function.

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Problem 4

a.

$$m_1 = \bar{X}, \qquad \mu_1' = \frac{1+\theta x}{2} I_{[-1,1]}(x)$$
$$= \frac{1}{2}\int_{-1}^{1} x(1+\theta x)\,dx$$
$$= \frac{1}{2}\left(\frac{x^2}{2} + \frac{\theta}{3}x^3\right)\Bigg|_{-1}^{1} = \frac{1}{2}\left(\frac{2\theta}{3}\right)$$
$$= \frac{\theta}{3}$$
$$\hat{\theta}_{MME} = 3\bar{X}$$

This estimator is not very good, since it can take a value outside of its range.

b.

$$L(\theta|x) = \frac{1+\theta x_1}{2}$$

The range of both x and θ are $[-1, 1]$. If $x > 0$, then likelihood increases with increasing θ, so maximum likelihood is when $\theta = 1$. If $x < 0$, likelihood decreases with increasing θ, so maximum likelihood is when $\theta = -1$. If $x = 0$, likelihood is a constant $1/2$ for all θ. Therefore the maximum likelihood expression is

$$\bar{\theta}_{MLE} = \begin{cases} 1 & X_1 > 0 \\ [-1,1] & X_1 = 0 \\ -1 & X_1 < 0 \end{cases}$$

c.

$$L(\theta|x_1 = 0.5, x_2 = -0.1, x_3 = 0.9, x_4 = -0.5) = \prod_{(i=1}^{4} \frac{1+\theta x}{2}$$
$$= \frac{1}{16}(1+.5\theta)(1-.1\theta)(1+.9\theta)(1-.5\theta)$$
$$= \frac{1}{16}(.0225\theta^4 - .2x^3 - .34x^2 + .8x + 1)$$
$$\frac{dL(\theta)}{dx} = \frac{1}{16}(.09\theta^3 - .6\theta^2 - .68\theta + .8)$$
$$\frac{d^2L(\theta)}{dx^2} = \frac{1}{16}(.27\theta^2 - 1.2\theta - .68)$$

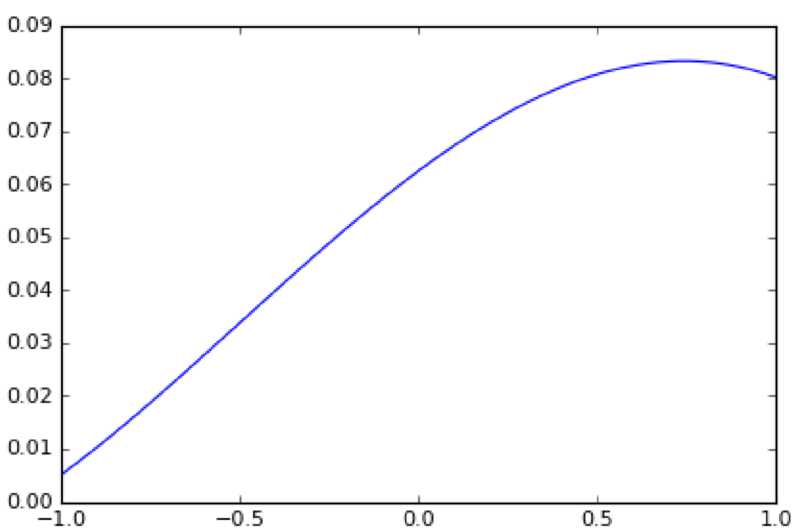
To use Newton's method to find the roots of the first derivative of the likelihood function, we will need the second derivative. Also, we need an intial estimate. Since $\bar{X}$ is 0.2, our initial estimate based on the MME is $3(0.2) = 0.6$. I will solve using a Python program written for STAT654, posted here:

```
# input function f as a function of x
def f(x):
        return (1/16)*(0.09*x**3-0.6*x**2-0.68*x+0.8)
#Derivative of f
def dfdx(x):
        return (1/16)*(0.27*x**2-1.2*x-0.68)
# number of iterations
it = 10
# initial guess
x_n = 0.6
# perform iteration
x_n = float(x_n)
for i in range(it):
    x_nplus = x_n - f(x_n)/dfdx(x_n)
    print("Iteration {1}: {4:.2f} = {0:.2f} - {2:.2f}/{3:.2f}".format(x_n,
                                                          i, f(x_n), dfdx(x_n), x_nplus))
    x_n = x_nplus
print("Value of x_n: {0:.5f}".format(x_n))
```

The result of this execution is:

```
Iteration 0: 0.75 = 0.60 - 0.01/-0.08
Iteration 1: 0.74 = 0.75 - -0.00/-0.09
Iteration 2: 0.74 = 0.74 - -0.00/-0.09
Iteration 3: 0.74 = 0.74 - -0.00/-0.09
Iteration 4: 0.74 = 0.74 - 0.00/-0.09
Iteration 5: 0.74 = 0.74 - 0.00/-0.09
Iteration 6: 0.74 = 0.74 - 0.00/-0.09
Iteration 7: 0.74 = 0.74 - 0.00/-0.09
Iteration 8: 0.74 = 0.74 - 0.00/-0.09
Iteration 9: 0.74 = 0.74 - 0.00/-0.09
Value of x_n: 0.74331
```

We can verify that this is a maxima by looking at the graph of the likelihood function on [1, 1]:



Thus $\hat{\theta}_{MLE} = 0.743$ is the maximum within $[-1, 1]$.

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Problem 5

$$m_1 = \bar{X} = 0.516\bar{6}$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = 1.3113$$

$$\begin{aligned}
\mu_1' &= E\left[\frac{1}{2\delta} I_{[\gamma-\delta,\gamma+\delta]}(u)\right] \\
&= \int_{\gamma-\delta}^{\gamma+\delta} \frac{1}{2\delta} u du \\
&= \frac{1}{4\delta}\left. u^2\right|_{\gamma-\delta}^{\gamma+\delta} = \frac{1}{4\delta}(4\gamma\delta) \\
&= \gamma
\end{aligned}$$

$$\begin{aligned}
\mu_2' &= E\left[\left(\frac{1}{2\delta}\right)^2 I_{[\gamma-\delta,\gamma+\delta]}(u)\right] \\
&= \int_{\gamma-\delta}^{\gamma+\delta} \frac{1}{4\delta^2} u du \\
&= \frac{1}{8\delta^2}(4\gamma\delta) \\
&= \frac{\gamma}{2\delta}
\end{aligned}$$

$$\begin{aligned}
\hat{\gamma}_{MLE} &= \bar{X} = 0.517 \\
\hat{\delta}_{MLE} &= \frac{\gamma}{2m_2} = \frac{0.516\bar{6}}{2 \cdot 1.3113} = 0.197
\end{aligned}$$