Homework 2

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Problem 1a

The Bayes classification function (f^*) for a piecewise function is the most likely outcome for each case. Thus for

$$P(Y=1|X=x) = \begin{cases} 0.9, & x < 0.2, \\ 0.2, & 0.2 < x < 0.8, \\ 0.9, & x > 0.8, \end{cases}$$

then

$$f^* = \begin{cases} 1, & x < 0.2, \\ -1, & 0.2 < x < 0.8, \\ 1, & x > 0.8. \end{cases}$$

The risk of the Bayes classification function is the probability the classification function not equalling the actual distribution of Y, or

$$R(f^*) = \begin{cases} 0.1, & x < 0.2, \\ 0.2, & 0.2 < x < 0.8, \\ 0.1, & x > 0.8. \end{cases}$$

Integrating the Bayes risk function over the space $0, 1$ gives $0.1 \cdot 0.2 + 0.2 \cdot 0.6 + 0.1 \cdot 0.2 = 0.16$.

Problem 1b

In order for the excess risk to be zero, the classifier must have the same sign as the Bayes classification function (f^*) in the entire domain of $\mathcal{X}$, which is $[0, 1]$. For the specific f^* given in Problem 1a, there are two changes of sign between 0 and 1. Since our function space ($\mathcal{F}$) is all polynomial functions of degree d, then this polynomial must have exactly two roots in the range $[0, 1]$.

For any polynomial with degree $d < 2$, there cannot be 2 roots, the classifier cannot have the same sign as f^* on the domain of $\mathcal{X}$, and thus the excess risk cannot be zero.

For the set of all polynomial with degree $d > 2$ there exists at least at least one polynomial with exactly two roots in the range $[0, 1]$. For $d = 2$ and f^* given above, any polynomial with roots 0.2 and 0.8 and no other roots in $[0, 1]$ will have excess risk equal to zero.