Math 1080: Spring 2019

Homework #1

Due Jan 18

Problem 1:

Let B be a 4x4 matrix to which we apply the following operations:

1. Double column 3
2. Add row 2 to row 1
3. Interchange columns 2 and 3
4. Halve row 4
5. Replace column 4 by sum of columns 1 and 3

Each of these operations can be performed by multiplying B on the left or on the right by a specific matrix E_k (where k stands for the operation above) Find the matrices E_k. Then find matrices A and C such that the result is obtained as a product ABC

Problem 2:

Consider the matrix

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

Show that Q is an orthogonal matrix. What transformation of $\mathbb{R}^3$ does it correspond to?

(Hint: Find the vector a that is invariant under Q. Pick a vector b orthogonal to a. Find the angle α between b and Qb. If this angle is independent of the choice of b, then Q corresponds to a rotation about a by the angle α. Think about other possibilities.)

Problem 3:

Find the 2x2 orthogonal matrix Q that corresponds to the reflection over the line $2x - 3y = 0$.

Problem 4:

Let u, v be two vectors and $A = I + uv^T$ a matrix. Show that if A is invertible, its inverse is the matrix $A^{-1} = I + \alpha uv^T$ and find the scalar α.When is A singular?

Problem 5:

(a) Compute the norms $\|w\|_1, \|w\|_2, \|w\|_\infty$ for the vector $w = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

(b) Compute the norms $\|A\|_1, \|A\|_2, \|A\|_\infty$ for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$

(c) Verify the inequalities $\|Aw\|_p \leq \|A\|_p \|w\|_p$.

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(1) 1. $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Order: BE_1

2. $E_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Order: E_2B

3. $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Order: BE_3

4. $E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$ Order: E_4B

5. $E_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Order: BE_5

Final order of operations: $\underbrace{E_4 A_2}_{A} \; B \; \underbrace{E_1 E_3 E_5}_{C}$

$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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(2) Q is orthogonal $\Rightarrow Q^TQ = I$ $\leftarrow$ WTS

$$Q = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \quad Q^T = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$ It is orthogonal

$$Q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2x - y + 2z \\ 2x + 2y - z \\ -x + 2y + 2z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -x - y + 2z \\ 2x - y - z \\ -x + 2y - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow (Q - I)a = 0 \Rightarrow a = \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, λ is a constant

Let $\lambda = 1 \Rightarrow a = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$a \perp b$ iff $a \cdot b = 0$

Let $b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. $a \cdot b_1 = 1 + 0 - 1 = 0 \checkmark$

$Qb_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\cos\theta = \frac{b_1 \cdot Qb_1}{\|b_1\|\,\|Qb_1\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Let $b_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $a \cdot b_2 = 0 + 1 - 1 = 0 \checkmark$ and linearly independent from b_1

$Qb_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\cos\theta = \frac{b_2 \cdot Qb_2}{\|b_2\|\,\|Qb_2\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$\boxed{\text{Q corresponds to a rotation about } a = \lambda\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ by the angle } \frac{\pi}{3}}$

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(3) $2x - 3y = 0$

$3y = 2x$

$y = \frac{2}{3}x$

$a = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is on the line ; find angle from x-axis $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = b$

$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|} = \frac{3}{\sqrt{13}} \Rightarrow \theta = 33.69°$

$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ is the standard reflection matrix

So, the 2×2 orthogonal matrix is $\boxed{\begin{bmatrix} 0.3846 & 0.9231 \\ 0.9231 & -0.3846 \end{bmatrix}} = A$

To check orthogonality, $A^T A = I \Rightarrow \begin{bmatrix} 0.3846 & 0.9231 \\ 0.9231 & -0.3846 \end{bmatrix}\begin{bmatrix} 0.3846 & 0.9231 \\ 0.9231 & -0.3846 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

(4) $A = I + uv^T$

$A^{-1} = I + \alpha uv^T$

$AA^{-1} = (I + uv^T)(I + \alpha uv^T)$

$= I + uv^T + \alpha uv^T + uv^T \alpha uv^T$

$= I + (1+\alpha)uv^T + \alpha uv^T uv^T$

$= I + (1 + \alpha + \alpha(v^T u))uv^T$

Need $(1 + \alpha(1 + v^T u)) = 0$

$\Rightarrow \alpha = \frac{-1}{1 + v^T u}$

If A is invertible, $AA^{-1} = I$ when $\alpha = \frac{-1}{1+v^T u}$, proving

$A^{-1} = I + \alpha uv^T$. ∎

A is singular when α doesn't exist, or when $v^T u = -1$.

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5. a. $||w||_1 = \sum |w_i| = |3| + |-1| + |5| = 9$

$||w||_2 = \sqrt{3^2 + (-1)^2 + 5^2} = \sqrt{35}$

$||w||_\infty = \max |w_i| = 5$

b. $||A||_1 = \max\{|2| + |-1|, |-1| + 0, |1| + |2|\} = 3$

$||A||_2 = \max\{|2| + |-1| + |1|, |-1| + 0 + |2|\} = 4$

$||A||_\infty \Rightarrow A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix}$

$\det(A^T A - \lambda I) = -\lambda^3 + 11\lambda^2 - 30\lambda \Rightarrow \lambda = 0, 5, 6$

$\max\{\sqrt{0}, \sqrt{5}, \sqrt{6}\} = \sqrt{6}$

c. $Aw = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$

$||Aw||_1 = |12| + |7| = 19$

$||Aw||_2 = \sqrt{(12)^2 + (7)^2} = \sqrt{193}$

$||Aw||_\infty = \max\{12, 7\} = 12$

1) $||Aw||_1 \leq ||A||_1 ||w||_1$

$19 \leq 3(9)$

$19 \leq 27$ ✓

2) $||Aw||_2 \leq ||A||_2 ||w||_2$

$\sqrt{193} \leq 4\sqrt{35}$

$13.89 \leq 23.66$ ✓

3) $||Aw||_\infty \leq ||A||_\infty ||w||_\infty$

$12 \leq \sqrt{6}(5)$

$12 \leq 12.25$ ✓