Math 1080: Spring 2019

Homework #4

Due Feb 15

In all problems below let A be $m \times n$ matrix, B be $n \times n$ matrix, and v be a vector in $\mathbb{R}^n$.

Problem 1:

Exactly how many flops are needed to perform the following lines of code?

a) `d = A*v;`
b) `C = A*B;`
c) `x = v'*B*v;` (in Matlab syntax `v'` represents the transpose of v.)
d) `x = A(1:n,:)*(B*v);`
e) `x = (A(1:n,:)*B)*v;` (this case differs from d) in the order of products)

Problem 2:

Exactly how many flops are needed to execute the following code segments?

a)
```
for k = 1:n
    a(k) = B(k,k)*v(n-k);
end
```

b)
```
for k = 1:n
    x = B(k,n-k+1:n)*v(n-k+1:n);
end
```

c)
```
for k = 1:n-1
    for j = n-k:n
        c = c + B(:,1:j)*A(n-j+1:n,k);
    end
end
```

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$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^n$

(1) a. $d = Av = \begin{bmatrix} a_1 \\ \hline \vdots \\ \hline a_m \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_1 v \\ \vdots \\ a_m v \end{bmatrix}$

$a_i v = \sum_{j=1}^{n} a_{ij} v_j$ = $(n-1)$ adds + (n) mults = $n+n-1 = 2n-1$ flops

Perform $a_i v$ m times $\Rightarrow (2n-1)m = \boxed{2mn - m}$

b. $C = AB = \begin{bmatrix} a_1 \\ \hline \vdots \\ \hline a_m \end{bmatrix} [b_1 | \dots | b_n] = C_{m \times n}$

$a_i b_j = \sum_{p=1}^{n} a_{ip} b_{pj}$ = $(n-1)$ adds + (n) mults = $2n-1$ flops

C will have $m \times n$ elements $\Rightarrow$ perform $a_i b_j$ $m \times n$ times

$\Rightarrow (2n-1)(mn) = \boxed{2mn^2 - mn}$

c. $x = v'Bv = [v_1 \dots v_n] \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$v' \Rightarrow \forall i,j \in [1,n]\ v_{ij} = v_{ji} \Rightarrow$ n flops

$v'B \Rightarrow$ dot product $a_i b_j = 2n-1$ flops n times $= (2n-1)n = 2n^2 - n$

$\Rightarrow (v'B)_{1 \times n}$

$(v'B)v \Rightarrow$ inner product $[v_1' \dots v_n'] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow 2n-1$ flops

$\Rightarrow$ n flops + $(2n^2 - n)$ flops + $(2n-1)$ flops

$= n + 2n^2 - n + 2n - 1 = 2n^2 + 2n - 1 = \boxed{2n(n+1) - 1}$

d. $x = A(1:n, :)(Bv)$

$\Rightarrow Bv$ is similar to Av in part (a), but replace m by n

$\Rightarrow 2n^2 - n$ where $(Bv) \in \mathbb{R}^{n \times 1}$.

$\Rightarrow$ Let $Bv = c_{n \times 1}$. $A(1:n, :)c$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$ flops

$\Rightarrow (2n^2 - n)$ flops + $(2n^2 - n)$ flops $= 2n^2 - n + 2n^2 - n$

$= \boxed{4n^2 - 2n}$ flops

→

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n$

1 cont'd

e. $x = (A(1:n,:)B)v$

$\Rightarrow A(1:n,:)B$ is the same as part (b), but replace m by n

$\Rightarrow 2n^3 - n^2$. Let $(A(1:n,:)B) = C_{n\times n}$

$\Rightarrow C_v$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$

$\Rightarrow (2n^3 - n^2)$ flops + $(2n^2 - n)$ flops = $2n^3 - n^2 + 2n^2 - n$

= $2n^3 + n^2 - n$ flops

(2) a. for $k = 1:n$ } loop n times
$a(k) = B(k,k) * v(n-k);$ } 1 mult
end

($B(k,k)$: scalar, $v(n-k)$: scalar)

$n * 1 =$ n flops

b. for $k = 1:n$ } loop n times
$x = B(k, n-k+1:n) * v(n-k+1:n)$
end

$$\sum_{k=1}^{n} \underbrace{\sum_{i=n-k+1}^{n} B_{ki} v_i}_{\substack{k-1 \text{ adds} \\ k \text{ mults}}\ \} 2k-1 \text{ flops}} = \sum_{k=1}^{n} 2k-1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\frac{n(n+1)}{2} - n$$

$$= n(n+1) - n = n^2$$

c. for $k = 1:n-1$
for $j = (n-k):n$ ($n\times j$ matrix; column vector with $j\times 1$ dimensions)
$c = c + B(:,1:j)A(n-j+1:n,k);$
end
end

($c = c + \ldots$: (n) adds; $B(:,1:j)A(n-j+1:n,k)$: outer product $n\times 1$ $\Rightarrow$ $(j-1)$ adds + (j) mults for $<>$ n times $\Rightarrow n(2j-1)$)

$$\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j-1) + n = \sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j) = 2n\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} j = 2n\sum_{k=1}^{n-1}\left(\sum_{j=1}^{n} j - \sum_{j=1}^{n-k-1} j\right)$$

$$= n\sum_{k=1}^{n-1} 2n(k+1) - k^2 - k = 2n^2\sum_{k=1}^{n-1}(k) + 2n^2\sum_{k=1}^{n-1} 1 - n\sum_{k=1}^{n-1} k^2 - n\sum_{k=1}^{n-1} k$$

$$= 2n^2\frac{(n-1)n}{2} + 2n^2(n-1) - n\frac{(n-1)(n)(2n-1)}{6} - n\frac{(n-1)n}{2}$$

$$= n^4 + n^3 - 2n^2 - \frac{n^2(n-1)(n+1)}{3} \text{ flops}$$

($\frac{n(n+1)}{2}$: $\sum_{j=1}^{n} j$; $\frac{(n-k-1)(n-k)}{2}$: $\sum_{j=1}^{n-k-1} j$; $\frac{2n(k+1)-k^2-k}{2}$)