Homework 3

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e. 2.

For the ridge regression function

$$|\mathbf{y}_{\text{train}} - \mathbf{\Phi}_{\text{train}} w|_2^2 + \lambda\, |w|_2^2 \,,$$

then we can solve for a minimum by taking the zeroes of the derivative.

$$\begin{aligned}
\frac{d}{dw}\left(|\mathbf{y}_{\text{train}} - \mathbf{\Phi}_{\text{train}} w|_2^2 + \lambda\, |w|_2^2\right) &= -2\mathbf{\Phi}_{\text{train}}^{\intercal}\left(\mathbf{y}_{\text{train}} - \mathbf{\Phi} w\right) + 2\lambda w = 0 \\
-2\mathbf{\Phi}_{\text{train}}^{\intercal}\mathbf{\Phi}_{\text{train}} w - 2\lambda w &= -2\mathbf{\Phi}_{\text{train}}^{\intercal}\mathbf{y}_{\text{train}} \\
w &= \left(\mathbf{\Phi}_{\text{train}}^{\intercal}\mathbf{\Phi}_{\text{train}} + \lambda \mathbf{I}_D\right)^{-1}\mathbf{\Phi}_{\text{train}}^{\intercal}\mathbf{y}_{\text{train}}
\end{aligned}$$

Using the singular value decomposition of $\mathbf{\Phi} = \mathbf{U}\mathbf{S}\mathbf{V}^{\intercal}$, and the spectral decomposition

$$\begin{aligned}
\mathbf{\Phi}\mathbf{\Phi}^{\intercal} &= \left(\mathbf{U}\mathbf{S}\mathbf{V}^{\intercal}\right)\left(\mathbf{U}\mathbf{S}\mathbf{V}^{\intercal}\right)^{\intercal} \\
&= \mathbf{V}\mathbf{S}^{\intercal}\mathbf{U}^{\intercal}\mathbf{U}\mathbf{S}\mathbf{V}^{\intercal} \\
&= \mathbf{V}\mathbf{S}^2\mathbf{V}^{\intercal}
\end{aligned}$$

Plugging this into the formula for w gives, and utilizing the property that the transpose of a square diagonal matrix is itself ($\mathbf{S}^{\intercal} = \mathbf{S}$),

$$\begin{aligned}
w &= \left(\mathbf{V}\mathbf{S}^2\mathbf{V}^{\intercal} + \lambda \mathbf{I}_D\right)\left(\mathbf{U}\mathbf{S}\mathbf{V}^{\intercal}\right)^{\intercal}\mathbf{y}_{\text{train}} \\
&= \mathbf{V}\left(\mathbf{S}^2 + \lambda \mathbf{I}_D\right)\mathbf{V}^{\intercal}\mathbf{V}\mathbf{S}^{\intercal}\mathbf{U}^{\intercal}\mathbf{y}_{\text{train}} \\
&= \mathbf{V}\left(\mathbf{S}^2 + \lambda \mathbf{I}_D\right)\mathbf{S}\mathbf{U}^{\intercal}\mathbf{y}_{\text{train}}
\end{aligned}$$

$\left(\mathbf{S}^2 + \lambda \mathbf{I}_D\right)\mathbf{S}$ resolves to a diagonal matrix whose trace is

$$\operatorname{tr}\left(\mathbf{S}_{\lambda}\right) = \sum_{j=1}^{D} \frac{d_j}{d_j^2 + \lambda}$$

yielding the final form

$$\mathbf{V}\mathbf{S}_{\lambda}\mathbf{U}^{\intercal}\mathbf{y}_{\text{train}}.$$

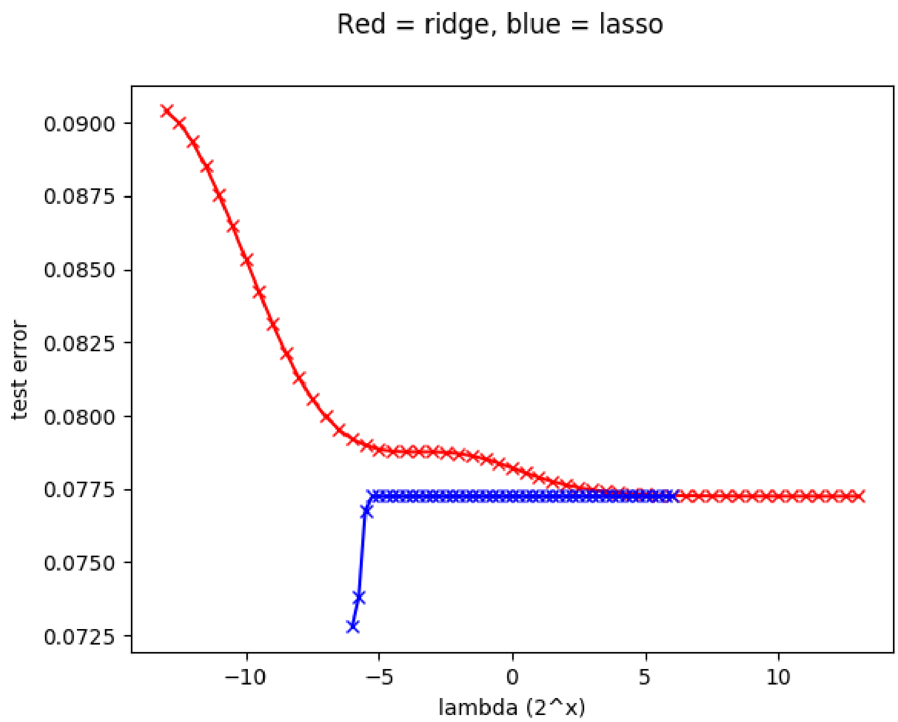
e. 3.

The computational complexity of multplying an $n \times m$ matrix by an $m \times p$ matrix is $O(nmp)$. Multiplying S_λ (a $D \times D$ matrix) by $\mathbf{U}^{\intercal}$ ($D \times n$) has a complexity of $O(nD^2)$ flops.

However, if we pre-compute $\mathbf{U}^{\intercal}\mathbf{y}_{\text{train}}$ only one time, then each matrix multiplication operation in the loop is $(D \times D) \times (D \times 1)$; by multiplying $\mathbf{U}^{\intercal}\mathbf{y}_{\text{train}}$ first by S_λ then by $\mathbf{V}$. These operations have a complexity of $O(D^2)$ flops.

i.)

The test errors for both methods combined are:



We can see from the graphs that both methods converge to a similar error rate as λ increases, but that Lasso regression holds that low error rate over a wider range of lambda values. Lasso regression also has a lower error rate at low values of lambda. Lasso regresion holds the lowest error rate of 0.072 when $\lambda = \sqrt{\frac{\log D}{n_{\text{train}}}} 2^{-6}$