Problem 1

a)

Eliminate variables in this order: W, O, M, F, G, B, H

$P(S = true|C = true) = \frac{P(S=true, C=true)}{P(C=true)}$

$P(S = true, C = true) = \sum_W \sum_F \sum_O \sum_M \sum_G \sum_B \sum_H P(W)P(O)P(F|W, O)P(c)P(M)P(G|c, M)$
$P(B|F, G)P(H|s, G)P(s|F, B)$

Let $\varphi_W(F, O) = \sum_W P(W)P(F|W, O)$

$= \sum_F \sum_O \sum_M \sum_G \sum_B \sum_H P(O)P(c)P(M)P(G|c, M)P(B|F, G)P(H|s, G)P(s|F, B)\varphi_W(F, O)$

Let $\varphi_O(F) = \sum_O P(O)\varphi_W(F, O)$

$= \sum_F \sum_M \sum_G \sum_B \sum_H P(c)P(M)P(G|c, M)P(B|F, G)P(H|s, G)P(s|F, B)\varphi_O(F)$

Let $\varphi_M(G) = \sum_M P(c)P(M)P(G|c, M)$

$= \sum_F \sum_G \sum_B \sum_H P(B|F, G)P(H|s, G)P(s|F, B)\varphi_O(F)\varphi_M(G)$

Let $\varphi_F(B, G) = \sum_F P(B|F, G)P(s|F, B)\varphi_O(F)$

$= \sum_G \sum_B \sum_H P(H|s, G)\varphi_M(G)\varphi_F(B, G)$

Let $\varphi_G(B, H) = \sum_G \varphi_M(G)\varphi_F(B, G)P(H|s, G)$

$= \sum_B \sum_H \varphi_G(B, H)$

Let $\varphi_B(H) = \sum_B \varphi_G(B, H)$

$= \sum_H \varphi_B(H)$

$= 0.313192$

So, we have $P(S = true|C = true) = \frac{P(S=true, C=true)}{P(C=true)} = \frac{0.313192}{0.4} = \mathbf{0.78298}$

b)

$P(F = true|G = true) = \frac{P(F=true, G=true)}{\sum_F P(F, G=true)}$

$P(F, G = true) = \sum_W \sum_O \sum_C \sum_M \sum_B \sum_S \sum_H P(W)P(O)P(F|W, O)P(C)P(M)P(g|C, M)$
$P(B|F, g)P(H|S, g)P(S|F, B)$

Let $\varphi_W(F, O) = \sum_W P(W)P(F|W, O)$

$= \sum_O \sum_C \sum_M \sum_B \sum_S \sum_H P(O)P(C)P(M)P(g|C, M)P(B|F, g)P(H|S, g)P(S|F, B)\varphi_W(F, O)$

Let $\varphi_C(M) = \sum_C P(C)P(g|C, M)$

$= \sum_O \sum_M \sum_B \sum_S \sum_H P(O)P(M)P(B|F, g)P(H|S, g)P(s|F, B)\varphi_W(F, O)\varphi_C(M)$

Let $\varphi_M = \sum_M P(M)\varphi_C(M) = 0.7066$

$= 0.7066 \sum_O \sum_B \sum_S \sum_H P(O)P(B|F, g)P(H|S, g)P(s|F, B)\varphi_W(F, O)$

Let $\varphi_O(F) = \sum_O P(O)\varphi_W(F,O)$

$= 0.7066 \sum_B \sum_S \sum_H P(B|F,g)P(H|S,g)P(s|F,B)\varphi_O(F)$

Let $\varphi_S(F,B,H) = \sum_S P(S|F,B)P(H|S,g)$

$= 0.7066 \sum_B \sum_H P(B|F,g)\varphi_O(F)\varphi_S(F,B,H)$

Let $\varphi_H(F,B) = \sum_H \varphi_S(F,B,H)$

$= 0.7066 \sum_B P(B|F,g)\varphi_O(F)\varphi_H(F,B)$

Let $\varphi_B(F) = \sum_B \varphi_H(F,B)P(B|F,G)$

$= 0.7066\varphi_O(F)\varphi_B(F)$

So, $P(F = true, G = true) = 0.71$, $P(F = false, G = true) = 0.29$, and $P(F = true|G = true) = \frac{P(F=true,G=true)}{G=true} = \frac{0.71}{0.71+0.29} = \mathbf{0.71}$

c)

$P(M = true|G = true) = \frac{P(M=true,G=true)}{\sum_M P(M,G=true)}$

$P(M, G = true) = \sum_S \sum_W \sum_O \sum_C \sum_F \sum_B \sum_H P(W)P(O)P(F|W,O)P(C)P(M)P(g|C,M)$
$P(B|F,g)P(H|S,g)P(S|F,B)$

Let $\varphi_W(F,O) = \sum_W P(W)P(F|W,O)$

$= \sum_S \sum_O \sum_C \sum_F \sum_B \sum_H P(O)P(C)P(M)P(g|C,M)P(B|F,g)P(H|S,g)P(S|F,B)\varphi_W(F,O)$

Let $\varphi_O(F) = \sum_O P(O)\varphi_W(F,O)$

$= \sum_S \sum_C \sum_F \sum_B \sum_H P(C)P(M)P(g|C,M)P(B|F,g)P(H|S,g)P(S|F,B)\varphi_O(F)$

Let $\varphi_B(F,S) = \sum_B P(B|F,g)P(S|F,B)$

$= \sum_S \sum_C \sum_F \sum_H P(C)P(M)P(g|C,M)P(H|S,g)\varphi_O(F)\varphi_B(F,S)$

Let $\varphi_H(S) = \sum_H P(H|S,g)$

$= \sum_S \sum_C \sum_F P(C)P(M)P(g|C,M)\varphi_O(F)\varphi_B(F,S)\varphi_H(S)$

Let $\varphi_C(M) = \sum_C P(C)P(g|C,M)$

$= \sum_S \sum_F P(M)\varphi_O(F)\varphi_B(F,S)\varphi_H(S)\varphi_C(M)$

Let $\varphi_F(S) = \sum_F \varphi_O(F)\varphi_B(F,S)$

$= \sum_S P(M)\varphi_H(S)\varphi_C(M)\varphi_F(S)$

Let $\varphi_S = \sum_S \varphi_H(S)\varphi_F(S) = 1$

$= P(M)\varphi_C(M)$

So, $P(M = true, G = true) = 0.0049$, $P(M = false, G = true) = 0.702$, and $P(M = true|G = true) = \frac{P(M=true,G=true)}{\sum_M P(M,G=true)} = \frac{0.0049}{0.0049+0.702} = \mathbf{0.00651}$

d)

$P(M = true|G = true, S = true) = \frac{P(M=true,G=true,S=true)}{\sum_M P(M,G=true,S=true)}$

$P(M, G = true, S = true) = \sum_W \sum_O \sum_C \sum_F \sum_B \sum_H P(W)P(O)P(F|W,O)P(C)P(M)P(g|C,M)$
$P(B|F,g)P(H|s,g)P(s|F,B)$

Let $\varphi_W(F,O) = \sum_W P(W)P(F|W,O)$

$= \sum_O \sum_C \sum_F \sum_B \sum_H P(O)P(C)P(M)P(g|C,M)P(B|F,g)P(H|s,g)P(s|F,B)\varphi_W(F,O)$

Let $\varphi_O(F) = \sum_O P(O)\varphi_W(F,O)$

$= \sum_C \sum_F \sum_B \sum_H P(C)P(M)P(g|C,M)P(B|F,g)P(H|s,g)P(s|F,B)\varphi_O(F)$

Let $\varphi_B(F) = \sum_B P(B|F,g)P(s|F,B)$

$= \sum_C \sum_F \sum_H P(C)P(M)P(g|C,M)P(H|s,g)\varphi_O(F)\varphi_B(F)$

Let $\varphi_H = \sum_H P(H|s,g) = 1$

$= \sum_C \sum_F P(C)P(M)P(g|C,M)\varphi_O(F)\varphi_B(F)$

Let $\varphi_C(M) = \sum_C P(C)P(g|C,M)$

$= \sum_F P(M)\varphi_O(F)\varphi_B(F)\varphi_C(M)$

Let $\varphi_F = \sum_F \varphi_O(F)\varphi_B(F) = 0.80915$

$= 0.80915P(M)\varphi_C(M)$

So, $P(M = true, G = true, S = true) = 0.00372209$, $P(M = false, G = true, S = true) = 0.57174539$, and $P(M = true|G = true, S = true) = \frac{P(M=true,G=true,S=true)}{\sum_M P(M,G=true,S=true)} = \frac{0.00372209}{0.00372209+0.57174539} = \mathbf{0.006468}$

e)

$P(W = true|G = true, S = true, B = false) = \frac{P(W=true,G=true,S=true,B=false)}{\sum_W P(W,G=true,S=true,B=false)}$

$P(W, G = true, S = true, B = false) = \sum_O \sum_C \sum_F \sum_M \sum_H P(W)P(O)P(F|W,O)P(C)P(M)$
$P(g|C,M)P(b|F,g)P(H|s,g)P(s|F,b)$

Let $\varphi_H(H) = \sum_H P(H|s,g) = 1$

$= \sum_O \sum_C \sum_F \sum_M P(W)P(O)P(F|W,O)P(C)P(M)P(g|C,M)P(b|F,g)P(s|F,b)$

Let $\varphi_C(M) = \sum_C P(C)P(G|C,M)$

$= \sum_O \sum_F \sum_M P(W)P(O)P(F|W,O)P(M)P(b|F,g)P(s|F,b)\varphi_C(M)$

Let $\varphi_M = \sum_M P(M)\varphi_C(M) = 0.7066$

$= 0.7066 \sum_O \sum_F P(W)P(O)P(F|W,O)P(b|F,g)P(s|F,b)$

Let $\varphi_O(F,W) = \sum_O P(O)P(F|W,O)$

$= 0.7066 \sum_F P(W)P(b|F,g)P(s|F,b)\varphi_O(F,W)$

Let $\varphi_F(W) = \sum_F P(b|F,g)P(s|F,b)\varphi_O(F,W)$

$= 0.7066P(W)\varphi_F(W)$

So, $P(W = true, G = true, S = true, B = false) = 0.011087$, $P(W = false, G = true, S = true, B = false) = 0.024137$, and $P(W = true|G = true, S = true, B = false) = \frac{P(W=true,G=true,S=true,B=false)}{\sum_W P(W,G=true,S=true,B=false)} = \frac{0.011087}{0.011087+0.024137} = \mathbf{0.3148}$

Problem 2

See code. Run "python3 decoder.py" to run the program. The matrix of counts that I found is as follows:

	barber	ilsung	fox	chain	fitzwilliam	quinceadams	grafvonunterhosen
david	5	2	9	15	15	10	15
anton	7	7	3	6	18	11	12
fred	5	4	9	9	12	7	20
jim	9	8	8	19	18	11	18
barry	12	7	11	9	6	9	15

Problem 3

We want to find

$\arg\max_{a,b,c} P(a|b)P(b|c)P(c)$
$= \arg\max_a \arg\max_b \arg\max_c P(a|b)P(b|c)P(c)$
$= \arg\max_a P(a|b) \arg\max_b P(b|c) \arg\max_c P(c)$

Let $\gamma(b) = \sum_c P(b|c)P(c)$ and $\gamma(a) = \sum_b P(a|b)\gamma(b)$

We get $\gamma(b = true) = 0.36$ and $\gamma(b = false) = 0.64$ with the following table:

	b = true	b = false
c = true	0.3	0.1
c = false	0.06	0.54

Then, we get $\gamma(a = true) = 0.198$ and $\gamma(a = false) = 0.642$ with the following table:

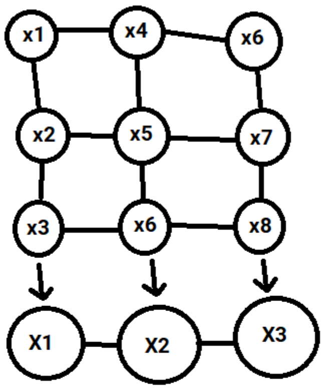
	a = true	a = false
b = true	0.09	0.21
b = false	0.108	0.432

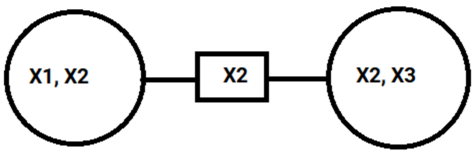
As such, we see $\arg\max_a \gamma(a)$ leads to **a = false**. After inspecting the 2nd table above, we see the value of b that contributes the most to $\gamma(a = false)$ is **b = false**, which is $\arg\max_b P(a = true|b)\gamma(b)$.

Finally, we want $\arg\max_c \gamma(b = false)$. When looking at the first table, the value of c that contributes the most to $\gamma(b = false)$ is when **c = false**.

So, using the max-product algorithm, through the use of backpointers, we have found argmax of the distribution occurs when **a = false, b = false, c = false**.

Problem 4





The generated junction tree is shown in the above diagram where $\phi(X_1, X_2) = \phi(x_1, x_2, x_3, x_4, x_5, x_6)$ and $\phi(X_2, X_3) = \phi(x_4, x_5, x_6, x_7, x_8, x_9)$. Let

$\phi(X_1, X_2) = \phi(x_1, x_2)\phi(x_2, x_3)\phi(x_1, x_4)\phi(x_2, x_5)\phi(x_3, x_6)\phi(x_4, x_5)\phi(x_5, x_6)$

$\phi(X_2, X_3) = \phi(x_4, x_7)\phi(x_5, x_8)\phi(x_6, x_9)\phi(x_7, x_8)\phi(x_8, x_9)$

$\phi(X_2) = 1$

We can now perform absorptions to calculate Z:

$\phi_1^*(X_2) = \sum_{X_1} \phi(X_1, X_2) = \sum_{x_1,x_2,x_3} = \phi(x_1, x_2)\phi(x_2, x_3)\phi(x_1, x_4)\phi(x_2, x_5)\phi(x_3, x_6)\phi(x_4, x_5)\phi(x_5, x_6)$

$\phi^*(X_2, X_3) = \phi(X_2, X_3)\frac{\phi_1^*(X_2)}{\phi(X_2)} = \phi(X_2, X_3)\phi^*(X_2) = \phi(X_2, X_3)\sum_{X_1} \phi(X_1, X_2)$

$\phi_2^*(X_3) = \sum_{X_2} \phi_1^*(X_2, X_3)$

$\phi^*(X_3, X_4) = \phi(X_3, X_4)\frac{\phi_2^*(X_3)}{\phi(X_3)}$

...

$\phi_{n-2}^*(X_{n-1}) = \sum_{X_{n-2}} \phi(X_{n-2}, X_{n-1})$

$\phi^*(X_{n-1}, X_n) = \phi(X_{n-1}, X_n)\frac{\phi_{n-2}^*(X_{n-1})}{\phi(X_{n-1})} = \phi(X_{n-1}, X_n)\sum_{X_{n-2}} \phi(X_{n-2}, X_{n-1}) \cdots \sum_{X_1} \phi(X_1, X_2)$

$Z = \sum_{X_{n-1},X_n} \phi^*(X_{n-1}, X_n)$
$= \sum_{X_{n-1},X_n} \phi(X_{n-1}, X_n)\sum_{X_{n-2}} \phi(X_{n-2}, X_{n-1}) \cdots \sum_{X_2} \phi(X_2, X_3)\sum_{X_1} \phi(X_1, X_2)$

First, we must calculate the big X potentials $\phi(X_i, X_{i+1})$, not to be confused with little x potentials in the original lattice $\phi(x_i, x_{i+1})$. Since each X_i stacks n variables (in an nxn lattice), it takes control of n binary variables. As such, it models 2^n possible variable configurations. In the JCT algorithm, since we have $\phi(X_i, X_{i+1})$, we must iterate over 2^n states for X_i and 2_n states for X_{i+1}, for a total of $2^n(2^n) = 2^{2n}$ possible state configurations.

As shown above, for $\phi(X_1, X_2)$, for each state configuration, we must multiply N-1 nodes' vertical edge potentials, which is 2 in our case, then N nodes' horizontal edge potentials, which is N, and finally N-1 node's vertical edge potentials again. This leads to a total of (N-1) + N + (N-1) nodes used for multiplications, or 3N-3 multiplications in total. Then, to generalize this, notice how above $\phi(X_2, X_3)$ has fewer edge potentials than $\phi(X_1, X_2)$. This will be the case for all $\phi(X_i, X_{i+1})$ where $i \neq 1$. In this case, instead of 3n-3 multiplications across 3n-2 nodes, we will instead have 2n-2 multiplications across 2n-1 nodes. As such, to actually generate each pairwise big X edge potential, or in other words initialize them, we require $2^{2n}((3n-3) + (2n-2)) = 2^{2n}(5n-5)$ multiplications.

Now, when we actually want to compute Z, we first want to look at the right-most summation provided above, which is $\sum_{X_1} \phi(X_1, X_2)$. In this case, a new function will be generated such that $\phi_1^*(X_2) = \sum_{X_1} \phi(X_1, X_2)$. For this function, X_2 will have 2^n states, and the summation will sum across 2^n numbers, requiring $2^n - 1$ addition operations. As such, since each state requires $2^n - 1$ additions, we get a total of $2^n(2^n - 1) = 2^{2n} - 2^n$ operations to compute this function. All functions we will compute after this can be represented as $\phi_i^*(X_{i+1}) = \sum_{X_i} \phi(X_i, X_{i+1})\phi_{i-1}^*(X_i)$. In this case, there are still $2^n - 1$ additions and 2^n multiplications for each of the 2^n states. As such, to compute these subsequent functions, we require $2^n(2^n + (2^n - 1)) = 2^n(2^{n+1} - 1) = 2^{2n+1} - 2^n$ calculations. We will have to perform these calculations for n-3 messages, all the way from $\phi_2^*(X_3)$ to $\phi_{n-2}^*(X_{n-1})$. Finally, when we eventually get to the final calculations of Z, we have to compute $\sum_{X_{n-1}, X_n} \phi(X_{n-1}, X_n)\phi_{n-2}^*(X_{n-1})$, which are the left-most values of that summation for Z as shown above. Since there is one multiplication per iteration of the loop, and the loop is iterating over two nodes with 2^n states each, there are a total of $2^n(2^n) = 2^{2n}$ loop iterations, leading to 2^{2n} multiplications, and it is easy to see there are $2^{2n} - 1$ additions, totaling $2^{2n} + (2^{2n} - 1) = 2^{2n+1} - 1$ operations. As such, since $\phi_1^*(X_2)$ requires $2^{2n} - 2^n$ operations, $\phi_i^*(X_{i+1})$ requires $2^{2n+1} - 2^n$ for a total of n-3 functions, and $\sum_{X_{n-1}, X_n} \phi(X_{n-1}, X_n)\phi_{n-2}^*(X_{n-1})$ requires $2^{2n+1} - 1$ operations, we get the following complexity for calculating Z:

$$
\begin{aligned}
&(2^{2n} - 2^n) + (n-3)(2^{2n+1} - 2^n) + (2^{2n+1} - 1) \\
&= 2^{2n} - 2^n + n2^{2n+1} - n2^n - 3(2^{2n+1}) + 3(2^n) + 2^{2n+1} - 1 \\
&= n2^{2n+1} - 3(2^{2n+1}) + 2^{2n+1} + 2^{2n} - 2^n - n2^n + 3(2^n) - 1 \\
&= 2^{2n+1}(n - 3 + 1) + 2^{2n} - 2^n(1 + n - 3) - 1 \\
&= 2^{2n+1}(n-2) + 2^{2n} - 2^n(n-2) - 1 \\
&= (n-2)(2^{2n+1} - 2^n) + 2^{2n} - 1
\end{aligned}
$$

For n = 10, we have $\mathbf{log(Z) = 186.7916}$.

Problem 5

a) The symptom marginals can be found below:

i	$p(s_i = 1)$	i	$p(s_i = 1)$
1	0.4418	21	0.7613
2	0.4567	22	0.6956
3	0.4414	23	0.5087
4	0.4913	24	0.4200
5	0.4939	25	0.3519
6	0.6575	26	0.3896
7	0.5046	27	0.3260
8	0.2687	28	0.4696
9	0.6491	29	0.5229
10	0.4907	30	0.7173
11	0.4226	31	0.5242
12	0.4291	32	0.3537
13	0.5450	33	0.5127
14	0.6330	34	0.5294
15	0.4295	35	0.3858
16	0.4588	36	0.4891
17	0.4276	37	0.6336
18	0.4043	38	0.5896
19	0.5821	39	0.4232
20	0.5896	40	0.5282

b) Instead of using the JCT algorithm, we can use the following:

$$
\begin{aligned}
&P(s_i) \\
&= \sum_{s \setminus s_i} \sum_{d_1 \ldots d_2 0} P(s_1, \ldots, s_{40}, d_1, \ldots, d_{20}) \\
&= \sum_{s \setminus s_i} \sum_{d_1 \ldots d_2 0} P(s_1 | Pa(s_1)) \ldots P(s_{40} | Pa(s_{40})) \prod_{j=1}^{20} P(d_j) \\
&= \sum_{d_1 \ldots d_{20}} P(s_i | Pa(s_i)) \prod_{j=1}^{20} P(d_j) \\
&= \sum_{Pa(s_i)} P(s_i | Pa(s_i)) P(Pa_1(s_i)) P(Pa_2(s_i)) P(Pa_3(s_i))
\end{aligned}
$$

In the above equations, s_i is the i-th symptom, d_j is the j-th disease, and $Pa_i(s_i)$ is the i-th parent disease of s_i. For each s_i, there are 3 parent diseases. For example, if the parents of s_4 are d_2, d_8, d_{22}, then the above probability evaluates to
$P(s_4) = \sum_{d_2, d_8, d_{22}} P(s_4 | d_2, d_8, d_{22}) P(d_2) P(d_8) P(d_{22})$.

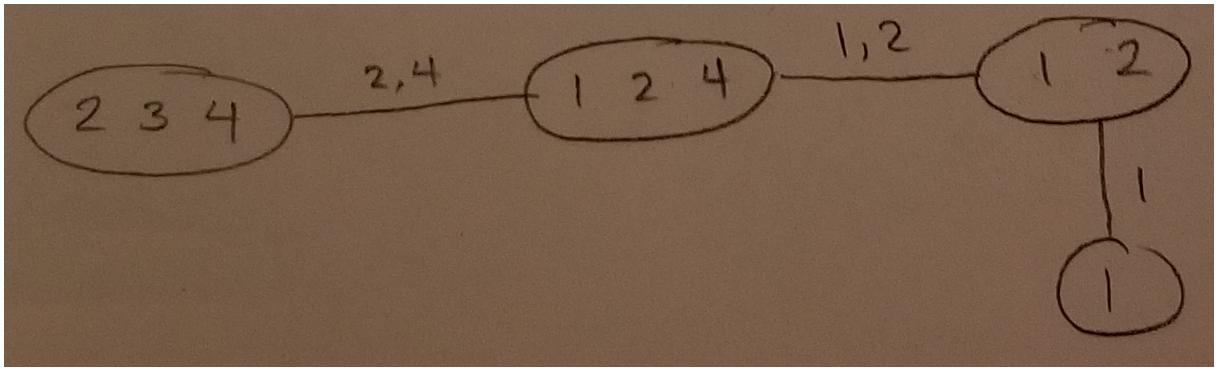
After implementing this method, it can be seen the max difference between the two sets of probabilities is 3.153e-14.

c) After setting $s_1, \ldots, s_5 = 1$ and $s_6, \ldots, s_10 = 2$, we compute the marginals of the diseases as follows:

j	$p(d_j = 1)$	j	$p(d_j = 1)$
1	0.0298	11	0.2873
2	0.3181	12	0.4898
3	0.9542	13	0.8996
4	0.3966	14	0.6196
5	0.4965	15	0.9205
6	0.4352	16	0.7061
7	0.1875	17	0.2012
8	0.7012	18	0.9085
9	0.0431	19	0.8650
10	0.6103	20	0.8839

Problem 6

a) The junction tree is as follows:



b) The normal way to calculate the marginal is as follows:

$P(x_1, x_2, x_3, x_4)$
$= \sum_{x_2} \sum_{x_3} \sum_{x_4} \phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1)$
$= \sum_{x_2} \sum_{x_4} \phi(x_1, x_2)\phi(x_4, x_1) \sum_{x_3} \phi(x_2, x_3)\phi(x_3, x_4)$
$= \sum_{x_2} \sum_{x_4} \phi(x_1, x_2)\phi(x_4, x_1)\phi_{x_3}(x_2, x_4)$
$= \sum_{x_2} \phi(x_1, x_2) \sum_{x_4} \phi(x_4, x_1)\phi_{x_3}(x_2, x_4)$
$= \sum_{x_2} \phi(x_1, x_2)\phi_{x_4}(x_1, x_2)$
$= \phi_{x_2}(x_1)$

Now, through the absorption procedure, we get the following:

Let $\phi(x_1, x_2, x_4) = \phi(x_1, x_2)\phi(x_4, x_1)$, $\phi(x_2, x_3, x_4) = \phi(x_2, x_3)\phi(x_3, x_4)$, and $\phi(x_2, x_4) = 1$.

$\phi_1^*(x_2, x_4) = \sum_{x_3} \phi(x_2, x_3, x_4)$

$\phi^*(x_1, x_2, x_4) = \phi(x_1, x_2, x_4)\frac{\phi_1^*(x_2,x_4)}{\phi(x_2,x_4)} = \phi(x_1, x_2, x_4)\frac{\sum_{x_3} \phi(x_2,x_3,x_4)}{\phi(x_2,x_4)} = \phi(x_1, x_2, x_4) \sum_{x_3} \phi(x_2, x_3, x_4) = P(x_1, x_2, x_4)$

$\phi_2^*(x_1, x_2) = \sum_{x_4} \phi^*(x_1, x_2, x_4)$

$\phi^*(x_1, x_2) = \phi(x_1, x_2)\frac{\phi_2^*(x_1,x_2)}{\phi(x_1,x_2)}$

$\phi_3^*(x_1) = \sum_{x_2} \phi^*(x_1, x_2)$

$\phi^*(x_1) = \phi(x_1)\frac{\phi_3^*(x_1)}{\phi(x_1)} = \phi_3^*(x_1)$

As such,w e have computed the marginal of x_1. To verify it is the correct result, we will expand the marginal below:

$\phi^*(x_1)$
$= \phi_3^*(x_1)$
$= \sum_{x_2} \phi^*(x_1, x_2)$
$= \sum_{x_2} \phi(x_1, x_2)\frac{\phi_2^*(x_1,x_2)}{\phi(x_1,x_2)}$
$= \sum_{x_2} \frac{\phi(x_1,x_2)}{\phi(x_1,x_2)} \sum_{x_4} \phi^*(x_1, x_2, x_4)$

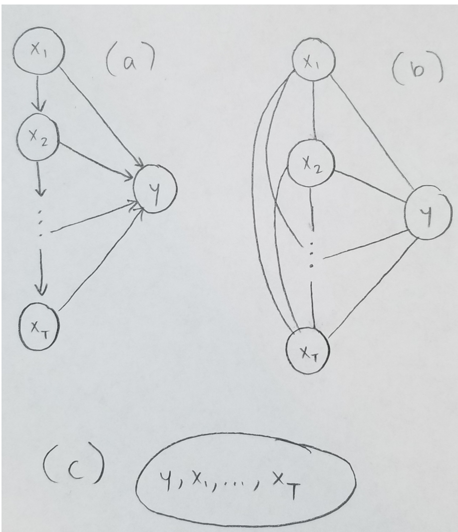
$= \sum_{x_2} \frac{\phi(x_1,x_2)}{\phi(x_1,x_2)} \sum_{x_4} \phi(x_1, x_2, x_4) \frac{\phi_1^*(x_2,x_4)}{\phi(x_2,x_4)}$
$= \sum_{x_2} \frac{\phi(x_1,x_2)}{\phi(x_1,x_2)} \sum_{x_4} \phi(x_1, x_2, x_4) \frac{\sum_{x_3} \phi(x_2,x_3,x_4)}{\phi(x_2,x_4)}$
$= \sum_{x_2} \sum_{x_3} \sum_{x_4} \frac{\phi(x_1,x_2)}{\phi(x_1,x_2)} \phi(x_1, x_2, x_4)\phi(x_2, x_3, x_4) \frac{1}{\phi(x_2,x_4)}$
$= \sum_{x_2} \sum_{x_3} \sum_{x_4} \frac{\phi(x_1,x_2)}{\phi(x_1,x_2)} \phi(x_1, x_2)\phi(x_4, x_1)\phi(x_2, x_3)\phi(x_3, x_4) \frac{1}{\phi(x_2,x_4)}$
$= \sum_{x_2} \sum_{x_3} \sum_{x_4} \phi(x_1, x_2)\phi(x_4, x_1)\phi(x_2, x_3)\phi(x_3, x_4)$
$= \sum_{x_2} \sum_{x_3} \sum_{x_4} \phi(x_1, x_2)\phi(x_2, x_3)\phi(x_3, x_4)\phi(x_4, x_1)$

This above equation is the exact same equation we got above while we were solving for the marginal of x_1. As such, we can safely confirm the absorption gives us the correct result.

Problem 7

a)

The following diagram shows (a) the bayesian network modeling the joint distribution, (b) the bayesian network creating a fully-connected network after moralization, and (c) the junction tree:



We want to figure out the runtime complexity of calculating $p(x_T)$ as given by the junction tree algorithm. The junction tree algorithm requires the moralization step. In moralization, two parents of a node must have a direct connection. As such, since all nodes are parents of y, all nodes must be connected to each other, leading to a fully-connected network. Since the network becomes fully-connected after moralization, the junction tree algorithm will give us one huge node containing all nodes in the graph, which is represented as $\phi(y, x_1, \ldots, x_T) = p(y|x_1, \ldots, x_T)p(x_1)\prod_{t=2}^{T} p(x_t|x_{t-1})$. Since all variables are binary, and there are $T+1$ nodes, there are a total of 2^{T+1} possible state configurations for this node potential. However, since we want just the potential for x_T, we only need to marginalize, or sum, over T nodes, leading to $2^T - 1$ additions for each of the two states of x_T. Since there are 2 states for x_T and we require $2^T - 1$ additions per state, we require $2(2^T - 1) = 2^{T+1} - 1$ calculations, or $O(2^T)$ time for the junction tree algorithm.

b)

We can reduce this time complexity significantly through variable elimination, as seen below:

$p(x_T)$
$= \sum_{y,x_1,\ldots,x_{T-1}} p(y|x_1, \ldots, x_T)p(x_1)\prod_{t=2}^{T} p(x_t|x_{t-1})$
$= \sum_{x_1,\ldots,x_{T-1}} p(x_1)\prod_{t=2}^{T} p(x_t|x_{t-1}) \sum_y p(y|x_1, \ldots, x_T)$

We know $\sum_y p(y|x_1, \ldots, x_T) = 1$.

$= \sum_{x_1,\ldots,x_{T-1}} p(x_1)\prod_{t=2}^{T} p(x_t|x_{t-1})$
$= \sum_{x_1,\ldots,x_{T-1}} p(x_1)p(x_2|x_1)\ldots p(x_{T-1})p(x_T)$

Let $\varphi_{x_i}(x_{i+1}) = \sum_{x_i} \varphi_{x_{i-1}}(x_i)p(x_{i+1}|x_i)$, with a special casing being $\varphi_{x_1} = \sum_{x_1} p(x_1)p(x_2|x_1)$

$= \sum_{x_2,\ldots,x_{T-1}} \varphi_{x_1}(x_2)p(x_3|x_2)\ldots p(x_{T-1})p(x_T)$
$= \sum_{x_3,\ldots,x_{T-1}} \varphi_{x_2}(x_3)p(x_4|x_3)\ldots p(x_{T-1})p(x_T)$
$= \sum_{x_{T-1}} \varphi_{x_{T-2}}(x_{T-1})p(x_T|x_{T-1})$
$= \varphi_{x_{T-1}}(x_T)$

Each function $\varphi_{x_i}(x_{i+1})$ has two states, and it sums over 2 values, requiring 1 multiplication for each summation. As such, for each state, there are 2 multiplications and 1 addition, totalling $2(2+1) = 6$ calculations to compute $\varphi_{x_i}(x_{i+1})$. Since we create $T-1$ of these tables, a total of $6(T-1)$ calculations are required to create all the tables. As such, with this new runtime of $6(T-1) = 6T-6 = O(T)$, we can see this alternative algorithm is linear in complexity.

Problem 8

a)

We eliminate the variables in the order x_{10}, x_9, x_7, x_6, x_5, x_4, x_3, x_2, x_0 as follows:

$p(x_1|x_{11}, x_8)$
$= \sum_{x_0,x_2,x_3,x_4,x_5,x_6,x_7,x_9,x_{10}} \phi(x_0, x_1, x_2)\phi(x_0, x_2, x_3)\phi(x_0, x_3, x_4)\phi(x_1, x_2, x_3)\phi(x_1, x_3, x_4)$
$\phi(x_2, x_3, x_4) \prod_{i=5}^{11} \phi(x_1, x_2, x_i)\phi(x_1, x_3, x_i)\phi(x_1, x_4, x_i)\phi(x_2, x_3, x_i)\phi(x_2, x_4, x_i)\phi(x_3, x_4, x_i)$

Let $\varphi_{11}(x_1, x_2, x_3, x_4) = \phi(x_1, x_2, x_{11})\phi(x_1, x_3, x_{11})\phi(x_1, x_4, x_{11})\phi(x_2, x_3, x_{11})\phi(x_2, x_4, x_{11})\phi(x_3, x_4, x_{11})$

Note: x_{11} is a given variable, so we do not need to sum over its values.

$= \sum_{x_0,x_2,x_3,x_4,x_5,x_6,x_7,x_9,x_{10}} \phi(x_0, x_1, x_2)\phi(x_0, x_2, x_3)\phi(x_0, x_3, x_4)\phi(x_1, x_2, x_3)\phi(x_1, x_3, x_4)$
$\phi(x_2, x_3, x_4)\varphi_{11}(x_1, x_2, x_3, x_4) \prod_{i=5}^{10} \phi(x_1, x_2, x_i)\phi(x_1, x_3, x_i)\phi(x_1, x_4, x_i)$
$\phi(x_2, x_3, x_i)\phi(x_2, x_4, x_i)\phi(x_3, x_4, x_i)$

Let $\varphi_{10}(x_1, x_2, x_3, x_4) = \sum_{x_{10}} \phi(x_1, x_2, x_{10})\phi(x_1, x_3, x_{10})\phi(x_1, x_4, x_{10})\phi(x_2, x_3, x_{10})\phi(x_2, x_4, x_{10})\phi(x_3, x_4, x_{10})$

$= \sum_{x_0,x_2,x_3,x_4,x_5,x_6,x_7,x_9} \phi(x_0, x_1, x_2)\phi(x_0, x_2, x_3)\phi(x_0, x_3, x_4)\phi(x_1, x_2, x_3)\phi(x_1, x_3, x_4)$
$\phi(x_2, x_3, x_4)\varphi_{10}(x_1, x_2, x_3, x_4)\varphi_{11}(x_1, x_2, x_3, x_4) \prod_{i=5}^{9} \phi(x_1, x_2, x_i)\phi(x_1, x_3, x_i)\phi(x_1, x_4, x_i)$
$\phi(x_2, x_3, x_i)\phi(x_2, x_4, x_i)\phi(x_3, x_4, x_i)$

Let $\forall_{i\in\{5,6,7,9\}}\varphi_i(x_1, x_2, x_3, x_4) = \sum_{x_i} \phi(x_1, x_2, x_i)\phi(x_1, x_3, x_i)\phi(x_1, x_4, x_i)\phi(x_2, x_3, x_i)$
$\phi(x_2, x_4, x_i)\phi(x_3, x_4, x_i)$

And $\varphi_8(x_1, x_2, x_3, x_4) = \phi(x_1, x_2, x_8)\phi(x_1, x_3, x_8)\phi(x_1, x_4, x_8)\phi(x_2, x_3, x_8)\phi(x_2, x_4, x_8)\phi(x_3, x_4, x_8)$
for a similar reason to x_{11} mentioned above.

$= \sum_{x_0,x_2,x_3,x_4} \phi(x_0, x_1, x_2)\phi(x_0, x_2, x_3)\phi(x_0, x_3, x_4)\phi(x_1, x_2, x_3)\phi(x_1, x_3, x_4)$
$\phi(x_2, x_3, x_4) \prod_{i=5}^{11} \varphi_i(x_1, x_2, x_3, x_4)$

Let $\varphi_4(x_0, x_1, x_2, x_3) = \sum_{x_4} \phi(x_0, x_3, x_4)\phi(x_1, x_3, x_4)\phi(x_2, x_3, x_4) \prod_{i=5}^{11} \varphi_i(x_1, x_2, x_3, x_4)$

$= \sum_{x_0,x_2,x_3} \phi(x_0, x_1, x_2)\phi(x_0, x_2, x_3)\phi(x_1, x_2, x_3)\varphi_4(x_0, x_1, x_2, x_3)$

Let $\varphi_3(x_0, x_1, x_2) = \sum_{x_3} \phi(x_0, x_2, x_3)\phi(x_1, x_2, x_3)\varphi_4(x_0, x_1, x_2, x_3)$

$= \sum_{x_0,x_2} \phi(x_0, x_1, x_2)\varphi_3(x_0, x_1, x_2)$

Let $\varphi_2(x_0, x_1) = \sum_{x_2} \phi(x_0, x_1, x_2)\varphi_3(x_0, x_1, x_2)$

$= \sum_{x_0} \varphi_2(x_0, x_1)$

Let $\varphi_0(x_1) = \sum_{x_0} \varphi_2(x_0, x_1)$

$= \varphi_0(x_1)$

Now, we will calculate the runtime of this algorithm. φ_{11} and φ_8 require 5 simple multiplications each. Then, $\forall_{i\in\{5,6,7,9,10\}}\varphi_i(x_1, x_2, x_3, x_4)$ requires 5 multiplications for both iterations of the loop, in addition to 1 addition, resulting in $2(5) + 1 = 11$ operations. For $\varphi_4(x_0, x_1, x_2, x_3)$, there are 9 multiplications for each iteration, and one addition, resulting in $2(9) + 1 = 19$ operations. For $\varphi_3(x_0, x_1, x_2)$ requires 2 multiplications for each iteration, and one addition, resulting in $2(2) + 1 = 5$ calculations. For $\varphi_2(x_0, x_1)$, we require there is one multiplication per loop iteration, and one addition, resulting in $2(1) + 1 = 3$ calculations. Finally, $\varphi_0(x_1)$ requires 1 addition. So, the total number of calculations is $2(5) + 5(11) + 19 + 5 + 3 + 1 = 93$ computations.

In terms of space, we need $2^4 = 16$ storage spaces for $\forall_{i \in \{5,\ldots,11\}} \varphi_i(x_1, x_2, x_3, x_4)$, $2^4 = 16$ storage space for $\varphi_4(x_0, x_1, x_2, x_3)$, $2^3 = 8$ storage spaces for $\varphi_3(x_0, x_1, x_2)$, $2^2 = 4$ spaces for $\varphi_2(x_0, x_1)$, and 2 spaces for $\varphi_0(x_1)$. In total, this means we need $7(16) + 16 + 8 + 4 + 2 = 142$ storage spaces for this variable elimination algorithm.

b)