James Hahn
Homework #5
MATH1080

1 a. absolute condition #

$\hat{K}(x) = |J(x)| = \boxed{\frac{2}{x}\ln(x)}$

relative condition #

$K(x) = |J| \frac{\|x\|}{\|F(x)\|} = \frac{2}{x}\ln(x)\frac{x}{\ln^2 x}$

$= \boxed{\frac{2}{\ln x}}$

Well-conditioned when x is large because $\lim_{x\to\infty} \frac{2}{\ln x} = 0$.
Also, x should be >> 10^6 because of the natural log.
Ill-conditioned for both, small and negative x.
I included the negative condition because $\ln(-x)$ is undef.

b. absolute condition #

$\delta f = \|x+\delta x\| - \|x\|$
$\leq \|x\| + \|\delta x\| - \|x\|$
$\leq \|\delta x\|$

$\hat{K}(x) = \frac{\|\delta f\|}{\|\delta x\|} \leq \frac{\|\delta x\|}{\|\delta x\|} = \boxed{1}$

relative condition #

$K(x) = \frac{\frac{\|\delta f\|}{\|F(x)\|}}{\frac{\|\delta x\|}{\|x\|}} \leq \frac{\frac{\|\delta x\|}{\|x\|}}{\frac{\|\delta x\|}{\|x\|}} = \boxed{1}$

Well-conditioned for all x because 1 is small.

c. absolute condition #

$J(x) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ a_{22} & -a_{12} & -a_{21} & a_{11} \end{bmatrix}_{2\times 4}$

$\hat{K}(x) = \|J(x)\|_\infty = \boxed{\max \sum_{i=1}^{m} J_{i,:}}$

Let $f(x) = [a_{11}+a_{22} \quad a_{11}a_{22}-a_{21}a_{12}]$
Let $x = [a_{11}\ a_{21}\ a_{12}\ a_{22}]$

relative condition #

$K(x) = \|J(x)\|_\infty \frac{\|x\|_\infty}{\|F(x)\|_\infty}$

$= \boxed{\max \sum_{i=1}^{m} J_{i,:} \frac{\max(x)}{\max(tr(x), det(x))}}$

Well-conditioned when all values of x are small
Ill-conditioned when $tr(x) = det(x) = 0$ and when **any value of x is large**

d. absolute condition #

$J(x,y) = \begin{bmatrix} y & x \\ 2x & 0 \\ 0 & 2y \\ y & x \end{bmatrix}$

Let $f(x,y) = [xy \quad x^2 \quad y^2 \quad xy]$

$\hat{K}(x,y) = \|J(x,y)\|_1 = \max\{2x+2y, 2x+2y\} = \boxed{|2x+2y|}$

relative condition #

$K(x,y) = \|J(x,y)\|_1 \frac{\|[x\ y]\|_1}{\|F(x,y)\|_1}$

$= |2x+2y| \frac{|x+y|}{|xy+x^2+y^2+xy|} = 2\frac{(x+y)^2}{(x+y)^2}$

$= \boxed{2}$

Well-conditioned for all values of x and y

(2) a. $\tilde{f}(x,y) = (fl(x) \otimes fl(x)) \ominus (fl(y) \otimes fl(y))$

$= \left([x(1+\varepsilon_1)\,x(1+\varepsilon_1)](1+\varepsilon_3) - [y(1+\varepsilon_2)\,y(1+\varepsilon_2)](1+\varepsilon_4)\right)(1+\varepsilon_5)$

$= \left(x^2(1+\varepsilon_1)^2(1+\varepsilon_3)(1+\varepsilon_5) - y^2(1+\varepsilon_2)^2(1+\varepsilon_4)(1+\varepsilon_5)\right)$ $|\varepsilon_1||\varepsilon_2||\varepsilon_3||\varepsilon_4||\varepsilon_5| \le \varepsilon_{mach}$

$= (x(1+2\varepsilon_6))^2 - (y(1+2\varepsilon_7))^2$ $|\varepsilon_6||\varepsilon_7| \le 2\varepsilon_{mach}$

$= \tilde{x}^2 - \tilde{y}^2$

$C = 2$

$= f(\tilde{x},\tilde{y})$ $\boxed{\text{Backwards stable}}$

b. $\tilde{f}(x,y) = [fl(x) \oplus fl(y)] \otimes [fl(x) \ominus fl(y)]$

$= [x(1+\varepsilon_1) \oplus y(1+\varepsilon_2)] \otimes [x(1+\varepsilon_1) \ominus y(1+\varepsilon_2)]$ $|\varepsilon_1||\varepsilon_2| \le \varepsilon_{mach}$

$= \left([x(1+\varepsilon_1) + y(1+\varepsilon_2)](1+\varepsilon_3)[x(1+\varepsilon_1) - y(1+\varepsilon_2)](1+\varepsilon_4)\right)(1+\varepsilon_5)$

$= [x(1+\varepsilon_1) + y(1+\varepsilon_2)][x(1+\varepsilon_1) - y(1+\varepsilon_2)](1+\varepsilon_3)(1+\varepsilon_4)(1+\varepsilon_5)$ $|\varepsilon_3||\varepsilon_4||\varepsilon_5| \le \varepsilon_{mach}$

$= ((x(1+\varepsilon_1))^2 - (y(1+\varepsilon_2))^2)(1+\varepsilon_3)(1+\varepsilon_4)(1+\varepsilon_5)$

$= (x(1+\frac{5}{2}\varepsilon_6))^2 - (y(1+\frac{5}{2}\varepsilon_7))^2$ $|\varepsilon_6||\varepsilon_7| \le \frac{5}{2}\varepsilon_{mach}$

$C = \frac{5}{2}$

$= \tilde{x}\tilde{x} - \tilde{y}\tilde{y} = \tilde{x}^2 - \tilde{y}^2$

$= f(\tilde{x},\tilde{y})$ $\boxed{\text{Backwards stable}}$

c. $\tilde{f}(x) = 1 \oslash [1 \oplus fl(x)]$

$= 1 \oslash ([1 + x(1+\varepsilon_1)](1+\varepsilon_2))$ $|\varepsilon_1||\varepsilon_2| \le \varepsilon_{mach}$

$= \dfrac{1}{(1+x(1+\varepsilon_1))(1+\varepsilon_2)}(1+\varepsilon_3)$ $|\varepsilon_3| \le \varepsilon_{mach}$

$= \dfrac{1}{1+x(1+\varepsilon_1)}\left(\dfrac{1+\varepsilon_3}{1+\varepsilon_2}\right)$

$= \dfrac{1}{1+\tilde{x}}(1+\varepsilon_4)$ $|\varepsilon_4| \le 2\varepsilon_{mach}$

$= f(\tilde{x})(1+\varepsilon_4) \Rightarrow$ NOT Backwords stable

$\|\tilde{f}(x) - f(\tilde{x})\| = f(\tilde{x})\varepsilon_4$

D

$\dfrac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = \varepsilon_4 \le 2\varepsilon_{mach}$

C

$\dfrac{\|\tilde{x} - x\|}{\|x\|} \le 1\,\varepsilon_{mach}$

$\boxed{\text{Stable}}$

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3) a. $f(x,y) = x^2 - y^2$

$J(x) = [2x \; -2y] \Rightarrow \|J(x)\| = \sqrt{4x^2+4y^2} = 2\|\mathbf{x}\|$

$K(x) = \|J(x)\| \frac{\|x\|}{\|f(x)\|} = 2\|\mathbf{x}\| \frac{\|\mathbf{x}\|}{|x^2-y^2|} = 2\frac{x^2+y^2}{|x^2-y^2|}$

$\boxed{\text{Accuracy} = \Delta \leq C K(x) \varepsilon_{mach} = 2K(x)\varepsilon_{mach}}$ since the algorithm is backwards stable (this is theorem 15.1 in the book).

b. $f(x,y) = x^2 - y^2$

This is the same algorithm as a) with the same conditioning numbers. Since it is also backwards stable, it has the same asymptotic accuracy: $\Delta \leq O(K(x)\varepsilon_{mach})$ where $K(x)$ can be found in part a). Since its C value is different, its actual accuracy is $\boxed{\Delta \leq \frac{5}{2}K(x)\varepsilon_{mach}}$

c. $f(x) = \frac{1}{1+x}$ $\quad |J(x)| = \frac{1}{(1+x)^2} \Rightarrow K(x) = |J(x)|\frac{|x|}{|f(x)|} = \frac{1}{(1+x)^2}\frac{|x|}{\left|\frac{1}{1+x}\right|}$

In problem 2 part c), we calculated

$\underbrace{\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} \leq 2\varepsilon_{mach}}_{D}$ and $\underbrace{\frac{\|\tilde{x} - x\|}{\|x\|} \leq 1\varepsilon_{mach}}_{C}$

$\boxed{\Delta \leq D\varepsilon_{mach} + CK(x)\varepsilon_{mach} = 2\varepsilon_{mach} + \frac{1}{(1+x)^2}\frac{|x|}{\left|\frac{1}{1+x}\right|}\varepsilon_{mach}}$

d. Additional question: Which is more accurate, a) or b)?

Answer: They both have the same asymptotic accuracy, but a) has the **better** true accuracy since it has a lower C value.