Math 1080: Spring 2019

Homework #3

Due Feb 1

Problem 1:

Show that the Householder reflector $F = I - 2ww^{\mathrm{T}}$, with $\|w\| = 1$, is symmetric and orthogonal. Find the eigenvalues and eigenvectors of F.

Problem 2:

Use Householder triangularization to find the QR factorization of

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Problem 3:

Use QR factorization to solve the least squares minimization problem $r = \min_x \|Ax - b\|_2$ for the following data. Provide both the vector x and the minimum residue r.

$$A = \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Problem 4:

(a) Show that the product of two upper triangular matrices is an upper triangular matrix.
(b) Show that the inverse of an upper triangular matrix (if it exists) is an upper triangular matrix.

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① Symmetry: $F^T = (I - 2ww^T)^T = I^T - 2(w^T)^T w^T = I - 2ww^T = F$ ∎

Orthogonal: $FF^T = FF$

$$= (I - 2ww^T)(I - 2ww^T)$$

$$= I - 4ww^T + 4w(w^Tw)w^T$$

$$= I - 4ww^T + 4ww^T$$

$$= I$$

$FF^T = I \Rightarrow F$ is orthogonal ∎

Eigenvectors / Eigenvalues:

$$Fx = \lambda x \Rightarrow x = Ix = F^2x = \lambda Fx = \lambda^2 x$$

$$\Rightarrow x = \lambda^2 x$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

$Fx = -x \Rightarrow x - 2ww^Tx = x$ $\quad (\lambda = -1)$

$$\Rightarrow x = w^Txw$$

$\Rightarrow$ $\boxed{x = \alpha w \text{ is an eigenvector for } \alpha \neq 0}$

$Fx = x \Rightarrow x - 2ww^Tx = x$ $\quad (\lambda = +1)$

$$\Rightarrow (w^Tx)w = 0$$

$\Rightarrow$ $\boxed{\text{Any vector orthogonal to } w \text{ is an eigenvector}}$

(2) $A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow$ Initialize $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$x = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$v_1 = 1\sqrt{20}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8.4721 \\ 2 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 8.4721 \\ 2 \\ 0 \end{bmatrix} \frac{1}{8.7050} = \begin{bmatrix} 0.9732 \\ 0.2298 \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 0.9732 \\ 0.2298 \\ 0 \end{bmatrix}\begin{bmatrix} 0.9732 & 0.2298 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & 0.447 & -0.447 \\ 0 & -1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0.973 \\ 0.230 \\ 0 \end{bmatrix}\begin{bmatrix} 0.973 & 0.230 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.894 & -0.447 & 0 \\ -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0.447 \\ -1 \end{bmatrix}$$

$$v_2 = 1(1.095)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.447 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.543 \\ -1 \end{bmatrix}$$

$$v_2 = \frac{1}{1.838}\begin{bmatrix} 1.543 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix}$$

$$R = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & 0.447 & -0.447 \\ 0 & -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix}\begin{bmatrix} 0.839 & -0.544 \end{bmatrix}\begin{bmatrix} 0.447 & -0.447 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.894 & -0.447 & 0 \\ -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0.839 \\ -0.544 \end{bmatrix}\begin{bmatrix} 0.839 & -0.544 \end{bmatrix}\begin{bmatrix} -0.447 & 0.894 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ -0.408 & 0.817 & 0.408 \end{bmatrix} \longrightarrow$$

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2 cont'd

$x = 0$

$v_3 = 0(0)[1] + 0 = 0$

$v_3 = 0/0 = 0$

Answer

$$R = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix} - 2(0)(0)(0) = \begin{bmatrix} -4.472 & -1.342 & -0.894 \\ 0 & -1.095 & 1.095 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ -0.408 & 0.817 & 0.408 \end{bmatrix} - 2(0)(0)[0\ 0\ 0] = \begin{bmatrix} -0.894 & -0.447 & 0 \\ 0.183 & -0.365 & 0.913 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = Q^T = \begin{bmatrix} -0.894 & 0.183 & 0 \\ -0.447 & -0.365 & 0 \\ 0 & 0.913 & 0 \end{bmatrix}$$

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Steps

1. $x = (A^TA)^{-1}A^Tb$
2. $r = \|Ax-b\|_2$

$$A = \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(1): $$x = \left(\begin{bmatrix} 2 & -2 & 0 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & -2 & 0 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -1 \\ -1 & 15 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.112 & 0.007 \\ 0.007 & 0.067 \end{bmatrix}\begin{bmatrix} 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.336 \\ 1.022 \end{bmatrix}$$

(2): $$r = \|Ax-b\| = \left\|\begin{bmatrix} 2 & 2 \\ -2 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.336 \\ 1.022 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right\|_2 = \left\|\begin{bmatrix} 1.716 \\ 0.396 \\ -1.978 \\ -2.642 \end{bmatrix}\right\|_2$$

$$= \sqrt{(1.716)^2 + (0.396)^2 + (-1.978)^2 + (-2.642)^2} = 3.741$$

$\longrightarrow$

4 a. Let $C = AB$ where A and B are upper triangular matrices.

So, A_{ij}, B_{ij} are nonzero if $i \leq j$

We know $C_{ij} = \sum_k A_{ik} B_{kj}$.

① If $i > k$, then $a_{ik} = 0$, so $a_{ik} b_{kj} = 0$.

② If $k > j$, then $b_{jk} = 0$, so $a_{ik} b_{kj} = 0$.

③ If $i \leq k \leq j$, then $a_{ik} b_{kj} \neq 0$.

So, $\forall i \leq j$, $C_{ij} \neq 0$ and $\forall j < i$, $C_{ij} = 0$.

By definition, C is upper triangular. ■

b. Let $A_{n \times n}$ be an invertible upper triangular matrix.

Therefore, A can be represented as the product of p upper triangular matrices, where p is an arbitrary integer, that equal the identity matrix.

So, $B_p B_{p-1} \cdots B_2 B_1 A = I_{n \times n}$

$\Rightarrow B_p B_{p-1} \cdots B_2 B_1 A A^{-1} = A^{-1}$

$\Rightarrow A^{-1} = B_p B_{p-1} \cdots B_2 B_1$

By part (a), since $\forall i$ B_i is upper triangular, their product A^{-1} is also upper triangular.

So, A^{-1} is upper triangular. ■