Math 1080: Spring 2019

Homework #2

Due Jan 25

Problem 1:

Find the orthogonal projector P onto range(A) where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$$

What is the nullspace of P? What is the image under P of the vector $[3 \quad 3 \quad 0]^T$?

Problem 2:

Let A be $m \times n$ matrix with $m > n$, and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization. Show that A has full rank if and only if all the diagonal entries of $\hat{R}$ are nonzero.

Problem 3:

Using Gram-Schmidt orthogonalization compute the QR factorization of the following matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & -4 \\ -2 & 6 & 5 \end{bmatrix}$$

Problem 4: Show that if P is a projector, then $\|P\|_2 \geq 1$.

(Hint: For (a), take an arbitrary vector and decompose as $x = Px + (I - P)x$. Use the triangle inequality to conclude that $\|x\|_2 \leq \|Px\|_2$. Now use the definition of $\|P\|_2$ to conclude the result.)

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① $P = A(A^TA)^{-1}A^T$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{12}\begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\text{null}(P) = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix} \xrightarrow{R_3 - R_2 - R_1} \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{by } 3]{\text{mult.}} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{2}{3}R_2]{\frac{1}{2}R_1 + \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_3 = t$, then $x_1 = -x_3 = -t$, $x_2 = -x_3 = -t$

So $x = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} \Rightarrow \text{span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\} = \text{null}(P)$

The image under P of $[3\ \ 3\ \ 0]^T = v$ is

$$Pv = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(2) WTS both directions:

1) if A has full rank, then all diagonal entries of R are nonzero.

2) if all diagonal entries of R are nonzero, then A has full rank.

First, we will show 1) ⇒ 2):

Let the diagonal entries of R be nonzero.

Let v be s.t. $Av = 0$

Then $\hat{Q}^T\hat{Q}\hat{R}c = \hat{R}c = 0$.

Because of back substitution, $c_k = 0 \ \forall k \in [1, n]$.

So, $c = 0$ and A is linearly indep. with full rank.

Now, we will show 2) ⇒ 1):

We will prove the contrapositive, or the fact that one nonzero diagonal entry in R implies a non-full rank of A, is true.

Let $A = \hat{Q}\hat{R}$ be reduced QR factorization of A.

Let k be the smallest number s.t. $r_{kk} = 0$.

If $k = 1$, then $a_1 = 0$ and A doesn't have full rank.

If $k > 1$, then a_k is a linear combination of $q_1 \cdots q_{k-1}$.

Since $r_{ii} \neq 0 \ \forall i \in [1, k-1]$, $\hat{R}_{1:k-1,1:k-1}$ has all nonzero diagonal entries and the 1^{st} to $(k-1)^{th}$ columns of A are lin. indep., and they are a basis for the first $(k-1)$ columns.

So, a_k is a linear combination of $a_{1:k-1}$ and A doesn't have full rank.

So, the contrapositive is true, therefore 2) is true.

We have shown both directions, so A has full rank iff all diagonal entries of $\hat{R}$ are nonzero. ■

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(3) $r_{11} = \|a_1\| = \sqrt{1+4+4} = 3$

$$q_1 = \frac{a_1}{r_{11}} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$r_{12} = q_1^T a_2 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = -6$$

$$v_2 = a_2 - r_{12} q_1 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + 6 \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$r_{22} = \|v_2\| = \sqrt{4+1+4} = 3$$

$$q_2 = \frac{v_2}{r_{22}} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$r_{13} = q_1^T a_3 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = 0$$

$$r_{23} = q_2^T a_3 = \begin{bmatrix} 2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = 6$$

$$v_3 = a_3 - r_{13} q_1 - r_{23} q_2 = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} - 0 \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} - 6 \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$r_{33} = \|v_3\| = \sqrt{4+4+1} = 3$$

$$q_3 = \frac{v_3}{r_{33}} = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & -1/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \qquad R = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$\longrightarrow$

(4) WTS $\|P\|_2 \geq 1$ where P is a projector.

Let x be an arbitrary vector.

Since P is a projector, $P^2 = P$.

$\|Px\| = \|P^2 x\|$ since $P^2 = P$

$\|P^2 x\| = \|P(Px)\|$

$\|P(Px)\| \leq \|P\| \, \|Px\|$ by Cauchy-Schwarz

$$\frac{\|P(Px)\|}{\|Px\|} \leq \|P\|$$

$$\frac{\|P^2 x\|}{\|Px\|} \leq \|P\|$$

$$\frac{\|Px\|}{\|Px\|} \leq \|P\| \quad \text{since } P^2 = P$$

$1 \leq \|P\|$

$\|P\|_2 \geq 1$ ■