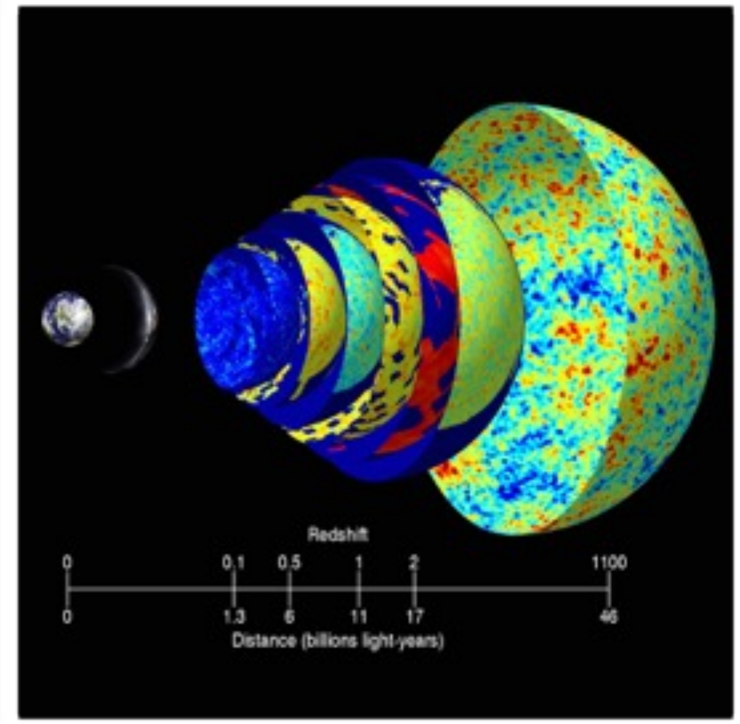
# Needlets
# for
# Spherical 3D data




Yabebal Fantaye
*With: C. Durastanti, F. Hansen, D. Marinucci, and I. Z. Pesenson*
*PhysRevD.90.103532*
*arxiv:1408.1095*

Image Credit: http://phys.org/news/2012-09-dark-energy-real-astronomers.html

# Outline

- 2D Needlet Formalism
- Radial 3D Harmonics
- Radial 3D Needlets
- Conclusion

# Spherical Harmonics

$$f(\omega) = \sum_{\ell \geq 0} \sum_{m=-l}^{l} a_{\ell,m} Y_{\ell,m}(\omega), \qquad \omega \in S^2$$

$$a_{\ell,m} = 4\pi \int_{S^2} \bar{Y}_{\ell,m}(\omega) f(\omega) \sigma(d\omega)$$

$$\widehat{C}_\ell = \frac{1}{2\ell + 1} \sum_m |a_{\ell m}|^2$$

# Harmonics vs Spatial information


Harmonics vs spatial localisation obeys the uncertainty principle. **Wavelets** allow a flexible compromise between localisation in both real and harmonic domain.

# Spherical needlets

Needlet center

Qadrature direction

$$\psi_{jk}(\hat{\gamma}) = \sqrt{\lambda_{jk}} \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} \overline{Y}_{\ell m}(\hat{\gamma}) Y_{\ell m}(\xi_{jk})$$

Needlets basis

Quadrature weight/ pixel area

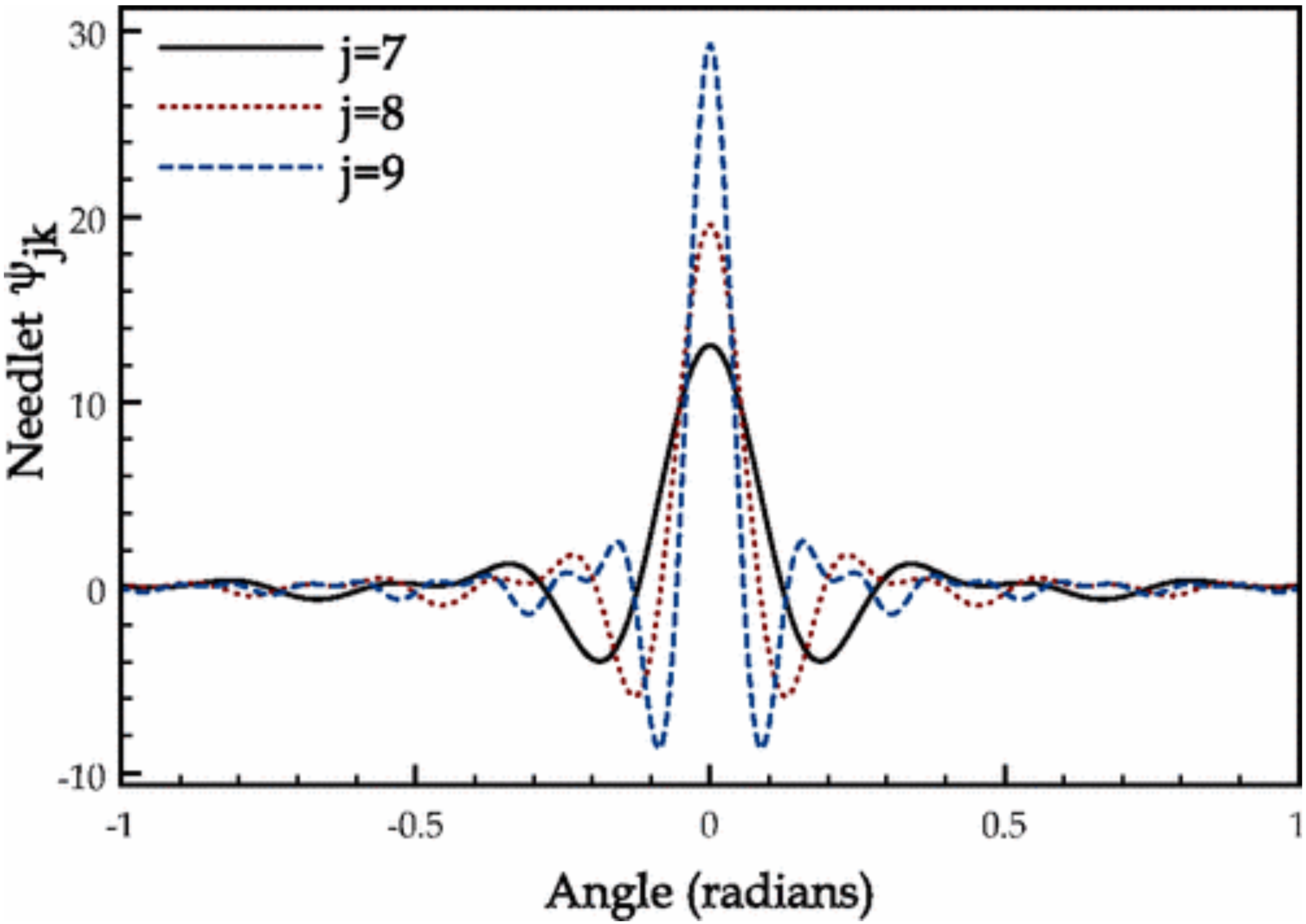
window function

Projection Operator

*j - the needlet scale*
*k - the pixel index*

# Spherical needlets

*Marinucci et. al. 2007*



# Needlet window function b(.)

1. $b^2(.)$ has support in $[\frac{1}{B}, B]$, and hence $b(\frac{\ell}{B^j})$ has support in $\ell \in [B^{j-1}, B^{j+1}]$
2. the function $b(.)$ is infinitely differentiable in $(0, \infty)$.
3. we have
$$\sum_{j=1}^{\infty} b^2(\frac{\ell}{B^j}) \equiv 1 \text{ for all } \ell > B.$$

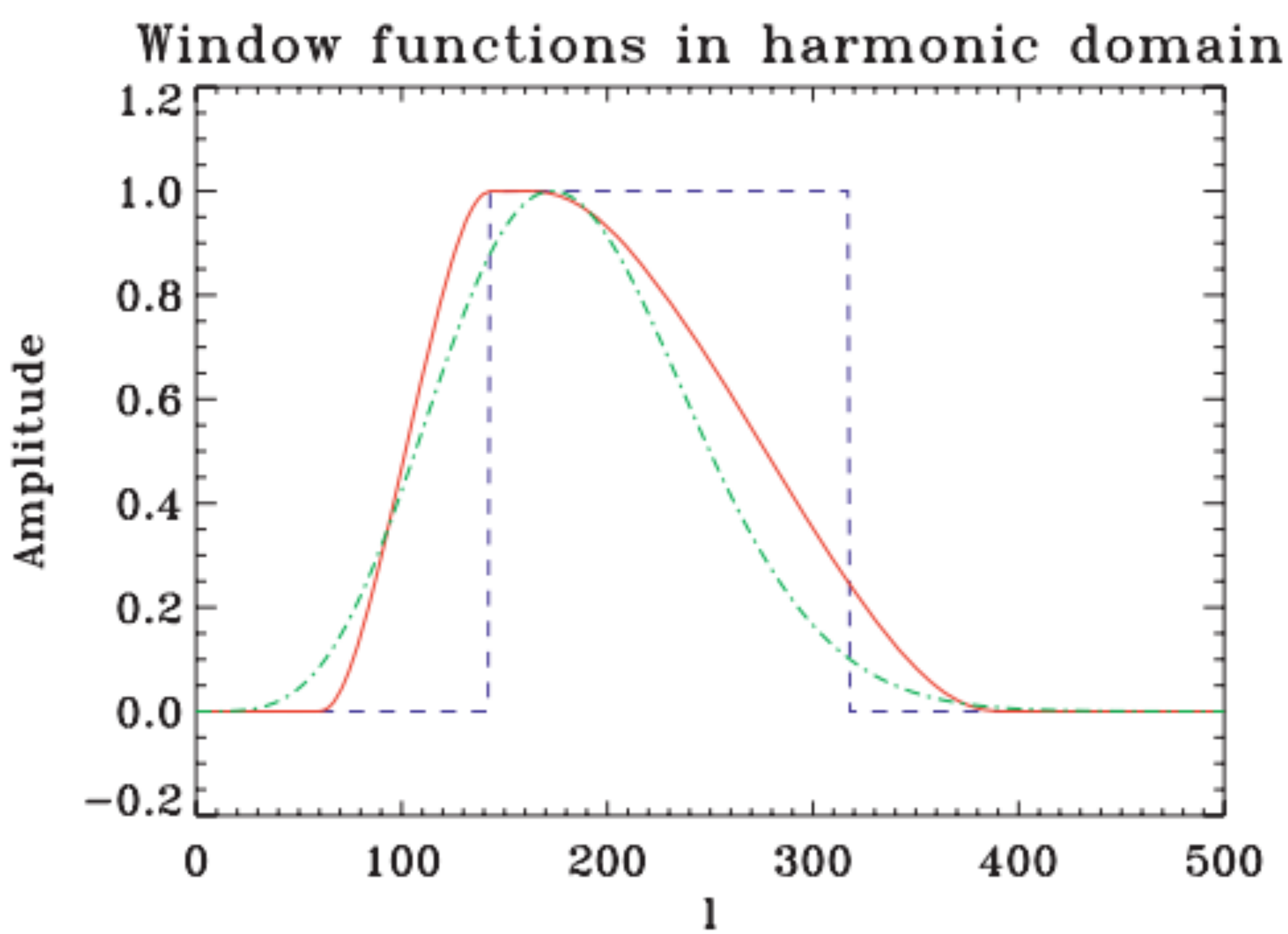
# Needlet decomposition

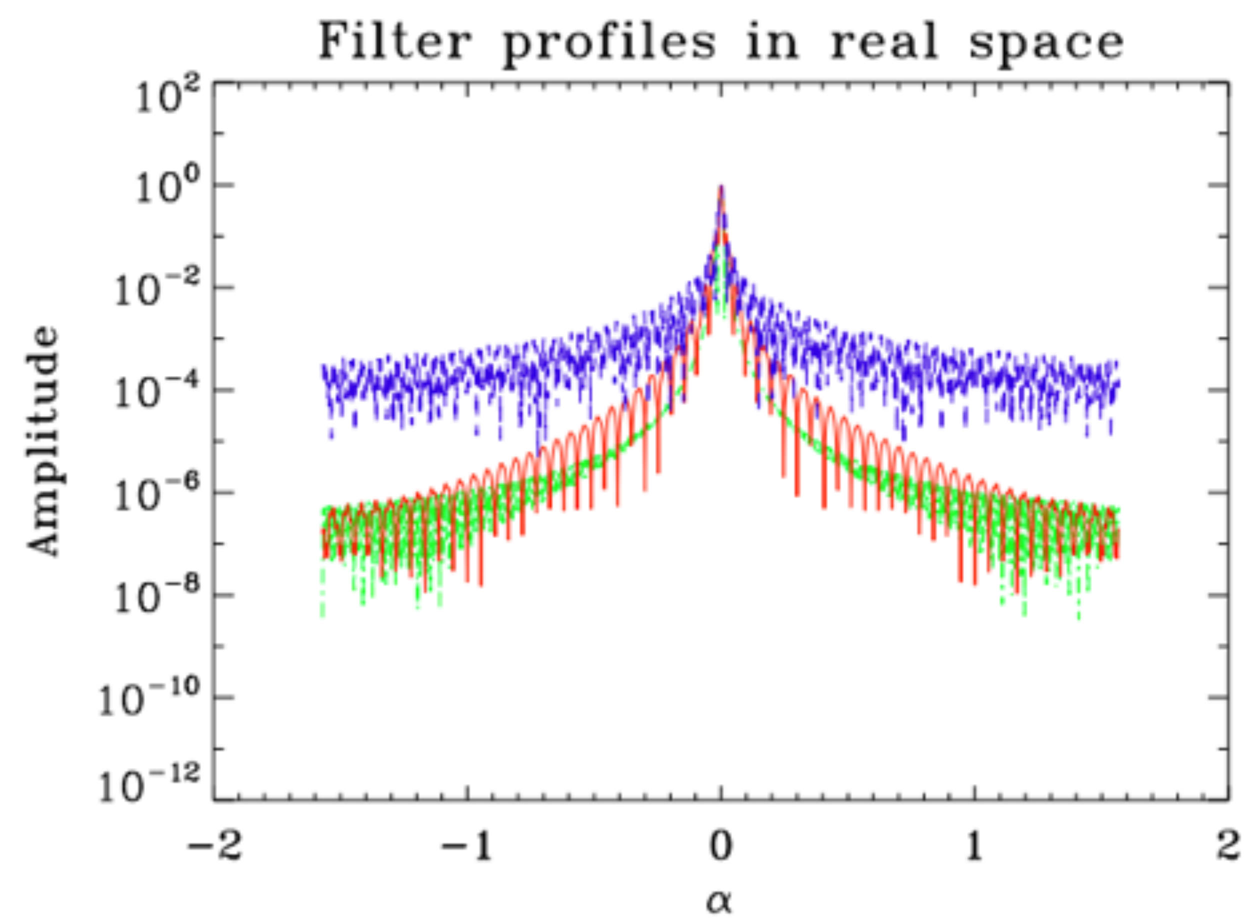
$$f(x) = \sum_{jk} \beta_{jk} \psi_{jk}(x)$$

Where,

$$\begin{aligned}
\beta_{jk} &= \int_{S^2} T(\hat{\gamma}) \psi_{jk}(\hat{\gamma}) d\Omega \\
&= \sqrt{\lambda_{jk}} \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} \left\{ \int_{S^2} T(\hat{\gamma}) \overline{Y}_{\ell m}(\hat{\gamma}) d\Omega \right\} Y_{\ell m}(\xi_{jk}) \\
&\boxed{= \sqrt{\lambda_{jk}} \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\xi_{jk}).} \qquad (3)
\end{aligned}$$

# Localization




Marinucci et. al. 2007

# Why needlets?

- No Tangent-Plane approximation
- Quasi-exponential localisation in pixel space
- Bounded support on multipoles
- Tight frame - exact reconstruction formula
- Minimal correlation in harmonic and real domain

# Application of needlets

- Feature detection
- Denoising - component separation
- Handling masks and missing data
- Power spectrum and bispectrum estimation
- And more …

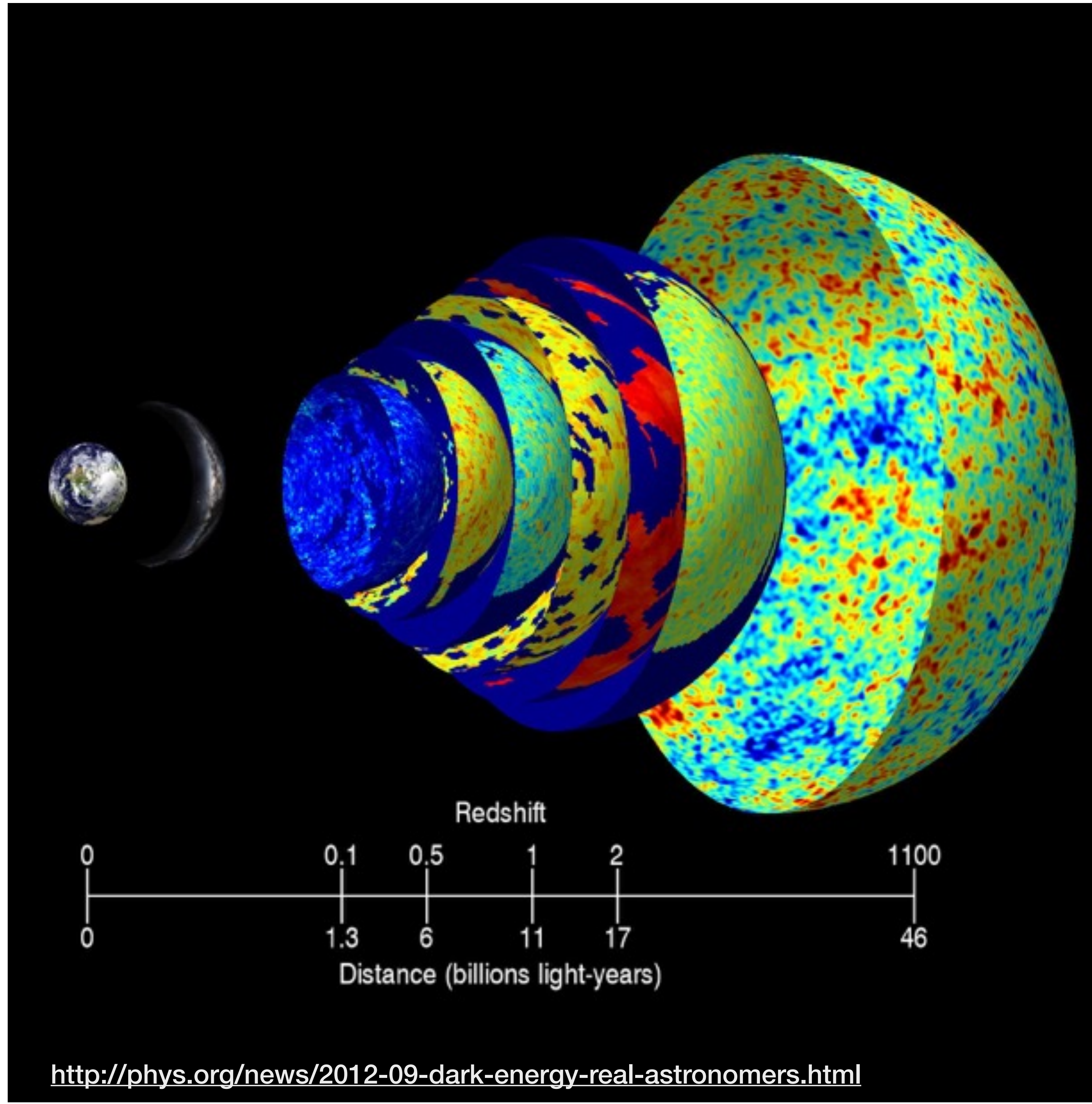
# Needlets preserve norm

$$\sum_{j,k} \beta_{jk}^2 \equiv \sum_{\ell=1}^{\infty} \frac{2\ell+1}{4\pi} \widehat{C}_\ell$$

Where, $$\widehat{C}_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2$$

# Radial 3D needlets - Setup

Our method envisages a data collection environment in which an observer located at the center of the ball is surrounded by concentric spheres with the same pixelization at different radial distances, for any given resolution.



# Motivation - Euclid, SKA, etc.

- Observations of the large scale structure has an inherent radial-angular asymmetry.
    - Data at different redshift have a different physical process; have different selection bias - at high z only bright objects can be observed; data accusation-pixelization are different.
- Our formalism is ideal for large scale structure experiments like Euclid, SKA, etc..

# 3D Radial needlet: formalism

- Given our setup, we can define a manifold $M = (0, 2\pi] \times S^2$ and a product space

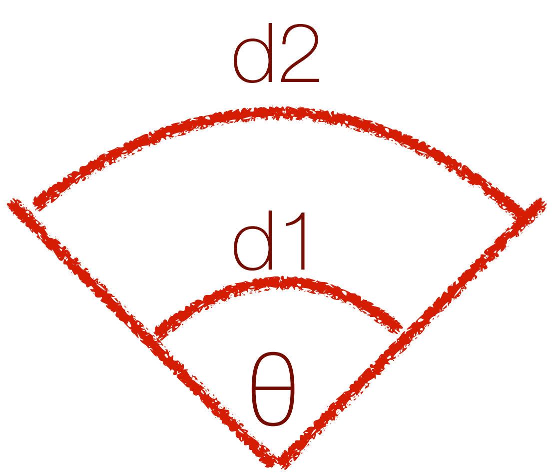
$$L^2(M, d\mu) = L^2((0, 2\pi], dr) \otimes L^2(S^2, d\sigma)$$

# New Metric

$$g_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta \end{pmatrix}$$

$$\downarrow$$

$$d_M(x_1, x_2) = \sqrt{(r_1 - r_2)^2 + d_{S^2}^2(\omega_1, \omega_2)}.$$

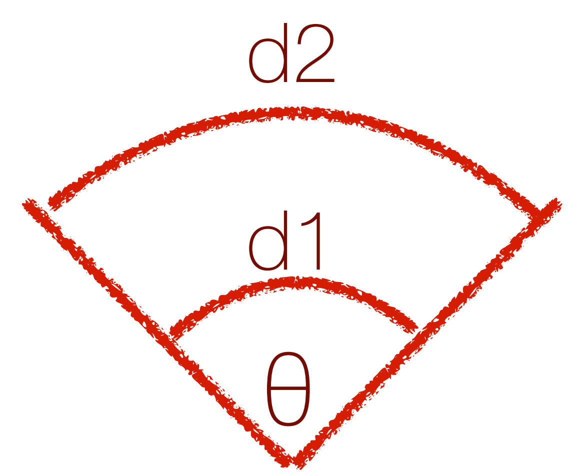



d2=d1

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\downarrow$$

$$d(x_1, x_2) = \sqrt{(r_1 - r_2)^2 + r_1 r_2 d_{S^2}^2(\omega_1, \omega_2)}$$




d2>d1

# 3D Harmonics basis

- The orthonormal basis of the new product space can be defined using the standard basis of the sphere and the radial line:

$$\frac{\partial^2}{\partial r^2}(2\pi)^{-\frac{1}{2}}\exp(inr) = -n^2(2\pi)^{-\frac{1}{2}}\exp(inr)$$

$$\Delta_{S^2} Y_{\ell,m} = -\ell(\ell+1)Y_{\ell,m}$$

# 3D Harmonics basis

- The Laplacian of our space can be constructed by combining the Laplacian operators of the sphere and the radial line

$$\Delta_M := \frac{\partial^2}{\partial r^2} + \Delta_{S^2}$$

$$\Delta_M(\exp(inr)Y_{\ell m}(\omega)) = -e_{n,\ell} \exp(inr)Y_{\ell m}(\omega)$$

where $e_{n,\ell} = (n^2 + \ell(\ell+1))$

**Define:** $u_{\ell,m,n}(r,\vartheta,\phi) = (2\pi)^{-\frac{1}{2}} \exp(inr)Y_{\ell,m}(\vartheta,\phi)$

# 3D Harmonics expansion

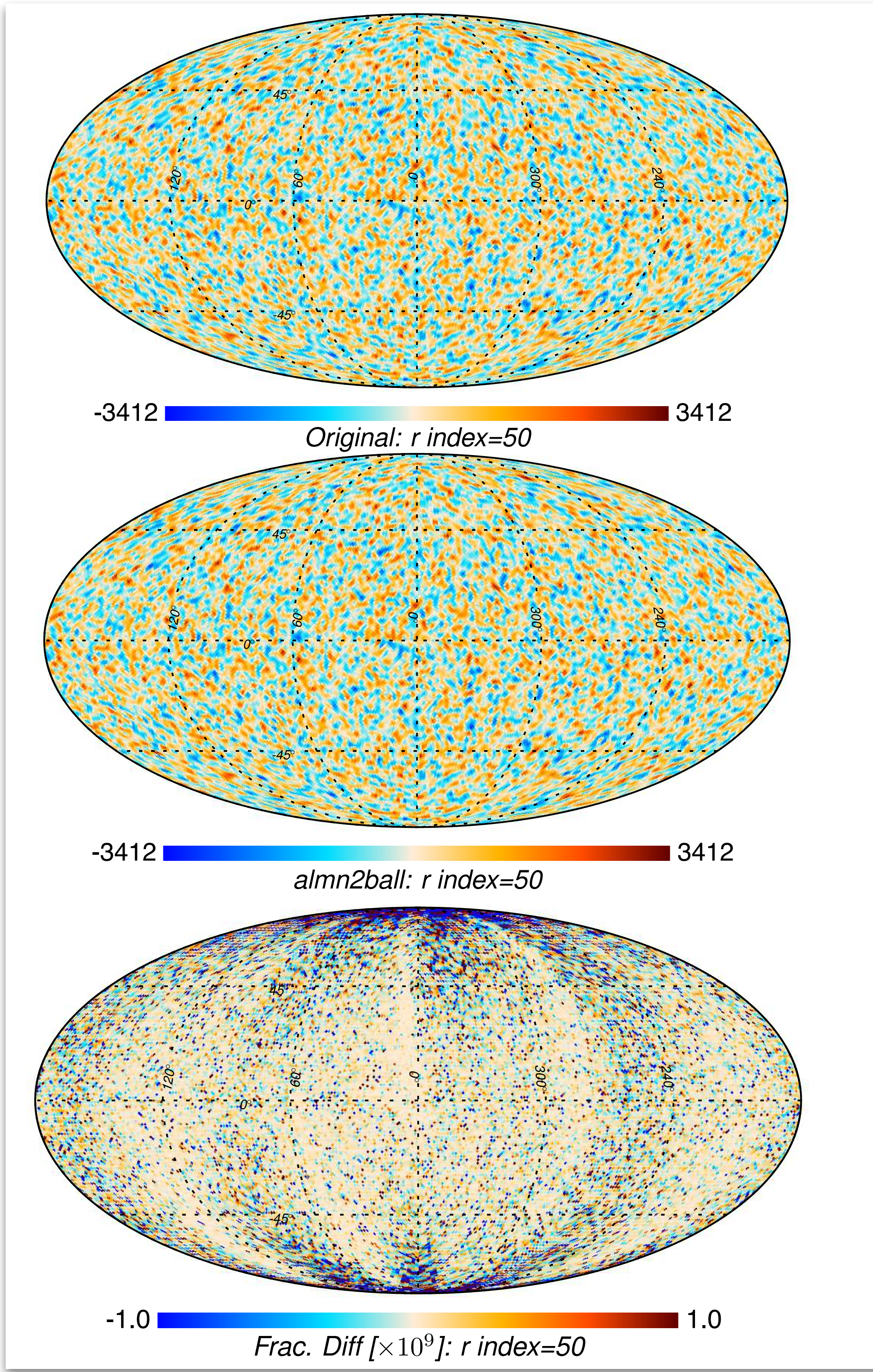
$$F(r,\vartheta,\phi)=\sum_{\ell\geq 0}\sum_{m=-\ell}^{\ell}\sum_{n\geq 0}a_{\ell,m,n}u_{\ell,m,n}(r,\vartheta,\phi)$$

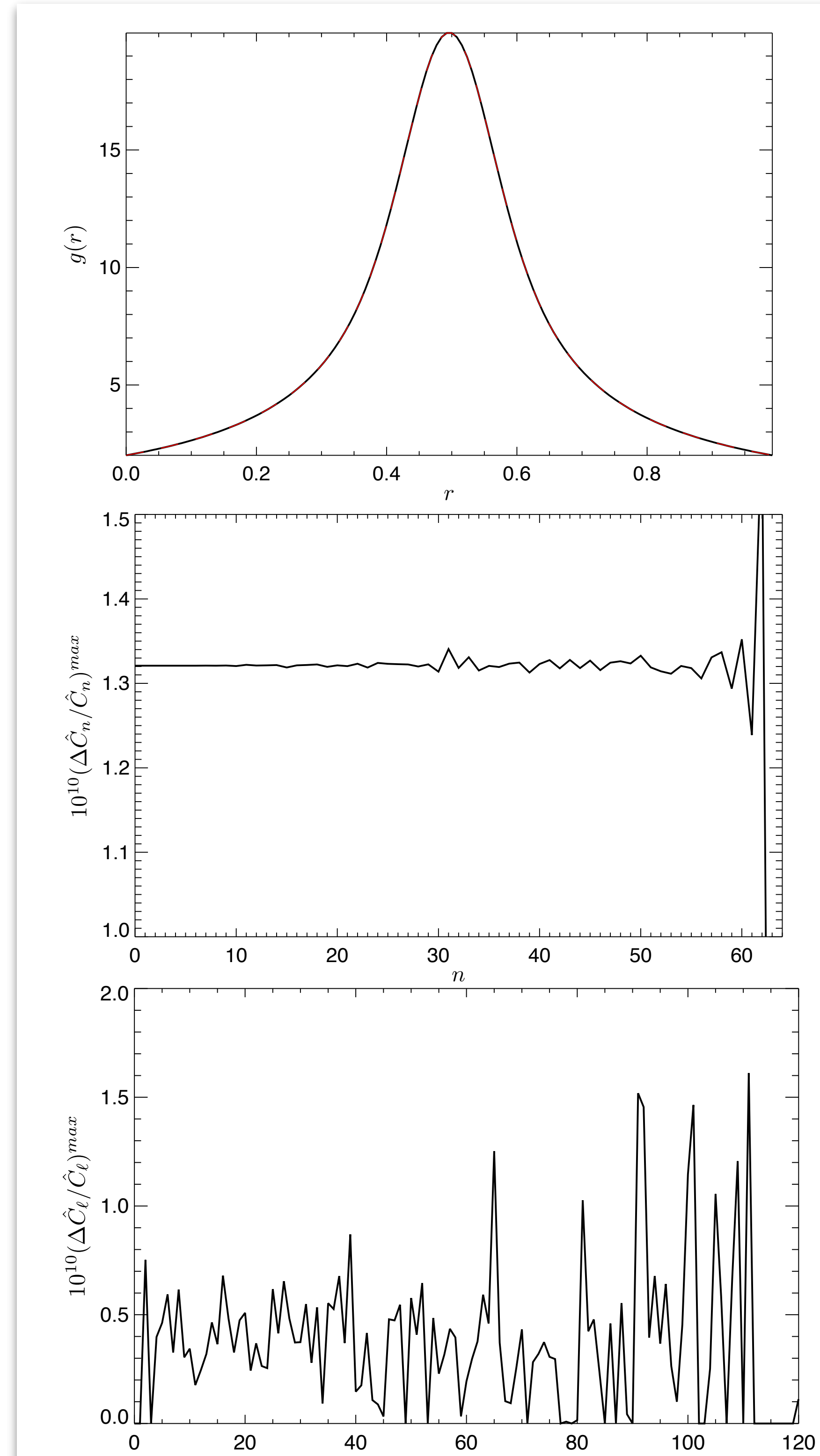
- Let's define angular and radial power spectra as

$$\hat{C}_\ell=\frac{1}{n_{\max}}\sum_n\frac{1}{(2\ell+1)}\sum_m|a_{\ell mn}|^2,$$

$$\hat{C}_n=\frac{1}{\ell_{\max}}\sum_l\frac{1}{(2\ell+1)}\sum_m|a_{\ell mn}|^2.$$

# ball2almn ↔ almn2ball Performance



# 3D needlets

3D Needlet

$$\Phi_{j,q,k}(x) = \sqrt{\lambda_{j,q,k}} \sum_{[\ell,n]_j} \sum_{m=-\ell}^{\ell} b\left(\frac{\sqrt{e_{\ell,n}}}{B^j}\right) \times \bar{u}_{\ell,m,n}(\xi_{j,q,k}) u_{\ell,m,n}(x),$$

3D Needlet expansion of a function

$$F(x) = \sum_{j\geq 0} \sum_{k=1}^{K_j} \sum_{q=1}^{Q_j} \beta_{j,q,k} \Phi_{j,q,k}(x)$$

3D Needlet expansion coefficients

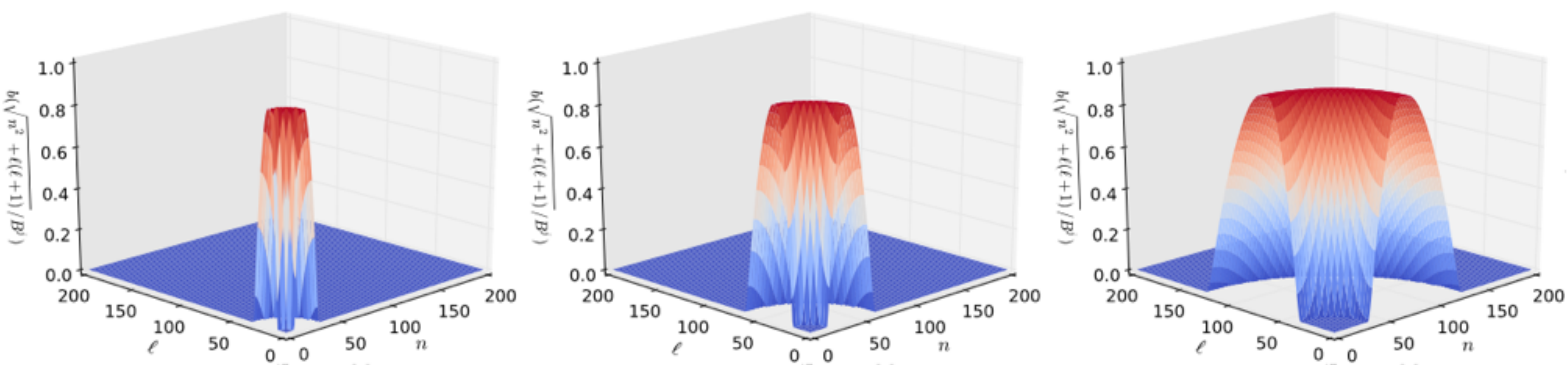
$$\beta_{j,q,k} = \sqrt{\lambda_{j,q,k}} \sum_{[\ell,n]_j} \sum_{m=-\ell}^{\ell} b\left(\frac{\sqrt{e_{\ell,n}}}{B^j}\right) a_{\ell,m,n} u_{\ell,m,n}(\xi_{j,q,k}).$$

# 3D Needlet window function

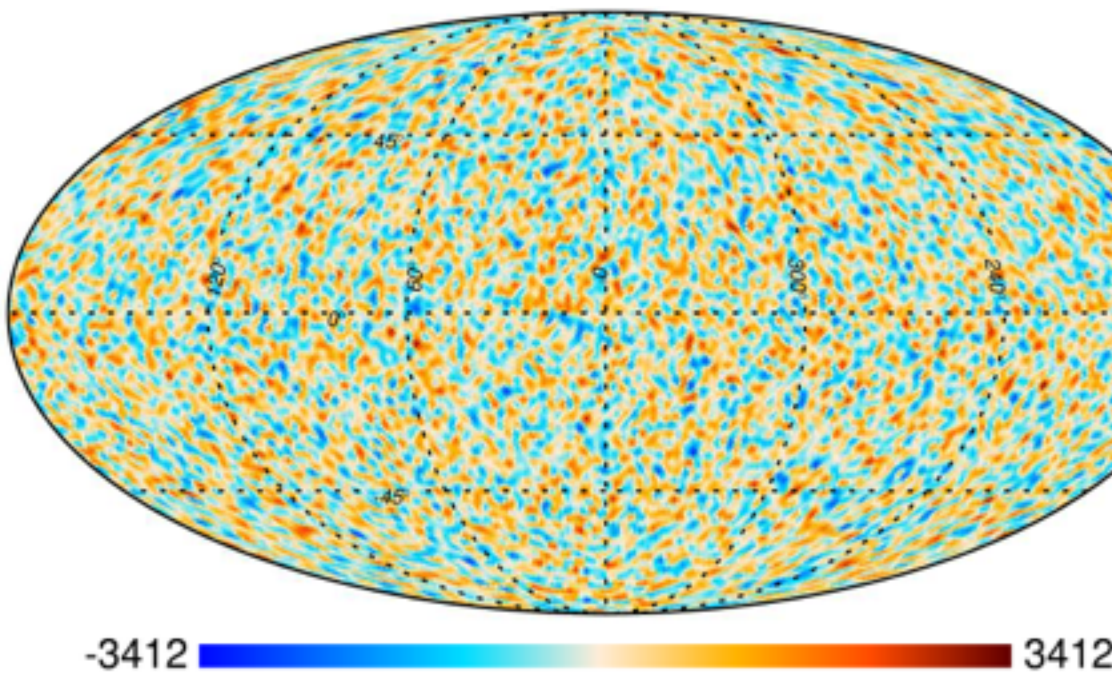
$$\sum_{j=-\infty}^{\infty} b^2\left(\frac{u}{B^j}\right) = 1, \quad \text{for all } u > 0.$$

$$e_{n,\ell} = (n^2 + \ell(\ell+1))$$

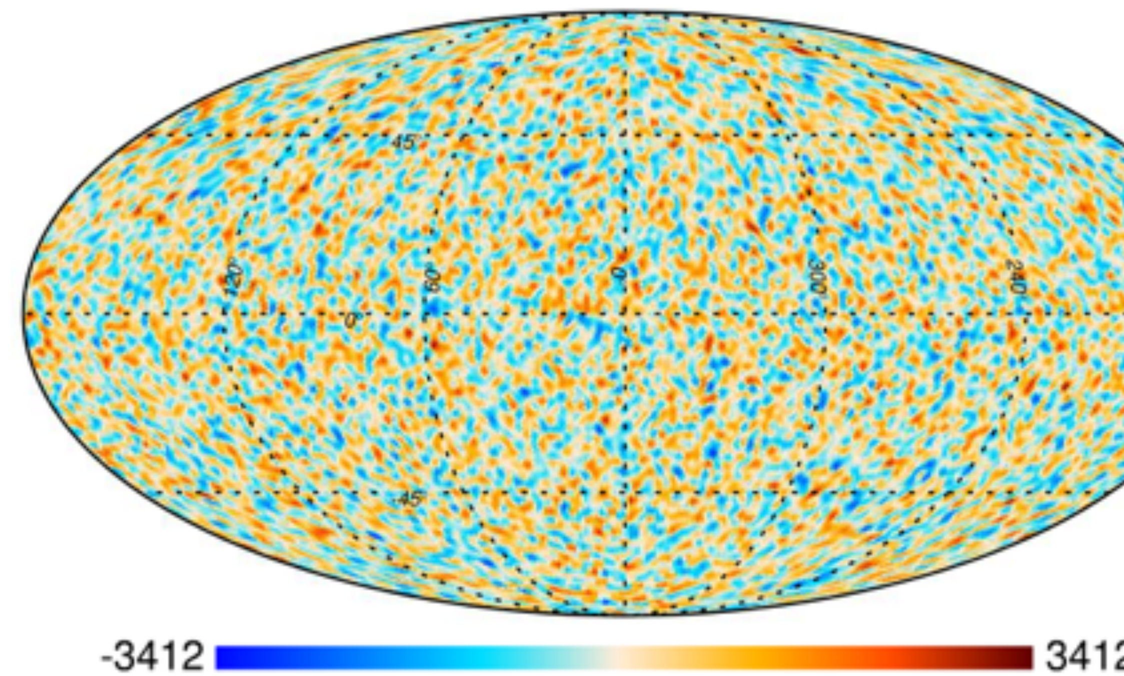
$$[\ell, n]_j = \{l, n:\ B^{2(j-1)} \leq e_{\ell,n} \leq B^{2(j+1)}\}$$



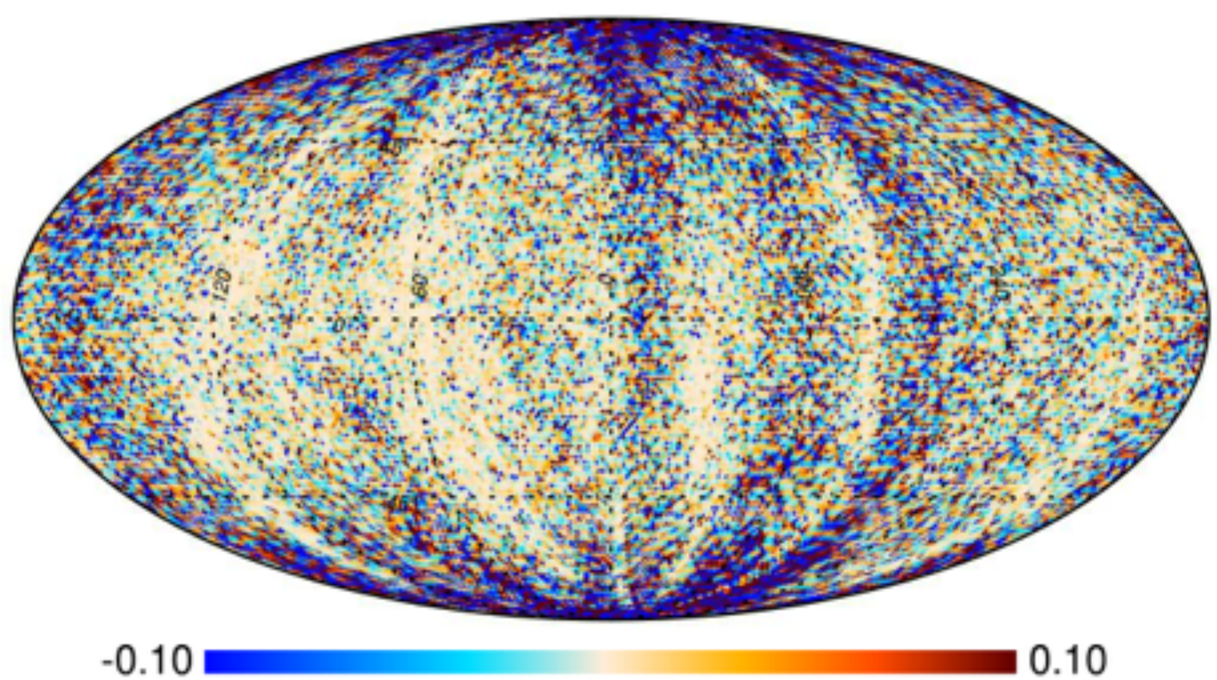
# ball2beta ↔ beta2ball Performance



Original: r index=50

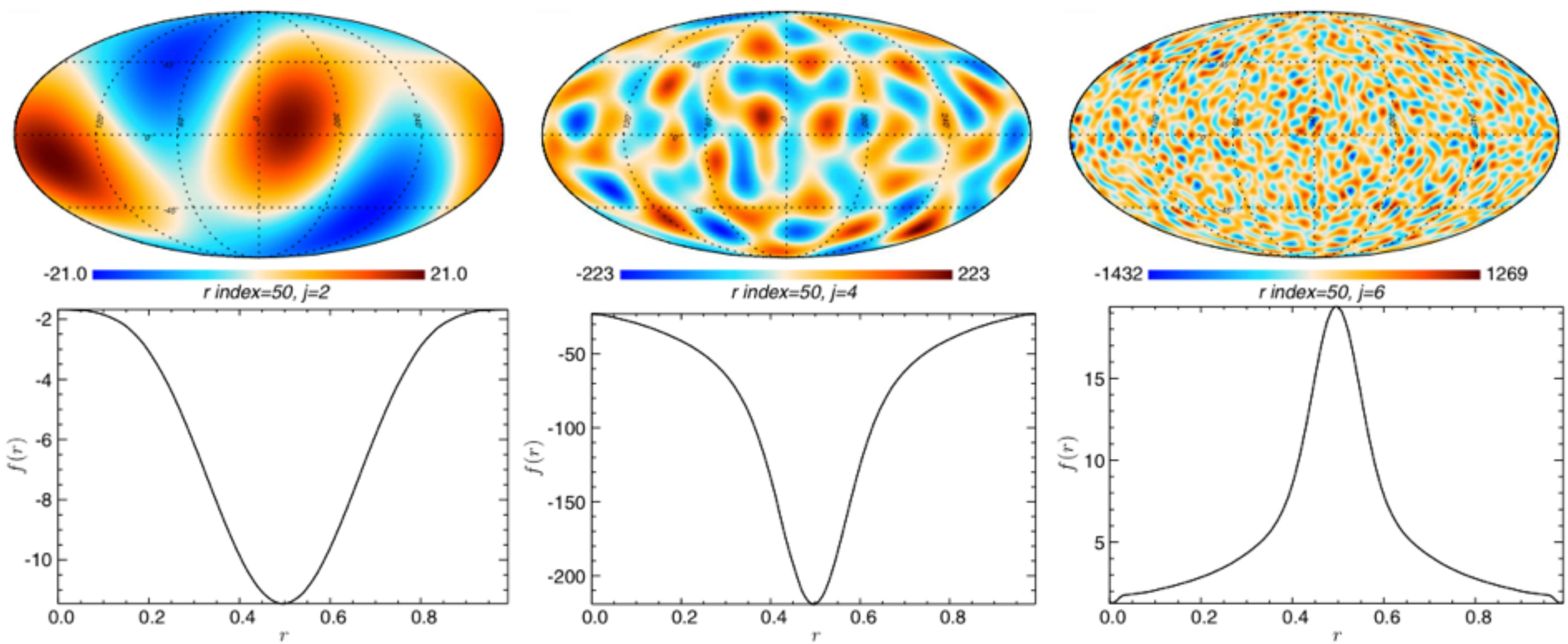


beta2ball: r index=50

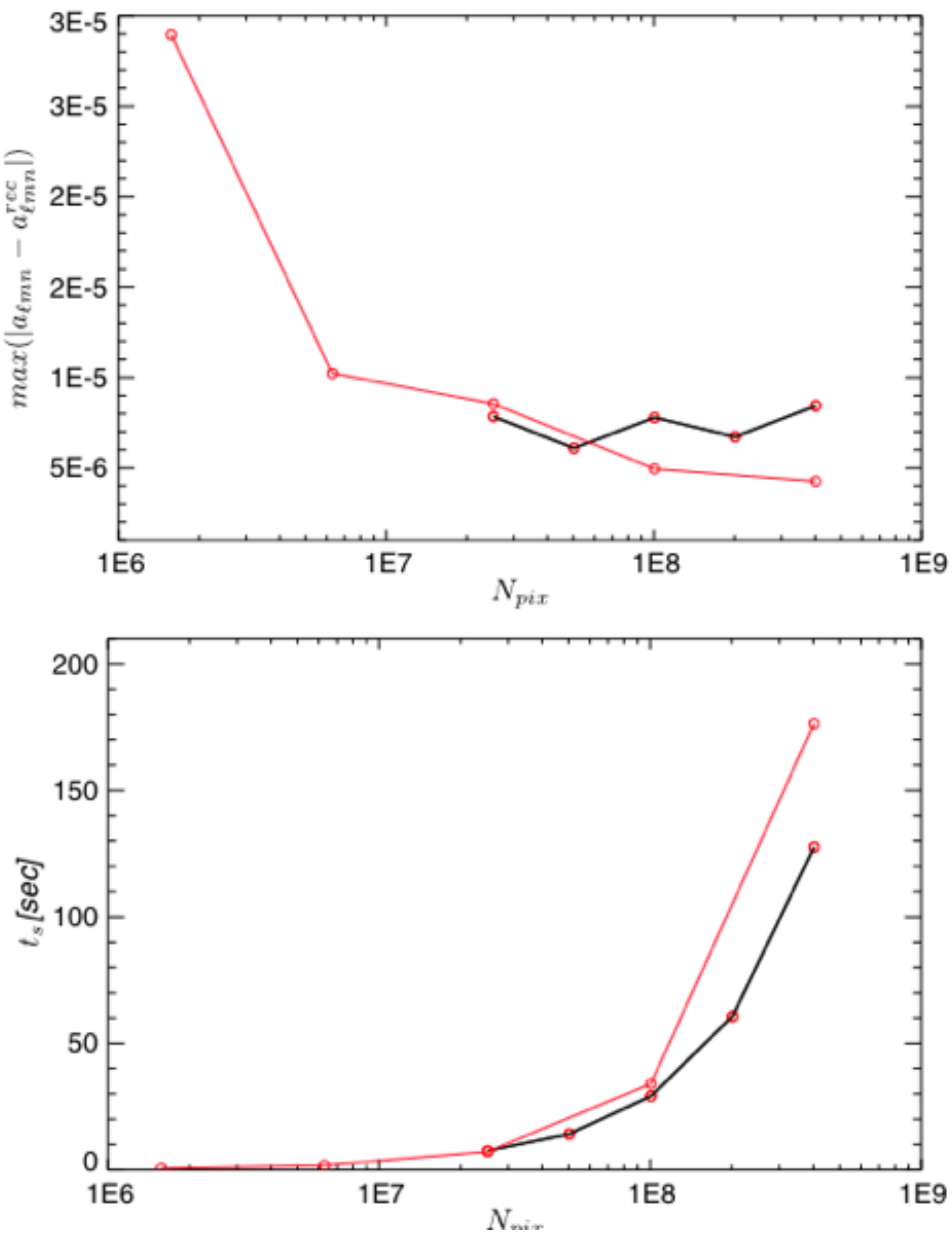


Frac. Diff [%]: r index=50

# 3D Needlet Components



# Speed and Accuracy




- Black curve: fixed Nside=256, and Nr=64,128,256,512,1024
- Red curve: fixed Nr=64, and Nside=64,128,256,512,1024
- Npix = 12Nside^2*Nr

# Conclusion

- 2D spherical wavelets, and especially needlets, have been used extensively in cosmological data analysis.

- We have developed a modular 3D harmonic code which naturally extends the Healpix package for large scale structure data.

- We have introduced a radial 3D needlet formalism that retains all the power and mathematical properties of the 2D needlets. We have implemented this formalism in a fast, accurate and parallelised 3D wavelet code called ***Radial 3D needlet,*** which is publicly available at https://github.com/yabebalFantaye/Radial3Dneedlet

- We believe our tool will give a convenient data analysis framework for LSS experiments such as EUCLID and SKA .

# Needlet window function b(.)

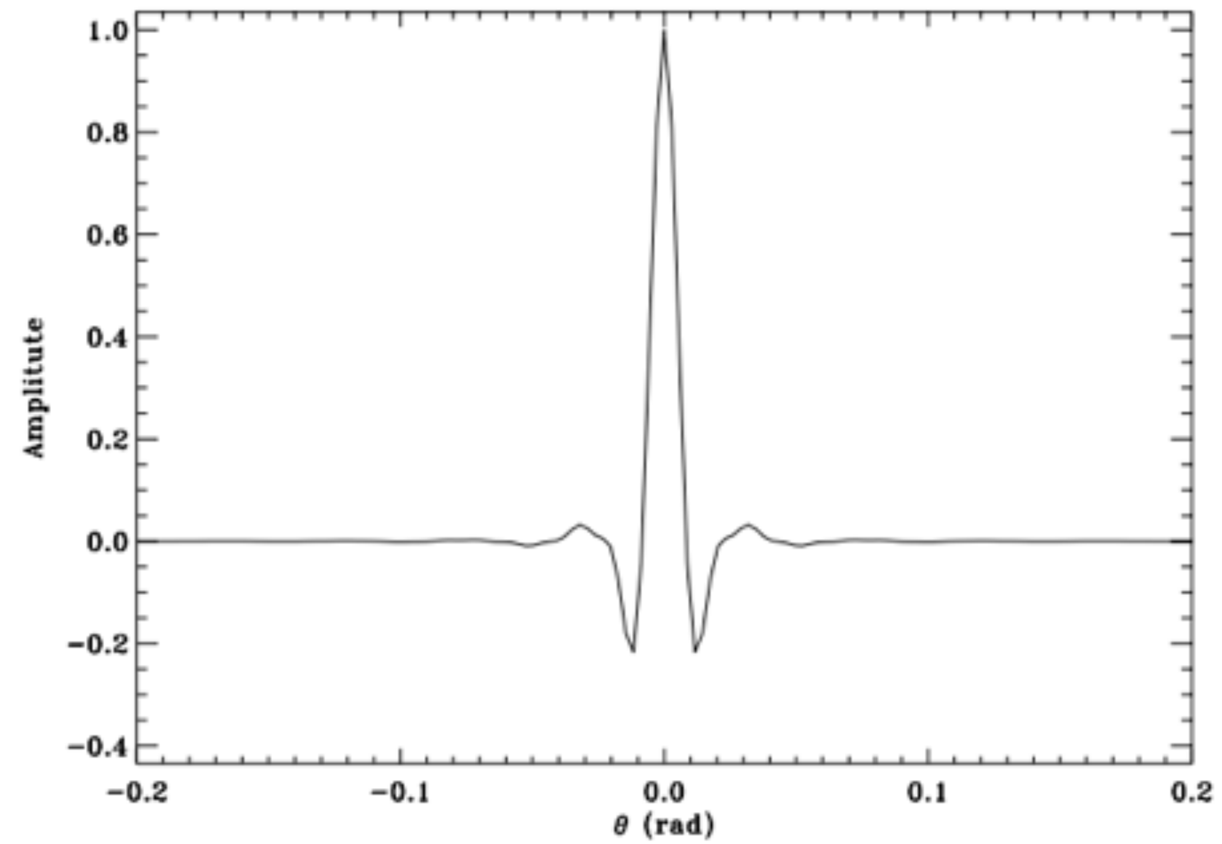
1. $b^2(.)$ has support in $[\frac{1}{B}, B]$, and hence $b(\frac{\ell}{B^j})$ has support in $\ell \in [B^{j-1}, B^{j+1}]$
2. the function $b(.)$ is infinitely differentiable in $(0, \infty)$.
3. we have
$$\sum_{j=1}^{\infty} b^2(\frac{\ell}{B^j}) \equiv 1 \text{ for all } \ell > B.$$

# Localization property

For any M, there exists a constant c_M such that for every $\xi \in \mathbb{S}^2$

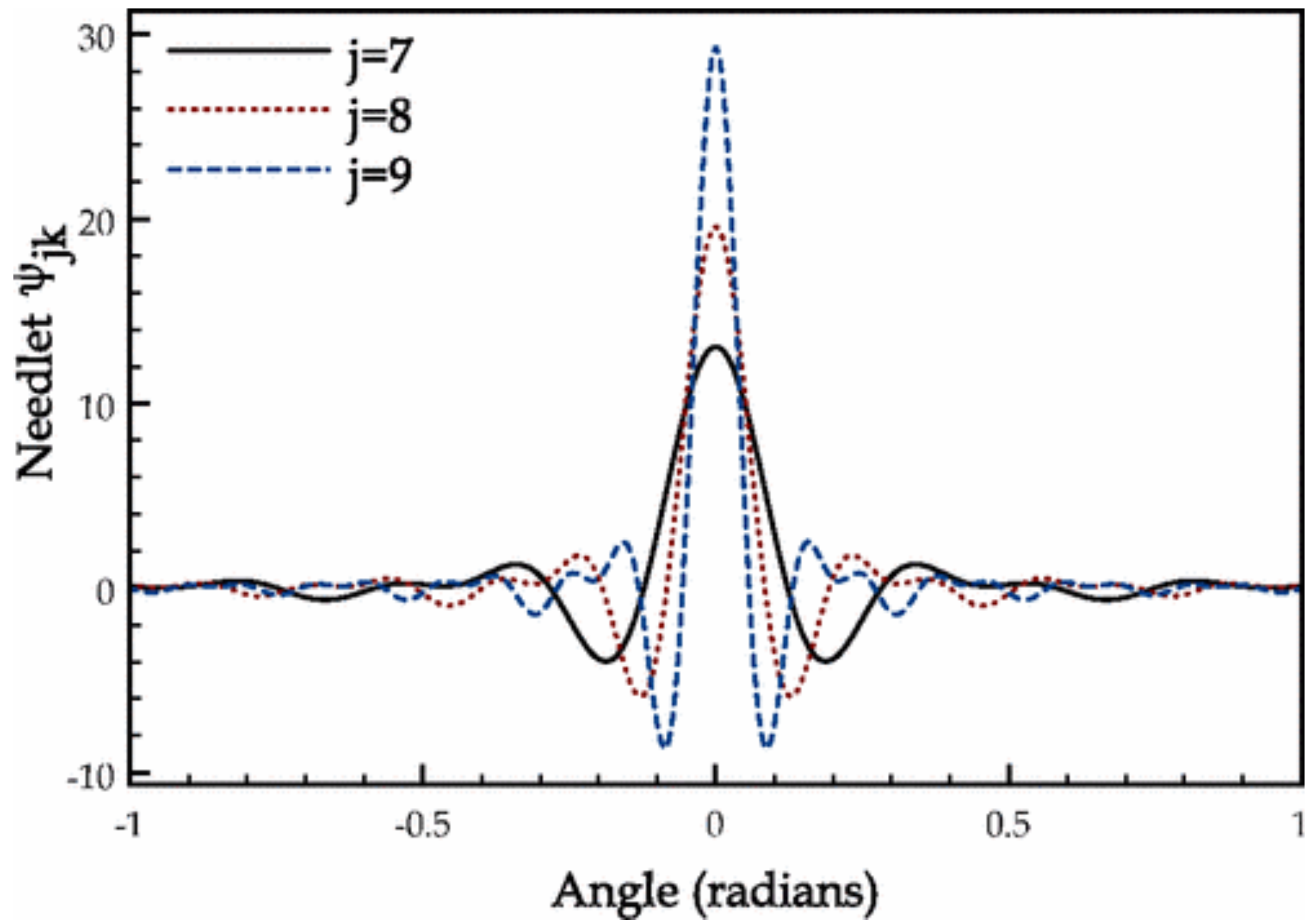
$$|\psi_{jk}(\xi)| \leq \frac{c_M B^j}{(1 + B^j \arccos\langle \xi_{jk}, \xi \rangle)^M}$$

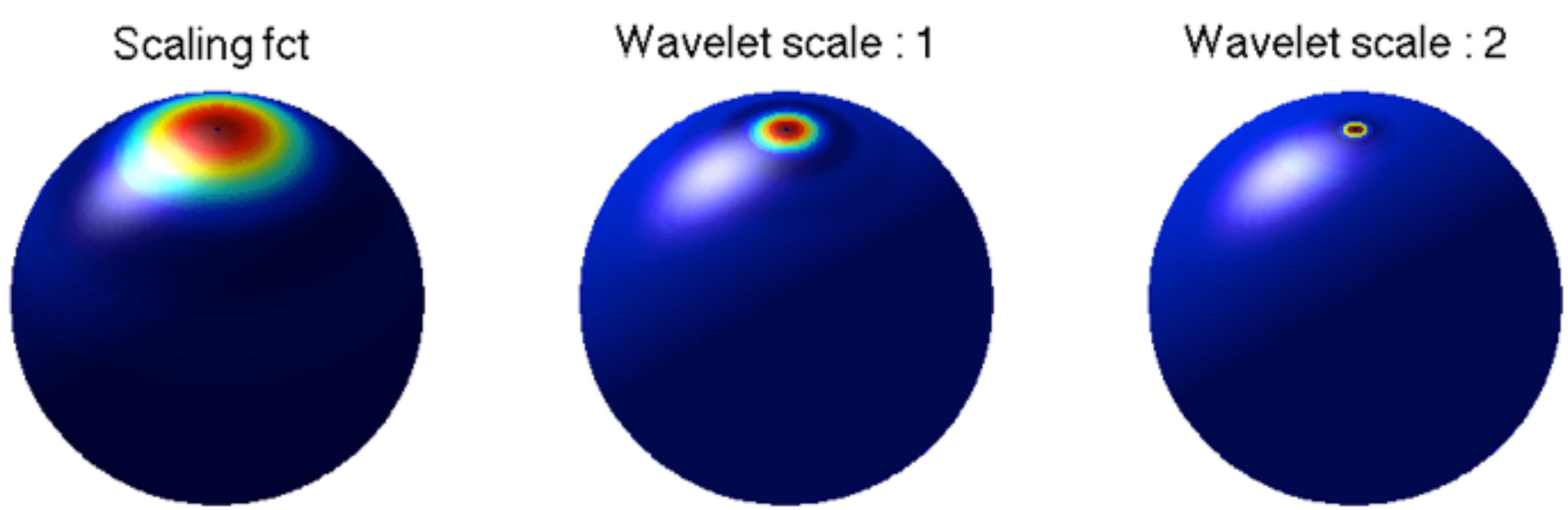
Needlet shape: Quasi-exponential localisation on real space



# Needlets at different scales

*Marinucci et. al. 2007*




http://www.jasonmcewen.org/

# Needlet coefficients

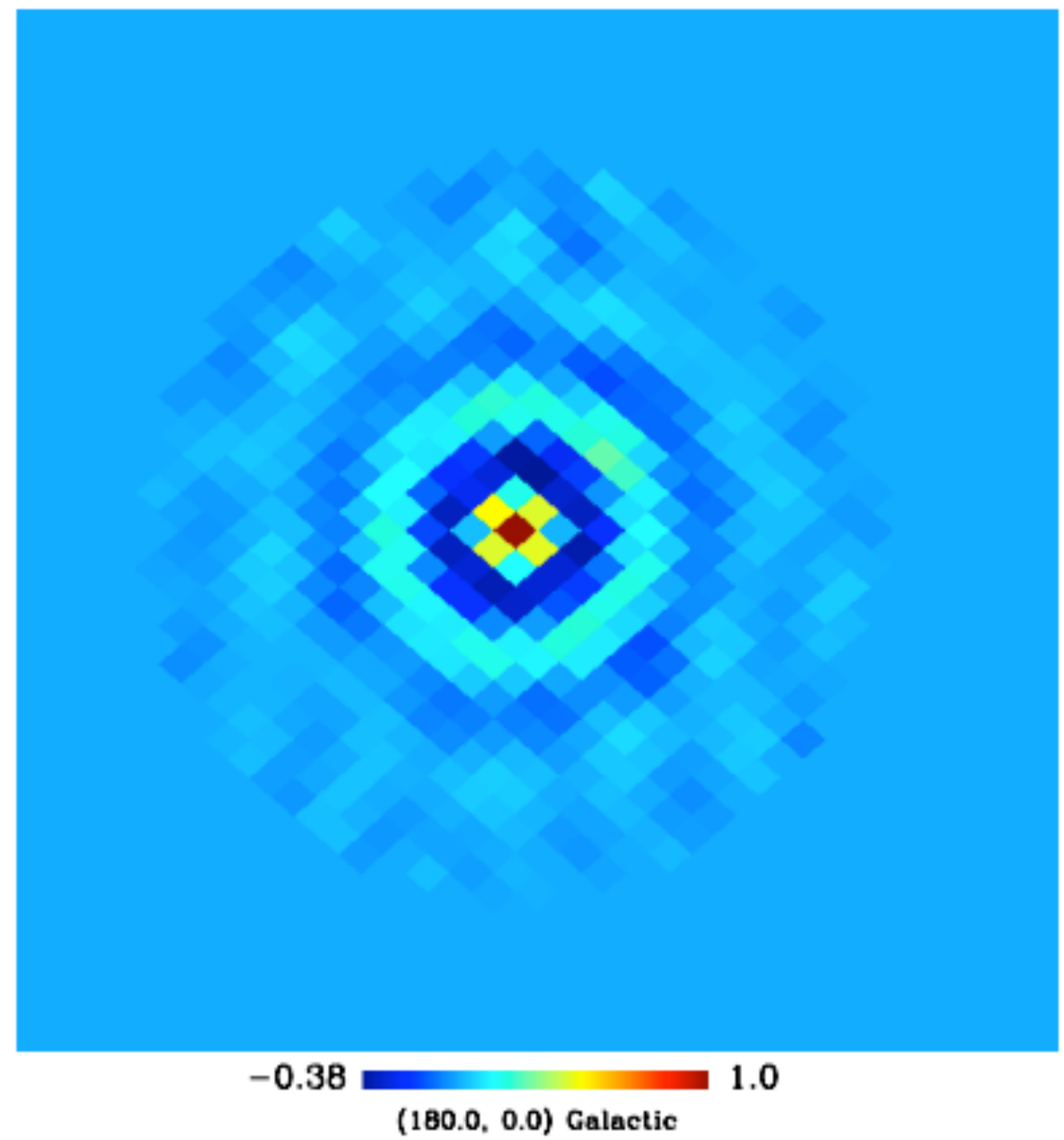
$$\begin{aligned}
\beta_{jk} &= \int_{S^2} T(\hat{\gamma})\psi_{jk}(\hat{\gamma})d\Omega \\
&= \sqrt{\lambda_{jk}}\sum_{\ell} b(\frac{\ell}{B^j}) \sum_{m=-\ell}^{\ell} \left\{\int_{S^2} T(\hat{\gamma})\overline{Y}_{\ell m}(\hat{\gamma})d\Omega\right\} Y_{\ell m}(\xi_{jk}) \\
&= \sqrt{\lambda_{jk}}\sum_{\ell} b(\frac{\ell}{B^j}) \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\xi_{jk}). \qquad (3)
\end{aligned}$$

# Uncorrelation inequality

The needlet coefficients at any finite distance are uncorrelated

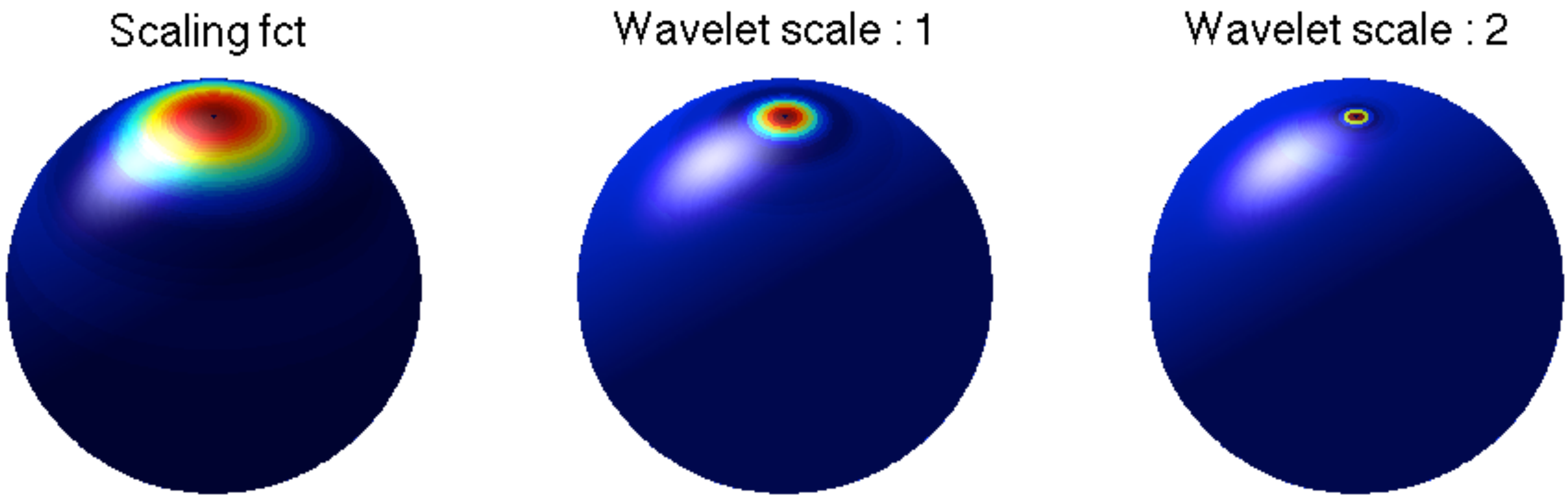
$$\left|Corr(\beta_{j,k}, \beta_{j,k'})\right| \leq \frac{C_M}{(1 + B^j d(\xi_{j,k}, \xi_{j,k'}))^M}$$

where, d(,) is geodesic distance




pixel space

# 2D Wavelets (S2LET)

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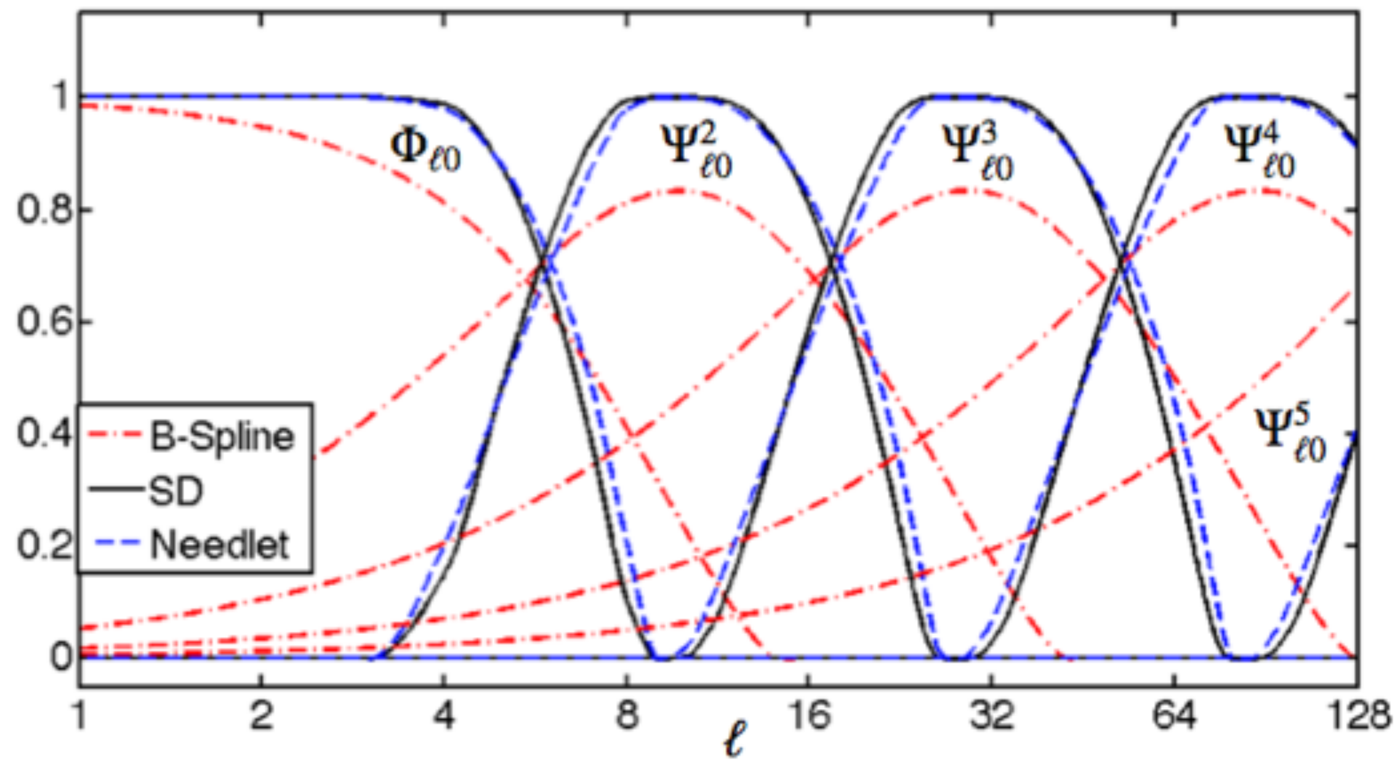
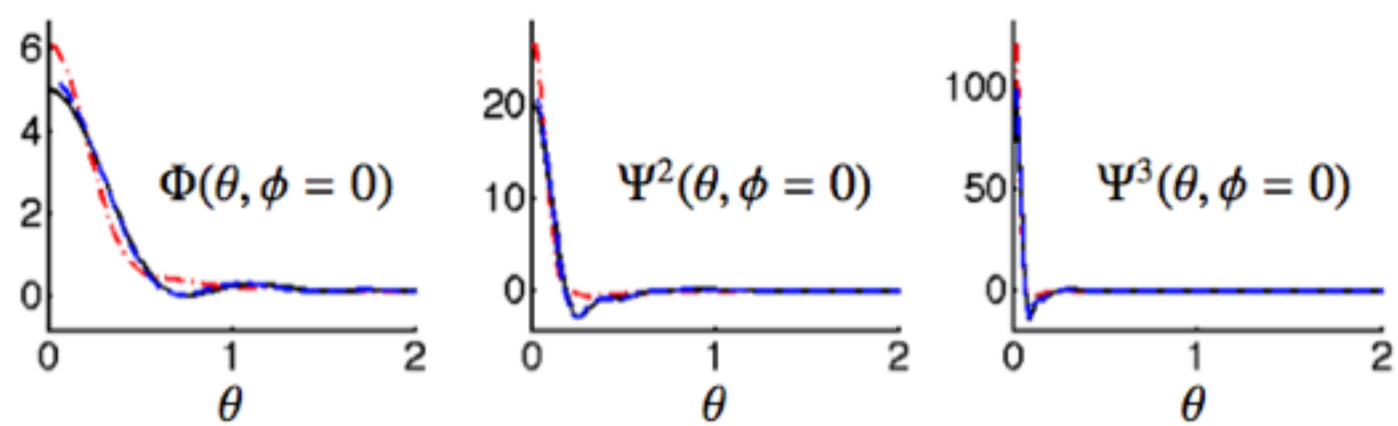
Real space kernels

# Types of Spherical wavelets

- Harmon
- Real spa




(a) Tiling of the harmonic line




(b) Angular profiles of the scaling function and the first wavelets



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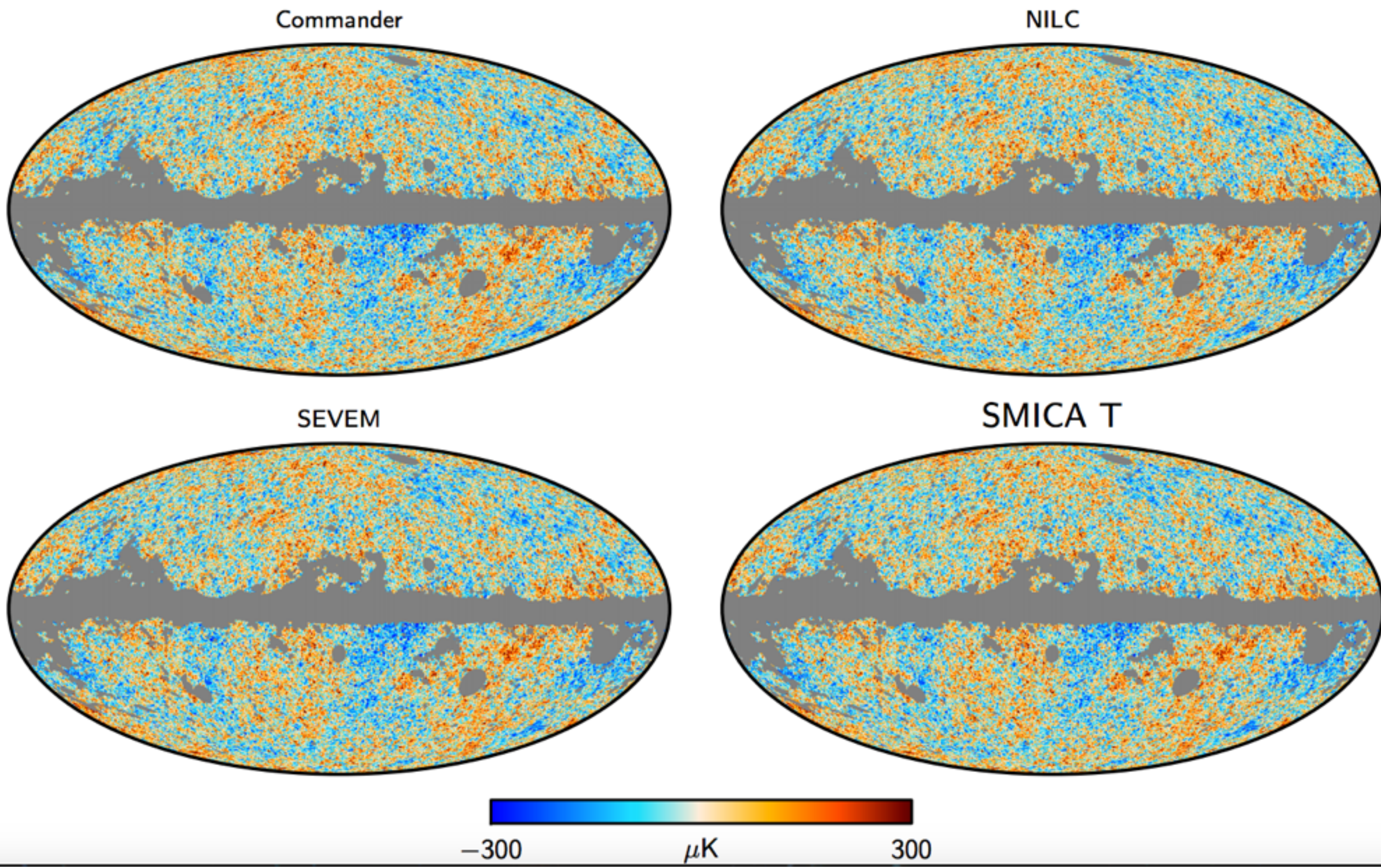
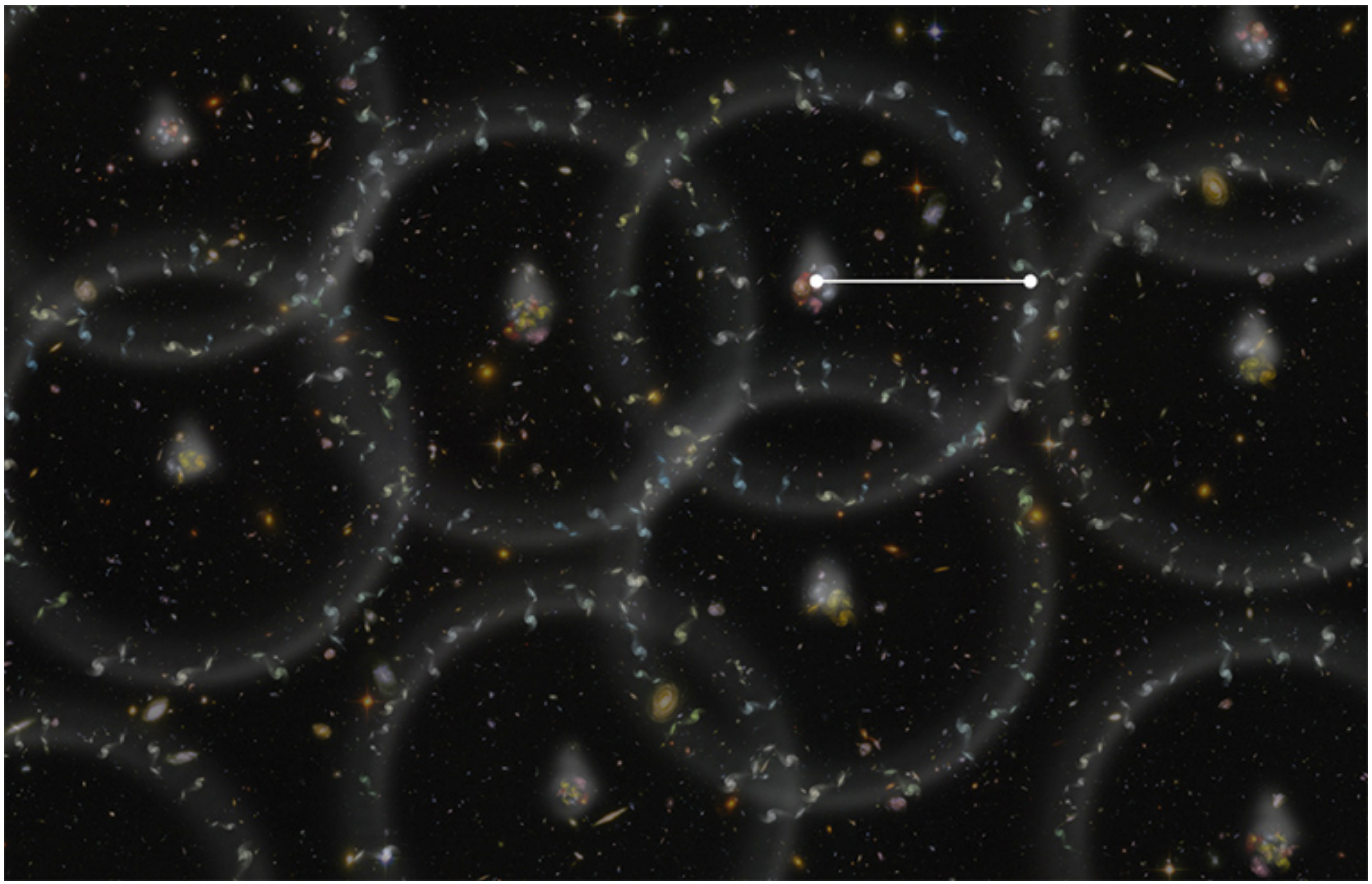
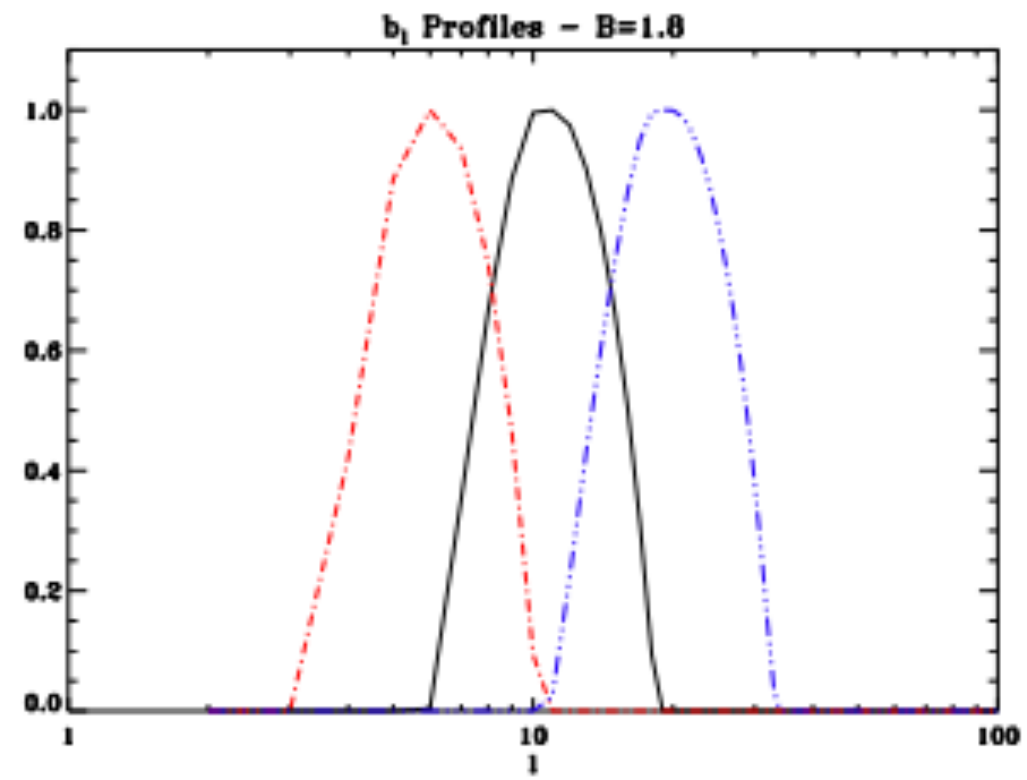
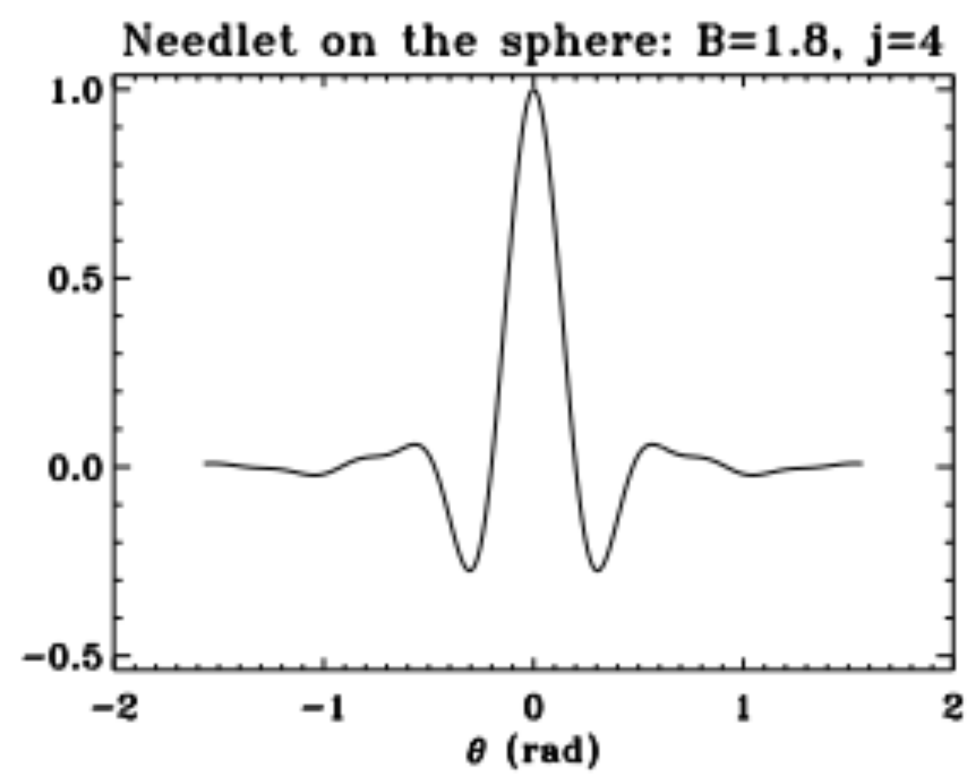
# Gaps on Maps



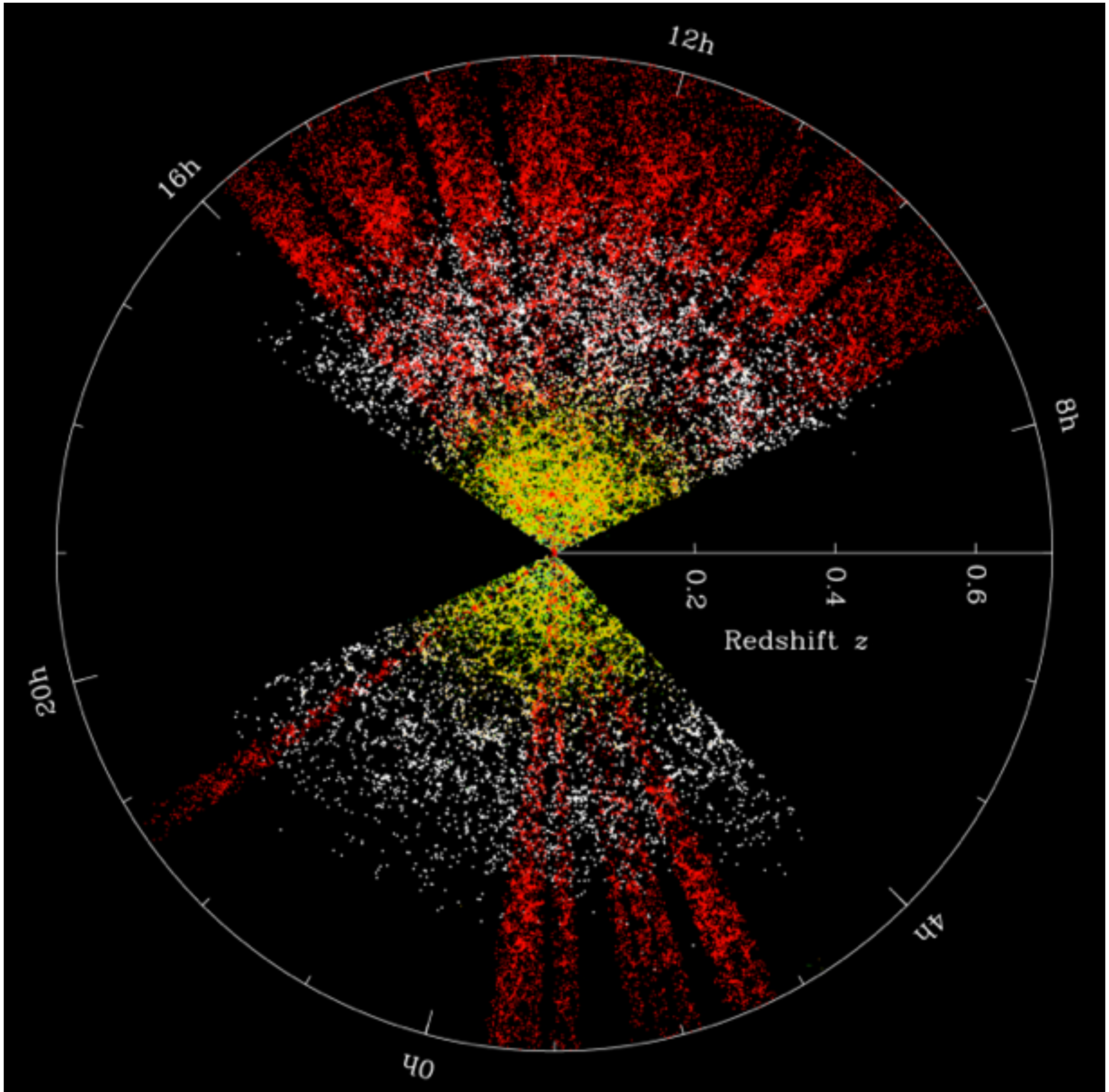


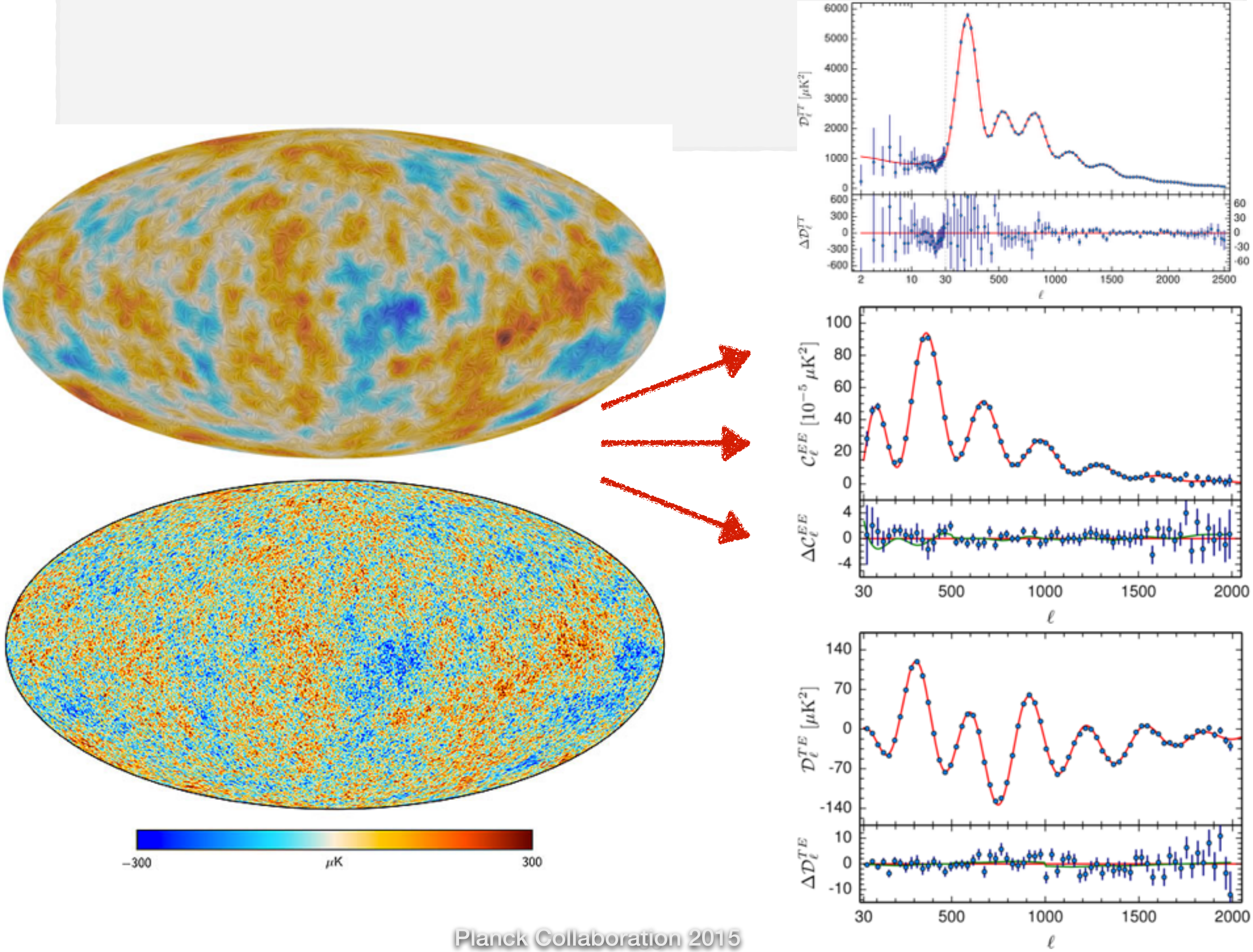


$b_l$ Profiles - B=1.8
1.0
0.8
0.6
0.4
0.2
0.0
1
10
100
l

Needlet on the sphere: B=1.8, j=4
1.0
0.5
0.0
−0.5
−2
−1
0
1
2
θ (rad)


12h
16h
8h
20h
4h
0h
0.2
0.4
0.6
Redshift z




Planck Collaboration 2015