

① Symmetry: $F^T = (I - 2ww^T)^T = I^T - 2(w^T)^T w^T = I - 2ww^T = F$ ■

Orthogonal: $FF^T = FF$

$$= (I - 2ww^T)(I - 2ww^T)$$

$$= I - 4ww^T + 4w(w^T w)w^T$$

$$= I - 4ww^T + 4ww^T$$

$$= I$$

$FF^T = I \Rightarrow F$ is orthogonal ■

Eigenvectors / Eigenvalues:

$$Fx = \lambda x \Rightarrow x = Ix = F^2 x = \lambda Fx = \lambda^2 x$$

$$\Rightarrow x = \lambda^2 x$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

$$Fx = -x \Rightarrow x - 2ww^T x = x \quad (\lambda = -1)$$

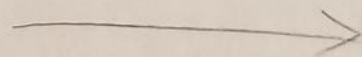
$$\Rightarrow x = w^T x w$$

$$\Rightarrow \boxed{x = \alpha w \text{ is an eigenvector for } \alpha \neq 0}$$

$$Fx = x \Rightarrow x - 2ww^T x = x \quad (\lambda = +1)$$

$$\Rightarrow (w^T x)w = 0$$

$$\Rightarrow \boxed{\text{Any vector orthogonal to } w \text{ is an eigenvector}}$$



$$\textcircled{2} \quad A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \text{Initialize } Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$V_1 = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8.4721 \\ 2 \\ 0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 8.4721 \\ 2 \\ 0 \end{bmatrix} \frac{1}{8.7050} = \begin{bmatrix} 0.9732 \\ 0.2298 \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0.9732 \\ 0.2298 \\ 0 \end{bmatrix} \begin{bmatrix} 0.9732 & 0.2298 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & 0.447 & -0.447 \\ 0 & -1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0.973 \\ 0.230 \\ 0 \end{bmatrix} \begin{bmatrix} 0.973 & 0.230 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.894 & -0.447 & 0 \\ -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0.447 \\ -1 \end{bmatrix}$$

$$V_2 = \frac{1}{1.095} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.447 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.543 \\ -1 \end{bmatrix}$$

$$V_2 = \frac{1}{1.838} \begin{bmatrix} 1.543 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix}$$

$$R = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & 0.447 & -0.447 \\ 0 & -1 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix} \begin{bmatrix} 0.839 & -0.544 \end{bmatrix} \begin{bmatrix} 0.447 & -0.447 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.894 & -0.447 & 0 \\ -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix} \begin{bmatrix} 0.839 & -0.544 \end{bmatrix} \begin{bmatrix} -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ -0.408 & 0.817 & 0.408 \end{bmatrix} \rightarrow$$

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James Hahn
MATH1080
Homework #3

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$$x = 0$$

$$v_3 = 0(0)[1] + 0 = 0$$

$$v_3 = 0/0 = 0$$

Answer

$$R = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix} - 2(0)(0)(0) = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ -0.408 & 0.817 & 0.408 \end{bmatrix} - 2(0)(0)[0 \ 0 \ 0] = \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = Q^T = \begin{bmatrix} -0.894 & 0.183 & 0 \\ -0.447 & -0.365 & 0 \\ 0 & 0.913 & 0 \end{bmatrix}$$

③ Steps

① $x = (A^T A)^{-1} A^T b$

② $r = \|Ax - b\|_2$

$$A = \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

①: $x = \left(\begin{bmatrix} 2 & -2 & 0 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & -2 & 0 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 9 & -1 \\ -1 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.112 & 0.007 \\ 0.007 & 0.067 \end{bmatrix} \begin{bmatrix} 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.336 \\ 1.022 \end{bmatrix}$$

②: $r = \|Ax - b\| = \left\| \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.336 \\ 1.022 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 1.716 \\ 0.396 \\ -1.978 \\ -2.642 \end{bmatrix} \right\|_2$

$$= \sqrt{(1.716)^2 + (0.396)^2 + (-1.978)^2 + (-2.642)^2} = 3.741$$

④ a. Let $C = AB$ where A and B are upper triangular matrices.

So, A_{ij}, B_{ij} are nonzero if $i \leq j$

We know $C_{ij} = \sum_k A_{ik} B_{kj}$.

① If $i > k$, then $a_{ik} = 0$, so $a_{ik} b_{kj} = 0$.

② If $k > j$, then $b_{kj} = 0$, so $a_{ik} b_{kj} = 0$.

③ If $i \leq k \leq j$, then $a_{ik} b_{kj} \neq 0$.

So, $\forall i \leq j, C_{ij} \neq 0$ and $\forall j < i, C_{ij} = 0$.

By definition, C is upper triangular. ■

b. Let $A_{n \times n}$ be an invertible upper triangular matrix.

Therefore, A can be represented as the product of p upper triangular matrices, where p is an arbitrary integer, that equal the identity matrix.

$$\text{So, } B_p B_{p-1} \cdots B_2 B_1 A = I_{n \times n}$$

$$\Rightarrow B_p B_{p-1} \cdots B_2 B_1 A A^{-1} = A^{-1}$$

$$\Rightarrow A^{-1} = B_p B_{p-1} \cdots B_2 B_1$$

By part (a), since $\forall i B_i$ is upper triangular, their product A^{-1} is also upper triangular.

So, A^{-1} is upper triangular. ■