Math 1080: Spring 2019

Homework #4

Due Feb 15

In all problems below let A be $m \times n$ matrix, B be $n \times n$ matrix, and v be a vector in $\mathbb{R}^n$.

Problem 1:

Exactly how many flops are needed to perform the following lines of code?

a) `d = A*v;`

b) `C = A*B;`

c) `x = v'*B*v;` (in Matlab syntax `v'` represents the transpose of v.)

d) `x = A(1:n,:)*(B*v);`

e) `x = (A(1:n,:)*B)*v;` (this case differs from d) in the order of products)

Problem 2:

Exactly how many flops are needed to execute the following code segments?

a)
```
for k = 1:n
    a(k) = B(k,k)*v(n-k);
end
```

b)
```
for k = 1:n
    x = B(k,n-k+1:n)*v(n-k+1:n);
end
```

c)
```
for k = 1:n-1
    for j = n-k:n
        c = c + B(:,1:j)*A(n-j+1:n,k);
    end
end
```

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$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n$

1. a. $d = Av = \begin{bmatrix} a_1 \\ \cdots \\ a_m \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_1 v \\ \vdots \\ a_m v \end{bmatrix}$

$a_i v = \sum_{j=1}^{n} a_{ij} v_j = (n-1)$ adds $+ (n)$ mults $= n+n-1 = 2n-1$ flops

Perform $a_i v$ m times $\Rightarrow (2n-1)m = 2mn - m$

b. $C = AB = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}[b_1 | \dots | b_n] = C_{m\times n}$

$a_i b_j = \sum_{p=1}^{n} a_{ip} b_{pj} = (n-1)$ adds $+ (n)$ mults $= 2n-1$ flops

C will have $m\times n$ elements $\Rightarrow$ perform $a_i b_j$ $m\times n$ times

$\Rightarrow (2n-1)(mn) = 2mn^2 - mn$

c. $x = v'Bv = [v_1 \dots v_n]\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix}$

~~$v' \Rightarrow \forall i,j \in [1,n]\ v_{ij} = v_{ji} \Rightarrow n$ flops~~

$v'B \Rightarrow$ dot product $a_i b_j = 2n-1$ flops n times $= (2n-1)n = 2n^2 - n$

$\Rightarrow (v'B)_{1\times n}$

$(v'B)v \Rightarrow$ inner product $[v_1' \dots v_n']\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow 2n-1$ flops

$\Rightarrow (2n^2-n)$ flops $+ (2n-1)$ flops

$= 2n^2 - n + 2n - 1 = 2n^2 + n - 1$

d. $x = A(1:n,:)(Bv)$

$\Rightarrow Bv$ is similar to Av in part (a), but replace m by n

$\Rightarrow 2n^2 - n$ where $(Bv) \in \mathbb{R}^{n\times 1}$.

$\Rightarrow$ Let $Bv = C_{n\times 1}$. $A(1:n,:)c$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$ flops

$\Rightarrow (2n^2-n)$ flops $+ (2n^2-n)$ flops $= 2n^2 - n + 2n^2 - n$

$= 4n^2 - 2n$ flops

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^n$

1 cont'd

e. $x = (A(1:n,:)B)v$

$\Rightarrow A(1:n,:)B$ is the same as part (b), but replace m by n

$\Rightarrow 2n^3 - n^2$. Let $(A(1:n,:)B) = C_{n \times n}$

$\Rightarrow Cv$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$

$\Rightarrow (2n^3 - n^2)$ flops $+ (2n^2 - n)$ flops $= 2n^3 - n^2 + 2n^2 - n$

$= \boxed{2n^3 + n^2 - n \text{ flops}}$

2 a.

```
for k = 1:n
    a(k) = B(k,k)* v(n-k);
end
```

} loop n times, 1 mult

B(k,k): scalar, v(n-k): scalar

$n * 1 = $ n flops

b.

```
for k = 1:n
    x = B(k, n-k+1:n)*v(n-k+1:n)
end
```

} loop n times

$$\sum_{k=1}^{n} \underbrace{\sum_{i=n-k+1}^{n} B_{ki} v_i}_{\substack{k-1 \text{ adds} \\ k \text{ mults}}} = \sum_{k=1}^{n} 2k - 1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n$$

$$= n(n+1) - n = n^2$$

k-1 adds, k mults } 2k-1 flops

c.

```
for k = 1:n-1
    for j = (n-k):n
        C = C + B(:,1:j)A(n-j+1:n,k);
    end
end
```

B(:,1:j): n×j matrix; A(n-j+1:n,k): column vector with j×1 dimensions

C + : (n) adds

$B(:,1:j)A(n-j+1:n,k)$: outer product n×1 $\Rightarrow$ (j-1) adds + (j) mults for <> n times $\Rightarrow n(2j-1)$

$$\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j-1) + n = \sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j) = 2n\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} j = 2n\sum_{k=1}^{n-1}\left(\underbrace{\sum_{j=1}^{n} j}_{\frac{n(n+1)}{2}} - \underbrace{\sum_{j=1}^{n-k-1} j}_{\frac{(n-k-1)(n-k)}{2}}\right)$$

$$\frac{n(n+1)}{2} - \frac{(n-k-1)(n-k)}{2} = \frac{2n(k+1) - k^2 - k}{2}$$

$$= n\sum_{k=1}^{n-1} 2n(k+1) - k^2 - k = 2n^2\sum_{k=1}^{n-1} k + 2n^2\sum_{k=1}^{n-1} 1 - n\sum_{k=1}^{n-1} k^2 - n\sum_{k=1}^{n-1} k$$

$$= 2n^2\frac{(n-1)n}{2} + 2n^2(n-1) - n\frac{(n-1)(n)(2n-1)}{6} - n\frac{(n-1)n}{2}$$

$$= \boxed{n^4 + n^3 - 2n^2 - \frac{n^2(n-1)(n+1)}{3} \text{ flops}}$$