Math 1080: Spring 2019

Homework #9 (due April 12)

Problem 1:

Determine one eigenvalue of the following matrix using Rayleigh Quotient iteration, starting with initial guess $v^{(0)} = [0 \quad 1]^{\mathrm{T}}$ and initial eigenvalue estimate $\lambda^{(0)} = \left(v^{(0)}\right)^T A v^{(0)}$. Terminate iteration after 3 steps, i.e., after you obtain $\lambda^{(3)}$. What is the approximate eigenvector $v^{(3)}$? What is the error of each $\lambda^{(k)}$?

$$A = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}$$

Problem 2:

Perform the first two iterations of the QR algorithm (i.e., compute $A^{(2)}$ and $\tilde{Q}^{(2)}$) for the following matrix. How close are the diagonal elements of $A^{(2)}$ to the eigenvalues of A?

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Problem 3:
Reduce the following matrix to Hessenberg form using Householder reflector.

$$A = \begin{bmatrix} 3 & -2 & 4 & 4 \\ -2 & 1 & 9 & -4 \\ 4 & 9 & 2 & -4 \\ 4 & -4 & -4 & 2 \end{bmatrix}$$

Problem 4:

Let Q and R be the QR factors of a symmetric tridiagonal matrix H. Show that the product $K = RQ$ is again a symmetric tridiagonal matrix.

(Hint: Prove the symmetry of K. Show that Q has Hessenberg form and that the product of an upper triangular matrix and a Hessenberg matrix is again a Hessenberg matrix. Then use the symmetry of K.)

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① $A = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}$

$v^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\lambda^{(0)} = (v^{(0)})^T A v^{(0)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3$

Solve for w: $(A - \lambda^{(0)} I) w = \begin{bmatrix} -3 & 2 \\ 2 & 0 \end{bmatrix} w = v^{(0)} \Rightarrow w = \begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix}$

$v^{(1)} = \frac{w}{\|w\|} = \frac{\begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix}}{0.9014} = \begin{bmatrix} -.5547 \\ 0.8320 \end{bmatrix}$

$\lambda^{(1)} = (v^{(1)})^T A v^{(1)} = -5.7689$

Solve for w: $(A - \lambda^{(1)} I) w = \begin{bmatrix} -.2311 & 2 \\ 2 & 2.7689 \end{bmatrix} w = v^{(1)} \Rightarrow w = \begin{bmatrix} .6897 \\ -.1977 \end{bmatrix}$

$v^{(2)} = \frac{w}{\|w\|} = \frac{\begin{bmatrix} .6897 \\ -.1977 \end{bmatrix}}{.7175} = \begin{bmatrix} .9613 \\ -.2755 \end{bmatrix}$

$\lambda^{(2)} = (v^{(2)})^T A v^{(2)} = -6.8316$

Solve for w: $(A - \lambda^{(2)} I) w = \begin{bmatrix} 0.8316 & 2 \\ 2 & 3.8316 \end{bmatrix} w = v^{(2)} \Rightarrow w = \begin{bmatrix} -5.2042 \\ 2.6445 \end{bmatrix}$

$v^{(3)} = \frac{w}{\|w\|} = \frac{\begin{bmatrix} -5.2042 \\ 2.6445 \end{bmatrix}}{5.8376} = \begin{bmatrix} -0.8915 \\ 0.4530 \end{bmatrix}$

$\lambda^{(3)} = (v^{(3)})^T A v^{(3)} = -6.9997$

The actual eigenvalues of A:

$\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix} = (6+\lambda)(3+\lambda) - 4 = \lambda^2 + 9\lambda + 14 \Rightarrow (\lambda+2)(\lambda+7) = 0$

$\Rightarrow \lambda = -2, -7$

Approximate eigenvector: $\begin{bmatrix} -.8915 \\ .4530 \end{bmatrix}$

Approximate eigenvalue: -6.9997

Real eigenvalue: -7

Errors at $\lambda^{(k)}$

$|\lambda^{(1)} + 7| = 1.2311$

$|\lambda^{(2)} + 7| = 0.1684$

$|\lambda^{(3)} + 7| = 0.0003$

(a) $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Let $A^{(0)} = A$

Let $Q^{(1)}R^{(1)} = A^{(0)}$;

$$Q^{(1)} = \begin{bmatrix} 0.9487 & 0.2673 & 0.1690 \\ -.3162 & 0.8018 & 0.5071 \\ 0 & -0.5345 & 0.8452 \end{bmatrix} \quad R^{(1)} = \begin{bmatrix} 3.1623 & -1.5811 & 0.3162 \\ 0 & 1.8708 & -2.4054 \\ 0 & 0 & 2.0284 \end{bmatrix}$$

$$A^{(1)} = R^{(1)}Q^{(1)} = \begin{bmatrix} 3.5 & -.5916 & 0 \\ -.5916 & 2.7857 & -1.0842 \\ 0 & -1.0842 & 1.7143 \end{bmatrix}$$

Let $Q^{(2)}R^{(2)} = A^{(1)}$:

$$Q^{(2)} = \begin{bmatrix} 0.9860 & 0.1542 & 0.0631 \\ -.1667 & 0.9125 & 0.3736 \\ 0 & -.3789 & 0.9254 \end{bmatrix} \quad R^{(2)} = \begin{bmatrix} 3.5496 & -1.0476 & 0.1807 \\ 0 & 2.8615 & -1.6389 \\ 0 & 0 & 1.1814 \end{bmatrix}$$

$$A^{(2)} = R^{(2)}Q^{(2)} = \begin{bmatrix} 3.6746 & -.4769 & 0 \\ -.4769 & 3.2321 & -.4476 \\ 0 & -.4476 & 1.0933 \end{bmatrix}$$

$$\tilde{Q}^{(2)} = Q^{(1)}Q^{(2)} = \begin{bmatrix} .8909 & .3262 & .3162 \\ -.4454 & .4907 & .7489 \\ 0.0891 & -.8080 & .5824 \end{bmatrix}$$

Eigenvalues of A:

$\det(A - \lambda I) = \lambda^3 + 8\lambda^2 - 19\lambda + 12 = -(\lambda-1)(\lambda-3)(\lambda-4) = 0$

$\Rightarrow \lambda = 1, 3, 4$

Difference between eigenvalues and diagonal of $\hat{A}$:

$$\left| \begin{bmatrix} 3.6746 \\ 3.2321 \\ 1.0933 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \right| = \begin{bmatrix} .3254 \\ .2321 \\ .0933 \end{bmatrix}$$

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(3) $A = \begin{bmatrix} 3 & -2 & 4 & 4 \\ -2 & 1 & 9 & -4 \\ 4 & 9 & 2 & -4 \\ 4 & -4 & -4 & 2 \end{bmatrix}$

$x = \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} \Rightarrow v = \text{sign}(x_1)\|x\| e_1 + x = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \\ 4 \end{bmatrix}$

$\tilde{Q} = I - 2\frac{vv^T}{v^Tv} = I - \begin{bmatrix} 1.333 & -.6667 & -.6667 \\ -.6667 & .3333 & .3333 \\ -.6667 & .3333 & .3333 \end{bmatrix} = \begin{bmatrix} -.3333 & .6667 & .6667 \\ .6667 & .6667 & -.3333 \\ .6667 & -.3333 & .6667 \end{bmatrix}$

$Q_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -.3333 & .6667 & .6667 \\ 0 & .6667 & .6667 & -.3333 \\ 0 & .6667 & -.3333 & .6667 \end{bmatrix} \Rightarrow A_1 = Q_1^T A Q_1 = \begin{bmatrix} 3 & 6 & 0 & 0 \\ 6 & -.38890 & -.8883 & 3.4444 \\ 0 & -.8883 & 13.1111 & -1.5556 \\ 0 & 3.4444 & -1.5556 & 4.2222 \end{bmatrix}$

$x = \begin{bmatrix} -.8883 \\ 3.4444 \end{bmatrix}, \quad v = \text{sign}(x_1)\|x\| e_1 + x = \begin{bmatrix} 3.5571 \\ 0 \end{bmatrix} + \begin{bmatrix} -.8883 \\ 3.4444 \end{bmatrix} = \begin{bmatrix} 2.6688 \\ 3.4444 \end{bmatrix}$

$\tilde{Q} = I - \frac{2vv^T}{v^Tv} = I - \begin{bmatrix} 1.7503 & .9683 \\ .9683 & 1.2497 \end{bmatrix} = \begin{bmatrix} .2497 & -.9683 \\ -.9683 & -.2497 \end{bmatrix}$

$Q_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & .2497 & -.9683 \\ 0 & 0 & -.9683 & -.2497 \end{bmatrix} \Rightarrow \boxed{H = Q_2^T A_1 Q_2 = \begin{bmatrix} 3 & 6 & 0 & 0 \\ 6 & -3.8890 & -3.5570 & 0 \\ 0 & -3.5570 & -2.3891 & -5.5524 \\ 0 & 0 & -5.5524 & 11.2776 \end{bmatrix}}$

$Q = Q_1 Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -.3333 & -.4791 & -.8120 \\ 0 & .6667 & .4892 & -.5623 \\ 0 & .6667 & -.7288 & .1563 \end{bmatrix}$

(4) Proof

Since H is symmetric, $QR = H = H^T = R^T Q^T$. So, we see

$K = RQ = Q^T QRQ = Q^T R^T Q^T Q = Q^T R^T = K^T$

So, since $K = K^T$, we have shown K is symmetric.

Next, we will show Q has Hessenberg form.

Since H is tridiagonal $m \times m$, it has Hessenberg form. So, $h_{ij} = 0$ when $i > j + 1$.

$\longrightarrow$

4 cont'd

In the QR factorization of H, the R is upper triangular by definition so $r_{ij}=0$ when $i>j$.

Now, let $i>j+1$. It follows that

$$h_{ij} = \sum_{k=1}^{m} q_{ik}r_{kj} = q_{i1}r_{1j} + \dots + q_{ij}r_{jj}$$

If $j=1$, the above relation only has one term $(h_{i1} = q_{i1}r_{11})$.

Since $r_{11} \neq 0$, we know $q_{i1}=0$ when $h_{i1}=0$, or when $i>2$.

If $j=2$, $h_{i2} = q_{i1}r_{12} + q_{i2}r_{22}$, which simplifies to $h_{i2} = q_{i2}r_{22}$ when $i>2$.

So, $q_{i2}=0$ when $h_{i2}=0$, or when $i>3$.

By induction, we easily see Q has zeros below the first subdiagonal, similar to H, so Q has Hessenberg form.

Next, we will show K is tridiagonal.

We know $r_{ij}=0$ when $i>j$ and Q is Hessenberg with $q_{ij}=0$ when $i>j+1$, so:

$$k_{ij} = \sum_{k=1}^{m} r_{ik}q_{kj} = \sum_{k=i}^{j+1} r_{ik}q_{kj}$$

So, when $i>j+1$, the above sum has no terms and $k_{ij}=0$.

So, we easily observe K is a Hessenberg matrix.

As such, since K is Hessenberg and symmetric, it follows that K is tridiagonal. ■