Math 1080: Spring 2019

Homework #8 (due April 5)

Problem 1:

Find the diagonalization $A = X\Lambda X^{-1}$ of the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 \\ -2 & 0 & 4 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}$$

Problem 2:

Find the Schur factorization $A = QTQ^T$ for the following matrix.

(Hint: Follow the proof of existence of the factorization.)

$$A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

Problem 3:

Calculate the Rayleigh quotients $r_k = r(x_k)$ for the following matrix A and given vectors x_k. How far is each r_k from the closest eigenvalue of A?

$$A = \begin{bmatrix} 4 & 6 & 1 \\ 6 & 4 & 6 \\ 1 & 6 & 4 \end{bmatrix},$$

$$x_1 = \begin{bmatrix} 1.5 \\ 2 \\ 1 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 1 \\ 2.1 \\ 1 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} 1 \\ 0 \\ -1.1 \end{bmatrix}, \qquad x_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 4:

Let A be a symmetric matrix and let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be its eigenvalues. Show that for any $x \neq 0$ the Rayleigh quotient $r(x) = \frac{x^T A x}{x^T x}$ obeys $\lambda_1 = \min_{x \neq 0} r(x)$ and $\lambda_n = \max_{x \neq 0} r(x)$.

(Hint: Use orthogonal diagonalization of the matrix A.)

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(1) $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 \\ -2 & 0 & 4 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ 0 & -1-\lambda & 3 & 1 \\ -2 & 0 & 4-\lambda & 0 \\ 2 & -2 & 1 & 2-\lambda \end{bmatrix} \Rightarrow \det(A-\lambda I) = \lambda^4 - 8\lambda^3 + 19\lambda^2 - 12\lambda$

$\lambda^4 - 8\lambda^3 + 19\lambda^2 - 12\lambda = 0 \Rightarrow \lambda(\lambda-1)(\lambda-3)(\lambda-4) = 0 \Rightarrow \lambda = 0, 1, 3, 4$

$\lambda_1 = 0: \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 1 \\ -2 & 0 & 4 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 - x_4 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{matrix} x_1 = 0 \\ x_3 = 0 \\ x_2 = x_4 \end{matrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ x_4 \\ 0 \\ x_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = V_1$

$\lambda_2 = 1: \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 3 & 1 \\ -2 & 0 & 3 & 0 \\ 2 & -2 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_3 = 0 \\ x_2 = 1/2\, x_4 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2\, x_4 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} = V_2$

$\lambda_3 = 3: \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 1 \\ -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & -1 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 1 \\ -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & -1 \end{bmatrix} Y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} y = 0 \Rightarrow \begin{cases} x_1 = 3/2\, x_4 \\ x_2 = 5/2\, x_4 \\ x_3 = 3x_4 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3/2\, x_4 \\ 5/2\, x_4 \\ 3x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/2 \\ 3 \\ 1 \end{bmatrix} = V_3$

$\lambda_4 = 4: \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -5 & 3 & 1 \\ -2 & 0 & 0 & 0 \\ 2 & -2 & 1 & -2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -7x_4 \\ x_3 = -12x_4 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -7x_4 \\ -12x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -12 \\ 1 \end{bmatrix} = V_4$

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$$X = \begin{bmatrix} 0 & 0 & 3/2 & 0 \\ 1 & 1/2 & 5/2 & -7 \\ 0 & 0 & 3 & -12 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

2) $A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 1 & 4 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 4-\lambda & -2 & 1 \\ -2 & 4-\lambda & 2 \\ 1 & 1 & 4-\lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = -\lambda^3 + 12\lambda^2 - 41\lambda + 30 = 0 \Rightarrow \lambda = 1, 5, 6$

$\lambda_2 = 5$: $A - \lambda I = \begin{bmatrix} -1 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 7/5 \\ 8/5 \\ -1 \end{bmatrix}$

We need to find three orthonormal vectors to x:

We get $y = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -29 \\ 26 \\ 1 \end{bmatrix} = z$

Now, normalize all 3 to get the basis: $Q = \begin{bmatrix} 7/\sqrt{138} & \frac{1}{\sqrt{11}} & -\frac{29}{\sqrt{1518}} \\ (4\sqrt{138})/69 & \frac{1}{\sqrt{11}} & (13\sqrt{1518})/759 \\ -5/\sqrt{138} & 3/\sqrt{11} & \frac{1}{\sqrt{1518}} \end{bmatrix}$

Now, $A = QTQ^T$, so $T = Q^T A Q$:

$$T = \begin{bmatrix} 7/\sqrt{138} & (4\sqrt{138})/69 & -5/\sqrt{138} \\ 1/\sqrt{11} & 1/\sqrt{11} & 3/\sqrt{11} \\ -29/\sqrt{1518} & (13\sqrt{1518})/759 & 1/\sqrt{1518} \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 7/\sqrt{138} & 1/\sqrt{11} & -29/\sqrt{1518} \\ (4\sqrt{138})/69 & 1/\sqrt{11} & (13\sqrt{1518})/759 \\ -5/\sqrt{138} & 3/\sqrt{11} & 1/\sqrt{1518} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -.3015 & .7443 \\ 0 & 6 & -.5959 \\ 0 & 0 & 5 \end{bmatrix}.$$

I checked and made sure $A = QTQ^T$ with an online calculator.

$$Q = \begin{bmatrix} 0.5959 & 0.3015 & -0.7443 \\ 0.6810 & 0.3015 & 0.6673 \\ -.4256 & 0.9045 & 0.0257 \end{bmatrix} \quad T = \begin{bmatrix} 1 & -.3015 & .7443 \\ 0 & 6 & -.5959 \\ 0 & 0 & 5 \end{bmatrix}$$

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(3) $A = \begin{bmatrix} 4 & 6 & 1 \\ 6 & 4 & 6 \\ 1 & 6 & 4 \end{bmatrix} \Rightarrow (A-\lambda I) = \begin{bmatrix} 4-\lambda & 6 & 1 \\ 6 & 4-\lambda & 6 \\ 1 & 6 & 4-\lambda \end{bmatrix}$

$\det(A-\lambda I) = -\lambda^3 + 12\lambda^2 + 25\lambda - 156 \Rightarrow \lambda = 3, -4, 13$

$r(x) = \frac{x^T A x}{x^T x}$

$x_1^T x_1 = 7.25$, $x_1^T A x_1 = 92 \Rightarrow r_1(x) = \frac{92}{7.25} = 12.6897$

$x_2^T x_2 = 6.41$, $x_2^T A x_2 = 78.04 \Rightarrow r_2(x) = \frac{78.04}{6.41} = 12.1747$

$x_3^T x_3 = 2.21$, $x_3^T A x_3 = 6.64 \Rightarrow r_3(x) = \frac{6.64}{2.21} = 3.0045$

$x_4^T x_4 = 3$, $x_4^T A x_4 = 38 \Rightarrow r_4(x) = \frac{38}{3} = 12.6667$

r_3 is very close to λ_1 ($|r_3 - \lambda_1| = 0.0045$)
r_1, r_2, and r_4 are all kind of close to λ_3 $\left(\begin{aligned} |r_1 - \lambda_3| &= 0.3103 \\ |r_2 - \lambda_3| &= 0.8253 \\ |r_4 - \lambda_3| &= 0.3333 \end{aligned} \right)$

(4) Proof

We know $A = Q\Lambda Q^T$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and where $q_1, \dots, q_n$ are orthogonal.

Let $x = \sum_{i=1}^{n} x_i q_i$ and the rayleigh quotient $r(x) = \frac{x^T A x}{x^T x}$.

Let $Ax = A\sum x_i q_i = \sum x_i A q_i$. We know $Aq_i = \lambda q_i$, so $\sum x_i A q_i = \sum x_i \lambda_i q_i$.

Finally, $x^T A x = x^T \sum x_i \lambda_i q_i$. Let's solve this for a simpler, two-dimensional case:

$$x^T \sum_{i=1}^{2} x_i \lambda_i q_i = (x_1 q_1 + x_2 q_2)^T (x_1 \lambda_1 q_1 + x_2 \lambda_2 q_2) + (x_1^T q_1)(x_1 \lambda_1 q_1)$$
$$= (x_1^T q_1)(x_1 \lambda_1 q_1) + (x_1^T q_1)(x_2 \lambda_2 q_2) + (x_2^T q_2)(x_1 \lambda_1 q_1) + (x_2^T q_2)(x_2 \lambda_2 q_2)$$
$$(*) = \lambda_1 (x_1^T q_1)(x_1 q_1) + \lambda_2 (x_1^T q_1)(x_2 q_2) + \lambda_1 (x_2^T q_2)(x_1 q_1) + \lambda_2 (x_2^T q_2)(x_2 q_2)$$

cont. →

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Let $\lambda' = \max\{\lambda_1, \lambda_2\}$

$$(*) = \lambda'\left[(x_1^T q_1)(x_1 q_1) + (x_1^T q_1)(x_2 q_2) + (x_2^T q_2)(x_1 q_1) + (x_2^T q_2)(x_2 q_2)\right]$$

$$= \lambda'\left(\sum x_i q_i\right)^T\left(\sum x_i q_i\right).$$ This is the numerator of $r(x)$.

Now, the denominator, $x^T x = (\sum x_i q_i)^T(\sum x_i q_i)$

So, $r(x) = \dfrac{\lambda'(\sum x_i q_i)^T(\sum x_i q_i)}{(\sum x_i q_i)^T(\sum x_i q_i)} = \lambda'$.

Since λ' is the max of $\{\lambda_1, \dots, \lambda_n\}$, which is λ_n, then $\max r(x) = \lambda_n$.

If we set $\lambda' = \min\{\lambda_1, \dots, \lambda_n\} = \lambda_1$, then $\min r(x) = \lambda_1$. ∎