CIS 606 **Homework 4** **Fall 2019**

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1. Exercise 5.2-b (Textbook page 144): Suppose that there is exactly one index i such that $A[i] = x$. What is the expected number of indices into A that we must pick before we find x and RANDOM-SEARCH terminates?

   In each iteration:

   $P(\text{A}[i] = x) = \frac{1}{n}$

   $P(\text{A}[i] \neq x) = \frac{n-1}{n}$

   From induction:

   $i = 1$: Finding x in 1st shot:

   $E(s, i = 1) = \frac{1}{n}$.

   $i = 2$: Finding x in 1st shot + 2nd shot:

   $E(s, i = 2) = (\frac{1}{n}) \cdot (1 - \frac{1}{n}) + (1 - \frac{1}{n}) \cdot (\frac{1}{n}) = \frac{n-1}{n} \cdot \frac{2}{n}$

   $i = 3$: Finding x in 1st shot + 2nd shot + 3rd shot:

   $E(s, i = 3) = (\frac{1}{n}) \cdot (1 - \frac{1}{n}) \cdot (1 - \frac{1}{n}) + (1 - \frac{1}{n}) \cdot (\frac{1}{n}) \cdot (1 - \frac{1}{n}) + (1 - \frac{1}{n}) \cdot (1 - \frac{1}{n}) \cdot (\frac{1}{n}) = (\frac{n-1}{n})^2 \cdot \frac{3}{n}$

   Doing it i ietrations:

   $E(s, i) = \sum_{i=1}^{\infty} (\frac{n-1}{n})^{i-1} \cdot \frac{i}{n} = \frac{1}{n-1} \cdot \sum_{i=1}^{\infty} (\frac{n-1}{n})^i \cdot i = \frac{1}{n-1} \cdot \sum_{i=0}^{\infty} (\frac{n-1}{n})^i \cdot i$

   From CLRS.A.8, page 1148:

   $\sum_{k=0}^{\infty} k \cdot x^k = \frac{x}{(1-x)^2}$, if $|x| < 1$

   Knowing $|\frac{n-1}{n}| < 1$, thus:

   $E(s, i) = \frac{1}{n-1} \cdot \sum_{i=0}^{\infty} (\frac{n-1}{n})^i \cdot i = \frac{1}{n-1} \cdot \frac{\frac{n-1}{n}}{(1 - \frac{n-1}{n})^2} = \frac{1}{n-1} \cdot \frac{\frac{n-1}{n}}{\frac{1}{n^2}} = n$

2. Exercise 5.2-c (Textbook page 144): Generalizing your solution to part (b), suppose that there are $k \geq 1$ indices i such that $A[i] = x$. What is the expected number of indices into A that we must pick before we find x and RANDOM-SEARCH terminates? Your answer should be a function of n and k.

   Generalizing previous example to a ranges of indecies $k > 1$:

   $P(\text{A}[i] = x) = \frac{k}{n}$

   $P(\text{A}[i] \neq x) = \frac{n-k}{n}$

   Now repeating for i ietrations:

   $E(s, i) = \sum_{i=1}^{\infty} (\frac{n-k}{n})^{i-1} \cdot \frac{k}{n} \cdot i = \frac{k}{n-k} \cdot \sum_{i=1}^{\infty} (\frac{n-k}{n})^i \cdot i = \frac{k}{n-k} \cdot \sum_{i=0}^{\infty} (\frac{n-k}{n})^i \cdot i$

   From CLRS.A.8, page 1148:

   $\sum_{k=0}^{\infty} k \cdot x^k = \frac{x}{(1-x)^2}$, if $|x| < 1$

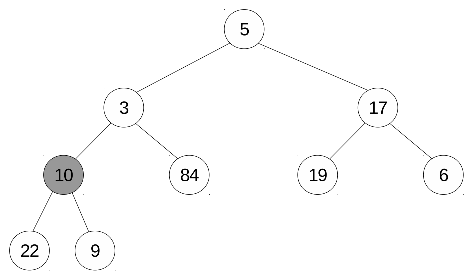
   Knowing $|\frac{n-k}{n}| < 1$, thus:

   $E(s, i) = \frac{k}{n-k} \cdot \sum_{i=0}^{\infty} (\frac{n-k}{n})^i \cdot i = \frac{k}{n-k} \cdot \frac{\frac{n-k}{n}}{(1 - \frac{n-k}{n})^2} = \frac{k}{n-k} \cdot \frac{\frac{n-k}{n}}{\frac{k^2}{n^2}} = \frac{n}{k}$

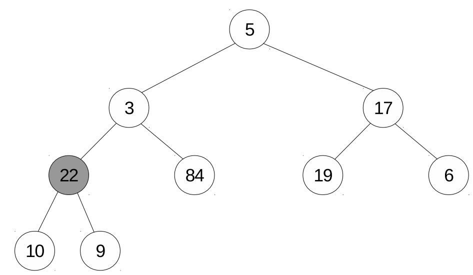
3. Exercise 6.3-1 (Textbook page 159) draw like Figure 6.3: Using Figure 6.3 as a model, illustrate the operation of BUILD-MAX-HEAP on the array $A = \langle 5, 3, 17, 10, 84, 19, 6, 22, 9 \rangle$.

   As heap A has 9 elements then the starting element will be START $= \lfloor \frac{9}{2} \rfloor = 4$

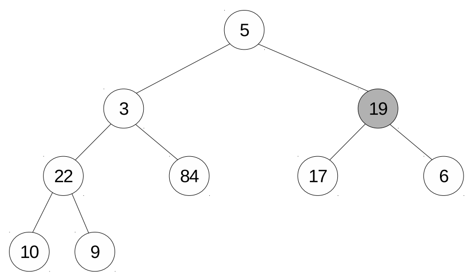
   Figure.1 shows the start point:



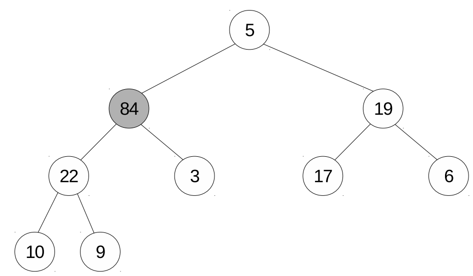
(a) Starting point



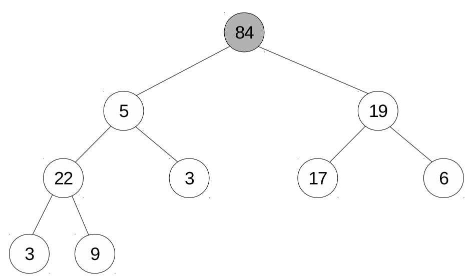
(b) After 1st iteration of BUILD-MAX-HEAP



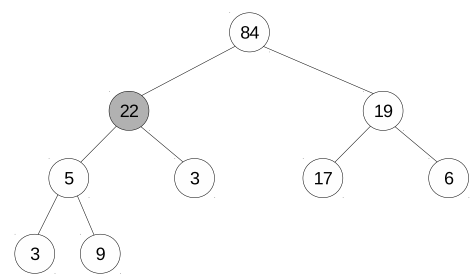
(c) After 2nd iteration of BUILD-MAX-HEAP



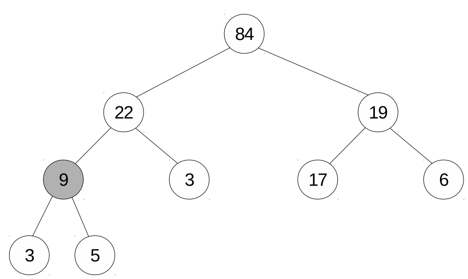
(d) After 3rd iteration of BUILD-MAX-HEAP



(e) After 4th iteration of BUILD-MAX-HEAP



(f) After 5th iteration of BUILD-MAX-HEAP



(g) After 6th iteration of BUILD-MAX-HEAP

Figure 1: Iteration of BUILD-MAX-HEAP

4. Exercise 6.5-9 (Textbook page 166): Give an $O(n \log k)$-time algorithm to merge k sorted lists into one sorted list, where n is the total number of elements in all the input lists. (*Hint*: Use a min-heap for k-way merging.)

   Figure.2 shows the suggested min-heap structure to sort array A, assuming array $A[1...n]$ with n elements is divided into k sorted sub-arrays a_{sorted}:

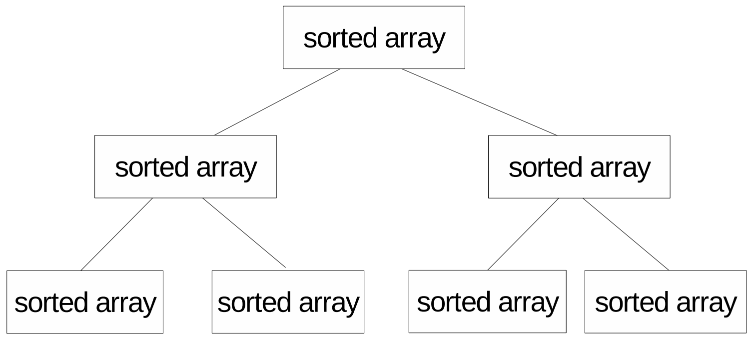


Figure 2: Structure of sorted array in min-heap

   In this structure, each *sorted subarray* is represented by its minimum element. Using the represnted minimum values of each subarray, a heap is generated. In this structure, the minimum element of the root *subarray* is the global minimum in the heap.

   After each MIN-HEAP procedure, minimum value of the *subarray* in the root is removed from heap, and replaced with the next element in the *subarray*. Now heap has to be adjusted with a new value at root (this is similar to inserting a new value to the root and min-heap).

   This algorithm is similar to min heap-sort algorithm. Based on the proof provided in CLRS p.163, time complexity of HEAP-EXTRACT-MAX with k elements is $O(\log k)$. Without loss of generality, time complexity of the HEAP-EXTRACT-MIN with k elements is $O(\log k)$ too. Every time HEAP-EXTRACT-MIN is implemented, only one element at the root *subarray* is removed. Thus HEAP-EXTRACT-MIN has to get executed n times. Therefore time complexity is $O(n \log k)$