PC351: Quantum Computing

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1 Introduction

- *Quantum information science* is the study of how laws of quantum mechanics can be used to perform processing tasks
- They will be able to solve or make efficient tasks such as optimizaation, NP-complete problems

1.1 Classical computation

In classical computing, a bit is a binary digit, represented as 0 or 1; and it is in only one state at any given time. In quantum computing, we need to represent that bits with vectors for their possible states. If we were to describe a *qubit* on a regular axis, it would be represented as:

$$\begin{aligned} |0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

and on a basis rotated 45°:

$$\begin{aligned} |0\rangle &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ |1\rangle &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

We use something known as *gates* to perform operations on the state vectors. The identity gate (I) works as such:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\Rightarrow I|0\rangle \Rightarrow |0\rangle \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\Rightarrow I|1\rangle \Rightarrow |1\rangle \end{aligned}$$

There is also the bit flip (not) gate:

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} &\Rightarrow X|0\rangle \Rightarrow |1\rangle \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\Rightarrow X|1\rangle \Rightarrow |0\rangle \end{aligned}$$

Most of the time we do no know the exact state of a qubit, so we represent them with probabilities:

$$\begin{aligned} |v\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \alpha|0\rangle + \beta|1\rangle \end{aligned}$$

And the probability that the qubit is 0 or 1 is $\alpha+\beta=1$.

Operators can be more general operations between any vector between the x and y axes.

Example 1. Hadamard gate, H:

$$\begin{aligned}
H|0\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow H|0\rangle \\
&= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\
H|1\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow H|1\rangle \\
&= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle
\end{aligned}$$

The gates can also be applied using all Dirac notation, which is a standard notation for describing quantum states. Here the action of a Hadamard gate being applied:

$$\begin{aligned}
H(\alpha|0\rangle + \beta|1\rangle) &= \alpha H|0\rangle + \beta H|1\rangle \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle + \alpha|1\rangle + \beta|0\rangle - \beta|1\rangle) \\
&= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle \\
&= \frac{1}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle
\end{aligned}$$

since $\alpha+\beta=1$.

The idea to find these probabilities is to find the measurement matrix such that:

$$\begin{aligned}
\begin{pmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\
&= M_0|v\rangle \\
M_1|v\rangle &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

1.2 Quantum Bits

A *quantum bit*, mostly referenced to as, a **qubit** can be represented with the general form:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$. Remember that $|\alpha|^2 + |\beta|^2 = 1$; where $P(0) = |\alpha|^2$ and $P(1) = |\beta|^2$.

2 Quantum Model

Let's talk about how we define a qubit state: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. We have to remember that α and β are complex numbers; so in the quantum case, we say that they are probability amplitudes. This isn't so bad, we can observe them!

The inability to directly observe the state is called a superposition. This sort of details that we are in a state that is a combination of both. (Ex. a qubit is as a state of 0,1 at the same time). We would say there is a superposition at $|0\rangle$ and $|1\rangle$.

There exists a *normalization condition* that says: $P(0)+P(1)=1$ and also $|\alpha|^2+|\beta|^2=1$.

Example 2. $|\psi\rangle=(1+i)|0\rangle+2|1\rangle$

$$\begin{aligned}
\alpha &= 1+i \\
|\alpha|^2 &= \alpha^*\alpha \\
&= (1-i)(1+i) \\
&= 1-i^2 \\
&= 2 \\
\beta &= 2 \\
|\beta|^2 &= \beta^*\beta \\
&= (2)(2) \\
&= 4
\end{aligned}$$

Based on the above, we can see that this state is not valid, as it does not pass the normalization condition: $|\alpha|^2+|\beta|^2=6\neq 1$.

Our condition is now $N=|\alpha|^2+|\beta|^2=6$. Let's define a new, normalized state:

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{N}}|\psi\rangle \\
&= \frac{1}{\sqrt{6}}[(1+i)|0\rangle+2|1\rangle]
\end{aligned}$$

We now have $\alpha=\frac{1+i}{\sqrt{6}}$ and $\beta=\frac{2}{\sqrt{6}}$ and the associated $|\alpha|^2=\frac{2}{6}$ and $|\beta|^2=\frac{4}{6}$. We have now just forced $|\alpha|^2+|\beta|^2=1$.

2.1 Complex numbers

First, let's define $|\alpha|=$ the magnitude and $i\,\theta=$ the phase. We can say that:

$$\begin{aligned}
\alpha &= \alpha_R+\alpha_{\text{Im}} \\
&= |\alpha|e^{i\theta}
\end{aligned}$$

Note: $e^{i\theta}=\cos(\theta)+i\sin(\theta)$; $|\alpha|\cos(\theta)=\alpha_R$; $|\alpha|\sin(\theta)=\alpha_{\text{Im}}$

We have:

$$\begin{aligned}
|\psi\rangle &= \alpha|0\rangle+\beta|1\rangle \\
&= |\alpha|e^{i\theta_\alpha}|0\rangle+|\beta|e^{i\theta_\beta}|1\rangle \\
&= e^{i\theta_\alpha}[|\alpha||0\rangle+|\beta|e^{i(\theta_\beta-\theta_\alpha)}|1\rangle]
\end{aligned}$$

We can then define: $e^{i(\theta_\beta-\theta_\alpha)}=e^{i\varphi}$. $|a|=\cos\left(\frac{\theta}{2}\right)$ and $|\beta|=\sin\left(\frac{\theta}{2}\right)$ and thus we have the identity:

$$|\alpha|^2+|\beta^2| = \cos^2\left(\frac{\theta}{2}\right)+\sin^2\left(\frac{\theta}{2}\right)=1$$

$$|\psi\rangle = e^{i\theta_\alpha}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle+\sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle\right]$$

at this point, we can drop the overall phase factor $e^{i\theta_\alpha}$.

$$\begin{aligned}P(0) &= \alpha^*\alpha\\ &= \left(e^{-i\theta_\alpha}\cos\left(\frac{\theta}{2}\right)\right)\left(e^{i\theta_\alpha}\cos\left(\frac{\theta}{2}\right)\right)\\ &= \cos^2\left(\frac{\theta}{2}\right)\\ P(1) &= \beta^*\beta\\ &= \left(e^{-i\theta_\alpha}\sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\right)\left(e^{i\theta_\alpha}\sin\left(\frac{\theta}{2}\right)e^{i\varphi}\right)\\ &= \sin^2\left(\frac{\theta}{2}\right)\end{aligned}$$

The standard qubit state is $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$. We say that this is the standard form of a qubit on the Bloch sphere.

All qubits can be represented as a vector of length 1. The surface of the Bloch sphere of all qubits is the space in which all gates will act.

Simple qubit gates (operators) are rotations that do not change the length of the vector. They for the set of *unitary* operators (matrices). They are defined as $U^\dagger U = 1 = I$.

2.2 Important unitary gates

$$\begin{aligned}X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &\rightarrow \text{bit flip gate}\\ Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} &\rightarrow i\,X\,Z\\ Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} &\rightarrow \text{phase flip gate}\\ I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} &\rightarrow \text{identity gate}\\ X^2, Y^2, Z^2 &= I\end{aligned}$$

$\dagger$ is called "dagger", and it works as such: $U^\dagger = (U^*)^T$.

Example 3. $M = \begin{pmatrix}0 & i\\ 1 & 1\end{pmatrix}$; $M^* = \begin{pmatrix}0 & -i\\ 1 & 1\end{pmatrix}$

$$\begin{aligned}(M^*)^T &= \begin{pmatrix}0 & 1\\ -i & 1\end{pmatrix} = M^\dagger\\ M^\dagger M &= \begin{pmatrix}0 & 1\\ -i & 1\end{pmatrix}\begin{pmatrix}0 & i\\ 1 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} \neq I\end{aligned}$$

Let's show a general U, such that:

$$U = \begin{pmatrix}U_{00} & U_{01}\\ U_{10} & U_{11}\end{pmatrix}$$

where $U^\dagger U = I$.

$$U = e^{i\delta}\left[\cos\left(\frac{\gamma}{2}\right)1 + i\sin\left(\frac{\gamma}{2}\right)(n_x X + n_y Y + n_z Z)\right]$$

This can be written $\forall$ gates of U.

Example 4. $|\psi\rangle = |0\rangle - i\,|1\rangle$

We have $|\alpha| = 1$, $|\beta| = (i)\,(-i) = 1$. Thus, $|\alpha|^2 + |\beta|^2 \neq 1$.

$$\begin{aligned} |\psi_N\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) \\ &= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle \end{aligned}$$

And the standard state of a qubit is $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + i\sin\left(\frac{\theta}{2}\right)|1\rangle$.

We can see:

$$\begin{aligned} \frac{\theta}{2} &= \frac{\pi}{4} \\ \theta &= \frac{\theta}{2} \end{aligned}$$

Next, solve for φ.

$$\begin{aligned} \varphi &= \theta_\beta - \theta_\alpha \\ \alpha &= |\alpha|\,e^{i\theta_\alpha} \\ &= \frac{1}{\sqrt{2}}\,e^{i0} \\ \theta_\alpha &= 0 \\ \beta &= \frac{-i}{\sqrt{2}} \\ &= |\beta|\,e^{i\theta_\beta} \\ |\beta| &= \sqrt{\beta^*\,\beta} \\ &= \sqrt{\left(\frac{i}{\sqrt{2}}\right)\left(\frac{-i}{\sqrt{2}}\right)} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

We must remember the ever so useful form: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$\begin{aligned} |\beta|\,e^{i\theta_\beta} &= |\beta|\cos(\theta) + i\,|\beta|\sin(\theta) \\ |\beta|\cos(\theta) &= 0 \\ |\beta|\sin(\theta) &= \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} &= \frac{1}{2}\sin(\theta_\beta) \\ \therefore \sin(\theta_\beta) &= -1 \\ \theta_\beta &= \frac{3\,\pi}{2} \end{aligned}$$

Thus, the direction is $\theta = \frac{\pi}{2}$ and $\varphi = \frac{3\,\pi}{2}$.

Gates on single qubits are:

1. Reversible rotations on the Bloch sphere.

2. 2×2 reversible rotation matrices; they have to be <u>unitary</u> ($U^\dagger U = 1$)

Any unitary matrix can be written as:

$$\begin{aligned} U &= \cos\left(\frac{\gamma}{2}\right) I + i\sin\left(\frac{\gamma}{2}\right) [n_x X + n_y Y + n_z Z] \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \cos\left(\frac{\gamma}{2}\right) I + i\sin\left(\frac{\gamma}{2}\right) \left[\begin{pmatrix} 0 & n_x \\ n_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\,n_y \\ i\,n_y & 0 \end{pmatrix} + \begin{pmatrix} n_z & 0 \\ 0 & -n_z \end{pmatrix}\right] \end{aligned}$$

Where γ is the *angle* of rotation and $\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ defines the *axis* of rotation.

Example 5. $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} &= \cos\left(\frac{\gamma}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\left(\frac{\gamma}{2}\right) \left[\begin{pmatrix} 0 & n_x \\ n_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\,n_y \\ i\,n_y & 0 \end{pmatrix} + \begin{pmatrix} n_z & 0 \\ 0 & -n_z \end{pmatrix}\right] \\ &= \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) + i\,n_z \sin\left(\frac{\gamma}{2}\right) & i\,n_x \sin\left(\frac{\gamma}{2}\right) + n_y \sin\left(\frac{\gamma}{2}\right) \\ i\,n_x \sin\left(\frac{\gamma}{2}\right) - i\,n_y \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) - i\,n_z \sin\left(\frac{\gamma}{2}\right) \end{pmatrix} \end{aligned}$$

We can then setup equations: 1, 2, 3, 4 respectively

$$\begin{aligned} 0 &= \cos\left(\frac{\gamma}{2}\right) + i\,n_z \sin\left(\frac{\gamma}{2}\right) \\ 1 &= i\,n_x \sin\left(\frac{\gamma}{2}\right) + n_y \sin\left(\frac{\gamma}{2}\right) \\ 1 &= i\,n_x \sin\left(\frac{\gamma}{2}\right) - i\,n_y \sin\left(\frac{\gamma}{2}\right) \\ 0 &= \cos\left(\frac{\gamma}{2}\right) - i\,n_z \sin\left(\frac{\gamma}{2}\right) \end{aligned}$$

This will give us a final answer of $\gamma = \pi$ and $\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

WTF Pt. 1

We know $|\psi\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and that $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ |\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \end{aligned}$$

And also $|\psi\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\begin{aligned} H|1\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ |\psi\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{aligned}$$

Not sure why this was mind-blowing at the time. But we ended with some nice states.

Note: The Hadamard gate always does rotations for 90°.

3 2 qubits, yeeeaaaahh

We will describe them generally as such:

$$\begin{aligned}|\psi_1\rangle &= \alpha_1|0\rangle + \beta_1|1\rangle \\ |\psi_2\rangle &= \alpha_2|0\rangle + \beta_2|1\rangle\end{aligned}$$

Combined description

We can use the *tensor product*:

$$\begin{aligned}|\psi_{12}\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \\ &= (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_2|0\rangle + \beta_2|1\rangle) \\ &= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_2\beta_1|10\rangle + \beta_1\beta_2|11\rangle \\ \therefore |\psi_{12}\rangle &= \begin{pmatrix}\alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \beta_1\alpha_1 \\ \beta_1\beta_2\end{pmatrix}\end{aligned}$$

We can then show the probabilities:

$$\begin{aligned}P_1(0) &= |\alpha_1|^2 \\ P_1(1) &= |\beta_1|^2 \\ P_2(0) &= |\alpha_2|^2 \\ P_2(1) &= |\beta_2|^2\end{aligned}$$

These can then be combined for probabilities of 00, 01, 10, 11.

Definition 6. *Let's define the* ***tensor*** *product ($\otimes$).*

$$\begin{aligned}|0\rangle \otimes |0\rangle &= |00\rangle \\ &= \begin{pmatrix}1 \\ 0\end{pmatrix} \otimes \begin{pmatrix}1 \\ 0\end{pmatrix} \\ &= \begin{pmatrix}1\begin{pmatrix}1 \\ 0\end{pmatrix} \\ 0\begin{pmatrix}1 \\ 0\end{pmatrix}\end{pmatrix} \\ &= \begin{pmatrix}1 \\ 0 \\ 0 \\ 0\end{pmatrix}\end{aligned}$$

In general:

$$\begin{pmatrix}a & b \\ c & d\end{pmatrix} \otimes \begin{pmatrix}e & f \\ g & h\end{pmatrix} = \begin{pmatrix}a\begin{pmatrix}e & f \\ g & h\end{pmatrix} & b\begin{pmatrix}e & f \\ g & h\end{pmatrix} \\ c\begin{pmatrix}e & f \\ g & h\end{pmatrix} & d\begin{pmatrix}e & f \\ g & h\end{pmatrix}\end{pmatrix}$$

Note: *If in the tensor product, the first vector is $|0\,\gamma\rangle$ then the resulting tensor product will look like: $\begin{pmatrix}\gamma \\ 0 \\ 0\end{pmatrix}$. The opposite also goes, $|1\gamma\rangle \rightarrow \begin{pmatrix}0 \\ 0 \\ \gamma\end{pmatrix}$.*

Note: A valid 2-qubit state will look like $|\psi\rangle = \sum_{ij}^{1} \alpha_{ij} |i\,j\rangle$. <u>Constraint</u>: only valid for:

$$\sum_{ij} |\alpha_{ij}|^2 = 1$$

Example 7. $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

$$\begin{aligned}
&= \sum_{ij} \alpha_{ij} |i\,j\rangle \\
\alpha_{00} &= \frac{1}{\sqrt{2}} \\
\alpha_{01} &= 0 \\
\alpha_{10} &= 0 \\
\alpha_{11} &= \frac{1}{\sqrt{2}} \\
\sum_{ij} |\alpha_{ij}|^2 &= \left(\frac{1}{\sqrt{2}}\right)^2 + 0 + 0 + \left(\frac{1}{\sqrt{2}}\right)^2 \\
&= 1
\end{aligned}$$

WTF entangled states: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

1. Individually $P(0) = P(1) = \frac{1}{2}$
2. Jointly, we always have perfect information. If qubit 1 is 0, then $P_2(0) = 1$. If qubit 1 is 1, then $P_2(1) = 1$.

Note: 1-qubit gates are rotations.

Note: If $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ then we are in a tensor product state, thus separable qubits.

If $|\psi_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$ then we are in an entangled state, thus unseparable (if separated, entangled properties are lost).

Example 8. $|\psi_1\rangle = |1\rangle$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$

$$\begin{aligned}
|\psi_{12}\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \\
&= (|1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= (0 + |1\rangle)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}|10\rangle - \frac{i}{\sqrt{2}}|11\rangle \\
&= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

Example 9. $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \therefore |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. This is a *Bell state* which represent the simplest forms of entanglement.

2-qubit gates

Tensor product gates:

$$U_{12} \;=\; U_1 \otimes U_2$$

Example 10. $U_{12} = X_1 Z_2 \Rightarrow X Z$. (Order of the gates is the your you apply them, you apply the first gate to the first qubit and the second gets applied to the second qubit).

Tensor product v-notation

$$\begin{aligned}U_1 &\otimes U_2\\ &= \begin{pmatrix} a & b\\ c & d\end{pmatrix} \otimes \begin{pmatrix} e & f\\ g & h\end{pmatrix}\\ &= \begin{pmatrix} a U_2 & b U_2\\ c U_2 & d U_2\end{pmatrix}\end{aligned}$$

We can then do:

$$\begin{aligned}X_1 &= \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\\ Z_2 &= \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\\ X_1 Z_2 &= \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}\end{aligned}$$

Thus, $X_1 Z_2 |00\rangle = |10\rangle$.

Apparently, the only thing I'll ever need

$$\begin{aligned}X|0\rangle &= |1\rangle\\ X|1\rangle &= |0\rangle\\ Z|0\rangle &= |0\rangle\\ Z|1\rangle &= -|1\rangle\\ Y &= -i X Z\\ Y_1 Z_2 |00\rangle &= X_1 Z_1 Z_2 |00\rangle\\ &= |10\rangle\\ &= -i|10\rangle\end{aligned}$$

Joint/non-separable 2-qubit gates

Let's start with U_{12}:

$$\begin{aligned}U_{12}|00\rangle &= |00\rangle\\ U_{12}|01\rangle &= |10\rangle\\ U_{12}|10\rangle &= |01\rangle\\ U_{12}|11\rangle &= |11\rangle\\ U_{12} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\end{aligned}$$

This is the *swap* gate.

Example 11.

$$\begin{aligned}
U_{12}|00\rangle &= |00\rangle\\
U_{12}|01\rangle &= |01\rangle\\
U_{12}|10\rangle &= |11\rangle\\
U_{12}|11\rangle &= |10\rangle\\
U_{12} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

This is *CNOT* gate. The first qubit is considered the *control* and the second, the *target*. In this example, if the control is 1, then flip the target.

Note: Any 2-qubit gate can be turned into a few single qubit gates and a CNOT.

Example 12. CNOT $H_1\, I_2|00\rangle$

$$\begin{aligned}
&= \mathrm{CNOT}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle\right)\\
&= \mathrm{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)\\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle
\end{aligned}$$

Measurements

Given a standard qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we know the probabilities, $P(0) = |\alpha|^2$ and $P(1) = |\beta|^2$.

If you get the outcome 0, then $|\psi_{\text{after}}\rangle = 0$, and the same for 1. These measurement operations are not reversible.

Given $|\psi\rangle = \left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$. U has $U^\dagger = (U^*)^T$ with $U^\dagger U = 1$.

$$\begin{aligned}
|\psi\rangle^\dagger &= (|\psi\rangle^*)^T\\
&= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\\
|\psi\rangle^\dagger|\psi\rangle &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}\begin{pmatrix} \alpha\\ \beta \end{pmatrix}\\
&= |\alpha|^2 + |\beta|^2 = 1
\end{aligned}$$

We can then define:

$$\begin{aligned}
|0\rangle &= \begin{pmatrix} 1\\ 0 \end{pmatrix}\\
\langle 0| &= \begin{pmatrix} 1 & 0 \end{pmatrix}\\
\langle 1| &= \begin{pmatrix} 0 & 1 \end{pmatrix}\\
\langle 0|\psi\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha\\ \beta \end{pmatrix}\\
\langle 1|\psi\rangle &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha\\ \beta \end{pmatrix}
\end{aligned}$$

We then have:

$$\begin{aligned} |\langle 0|\psi\rangle|^2 &= |\alpha|^2 \\ |\langle 0|\psi\rangle|^2 &= (\langle 0|\psi\rangle)\,(\langle 0|\psi\rangle^*) \\ &= \langle 0|\psi \times \psi|0\rangle \end{aligned}$$

Keep in mind $\langle\psi|0\rangle = \alpha^*$.

$$\begin{aligned} &= \langle\psi|0 \times \psi|0\rangle \\ &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \\ &= |\alpha|^2 \end{aligned}$$

Basically, we've said:

$$\begin{aligned} \langle\psi|0 \times 0|\psi\rangle &= |\alpha|^2 \\ &= P(0) \end{aligned}$$

This is the *projection operator*, or measurement operator.

$$\begin{aligned} |0 \times 0| &= \text{outer product} \\ &= M_0 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \end{aligned}$$

Likewise:

$$\begin{aligned} \langle\psi|1 \times 1|\psi\rangle &= |\beta|^2 \\ &= P(1) \\ |1 \times 1| &= M_1 \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \end{aligned}$$

We also have:

$$\begin{aligned} M_0^\dagger M_0 &\neq 1 \\ M_1^\dagger M_1 &\neq 1 \end{aligned}$$

These operators are non-reversible.

$$\begin{aligned} M_0|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \end{aligned}$$

Note: We need to normalize.

$$\begin{aligned}
|\psi_{\text{after}}\rangle &= \frac{M_0|\psi\rangle}{\sqrt{N}} \\
&= |0\rangle \\
\text{with} & \\
M_0|\psi\rangle &= |0\times 0||\psi\rangle \\
&= \alpha\,|0\rangle
\end{aligned}$$

Given $|\psi_{12}\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$, now we have:

$$\begin{aligned}
P(00) &= |\alpha_{00}|^2 \\
&= \langle\psi|00\times 00|\psi\rangle
\end{aligned}$$

With the general form being:

$$\begin{aligned}
P(i\,j) &= \langle\psi|i\,j\times i\,j|\psi\rangle \\
\text{with} & \\
M_{ij} &= |i\,j\times i\,j| \\
|\psi_{\text{after}}\rangle &= \frac{M_{ij}\,|\psi\rangle}{\sqrt{N}}
\end{aligned}$$

Following, with the probabilities:

$$\begin{aligned}
P_1(0) &= |\alpha_{00}|^2 + |\alpha_{01}|^2 \\
&= \langle\psi|M_{00}|\psi\rangle + \langle\psi|M_{01}|\psi\rangle \\
&= \langle\psi|(M_{00}+M_{01})|\psi\rangle
\end{aligned}$$

If qubit 1 is 0:

$$\begin{aligned}
|\psi_{\text{after}}\rangle &= \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{N}} \\
&= \frac{(M_{00}+M_{01})\,|\psi\rangle}{\sqrt{N}}
\end{aligned}$$

Superdense Coding

Alice wants to send Bob one of four messages: 00, 01, 10, 11.

1. She and Bob share 1 qubit each of an entangled pair
2. If $\alpha = 1$, Alice performs Z on her qubit
3. If $\beta = 1$, Alice performs X on her qubit
4. Alice sends Bob her qubit
5. Bob performs CNOT with Alice's qubit as the control
6. Bob performs H on her qubit
7. Bob measures both qubits, his outcome with be $\alpha\beta$

Suppose Alice wants to send 10. $\alpha=1$, $\beta=0$.

0. We always start with $|00\rangle \rightarrow$ state preparation $\rightarrow$ Ex. $|\psi_1\rangle\,|\psi_2\rangle$. Measure each qubit. If outcome is 0, do nothing. Is outcome is 1, flip (X).

1.

$$\begin{aligned}
\mathrm{CNOT}\,H_1|0_1 0_2\rangle &= \mathrm{CNOT}\left(\frac{1}{\sqrt{2}}\,(|0_1\rangle + |0_2\rangle)\,|0_2\rangle\right) \\
&= \mathrm{CNOT}\,\frac{1}{\sqrt{2}}\,(|0_1 0_2\rangle + |1_1 0_2\rangle) \\
|\psi_{12}\rangle &= \frac{1}{\sqrt{2}}\,(|0_1 0_2\rangle + |1_1 1_2\rangle)
\end{aligned}$$

Note: Alice has qubit 1, Bob has 2.

2.

$$\begin{aligned}
|\psi'\rangle &= Z_1|\psi_{12}\rangle \\
&= \frac{1}{\sqrt{2}}\,(Z_1|0_1 0_2\rangle + Z_1\,|0_1 0_2\rangle) \\
&= \frac{1}{\sqrt{2}}\,(|0_1 0_2\rangle - |1_1 1_2\rangle)
\end{aligned}$$

3. (5.)

$$\begin{aligned}
|\psi''\rangle &= \mathrm{CNOT}_1|\psi'\rangle \\
&= \frac{1}{\sqrt{2}}\,(\mathrm{CNOT}_1|0_1 0_2\rangle - \mathrm{CNOT}|1_1 1_2\rangle) \\
&= \frac{1}{\sqrt{2}}(|0_1 0_2\rangle - |1_1 0_2\rangle)
\end{aligned}$$

4. (6.)

$$\begin{aligned}
|\psi'''\rangle &= H_1|\psi''\rangle \\
&= \frac{1}{\sqrt{2}}\,(H_1|0_1 0_2\rangle - H_1|1_1 0_2\rangle) \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\,(|0_1\rangle + |1_1\rangle)\,|0_2\rangle - \frac{1}{\sqrt{2}}\,(|0_1\rangle - |1_1\rangle)\,|0_2\rangle\right) \\
&= \frac{1}{2}\,(|1_1 0_2\rangle + |1_1 0_2\rangle) \\
&= |1_1 0_2\rangle
\end{aligned}$$

Note: When you measure entanglement, entanglement properties are gone.