

① $P = A(A^T A)^{-1} A^T$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{12} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\text{null}(P) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \xrightarrow{R_3 - R_2 - R_1} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{mult. by 3}} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

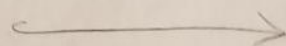
$$\xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{2}{3}R_2} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 + \frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_3 = t$, then $x_1 = -x_3 = -t$, $x_2 = -x_3 = -t$

$$\text{So } x = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} \Rightarrow \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} = \text{null}(P)$$

The image under P of $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T = v$ is

$$Pv = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$



② WTS both directions:

1) if A has full rank, then all diagonal entries of R are nonzero.

2) if all diagonal entries of R are nonzero, then A has full rank.

First, we will show $1) \Rightarrow 2)$:

Let the diagonal entries of R be nonzero.

Let v be s.t. $Av = 0$.

Then $\hat{Q}^T \hat{Q} \hat{R} c = \hat{R} c = 0$.

Because of backsubstitution, $c_k = 0 \quad \forall k \in [1, n]$.

So, $c = 0$ and A is linearly indep. with full rank.

Now, we will show $2) \Rightarrow 1)$:

We will prove the contrapositive, or the fact that one nonzero diagonal entry in R implies a non-full rank of A , is true.

Let $A = \hat{Q} \hat{R}$ be reduced QR factorization of A .

Let k be the smallest number s.t. $r_{kk} = 0$.

If $k=1$, then $a_1 = 0$ and A doesn't have full rank.

If $k > 1$, then a_k is a linear combination of q_1, \dots, q_{k-1} .

Since $r_{ii} \neq 0 \quad \forall i \in [1, k-1]$, $\hat{R}_{1:k-1, 1:k-1}$ has all nonzero diagonal entries and the 1^{st} to $(k-1)^{\text{th}}$ columns of A are lin. indep. ^{and} they are a basis for the first $(k-1)$ columns.

So, a_k is a linear combination of $a_{1:k-1}$ and A doesn't have full rank.

So, the contrapositive is true, therefore $2)$ is true.

We have shown both directions, so A has full rank iff all diagonal entries of \hat{R} are nonzero. ■

$$(3) \quad r_{11} = \|a_1\| = \sqrt{1+4+4} = 3$$

$$q_1 = \frac{a_1}{r_{11}} = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$r_{12} = q_1^T a_2 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = -6$$

$$v_2 = a_2 - r_{12} q_1 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + 6 \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$r_{22} = \|v_2\| = \sqrt{4+1+4} = 3$$

$$q_2 = \frac{v_2}{r_{22}} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$r_{13} = q_1^T a_3 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = 0$$

$$r_{23} = q_2^T a_3 = \begin{bmatrix} 2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} = 6$$

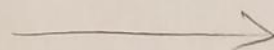
$$v_3 = a_3 - r_{13} q_1 - r_{23} q_2 = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} - 0 \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} - 6 \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$r_{33} = \|v_3\| = \sqrt{4+4+1} = 3$$

$$q_3 = \frac{v_3}{r_{33}} = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & -1/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$R = \begin{bmatrix} 3 & -6 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$



④ WTS $\|P\|_2 \geq 1$ where P is a projector.

Let x be an arbitrary vector.

Since P is a projector, $P^2 = P$.

$$\|Px\| = \|P^2x\| \quad \text{since } P^2 = P$$

$$\|P^2x\| = \|P(Px)\|$$

$$\|P(Px)\| \leq \|P\| \|Px\| \quad \text{by Cauchy-Schwarz}$$

$$\frac{\|P(Px)\|}{\|Px\|} \leq \|P\|$$

$$\frac{\|P^2x\|}{\|Px\|} \leq \|P\|$$

$$\frac{\|Px\|}{\|Px\|} \leq \|P\| \quad \text{since } P^2 = P$$

$$1 \leq \|P\|$$

$$\|P\|_2 \geq 1 \quad \blacksquare$$