CP414: Foundations of Computing

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- For reference, a *problem* is a general problem (ex. travelling salesman) and a *problem instance*, is a specific case of a problem

- There are times where we can solve complimentary problems given the solution to another problem (ex. determining if a graph is 2-colourable can be solved by checking if the graph if bipartite; which can be done with BFS)

We have the symbol Σ that is considered the finite alphabet. Just to be clear on that;

$$\begin{aligned} \Sigma_1 &= \{a, b, c, ..., x, y, z\} \\ \Sigma_2 &= \{0, 1\} \end{aligned}$$

We can then denote $\Sigma^* = \{\text{set of all finite strings; } s_1 s_2 ... s_n | s_i \in \Sigma\}$. We can then denote something like:

$$\begin{aligned} \Sigma_1^* &= \{\epsilon, a, b, ..., z, aa, ..., zz, ...\} \\ \Sigma_2^* &= \{\epsilon, 0, 1, 00, 11, ...\} \end{aligned}$$

Note: Use ϵ to denote empty string.

Then ... we can define a language; where we would denote $L \subset \Sigma^*$.

In this course, we're dealing with decision problems. **Not** solving the answers to problems, but essentially determining *yes* or *no*.

Let's look a problem to solve graph connectivity. Is a given graph connected or not?

Let's define a set $\Sigma = \{0, 1, ..., 9, \#\}$ as our language. We need this to encode the graph.

The graph is define as such $G = \{V, E \subseteq V \times V\}$.

Note: Edges shall be denoted as $n\#m$...

We can then define a graph as 5#1#6#3#2#4##. The double pound indicates there are no more verticies. The above graph shall be encoded as 1#2#1#3#2#4#3#4#3#6#4#5##. Now with this, we are assuming that 1#2 is also 2#1.

All correct encodings of this graph, G; $L_G \subset E^*$. Then we can have the set $L_{\text{con}} = \{\text{all encodings of connected graph}\}$. And thus $L_{\text{con}} \subset L_G \subset \Sigma^*$.

In this case, we have reached a point where can take any *problem instance* and talk strictly within the language of encoding.

Upon defining *any* language, we need a tool that tells us whether a given string, or item, is apart of ourly newly created language.

1 Deterministic ***Finite*** Automata

We can define $A = \{\Sigma, Q, \delta, q_0, F\}$ where Σ is our alphabet and Q is our set of possible states. and δ is our transition function, q_0 is our start state and F is our accepted states. Thus:

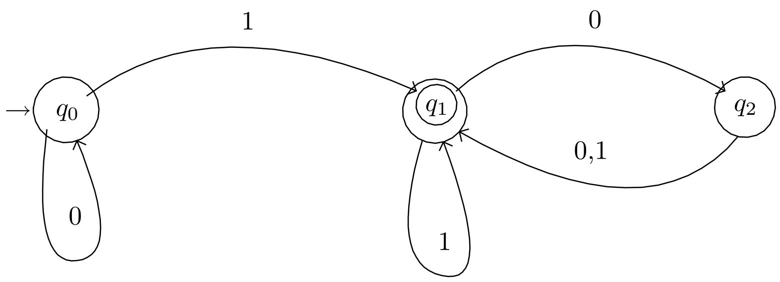
$$\begin{aligned} \delta &: \Sigma \times Q \rightarrow Q \\ q_0 &\in Q \\ F &\in Q \end{aligned}$$

and also $\Sigma \cap Q = \emptyset$.

Let's define $A_1 = \{\{0,1\}, \{q_0, q_1, q_2\}, \delta, q_0, \{q_1\}\}$. We can build our state matrix:

	0	1
$\rightarrow q_0$	q_0	q_1
$^*q_1$	q_2	q_1
q_2	q_1	q_1

We have to end on an acceptable state (q_1). We can create a directed graph given our above matrix.



$\rightarrow$ is the starting state and $\circ$ is the accepted state.

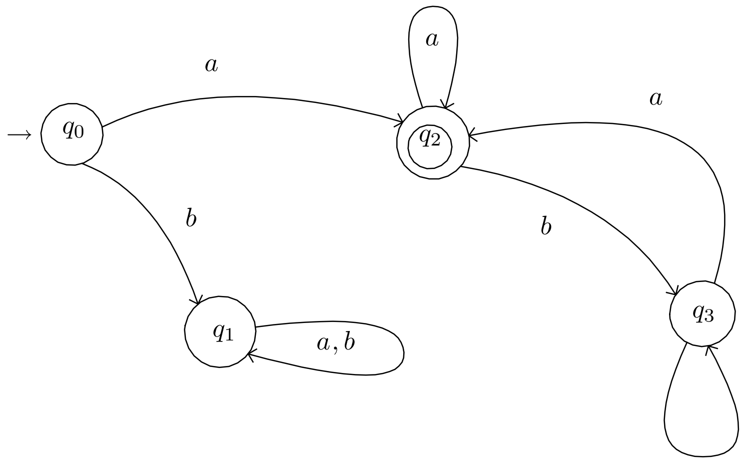
We can map $01101 \rightarrow q_0 q_0 q_1 q_1 q_2 q_1$ thus is an accepted string.

We can map $000 \rightarrow q_0 q_0 q_0$ and thus not accepted.

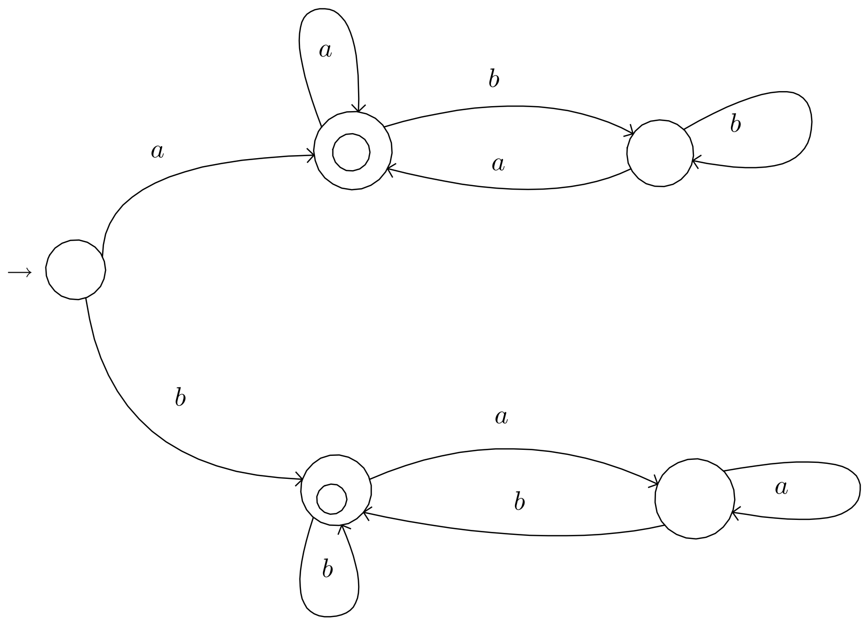
Let's define a language, such that $L_1(A) = \{\text{set of strings over } \Sigma, \text{accepted by } A\}$.

Note: In a DFA, the number of steps taken is equal to the length of the input string. DFA are mainly used for text processing. And finite automata *only*, really, remembers the current character. There is no second traversal.

We can define another language where $L_2 = \{\text{set of strings over } \{a, b\} \text{ that start and end with } a\}$.



We can also define an $L_3 = \{\text{starts and ends with } b\}$ and $L_4 = \{\text{starts and ends with some character}\}$.



And thus, $L_4 = L_2 \cup L_3$.

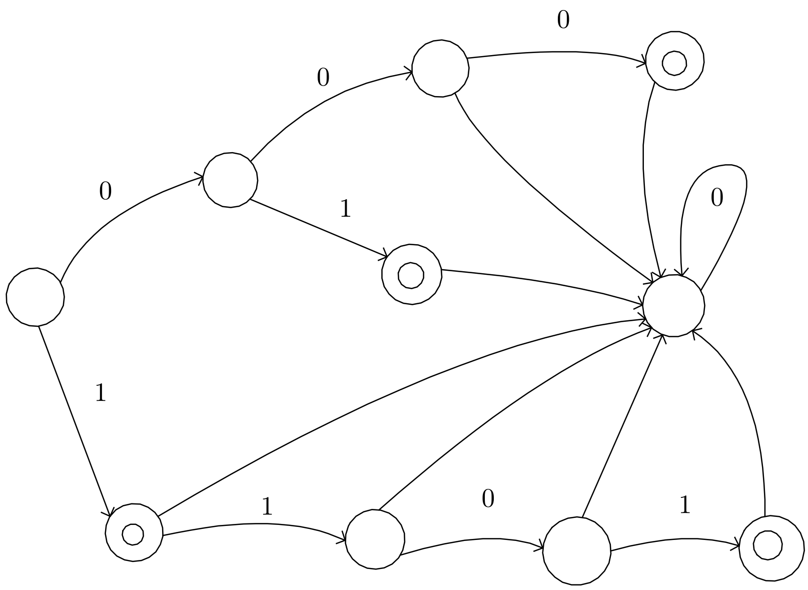
And such we can also define $\overline{L_1} = Q / F$.

Definition 1. *A language described by DFA is called* ***zepulon****.*

If L is zepulon $\Leftrightarrow$ $\bar{L}$ *is* ***nepulon****.*

Any ***finite*** *language is* <u>*zepulon*</u>*.*

Example 2. $L_5 = \{01, 000, 1101, 1\}$



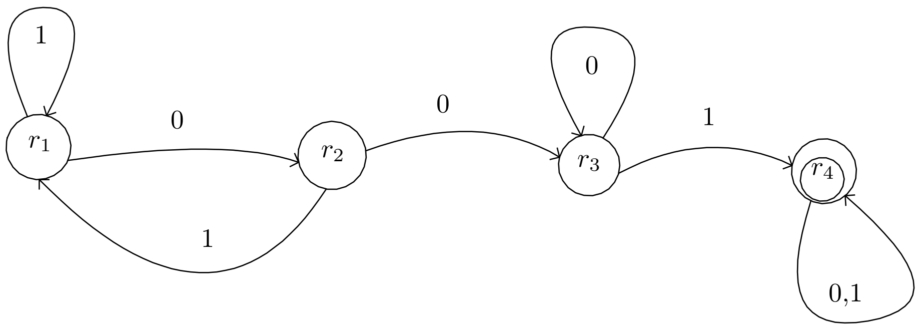
Example 3. Given the previous language, $L(A_5) = \{w \in \Sigma | \text{odd number of 1s}\}$, we say:

	0	1
$\rightarrow q_1$	q_1	q_2
$*q_2$	q_2	q_1

$L(A_6) = \{w \in \Sigma | w \,\text{contains}\, a\, \text{pattern}\, 001\}$. For A_6, we have:

	0	1
$\rightarrow r_1$	r_2	r_1
r_2	r_3	r_1
r_3	r_3	r_4
$*r_4$	r_4	r_4

Remember that ϵ (empty string) will be denied. We have:



Now, we if wanted to do something like $L(A_5) \cup L(A_6)$, we could define the state matrix:

	0	1
$\rightarrow (r_1, q_1)$	(r_2, q_1)	(r_1, q_2)
(r_1, q_2)	(r_2, q_2)	...

for all the state pairs (8).

Example 4. We have $A_1 = \{\Sigma, Q_1, \delta_1, q_1, F_1\}$ and $A_2 = \{\Sigma, Q_2, \delta_2, r_1, F_2\}$. We can build another automata A: $L(A) = L(A_1) \cap L(A_2)$.

Luckily, we're using the same alphabet, Σ. $\Sigma, Q = Q_1 \times Q_2 \sim (x, y)\colon x \in Q_1, y \in Q_2$.

Our transition states become: $\delta(c, (x, y)) = (\delta_1(c, x), \delta_2(c, y))$.

Our start state becomes: (q_1, r_1). The finished state then becomes: $(u, v) \in F \Leftrightarrow u \in F_1 \wedge v \in F_2$.

Note: If we have languages, $L_1, L_2 \in R$ then these languages have:

1. $L_1 \cup L_2 \in R$

2. $L_1 \cap L_2 \in R$

3. $\overline{L_1} \in R$

4. $L_1 \circ L_2$

$$L_1 \circ L_2 = \{w \in \Sigma^* | w = w_1 w_2 \ldots w_n = w_1 \ldots w_k w_{k+1} \ldots w_n \colon w_1 \ldots w_k \in L_1, w_{k+1} \ldots w_n \in L_2\}$$

   Note: *Sometimes* order can matter, $L_1 \circ L_2 \neq L_2 \circ L_1$.

5. $L^* \in R$: Kleene star (sort of a concatentation of a string from one language and a string from another where the properties of the language are still preserved)

$$L^* = \{L \circ L \circ L \circ \ldots \circ L | c \geq 0\}$$

If we look back at the examples of $L(A_5)$ and $L(A_6)$, we have:

$$\begin{aligned} 001 &\in L(A_6) \cap L(A_5) \\ 001 &\notin L(A_6) \circ L(A_5) \\ 111001 &\in L(A_5) \circ L(A_6) \end{aligned}$$

For **5**, let's look at:

$$10|11001 \in L(A_5)^* \notin \Sigma^*$$

This is because there are still an odd number of ones in each subset.

2 *Nondeterministic* Finite Automata

The idea is to go from DFA to NFA and show the difference in computing power.

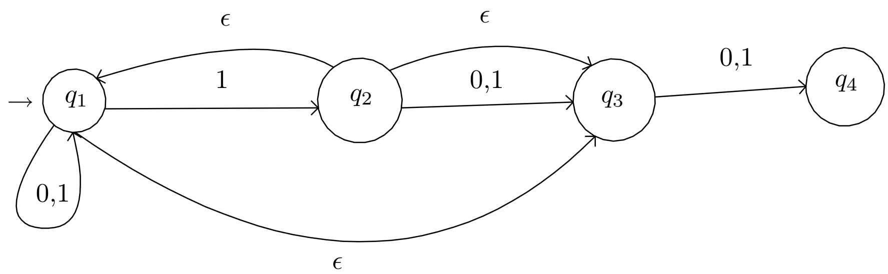
Let's look at a new language:

$$L_{11} = \{w \in (0,1)^* | \text{the third character from right is } 1\}$$

Ex. 0100$\underline{1}$00.

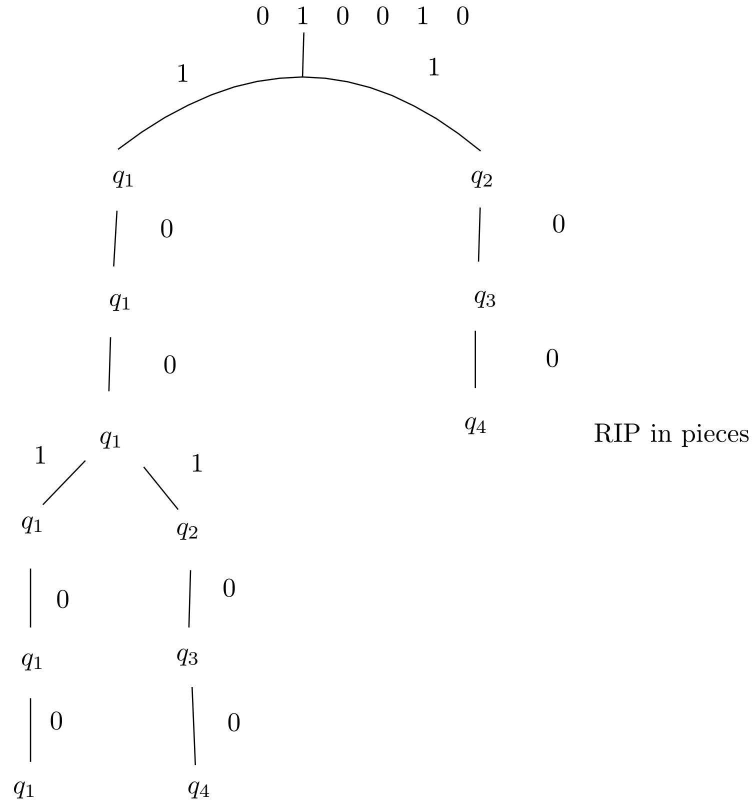
We can then define a NFA; NFA $= \{\Sigma, Q, \delta, q_0, F\}$ and really the only difference to distinguish an NFA is the transition function, δ.

Example 5. We have $L_{11} = \{w \in (0,1)^* | \text{3rd number from right is } 1\}$. We can have the automata:



In the above automata, we can build the path ϵ (empty string) where we can either go back to q_1.

Example 6. We can look at 0100100.



The transition function will now look like:

$$\delta \ : \ Q \times \Sigma_\epsilon \rightarrow P(Q)$$

(the new alphabet is extended to include ϵ). Rather, we map our input to a subset of states.

Example 7. We take look at: