On the geometry and the deformation of shape represented by a piecewise continuous Bézier curve with application to shape optimization

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Abstract. In this work, we develop a framework based on piecewize Bézier curves to plane shapes deformation and we apply it to shape optimization problems. We describe a general setting and some general result to reduce the study of a shape optimization problem to a finite dimensional problem of integration of a special type of vector field. We show a practical problem where this approach leads to efficient algorithms.

In all the text below, $E = \mathbb{R}^2$. In this text, we will define a set of manifolds, each point of such a manifold is a parametrized curves in E.

1 Bézier curves

Bézier curves are usual objects in Computer Aided Geometric Design (CAGD) and have natural and straightforward generalization for surfaces and higher dimension geometrical objects. We focus here on curves even if a lot of results have natural generalization in higher dimension. This section has aim to fix notation and make the paper as self contained as possible.

1.1 Basic definitions

Given $P_0, P_1, \ldots, P_D \in E$, we define:

$$B\left(\left(P_0, \ldots, P_D\right), t\right) = (1-t)\, B\left(\left(P_0, \ldots, P_{D-1}\right), t\right) + tB\left(\left(P_1, \ldots, P_D\right), t\right)$$

with $B\left((P), t\right) = P$ for every $P \in E$. The associated Bézier curve is $\left\{B\left(\left(P_0, \ldots, P_D\right), t\right) \mid t \in 0, 1\right\}$ and the list $\left(P_0, \ldots, P_D\right)$ is called the control polygon and the points $P_0, \ldots, P_D$ are called the control points.

This process associates to every set of points a parametrized curve. It is a polynomial parametrized curve and its degree is bounded :

Proposition 1. *Let $P_0, \ldots, P_D \in E$, then $B\left(\left(P_0, \ldots, P_D\right), t\right)$ is a polynomial parametrization and its coordinates have degree at most D.*

1.2 Bernstein's polynomials

Definition 1. *Let D be an integer and $i \in \{0, \ldots, D\}$, we define the Bernstein polynomial $b_{i,D}(t) := \binom{i}{D}(1-t)^{D-i}t^i$.*

Notation 1 *We denote $\mathbb{R}[t]_D$ the set of polynomial of degree less or equal to D. The set $\mathbb{R}[t]_D$ has a natural $\mathbb{R}$-vector space structure, its dimension is $D+1$ and $\{1, t, \ldots, t^D\}$ is a basis of this vector space.*

Proposition 2. *The set $\{b_{0,D}, \ldots, b_{D,D}\}$ is a basis of $\mathbb{R}[t]_N$.*

Proposition 3. *Let $P_0, \ldots, P_D \in E$, then $B\left(\left(P_0, \ldots, P_D\right), t\right) = \sum_{i=0}^{N} P_i b_{i,D}(t)$ for all $t \in [0, 1]$.*

Corollary 1. *Every polynomially parametrized curve can be represented as a Bézier curve.*

1.3 Interpolation

Since a Bézier curve of degree D is defined using $D+1$ control points, one can hope to associate $D+1$ control points from a sampling of $D+1$ points on a curve. The following result shows that this is possible. But in fact, we do not have one Bézier curve of degree D but many ones. Each such curve is associated to a particular sampling of the parameter interval $[0, 1]$.

Proposition 4. *Let $M_0, \ldots, M_D \in E$, then there exists Bézier curves of degree D passing through these points.*

Lemma 1. *Let $t_0 = 0 < t_1 < \cdots < t_D = 1$, then there exists one and only one Bézier curve $B\left(\left(P_0, \ldots, P_D\right), t\right)$ of degree D such that $B\left(\left(P_0, \ldots, P_D\right), t_i\right) = M_i, \forall i \in \{0, \ldots, D\}$.*

Proof. Denote M the $2 \times (D+1)$ matrix built with the coordinate of M_i as i^{th} row, i.e. $M = (M_0, \ldots, M_D)^t$, and denote P the $2 \times (D+1)$ matrix built with the coordinate of P_i as i^{th} row, i.e. $P = (P_0, \ldots, P_D)^t$. We consider the following matrix associated to $\boldsymbol{t} = \left(t_0, \ldots, t_D\right)$:

$$B_{\boldsymbol{t},D} = \begin{pmatrix} b_{0,D}(0) & b_{1,D}(0) & \cdots & b_{D,D}(0) \\ b_{0,D}(t_1) & b_{1,D}(t_1) & \cdots & b_{D,D}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ b_{0,D}(1) & b_{1,D}(1) & \cdots & b_{D,D}(1) \end{pmatrix}. \tag{1}$$

The matrix of equation 1 is invertible (it is the Vandermonde matrix express in the Bernstein basis) and clearly if P is such that $B_{\boldsymbol{t},D}P = M$, then $B([P_0, \ldots, P_D], t)$ give the wanted curve for the proof of the lemma.

Remark that once $\boldsymbol{t}$ is know one can compute $B_{\boldsymbol{t},D}^{-1}$ once for all and that is possible to take advantage of its Vandermonde-like structure in order to improve the cost of the multiplication of a vector by $B_{\boldsymbol{t},D}$. Generally, we use a regular subdivision ($t_i = \frac{i}{D}$) but there are more suitable choices in regard of the stability of the computation.

2 Piecewize Bézier curves

2.1 Basics on piecewize Bézier curves

Let $P_{0,0},\ldots,P_{0,N}$ and $P_{1,0},\ldots,P_{1,D}\in E$ such that $P_{0,N}=P_{1,0}$, we define the following parametrization of a curve:

$$\Gamma:\begin{cases}[0,1]\to E\\ t\mapsto\begin{cases}B\left(\left(P_{0,0},\ldots,P_{0,D}\right),2t\right)\text{for}t\in[0,1/2]\\ B\left(\left(P_{1,0},\ldots,P_{1,D}\right),2t-1\right)\text{for}t\in[1/2,1]\end{cases}\end{cases}.$$

When $N=D$ we say that this parametrization is uniform with respect to the degree and often, we simply say uniform when it does not introduce ambiguity. The curves parametrized by $B\left(\left(P_{0,0},\ldots,P_{0,D}\right),t\right)$ and $B\left(\left(P_{1,0},\ldots,P_{1,D}\right),t\right)$ are called the patches of $\mathcal{C}=\Gamma([0,1])$. The set of control points of the patches of $\mathcal{C}$ are called the control points of $\mathcal{C}$. This is a continuous curve.

More generally, if $P_{0,0},\ldots,P_{0,N_1},P_{1,0},\ldots,P_{1,N_2},\ldots,P_{l,0},\ldots,P_{l,N_l}\in E$, such that $P_{i,N_i}=P_{i+1,0}$ for all $i\in\{0,\ldots,l-1\}$, we define:

$$\Gamma\left(\left(\left(P_{0,0},\ldots,P_{0,D}\right),\ldots,\left(P_{N,0},\ldots,P_{N,D}\right)\right),t\right)=B\left(\left(P_{i,0},\ldots,P_{i,D_i}\right),\frac{i}{l+1}+(l+1)\,t\right)\tag{2}$$

for $t\in\left[\frac{i}{(l+1)},\frac{(i+1)}{(l+1)}\right]$ and for all $i\in\{0,\ldots,l\}$. This defines a continuous parametrization. The curves parametrized by $B\left(\left(P_{i,0},\ldots,P_{i,D_i}\right),t\right)$ are called the patches of $\mathcal{C}=\Gamma([0,1])$. Furthermore, if $P_{l,D_l}=P_{0,0}$ we say that the curve $\mathcal{C}$ is closed or that it is a loop.

We denote $\mathcal{B}_{N,D}$ the set of uniform piecewize Bézier curves built from N patches of degree D. This clearly a finite dimensional subvariety of $\mathcal{C}^0([0,1],E)$ as the image of the following map:

$$\Psi_{N,D}:\begin{cases}\left(E^{D+1}\right)^{N+1}\longrightarrow\mathcal{C}^0([0,1],E)\\ \left(\left(P_{i,j},j=0\ldots D\right),i=0\ldots N\right)\longmapsto\Gamma\left(\left(\left(P_{i,j},j=0\ldots D\right),i=0\ldots N\right),t\right)\end{cases}\tag{3}$$

Clearly, $\Psi_{N,D}$ is onto from $\left(E^{D+1}\right)^{N+1}$ to $\mathcal{B}_{N,D}\subset\mathcal{C}^0([0,1],E)$. It not very difficult to check that $\Psi_{N,D}$ is almost always one-to-one from $\left(E^{D+1}\right)^{N+1}$ to $\mathcal{B}_{N,D}$. So, $\Psi_{N,D}$ is almost everywhere a diffeomorphism between $\left(E^{D+1}\right)^{N+1}$ and $\mathcal{B}_{N,D}$. This embed $\mathcal{B}_{N,D}$ with a manifold structure (even a submanifold structure in $\mathcal{C}^0([0,1],E)$).

The density of polynomials in the set of continuous functions imply that for each $\Phi:[0,1]\longrightarrow E$ there exists $\left(\Gamma_n(t)\right)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|\Phi-\Gamma_n\|_2=0$, in a way that considering Bézier curves is not a drastic restriction.

2.2 Sampling map and retraction to $\boldsymbol{\Psi_{N,D}}$

Definition 2. *Let $t_0=0<t_1<\ldots<t_D=1$, we denote $\boldsymbol{t}=(t_0,\ldots,t_D)$ the associated subdivision of $[0,1]$, then we define the sampling map $\mathcal{S}_{\boldsymbol{t}}:\mathcal{B}_{1,D}\longrightarrow E^{D+1}$ by $\mathcal{S}_{\boldsymbol{t}}(\Gamma)=\left(\Gamma(t_0),\ldots,\Gamma(t_D)\right)$.*

Proposition 5. *The following diagram is commutative:*

$$\begin{array}{ccc} E^{D+1} & \overset{\Psi_{1,D}}{\longrightarrow} & \mathcal{C}^0\left([0,1],E\right) \\ & \underset{B_{\boldsymbol{t},D}}{\searrow} & \downarrow \mathcal{S}_{\boldsymbol{t}} \\ & & E^{D+1} \end{array} \tag{4}$$

and $\Psi_{N,D}$ is an invertible linear isomorphism between $\mathcal{B}_{1,D} = \operatorname{Im}\left(\Psi_{1,D}\right)$ and E^{D+1} and its inverse is $\Psi_{N,D}^{-1} = B_{\boldsymbol{t},D}^{-1} \circ \mathcal{S}_{\boldsymbol{t}}$.

Proof. Let $\Gamma\left(t\right) = \sum_{j=1}^{N}\sum_{i=0}^{D} P_{j,i} b_{i,D}\left(t\right)$ *i.e.* $\Gamma = \Psi_{N,D}\left(\left(P_{0,0},\ldots,P_{0,D}\right),\ldots,\left(P_{N,0},\ldots,P_{N,D}\right)\right)$, then clearly $\mathcal{S}_{\boldsymbol{t}}\left(\Gamma\right) = B_{\boldsymbol{t},D}\boldsymbol{P}$ where $\boldsymbol{P} = \begin{pmatrix} P_{0,0}^t \\ \vdots \\ P_{N,D}^t \end{pmatrix}$, and so $\mathcal{S}_{\boldsymbol{t}} \circ \Psi_{N,D}\left(\boldsymbol{P}\right) = B_{\boldsymbol{t},N}\left(\boldsymbol{P}\right)$. The remainder of the theorem is a consequence of the fact that $B_{\boldsymbol{t},N}$ is a linear isomorphism.

Proposition 6. *Let $t_{1,0} = 0 < t_{1,1} < \cdots < t_{1,D} = 1/N = t_{2,0} < t_{2,1} < \cdots < t_{2,D} = 2/N = t_{3,0} < \cdots < t_{N,D} = 1$, we denote $\boldsymbol{t} = \left(\boldsymbol{t}_1,\ldots,\boldsymbol{t}_N\right)$ where $\boldsymbol{t}_i = \left(t_{0,i},\ldots,t_{D,i}\right)$ and we define the sampling map $\mathcal{S}_{\boldsymbol{t},N} : \mathcal{B}_{N,D} \longrightarrow \left(E^D\right)^N$ by $\mathcal{S}_{\boldsymbol{t},N}\left(\Gamma\right) = \mathcal{S}_{\boldsymbol{t}_1} \times \cdots \times \mathcal{S}_{\boldsymbol{t}_N}\left(\Gamma\right) = \left(\Gamma\left(t_{1,0}\right),\ldots,\Gamma\left(t_{N,D}\right)\right)$. Then $\mathcal{S}_{\boldsymbol{t},N}$ is a linear isomorphism between $\mathcal{B}_{N,D}$ and $\left(E^{D+1}\right)^{N+1}$.*

Proof. It is a simple consequence of the fact that a cartesian product of isomophisms is an isomorphisms. The inverse map is the cartesian product of the inverse of the component maps.

The proposition 6 is important since it allows to give to $\mathcal{B}_{N,D}$ a vector space structure isomorphic to $\left(E^{D+1}\right)^{N+1}$ (and so, of finite dimension). For instance, it allows to transport distance and so on in $\mathcal{B}_{N,D}$.

In fact, we focus here into a speciale type of sampling. We consider such an sampling where $t_{i,0} = \frac{i}{N}$ and $t_{i,D} = \frac{i+1}{N}$ and $t_{i,j} = t_{i,0} + \frac{j}{ND}$. We will call this kind of sampling a regular sampling and we will omit the subscript $\boldsymbol{t}$ everywhere using these samplings. We use these sampling to simply the presentation, but all the results has equivalent statements with general sampling. Representing each patch by its control polygon, the matrix of $\mathcal{S}_{\boldsymbol{t},N}$ is N times the cartesian production of the map $B_{1,D}$ with itself: $B_{1,D} \times \cdots \times B_{1,D}$. This gives us an easy way to solve the following interpolation problem.

Problem 1. Given $M_{0,0},\ldots,M_{0,D},\ldots,M_{N,0},\ldots,M_{N,D} \in E$, find $\Gamma \in \mathcal{B}_{N,D}$ such that $\mathcal{S}_N\left(\Gamma\right) = \begin{pmatrix} M_{0,0}^t \\ \vdots \\ M_{N,D}^t \end{pmatrix}$.

Proposition 7. *The solution of problem 1 is given by the image by $\Psi_{N,D}$ of:*

$$\begin{pmatrix} B_{1,D}^{-1} & & \\ & \ddots & \\ & & B_{1,D}^{-1} \end{pmatrix} \begin{pmatrix} M_{0,0}^{t} \\ \vdots \\ M_{N,D}^{t} \end{pmatrix}. \tag{5}$$

The proposition 7 implies that $\chi_{\boldsymbol{t},D} = B_{t,D}^{-1} \circ \mathcal{S}_{\boldsymbol{t}} : \mathcal{B}_{N,D} \longrightarrow E^{D+1}$ is such that $\Psi_{1,D} \circ \chi_{\boldsymbol{t},D} = \mathrm{Id}_{E^{D+1}}$. It is easy to extend this result to $\Psi_{N,D}$ using $B_{N,D} = B_{1,D} \times \cdots \times B_{1,D}$ satisfying $B_{N,D}^{-1} = B_{1,D}^{-1} \times \cdots \times B_{1,D}^{-1}$.

This approach allows to project any element of $\mathcal{C}^0\left([0,1], E\right)$ on $\mathcal{B}_{N,D}$ using $\mathcal{S}_N$. Let $\Lambda \in \mathcal{C}^0\left([0,1], E\right)$, then denoting $\boldsymbol{M} = \left(\Gamma(0), \Gamma(\frac{1}{ND}), \ldots, \Gamma(\frac{ND-1}{ND}), \Gamma(1)\right)^t$ we have $\boldsymbol{P} = \mathcal{S}_N^{-1}\left(\boldsymbol{M}\right) \in \mathcal{B}_{N,D}$ is such that $\Psi_{N,D}\left(\boldsymbol{P}\right) = B\left(\boldsymbol{P}, t\right)$ coincides with $\Lambda\left([0,1]\right)$ on at least $(D+1)$ points counted with multiplicities on each patch. This is only the fact that $\chi_{\boldsymbol{t},D}$ can be extend to $\mathcal{C}^0\left([0,1], E\right)$.

The main claim is that instead of working directly with $\mathcal{B}_{N,D}$, it is easier to work on the "set of control polygons", namely E^{D+1} using sampling and interpolation giving linear isomorphism between control polygons and sampling points on the curves. In what follows, we will always take this point of view.

2.3 Tangent space $\boldsymbol{T\mathcal{B}_{N,D}}$ and deformation of curve

Recall that $\Psi_{N,D}$ define a linear isomorphism between the "space of control polygons" $\left(E^{D+1}\right)^{N+1}$ and the space of piecewize Bézier curves $\mathcal{B}_{N,D}$. We already saw that for any $\gamma\left(t\right) \in \mathcal{B}_{N,D}$ then $\boldsymbol{P} \in \left(E^{D+1}\right)^{N+1}$ such that $\Psi_{N,D}\left(\boldsymbol{P}\right) = \gamma\left(t\right)$ is given by $B_{N,D}^{-1} \circ \mathcal{S}_N\left(\gamma\right)$. This give the following proposition:

Proposition 8. *We have that $T\Psi_{N,D} : T\left(E^{D+1}\right)^{N+1} \longrightarrow T\mathcal{B}_{N,D}$ is such that from any $\gamma \in \mathcal{B}_{N,D}$ we have $T\Psi_{N,D}^{-1}\left(\gamma\right) : T_\gamma\mathcal{B}_{N,D} \longrightarrow T_{\chi_{N,D}(\gamma)}\left(E^{D+1}\right)^{N+1}$ is given by $T\Psi_{N,D}\left(\chi_{N,D}\left(\gamma\right)\right)^{-1}\left(\varepsilon\right) = B_{N,D}^{-1} \circ \mathcal{S}_N\left(\varepsilon\right) = \chi_{N,D}\left(\varepsilon\right)$ for any $\varepsilon\left(t\right) \in T_\gamma\mathcal{B}_{N,D}$ and this is a linear isomorphism.*

An element of $\varepsilon\left(t\right) \in T_\gamma\mathcal{B}_{N,D}$ is called a deformation curve. In fact, this proposition allows to express, given a piecewize Bézier curve and a deformation, how to deform its control polygon. This is an essential step proving that manipulating piecewize Bézier curve, it is enough to manipulate its control polygon. This is the object of the following lemma.

Lemma 2. *Let $\boldsymbol{P} \in \left(E^{D+1}\right)^{N+1}$, $\gamma\left(t\right) = \Psi_{N,D}\left(\boldsymbol{P}\right) = B\left(\boldsymbol{P}, t\right) \in \mathcal{B}_{N,D}$ and $\varepsilon\left(t\right) \in T_\gamma\mathcal{B}_{N,D}$, then:*

i. $\varepsilon\left(t\right) = \Psi_{N,D}\left(\chi_{N,D}\left(\varepsilon\right)\right)$.
ii. $\gamma\left(t\right) + \varepsilon\left(t\right) = \Psi_{N,D}\left(\boldsymbol{P} + \chi_{N,D}\left(\varepsilon\right)\right)$.

This lemma explain how to lift a deformation from the space of curves to the space of control polygons. The vector space structure of both the space of control polygons $\left(E^{D+1}\right)^{N+1}$ and of piecewize Bézier curves $\mathcal{B}_{N,D}$ allows to avoid the use of computationally difficult concept as exponential map between manifold and its tangent space and so on. This structure has also to define a simple notion of distance between two such curves.

3 Applications to shape optimization

In this section, we show how the preceding formalism can be exploited in the context of shape optimization. An application to a problem of image segmentation is presented to illustrate our purpose.

3.1 Shape optimization problem

A shape optimization problem consists in, given a set of admissible shapes $\mathcal{A}$ and a functional $F : \mathcal{A} \to \mathbb{R}^+$ find a shape $\alpha \in \mathcal{A}$ such that for all other shapes $\beta \in \mathcal{A}$, we have $F(\alpha) \leqslant F(\beta)$. Generally, one try to give to the space of admissible shape a structure of manifold in a way to be able to compute a "shape gradient" , $\nabla F(\beta)$, expressing the evolution of the criterium F with respect to a deformation of the shape β. It is to say that $\nabla F(\beta)$ associates to every point $M \in \beta$ a deformation vector $\nabla F(\beta)(M) \in T_M E$. The computation of such a gradient can require sophisticated computation since very often, even the computation of the criterium itself require to solve a system partial differential equations. Many problem can be expressed as a shape optimization problem. Classical approach to solve this kind of problem is to use $\nabla F(\beta)$, when it is computable, in a gradient method to find a local minimum.

To keep the presentation as simple as possible, we focus on geometric optimisation, i.e. the topology of the shape is fixed, in the case where the frontier of the admissible shapes are continuous Jordan curves. But the framework presented here can be extended to topological optimization as it is shown in [5] for a special application on a problem of image segmentation. The case treated here received attention because of its deep links with images segmentation and shape recognition (see [5,3,2] for instance).

We denote $\mathcal{C}^0_J([0,1], E)$ the set of function parametrizing a Jordan curve and $\mathcal{B}^c_{N,D} = \{\gamma \in \mathcal{B}_{N,D} \mid \gamma(t) = \gamma(s) \,\text{with} s \neq t \Leftrightarrow (t=0 \,\text{and} s=1) \,\text{or}\, (t=1 \text{and} s=0)\}$. We have $\mathcal{B}^c_{N,D} \subset \mathcal{C}^0_J([0,1], E)$. We denote:

$$H_{N,D} = \left\{ \left(\left(P_{i,j}, j = 0 \ldots D \right), i = 0 \ldots N \right) \in \left(E^D\right)^N \mid P_{0,0} = P_{N,D} \right\}$$

which is a linear subspace of $\left(E^{D+1}\right)^{N+1}$. We then denote $\Psi^c_{N,D} = \Psi_{N,D} \left|_{H_{N,D}}\right.$. As above, $\Psi^c_{N,D}$ define a linear isomorphism between $H_{N,D}$ and $\mathcal{B}^c_{N,D}$ using $\mathcal{S}_N \left|_{\mathcal{B}^c_{N,D}}\right.$ and the same $B_{N,D}$ to define its converse explicitly.

3.2 Vector field on $\mathcal{B}_{N,D}$ lifted from the shape gradient

Let ∇F be a shape, then for each $\alpha \in \mathcal{C}^0_J([0,1], E)$ and for any $M \in \alpha([0,1])$, ∇F associate to M an element $\nabla F(\alpha)(M) \in T_M E$. Consider now $\alpha \in \mathcal{B}^{c}_{N,D}$ and $((M_{0,0}, \ldots, M_{0,D}), \ldots, (M_{N,0}, \ldots, M_{N,D})) = \mathcal{S}_N(\alpha)$, then $M_{0,0} = M_{N,D}$. We

$$\mathcal{T}_{N,F}(\alpha) = ((\nabla F(\alpha)(M_{i,j}), j = 0 \ldots D), i = 0 \ldots D)$$

This representes the sampling of the deformation of the curve implied by the shape gradient ∇F to α. It is not difficult to see that $\mathcal{T}_{N,F}(\alpha) \in T_{\mathcal{S}_N(\alpha)}\left(\left(E^D\right)^N\right)$.

Theorem 1. *To each shape gradient ∇F the map $B^{-1}_{N,D} \circ \mathcal{T}_{N,F}$ associate a vector field on $H_{N,D}$ which correspond to a vector field V_F on $\mathcal{B}^{c}_{N,D}$ through $T\Psi^{c}_{N,D}$.*

This theorem allows to interpret gradient descent method for shape optimization as a algorithm for integrating a vector field in a finite dimensional space. From this point of view, gradient descent method correspond to the most naive method to integrate this vector field, namely the Euler method. Clearly, this approach suggests to use better algorithm for vector field integration.

3.3 Geometry of the vector field and local extrema of shape cost functional

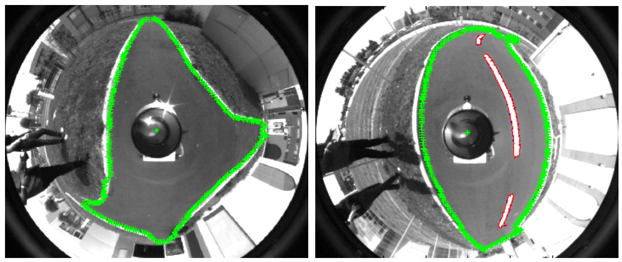
Proposition 9. *Let $\alpha \in \mathcal{C}^0_J([0,1], E)$ be such that $\nabla F(\alpha) = \mathbf{0}$, i.e. $\nabla F(\alpha)(M) = \mathbf{0}$ for all $M \in \alpha([0,1])$ and let $\gamma \in \mathcal{B}_{N,D}$ such that $\gamma\left(\frac{i}{ND}\right) = \alpha\left(\frac{i}{ND}\right)$ for $i \in \{0, \ldots, ND\}$, i.e. $\gamma = \Psi_{N,D}\left(B^{-1}_{N,D} \circ \mathcal{S}_N(\alpha)\right)$, then $\nabla F(\gamma)\left(\gamma\left(\frac{i}{ND}\right)\right) = 0$ for $i \in \{0, \ldots, ND\}$ and then $V_F\left(B^{-1}_{N,D} \circ \mathcal{S}_N(\alpha)\right) = \mathbf{0}$. It is to say that a local extremum of F induces a local extremum of its restriction to $\mathcal{B}_{N,D}$ and that this extremum is "lifted" on a singularity of the vector field V_F on $H_{N,D}$.*

Proof. The deformation curve of γ induces by the gradient of F vanishes at at least $(D+1)(N+1)$ points, but it is a "Bézier curve" of degree D, so it a zero polynomial. So, its control polygon is reduce to the origin and then $B^{-1}_{N,D} \circ \mathcal{S}_N(\alpha)$ is a singularity of V_F.

In fact, the vector field V_F is associated to the gradient of the function $F \circ \Psi_{N,D}$. This is an heavy constrain on the vector field. For instance, it is easy easy to see that $\boldsymbol{P}$ is an attractive singularity of V_F if and only if $\Psi_{N,D}(\boldsymbol{P})$ is a local minimum of $F\left|_{H_{N,D}}\right.$.

3.4 Application to a problem of images segmentation

In this section, we sketch an application to a problem of images segmentation. It is a problem of omnidirectional vision. Previous methods tried with some success but does not allowed a full real time treatment. There all based on snake-like algorithms (see [9]). The gradient use to detect edges is a classical one based on a Canny filter and is combined with a balloon force. The best previously known method is such that propagation of the contour were done using the fast marching algorithm for level set method. This a typical formulation of image segmentation as a shape optimization problem. In [6], we use piecewize Bézier curves to contour propagation and achieve a very fast segmentation algorithm allowing real time treatment even with sequential algorithm (no use of parallelism or special hardware architecture) on a embedded system.



It is very interesting to see that, with few algorithmic modification, it is also possible to treat change of topology, i.e. curves with several connected components as it is shown in the following figure.

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