Math 1080: Spring 2019

Homework #6

due Monday, March 4

Problem 1: Compute LU factorization of the matrix

$$A = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 3 & -1 \\ -2 & 0 & 3 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}$$

Problem 2:

Solve the following system of equations by both LU factorization and QR factorization:

$$\begin{aligned} 2x_1 - x_3 &= -7 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + x_2 + 3x_3 &= 2 \end{aligned}$$

Problem 3:

Let A be nonsingular $n \times n$ matrix. Show that A has LU factorization $A = LU$ (no pivoting) with the diagonal terms of the matrix U all nonzero if and only if for each $1 \leq k \leq n$ the upper left $k \times k$ submatrix $A_{1:k,1:k}$ is nonsingular.

(Hint: Use induction argument).

Problem 4:

Compute the LU factorization with partial pivoting, (i.e., find P, L, U such that $PA = LU$) for the following matrix

$$A = \begin{bmatrix} -1 & -2 & -1 & -2 \\ 3 & -1 & 1 & -1 \\ 3 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

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(1) $A = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 3 & -1 \\ -2 & 0 & 3 & 0 \\ 2 & -2 & 1 & 2 \end{bmatrix}$

$L_1 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 2/3 & 0 & 1 & \\ -2/3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L_1 A = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 5/3 & -4/3 \\ 0 & -2 & 7/3 & 10/3 \end{bmatrix} = \tilde{A}_1$

$L_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ 0 & -2 & 0 & 1 \end{bmatrix} \Rightarrow L_2 \tilde{A}_1 = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 5/3 & -4/3 \\ 0 & 0 & -11/3 & 16/3 \end{bmatrix} = \tilde{A}_2$

$L_3 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 11/5 & 1 \end{bmatrix} \Rightarrow L_3 \tilde{A}_2 = \begin{bmatrix} 3 & 0 & -2 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 5/3 & -4/3 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = U$

$L = L_1^{-1} L_2^{-1} L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/3 & 0 & 1 & 0 \\ 2/3 & 2 & -11/5 & 1 \end{bmatrix}$

LU

(2) (1) Find $LU = A$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & 0 \\ -1 & 1 & \\ -1/2 & 0 & 1 \end{bmatrix} \Rightarrow L_1 A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 1 & 7/2 \end{bmatrix} = \tilde{A}_1$$

$$L_2 = \begin{bmatrix} 1 & & \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \Rightarrow L_2 \tilde{A}_1 = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 3/2 \end{bmatrix} = U$$

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

(2) Find $QR = A$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$$

$$a_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \|a_1\| = \sqrt{4+4+1} = 3$$

$$v_1 = a_1 - \text{sign}(a_{11})\|a_1\| e_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$H_1 = I - 2\frac{v_1 v_1^T}{v_1^T v_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$H_1 \cdot A = \begin{bmatrix} 3 & 5/3 & 8/3 \\ 0 & -4/3 & -11/3 \\ 0 & -2/3 & -1/3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -4/3 & -11/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix} \Rightarrow \|a_2\| = \sqrt{16/9 + 4/9} = 1.4907$$

$$v_2 = a_1 - \text{sign}(a_{11})\|a_1\| e_1 = \begin{bmatrix} -4/3 \\ -2/3 \end{bmatrix} + (1.4907)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.824 \\ -2/3 \end{bmatrix}$$

$$H_2 = I - 2\frac{v_1 v_1^T}{v_1^T v_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{8.4197}\begin{bmatrix} -2.824 \\ -2/3 \end{bmatrix}\begin{bmatrix} -2.824 & -2/3 \end{bmatrix} = \begin{bmatrix} -.894 & -.447 \\ -.447 & .894 \end{bmatrix}$$

$$H_2 A_2 = \begin{bmatrix} 1.4907 & 3.4286 \\ 0 & 1.3416 \end{bmatrix}$$

2 cont'd

$$H_2 H_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -.894 & -.447 \\ 0 & -.447 & .894 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 2/3 & -1/3 & -2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5/3 & 7/3 \\ 0 & 1.4907 & 3.4286 \\ 0 & 0 & 1.3416 \end{bmatrix} = R$$

$$H_1 H_2 = \begin{bmatrix} 2/3 & -.7454 & 0 \\ 2/3 & .5963 & -.4472 \\ 1/3 & .2981 & .8944 \end{bmatrix} = Q$$

$$QR = \begin{bmatrix} 2/3 & -.7454 & 0 \\ 2/3 & .5963 & -.4472 \\ 1/3 & .2981 & .8944 \end{bmatrix} \begin{bmatrix} 3 & 5/3 & 7/3 \\ 0 & 1.4907 & 3.4286 \\ 0 & 0 & 1.3416 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} = A \checkmark$$

LU

3 Solve $Ly = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} y = \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix}$ by back substitution

$\Rightarrow y_1 = -7, \quad y_2 = 1 - (-7)(1) = 8, \quad y_3 = 2 - (-7)(1/2) - 8(1/2) = 3/2$

4 Solve $Ux = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 3/2 \end{bmatrix} x = \begin{bmatrix} -7 \\ 8 \\ 3/2 \end{bmatrix}$

$\Rightarrow x_3 = \frac{3/2}{3/2} = 1, \quad x_2 = \frac{(8 - x_3(4))}{2} = 2, \quad x_1 = \frac{-7 + x_3}{2} = -3$

$x = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix} \checkmark$

QR

5 $c = Q^T b = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -.7454 & .5963 & .2981 \\ 0 & -.4472 & .8944 \end{bmatrix} \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -10/3 \\ 6.4103 \\ 1.3416 \end{bmatrix}$

6 Solve $Rx = \begin{bmatrix} 3 & 5/3 & 7/3 \\ 0 & 1.4907 & 3.4286 \\ 0 & 0 & 1.3416 \end{bmatrix} x = \begin{bmatrix} -10/3 \\ 6.4103 \\ 1.3416 \end{bmatrix} = c$ with backsubstitution

$\Rightarrow x_3 = \frac{1.3416}{1.3416} = 1, \quad x_2 = \frac{6.4103 - x_3(3.4286)}{1.4907} = 2$

$x_1 = \frac{-10/3 - x_3(7/3) - 5/3(x_2)}{3} = -3$

$x = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(3) Proof

Let $A_{1\times1}$ be a nonsingular matrix with trivial LU factorization

$$A = [a_{11}] = [1][a_{11}] = LU$$

Let $n=2$ and $A \in \mathbb{R}^{n\times n}$.

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{bmatrix} = LU$$

It's easy to see $a_{22} - \frac{a_{21}}{a_{11}} a_{12} = \frac{\det A}{a_{11}}$

Since A is nonsingular, $\det A \neq 0$ and $a_{11} \neq 0$ (as shown in the assumption with $A_{1\times1}$).

So, $U_{22} \neq 0$, which verifies $\det(A) = \det(L)\det(U)$

$$\underset{\text{nonzero}}{\det(A)} = \underset{1}{\det(L)}\ \underset{\text{must be nonzero}}{\det(U)}$$

Also, as verification, $\begin{bmatrix} \frac{a_{21}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} - \frac{a_{21}}{a_{11}} a_{12} \end{bmatrix} = a_{22}$.

Now, let $n=k$ and $A \in \mathbb{R}^{k\times k}$.

$$\Rightarrow A = \begin{bmatrix} \boxed{B} & w \\ u^T & a_{nn} \end{bmatrix} = \begin{bmatrix} L_1U_1 & w \\ u^T & a_{nn} \end{bmatrix} = \begin{bmatrix} L_1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} U_1 & x \\ 0 & u_{nn} \end{bmatrix} = LU$$

To verify, $LU = \begin{bmatrix} L_1U_1 & L_1x \\ z^TU_1 & z^Tx + u_{nn} \end{bmatrix}$ so

$L_1U_1 = L_1U_1$

$w = L_1x \Rightarrow x = L_1^{-1}w$

$z^TU_1 = u \Rightarrow z^T = u^TU_1^{-1}$

$z^Tx + u_{nn} = a_{nn} \Rightarrow u^TU_1^{-1}L_1^{-1}w + u_{nn} = a_{nn}$

$\Rightarrow u_{nn} = a_{nn} - u^TU_1^{-1}L_1^{-1}w$

Since A is nonsingular, any U_1 diagonal entry has to be nonsingular because U_1 is upper triangular.

So, we have shown, by induction, that if $A \in \mathbb{R}^{n\times n}$ is nonsingular, it has LU factorization with nonzero diagonal terms in U. ∎

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(4) $A = \begin{bmatrix} -1 & -2 & -1 & -2 \\ 3 & -1 & 1 & -1 \\ 3 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow P_1 A = \begin{bmatrix} 3 & -1 & 1 & -1 \\ -1 & -2 & -1 & -2 \\ 3 & 0 & 2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & & \\ 1/3 & 1 & & 0 \\ -1 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L_1 P_1 A = \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & -7/3 & -2/3 & -7/3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} = \tilde{A}_1$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow P_2 \tilde{A}_1 = \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & -7/3 & -2/3 & -7/3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 3/7 & 1 & \\ 0 & 3/7 & 0 & 1 \end{bmatrix} \Rightarrow L_2 P_2 \tilde{A}_1 = \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & -7/3 & -2/3 & -7/3 \\ 0 & 0 & 5/7 & -1 \\ 0 & 0 & -9/7 & -1 \end{bmatrix} = \tilde{A}_2$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow P_3 \tilde{A}_2 = \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & -7/3 & -2/3 & -7/3 \\ 0 & 0 & -9/7 & -1 \\ 0 & 0 & 5/7 & -1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & & & \\ 0 & 1 & & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & 5/9 & 1 \end{bmatrix} \Rightarrow L_3 P_3 \tilde{A}_2 = \begin{bmatrix} 3 & -1 & 1 & -1 \\ 0 & -7/3 & -2/3 & -7/3 \\ 0 & 0 & -9/7 & -1 \\ 0 & 0 & 0 & -14/9 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & & & \\ -1/3 & 1 & & 0 \\ 0 & -3/7 & 1 & \\ 1 & -3/7 & -5/9 & 1 \end{bmatrix}$$

$$PLU = A$$

$$P = P_3 P_2 P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$