ECE/ML/CS/ISYE 8803

Probabilistic Graphical Models

Module 3 (Part B):

Undirected Graphical Models

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Overview

- Sanity Check
- From BN to MN
- From MN to BN

Read Chapter 4 of K&F

Sanity Check (I)

Consider representing a given P using a graphical model. Which of the following is correct?

- ○ P can always be represented as a BN but maybe not an MN.
- ○ P always can be represented as an MN but maybe not a BN.
- ○ P can always be represented both as a BN and as an MN.
- ○ It's possible that P can be represented as neither a BN nor an MN.

-Koller

Sanity Check (II)

Consider representing a given P using a graphical model, where our goal is to capture I(P) exactly. Which of the following is correct?

- I(P) can always be represented as a BN but maybe not an MN.
- I(P) always can be represented as an MN but maybe not a BN.
- I(P) can always be represented both as a BN and as an MN.
- It's possible that I(P) can be represented using neither a BN nor an MN.

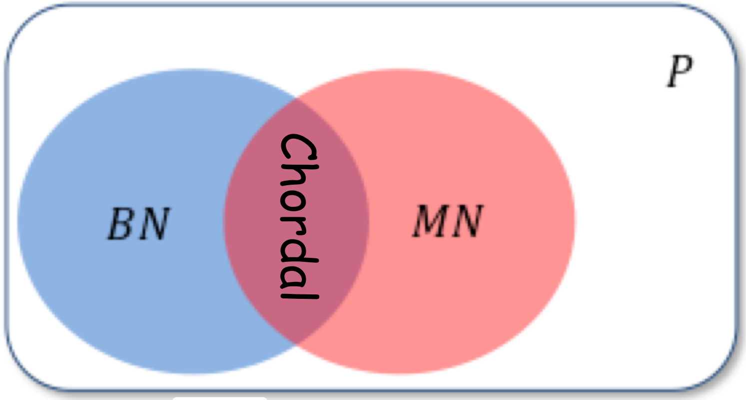
-Koller

Which Graph Should I Use?

- Causality in the problem in hand: use BN
- No clear understanding of any causality (Interaction undirected): use MN
- Often use combination of both directed and undirected edges
- Also taking into account which network is easier to learn in a the particular problem in hand.

Conversion between MN and BN

P-map property of graphical models



- Conversion using P-map will limit to common area in above.
- Focus on conversion via the (minimal) I-map property instead.
  - Recall: G is a minimal I-map for P if
    - $I(G) \subseteq I(P)$
    - Removal of a single edge in G render it not an I-map

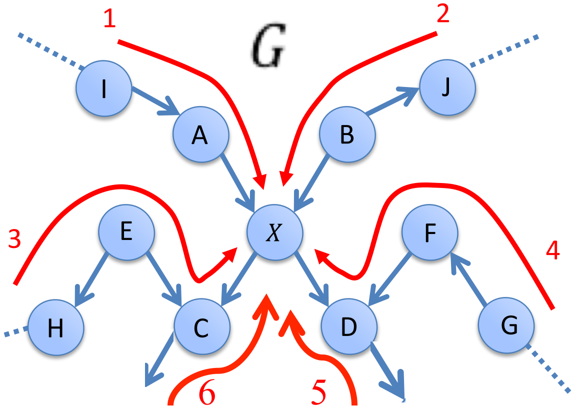
From BN to MN

- Goal: build a Markov network H capable of representing any distribution P that factorizes over G
  - Equivalent to requiring I(H)⊆I(G)

- Approach is motivated by local Markov independencies:

- Markov Blanket for BN?
  - MB_X in BN is the set of nodes consisting of X's parents, X's children and other parents of X's children

All active trails to X should be blocked by Markov blanket



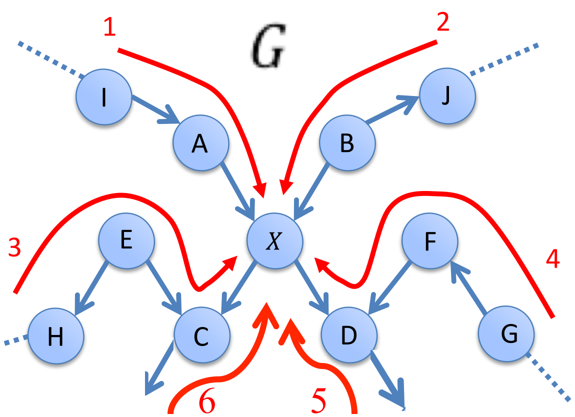
$MB_X = \{A, B, C, D, E, F\}$

$X \perp V - \{X\} - MB_X \mid MB_X$

From BN to MN via moralization

- Moral graph $M(G)$ of a BN G is an undirected graph that contains an undirected edge between X and Y if
  - There is a directed edge between them in the either direction
  - X and Y are parents of a common children

- Moral graph insure that MB_X is the set of neighbors in undirected graph $M(G)$



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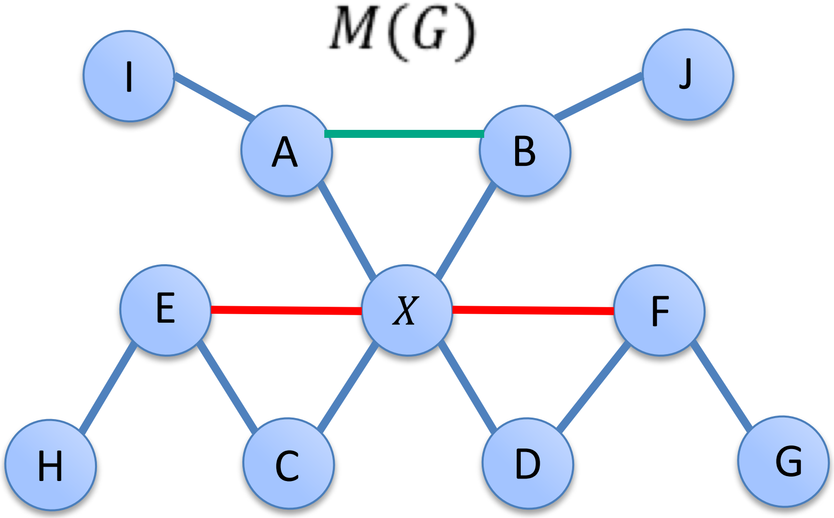
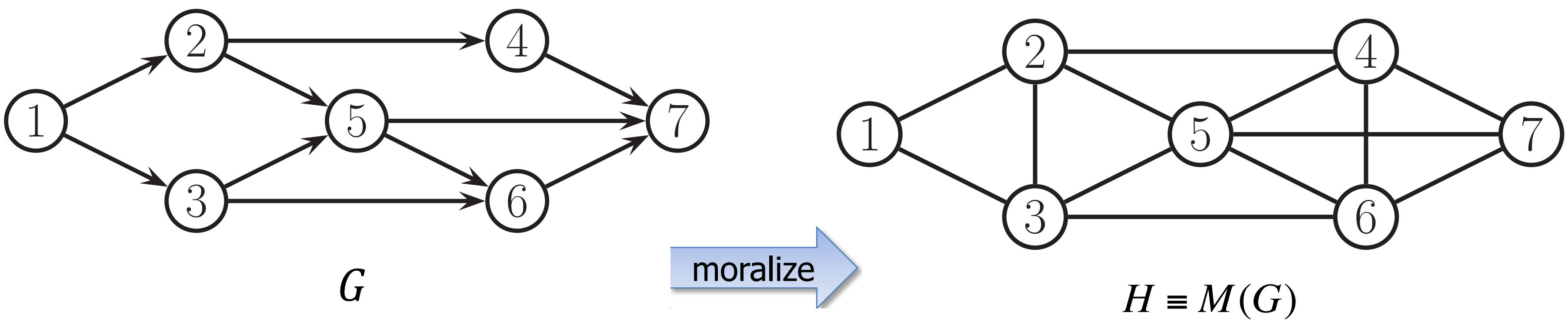


From BN to Minimal I-map MN via moralization

- Moral graph H $M(G)$ of any BN G is a minimal I-map for G: $I(H) \subseteq I(G)$
  - Moralization turns each (X, Pa_X) into a fully connected component
  - CPTs associated with BN can be used as clique potentials



- The moral graph loses some independence relation
  - immoral v-structures

From BN to Perfect I-map MN via moralization

- If BN G is already moral, then its moral graph $M(G)$ is a perfect I-map of G

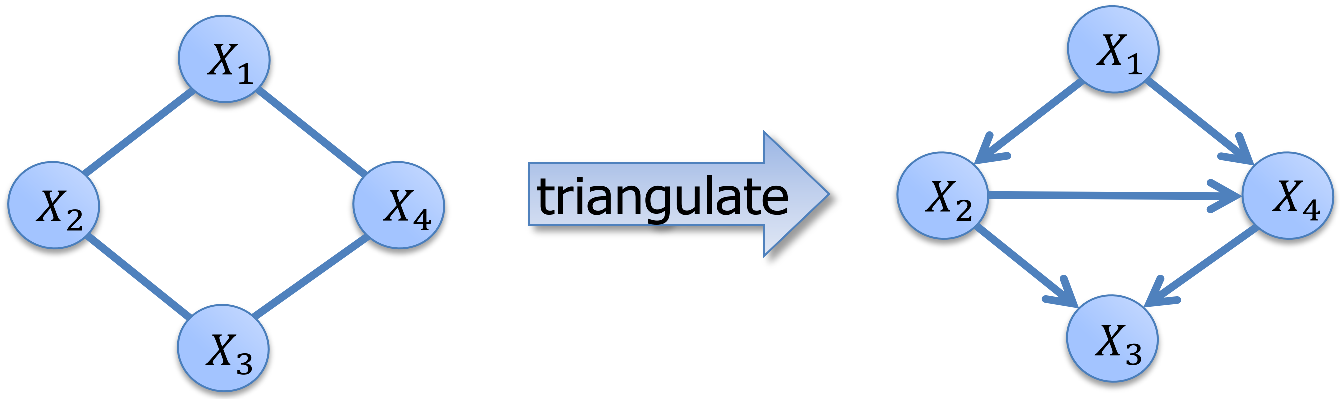
- Proof Sketch:
  - $I\big(M(G)\big) \subseteq I(G)$
  - The only independence relations that are potentially lost from G to $M(G)$ are those arising from V-structures
  - Since G has no V-structures (already moral), no independence are lost in $M(G)$

From MN to BN: $I(G) \subseteq I(H)$

Transformation is harder and resulting BN can be more complex than MN.

- Any BN I-map for an MN must add triangulating edges into the graph

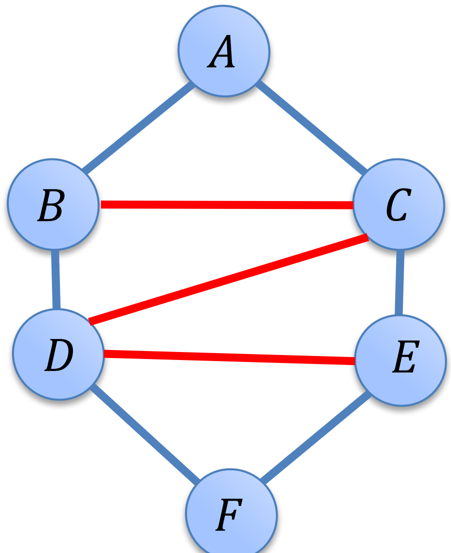
- Intuition:
  - V-structures in BN introduce immoralities
  - These immoralities were not present in a Markov networks
  - Triangulation eliminates immoralities



Chordal Graph

- Let $X_1 - X_2 - \cdots - X_k - X_1$ be a loop in a graph. A chord in a loop is an edge connecting non-consecutive X_i and X_j

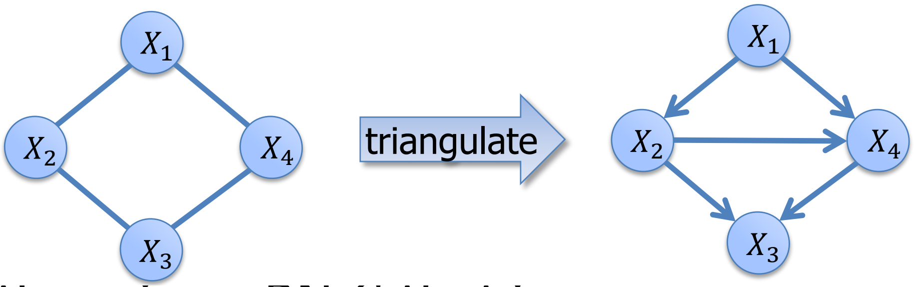
- An undirected graph G is chordal if any loop $X_1 - X_2 - \cdots - X_k - X_1$ for $k \geq 4$ has a chord



- A directed graph G is chordal if its underlying undirected graph is chordal

From MN to Minimal I-map BN: $I(G) \subseteq I(H)$

- Let H be an MN, and G be any BN minimal I-map for H. Then G can have no immoralities
  - Intuitive reason: immoralities introduce additional independencies that are not in the original MN
- Let G any BN minimal I-map for H. Then G is necessarily chordal!
  - Because any non-triangulated loop of length at least 4 in a Bayesian network necessarily contains an immorality
  - Triangulation eliminates immoralities



- Let H be a non-chordal MN. Then there is no BN G that is a perfect I-map for H
- *Let $\mathcal{H}$ be a chordal Markov network. Then there is a Bayesian network $\mathcal{G}$ such that $\mathcal{I}(\mathcal{H}) = \mathcal{I}(\mathcal{G})$.*

From MN to BN via Trangularization

- Construction algorithm
  - Use Markov network as template for independencies I(H)
  - Fix ordering of nodes
  - Add each node along with its minimal parent set according to the independencies defined in the distribution
  - Trangulate (the process loses some independencies of MN)

Example: Construction of Minimal I-map BN for a nonchordal MN network.