Exercise 1 (5 points):

1. Find the inverse Laplace transform of the flowing functions:

$$F(x) = \frac{3x-2}{(x+2)\left(x-\frac{1}{2}\right)} \qquad G(x) = \frac{-x}{(x+2)\left(x-\frac{1}{2}\right)}$$

2. Using the Laplace transform, solve the following system of linear differential equations with initial conditions:

$$(S): \quad \begin{cases} u''(t) + 3v''(t) - u(t) = 0, & \forall t \geq 0, \\ u'(t) + 3v'(t) - 2v(t) = 0, & \forall t \geq 0, \\ u(0) = 0, \quad v(0) = 0, \quad u'(0) = 0, \quad v'(0) = -\frac{2}{3}. \end{cases}$$

Exercise 2 (3.5 points):

1. Given that, for $\alpha > 0$, $\mathcal{F}\{e^{-\alpha|t|}\}(x) = \frac{2\alpha}{x^2+\alpha^2}$, compute the following integral:

$$\int_0^{+\infty} \frac{\cos(xt)}{1+x^2}\, dx.$$

2. Find a solution $y \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ to the integral equation:

$$y(t) + 3\int_{-\infty}^{+\infty} y(t-u)e^{-|u|}\, du = e^{-|t|}, \quad \forall t$$

Exercise 3 (4.5 points): :

Let $f :]0,+\infty[\to \mathbb{R}$ be a continuous function such that

$$\lim_{t\to 0+} t^{\alpha} f(t) = 0, \qquad \lim_{t\to+\infty} t^{\beta} f(t) = 0, \qquad \alpha < \beta$$

for some $\alpha, \beta \in \mathbb{R}$. We define the Mellin transform of f as

$$\mathcal{M}\{f\}(x) = \int_0^{\infty} f(t)\, t^{x-1}\, dt.$$

1. Show that $\mathcal{M}\{f\}$ is continuous on $]\alpha, \beta[$.
2. Suppose that f is C^1 function. Prove the identity

$$\mathcal{M}\{f'\}(x) = -(x-1)\,\mathcal{M}\{f\}(x-1).$$

   Specify precisely the set of real values x for which this formula holds.

3. Express the Mellin transform $\mathcal{M}\{f\}(x)$ as a Laplace transform of a suitable function derived from f.