Daniel Hartig
Stat 652
Homework 5

Problem 1

a.

$$P(X_i < x) = F(x) = \int e^{\theta - x} I_{(\theta,\infty)}(x) dx = \begin{cases} 1 - e^{\theta - x} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

$$P(X_i > x) = 1 - F(x) = \begin{cases} e^{\theta - x} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

$$P(X_{(1)} > x) = \sum_{i=1}^{n} P(X_i > x) = [P(X_i > x)]^n = \begin{cases} e^{n(\theta - x)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

$$P(X_{(1)} < x) = F_{X_{(1)}}(x) = \begin{cases} 1 - e^{n(\theta - x)} & x \geq \theta \\ 0 & x < \theta \end{cases}$$

$$f_{X_{(1)}}(x) = ne^{n(\theta - x)} I_{(\theta,\infty)}(x)$$

$$\begin{aligned} E(X_{(1)}) &= \int_\theta^\infty xne^{n(\theta-x)}dx = ne^{n\theta} \int_\theta^\infty xe^{-nx}dx \\ &= ne^{n\theta}\left[\frac{-(nx+1)e^{-nx}}{n^2}\right]\Bigg|_\theta^\infty \\ &= ne^{n\theta}\frac{(n\theta+1)e^{-n\theta}}{n^2} \\ &= \frac{n\theta+1}{n} = \theta + \frac{1}{n} \end{aligned}$$

b.

$$\begin{aligned} E\left(X_{(1)}^2\right) &= ne^{n\theta}\int_\theta^\infty x^2 e^{-nx}dx \\ &= ne^{n\theta}\left[\frac{n^2x^2+2nx+2}{n^3}e^{-nx}\right]\Bigg|_\theta^\infty \quad = \theta^2 + \frac{2}{n}\theta + \frac{2}{n^2} \end{aligned}$$

$$\begin{aligned} E\left[(X_{(1)}-\theta)^2\right] &= E\left[X_{(1)}^2 - 2\theta X_{(1))} + \theta^2\right] \\ &= E\left(X_{(1)}^2\right) - 2\theta E(X_{(1))}) + \theta^2 \\ &= \theta^2 + \frac{2}{n}\theta + \frac{2}{n^2} - 2\theta\left(\theta + \frac{1}{n}\right) + \theta^2 \\ &= \frac{2}{n^2} \end{aligned}$$

c.

$$\hat{\theta}_{MLE} = min\,(X_1, ..., X_n)$$

To choose between the location-equivariant estimator and the scale-equivariant estimator, we assess both

$$\begin{aligned} \hat{\theta}_{MLE}(X_1+c, ..., X_n+c) &= min(X_1+c, ..., X_n+c) \\ &= min(X_1, ..., X_n) + c \end{aligned}$$

$$\begin{aligned} \hat{\theta}_{MLE}(cX_1, ..., cX_n) &= min(cX_1, ..., cX_n) \\ &= c \cdot min(X_1, ..., X_n) \end{aligned}$$

Either the location-equivariant or scale-equivariant version of the Pittman estimator is valid. We choose the location equivariant for simplicity of calculation. The likelihood function is

$$L(\theta|\mathbf{x}) = \exp\left(\sum_{i=1}^{n} \theta - x_i\right)$$

The range of the parameter θ is $(-\infty, infty)$. However, since we also have the constraint $x \geq \theta$, θ must be smaller than the smallest x_i in our sample. Therefore we set the bounds of θ at $(-\infty, X_{(1)})$.

$$\begin{aligned}
\int_{-\infty}^{x_{(1)}} \theta \exp\left(\sum_{i=1}^{n} \theta - x_i\right) d\theta &= \exp\left(-\sum_{i=1}^{n} x_i\right) \int_{-\infty}^{x_{(1)}} \theta e^{n\theta} d\theta \\
&= \exp\left(-\sum_{i=1}^{n} x_i\right) \left[\frac{n\theta - 1}{n^2} e^{n\theta}\right]\Bigg|_{-\infty}^{x_{(1)}} \\
&= \exp\left(-\sum_{i=1}^{n} x_i\right) \frac{nx_{(1)} - 1}{n^2} \exp\left(nx_{(1)}\right) \\
&= \frac{nx_{(1)} - 1}{n^2} \exp\left(-\sum_{i=1}^{n} x_{(1)} - x_i\right)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{x_{(1)}} \exp\left(\sum_{i=1}^{n} \theta - x_i\right) d\theta &= \exp\left(-\sum_{i=1}^{n} x_i\right) \int_{-\infty}^{x_{(1)}} e^{n\theta} d\theta \\
&= \exp\left(-\sum_{i=1}^{n} x_i\right) \left[\frac{e^{n\theta}}{n}\right]\Bigg|_{-\infty}^{x_{(1)}} \\
&= \exp\left(-\sum_{i=1}^{n} x_i\right) \frac{1}{n} \exp\left(nx_{(1)}\right) \\
&= \frac{1}{n} \exp\left(-\sum_{i=1}^{n} x_{(1)} - x_i\right)
\end{aligned}$$

$$\begin{aligned}
\hat{\theta}_{PL} &= \frac{\int \theta L(\theta|\mathbf{x}) d\theta}{\int L(\theta|\mathbf{x}) d\theta} \\
&= \frac{\frac{nx_{(1)}-1}{n^2} \exp\left(-\sum_{i=1}^{n} x_{(1)} - x_i\right)}{\frac{1}{n} \exp\left(-\sum_{i=1}^{n} x_{(1)} - x_i\right)} \\
&= \frac{nx_{(1)} - 1}{n} = x_{(1)} - \frac{1}{n}
\end{aligned}$$

d.

$$\begin{aligned}
E\left[(X_{(1)} - \frac{1}{n} - \theta)^2\right] &= E\left[X_{(1)}^2 + \frac{1}{n^2} + \theta^2 - 2\theta X_{(1))} - \frac{2}{n} X_{(1))} + \frac{2\theta}{n}\right] \\
&= E\left(X_{(1)}^2\right) + \frac{1}{n^2} + \theta^2 - 2\theta E(X_{(1))}) - \frac{2}{n} E(X_{(1))}) + \frac{2\theta}{n} \\
&= \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} + \frac{1}{n^2} + \theta^2 - 2\theta\left(\theta + \frac{1}{n}\right) - \frac{2}{n}\left(\theta + \frac{1}{n}\right) + \frac{2\theta}{n} \\
&= \frac{1}{n^2}
\end{aligned}$$

Problem 2

a.

$$\pi(\theta) = I_{(0,1)}(\theta)$$

For the following equations, we will represent the summation over the set $i = 1, 2, ..., n$ with just the $\sum$ symbol.

$$f(\mathbf{x}|\theta) = \theta^n (1 - \theta)^{-n + \sum x_i}$$

$$\begin{aligned}
f(\mathbf{x}|\theta)\pi(\theta) &= \theta^n (1 - \theta)^{-n + \sum x_i} I_{(0,1)}(\theta) \\
\int_0^1 f(\mathbf{x}|\theta)\pi(\theta) d\theta &= \int_0^1 \theta^n (1 - \theta)^{-n + \sum x_i} d\theta \\
&= \frac{\Gamma(n+1)\Gamma(1 - n + \sum x_i)}{\Gamma(2 + \sum x_i)} \int_0^1 \frac{\Gamma(2 + \sum x_i)}{\Gamma(n+1)\Gamma(1 - n + \sum x_i)} \theta^n (1 - \theta)^{-n + \sum x_i} d\theta
\end{aligned}$$

The portion inside the integral is the density function of beta distribution with parameters $\alpha = n + 1$ and $\beta = 1 - n + \sum x_i$. The beta distribution integrates to 1 over the range $(0, 1)$.

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_0^1 f(\mathbf{x}|\theta)\pi(\theta)d\theta} \\
&= \frac{\theta^n(1-\theta)^{-n+\sum x_i}}{\frac{\Gamma(n+1)\Gamma(1-n+\sum x_i)}{\Gamma(2+\sum x_i)}} \\
&= \frac{\Gamma(2+\sum x_i)}{\Gamma(n+1)\Gamma(1-n+\sum x_i)}\theta^n(1-\theta)^{-n+\sum x_i}
\end{aligned}$$

The density function of the posterior distribution is itself the same beta distribution from above.

c.

Following the derivation of the Bayes minimizer based on the squared error loss function from page 7.3.36 of the notes, we see that $\hat{\theta}(\mathbf{x}) = E(\theta|\mathbf{x})$. Since $\pi(\theta|\mathbf{x})$ has as gamma distribution,

$$\begin{aligned}
\hat{\theta}(\mathbf{x}) = E(\theta|\mathbf{x}) &= \frac{\alpha}{\alpha+\beta} \\
&= \frac{n+1}{2+\sum_{i=1}^n x_i}
\end{aligned}$$

Problem 3

For this problem, there is a single sample observation which we will denote by x.

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-\theta^2}{2}\right)$$

$$f(\mathbf{x}|\theta) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-[x-\theta]^2}{2}\right)$$

$$f(\mathbf{x}|\theta)\pi(\theta) = \frac{1}{2\pi}\exp\left(\frac{-x^2}{2} + x\theta + \theta^2\right)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\mathbf{x}|\theta)\pi(\theta)d\theta &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\exp\left(\frac{-x^2}{2} + x\theta - \theta^2\right)d\theta \\
&= \frac{1}{2\sqrt{\pi}}\exp\left(\frac{-x^2}{4}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left(\frac{-x^2}{4} + x\theta - \theta^2\right)d\theta \\
&= \frac{1}{2\sqrt{\pi}}\exp\left(\frac{-x^2}{4}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left(-\left[[\theta - \frac{x}{2}\right]^2\right)d\theta
\end{aligned}$$

The portion inside the integral is the density function of a normal distribution with variable θ, $\mu = x/2$, $\sigma^2 = 1/2$. Over the range $(-\infty, \infty)$, the normal distribution integrates to 1.

$$\int_{-\infty}^{\infty} f(\mathbf{x}|\theta)\pi(\theta)d\theta = \frac{1}{2\sqrt{\pi}} \exp\left(\frac{-x^2}{4}\right)$$

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \frac{f(\mathbf{x}|\theta)\pi(\theta)}{\int_{-\infty}^{\infty} f(\mathbf{x}|\theta)\pi(\theta)d\theta} \\
&= \frac{\frac{1}{2\pi}\exp\left(\frac{-x^2}{2} + x\theta + \theta^2\right)}{\frac{1}{2\sqrt{\pi}}\exp\left(\frac{-x^2}{4}\right)} \\
&= \frac{1}{\sqrt{\pi}} \exp\left(-\left[\theta - \frac{x}{2}\right]^2\right)
\end{aligned}$$

The density function of the posterior distribution is itself the same normal distribution from above. As shown in the assignment, the posterior expected loss is given by

$$\int e^{(\hat{\theta}-\theta)/2}\pi(\theta|\mathbf{x})d\theta - \frac{1}{2}\hat{\theta}\int \pi(\theta|\mathbf{x})d\theta + \frac{1}{2}\int \theta\pi(\theta|\mathbf{x})d\theta - \int \pi(\theta|\mathbf{x})d\theta$$

with all integrals over the range of $\theta \in (-\infty, \infty)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{(\hat{\theta}-\theta)/2}\pi(\theta|\mathbf{x})d\theta &= \int_{-\infty}^{\infty} e^{(\hat{\theta}-\theta)/2}\frac{1}{\sqrt{\pi}}\exp\left(-\left[\theta - \frac{x}{2}\right]^2\right)d\theta \\
&= \frac{1}{\sqrt{\pi}}\exp\left(\frac{\hat{\theta}}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{\theta}{2}\right)\exp\left(-\theta^2 + x\theta - \frac{x^2}{4}\right)d\theta \\
&= \frac{1}{\sqrt{\pi}}\exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right)\int_{-\infty}^{\infty}\exp\left(-\left[\theta^2 - \theta\left(x - \frac{1}{2}\right) + \frac{x^2 - x + 1/4}{4}\right]\right) \\
&= \exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\exp\left(-\left[\theta - \left(\frac{x}{2} - \frac{1}{4}\right)\right]^2\right)
\end{aligned}$$

The portion inside the integral is the density function of a normal distribution with variable θ, $\mu = x/2 - 1/4$, $\sigma^2 = 1/2$. Over the range $(-\infty, \infty)$, the normal distribution integrates to 1. The posterior distribution is a valid pdf, therefore over the range $(-\infty, \infty)$ it integrates to 1. The integral of $\theta\pi(\theta|\mathbf{x})$ over $\theta \in (-\infty, \infty)$ is $E(\theta|\mathbf{x})$.

$$\int e^{(\hat{\theta}-\theta)/2}\pi(\theta|\mathbf{x})d\theta - \frac{1}{2}\hat{\theta}\int \pi(\theta|\mathbf{x})d\theta + \frac{1}{2}\int \theta\pi(\theta|\mathbf{x})d\theta - \int \pi(\theta|\mathbf{x})d\theta = \exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right) - \frac{1}{2}\hat{\theta} + \frac{1}{2}E(\theta|\mathbf{x}) - 1$$

To minimize the posterior expected loss function, we take its derivative with respect to $\hat{\theta}(\mathbf{x})$,

$$\frac{d}{d\hat{\theta}}\left[\exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right) - \frac{1}{2}\hat{\theta} + \frac{1}{2}E(\theta|\mathbf{x}) - 1\right] = \frac{1}{2}\exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right) - \frac{1}{2}$$

and set the resulting expression equal to zero:

$$\begin{aligned}
\frac{1}{2}\exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right) - \frac{1}{2} &= 0 \\
\exp\left(\frac{\hat{\theta}}{2}\right)\exp\left(-\frac{x}{4} + \frac{1}{16}\right) &= 1 \\
\exp\left(\frac{\hat{\theta}}{2}\right) &= \exp\left(\frac{x}{4} - \frac{1}{16}\right) \\
\frac{\hat{\theta}}{2} &= \frac{x}{4} - \frac{1}{16} \\
\hat{\theta} &= \frac{x}{2} - \frac{1}{8}
\end{aligned}$$

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All functions of the form $\exp(a\hat{\theta})$ are increasing on $(-\infty, \infty)$ if a is positive. Since the derivative of the posterior loss function is zero at the point above, and it is increasing, then the derivative of the posterior loss function must be negative for $\hat{\theta} < x/2 - 1/8$ and positive for $\hat{\theta} > x/2 - 1/8$. Therefore, this point is the global minimum, and the Bayes estimate for $\hat{\theta}$.

Problem 4

a.

The UMVUE for θ from a series of Bernoulli(θ) distributions is

$$\hat{\theta}_U = \frac{T(\mathbf{X})}{n}$$

as demonstrated in HW4, Problem 2. The sufficient statistic $T(\mathbf{X})$ itself has a binomial distribution, so

$$E(T(\mathbf{X})) = n\theta; \qquad \mathrm{Var}(T(\mathbf{X})) = n\theta(1-\theta); \qquad E\left(T(\mathbf{X})^2\right) = \mathrm{Var}(T(\mathbf{X})) + [E(T(\mathbf{X}))]^2 = n\theta - n\theta^2 + n^2\theta^2 = n\theta + n(n-1)\theta^2$$

The MSE for Mr. B's estimate of $\hat{\theta}_B = 9/19$ is

$$MSE_B = \sum_{i=1}^{n} \left(\frac{10}{19} - \frac{9}{19}\right)^2 = \frac{n}{19^2}.$$

The MSE for $\hat{\theta}_U = T(\mathbf{X})/n$ is

$$\begin{aligned} MSE_U = E\left(\left[\hat{\theta}_U - \theta\right]^2\right) &= \frac{1}{n^2} E\left(T(\mathbf{X})^2\right) - \frac{2\theta}{n} E(T(\mathbf{X})) + \theta^2 \\ &= \frac{1}{n^2} n\theta + \frac{1}{n^2} n(n-1)\theta^2 - \frac{2\theta}{n} n\theta + \theta^2 \\ &= \frac{1}{n}\theta + \left(\frac{n-1}{n} - 2 + 1\right)\theta^2 = \frac{\theta(1-\theta)}{n} \end{aligned}$$

Plugging that actual value of $\theta = 10/19$, we get

$$MSE_U = \frac{1}{n}\frac{10}{19}\frac{9}{19} = \frac{90}{n19^2}$$

To solve for $MSE_U < MSE_B$,

$$\begin{aligned} MSE_U &< MSE_B \\ \frac{90}{n19^2} &< fracn19^2 \\ 90 &< n^2 \end{aligned}$$

Therefore, the sample size must be $n \geq 10$ to fulfull the inequality above and ensure that Mr. U's unbiased estimator has a smaller MSE than Mr. B's.

b.

The distribution of the sufficient statistic $T(\mathbf{X})$ is binomial, and will thus yield a integer. Given a sample size of ten, the only integer which, when divided by 10, give a closer estimate than $9/19 = 0.4736$ is $5/10 = 0.5$. The distance from $6/10 = 0.6$ to $10/19 = 0.526$ is greater than that between $9/19$ and $10/19$. For a binomial random variable with sample size 10 and parameter $\theta = 10/19 = 0.526$, the probability that there will be exactly 5 successes is

$$P(T(\mathbf{X}) = 5) = \binom{10}{5}\left(\frac{10}{19}\right)^5\left(\frac{9}{19}\right)^5 = 0.243$$