Math 1080: Spring 2019

Homework #7 (due March 20)

Problem 1:

Show that if $A = \begin{bmatrix} a_{11} & w^{\mathrm{T}} \\ w & K \end{bmatrix}$ is symmetric and positive definite, then $a_{11} > 0$ and both K and $K - ww^{\mathrm{T}} / a_{11}$ are symmetric and positive definite.

(Hint: Use the definition of positive definite matrix. Assume $x = \begin{bmatrix} \beta \\ y \end{bmatrix}$ where β is a scalar and y is an n-1 dimensional vector.)

Problem 2:

Use necessary conditions to test positive definiteness of the symmetric matrix A. If conditions are satisfied, compute the Cholesky factorization:

a) $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}$

Problem 3:

Solve the following system of equations by Cholesky factorization (if the coefficient matrix is positive definite) or by Gaussian elimination (otherwise)

$$\begin{aligned} 4x_1 + 2x_2 \qquad\qquad - 2x_4 &= 6 \\ 2x_1 + 10x_2 - 6x_3 + 2x_4 &= 36 \\ -6x_2 + 8x_3 \qquad &= -30 \\ -2x_1 + 2x_2 \qquad\qquad + 4x_4 &= 6 \end{aligned}$$

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MATH1080
Homework #7

① Proof

If A is symmetric, then $A = A^T = \begin{bmatrix} a_{11} & w^T \\ w & K^T \end{bmatrix}$, so $K = K^T$ and

$\left(K - \frac{ww^T}{a_{11}}\right)^T = K - \frac{ww^T}{a_{11}}$.

So, both K and $K - \frac{ww^T}{a_{11}}$ are symmetric.

Since A is positive definite, we know $\forall x \neq 0 \; x^T A x > 0$.

If we assume $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, then $0 < x^T A x = \begin{bmatrix} x_1 & 0^T \end{bmatrix} \begin{bmatrix} a_{11} & w^T \\ w & K \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$= a_{11}(x_1)^2$ for all $x_1 \neq 0$. Thus, $a_{11} > 0$.

Now, assume $x = \begin{bmatrix} 0 \\ y \end{bmatrix}$, so $0 < x^T A x = \begin{bmatrix} 0 & y^T \end{bmatrix} \begin{bmatrix} a_{11} & w^T \\ w & K \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = y^T K y$

for all $y \neq 0$, so K is positive definite.

Finally, assume $x = \begin{bmatrix} x_1 \\ y \end{bmatrix}$, so $\forall x \neq 0 \; 0 < x^T A x = \begin{bmatrix} x_1 & y^T \end{bmatrix} \begin{bmatrix} a_{11} & w^T \\ w & K \end{bmatrix} \begin{bmatrix} x_1 \\ y \end{bmatrix}$

$= \begin{bmatrix} x_1 & y^T \end{bmatrix} \begin{bmatrix} a_{11}x_1 + w^T y \\ w x_1 + K y \end{bmatrix} = a_{11}(x_1)^2 + 2x_1 w^T y + y^T K y$.

Now, if we let $x_1 = -\frac{y^T w}{a_{11}}$ and $y^T w = w^T y$, then $\forall y \neq 0$

$0 < a_{11} x_1^2 + 2x_1 w^T y + y^T K y = y^T \left(K - \frac{ww^T}{a_{11}}\right) y$.

So, $K - \frac{ww^T}{a_{11}}$ is positive definite.

② a. $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

$\boxed{a_{ii} > 0 \quad \checkmark}$

$\boxed{\frac{a_{ii} + a_{jj}}{2} > a_{ij} \quad \checkmark}$

$\boxed{\text{max element on diagonal} \quad \checkmark}$

$R_1 = \begin{bmatrix} \sqrt{3} & -\sqrt{3}/3 & -\sqrt{3}/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad K_1 = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} -\sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{bmatrix} = \begin{bmatrix} 8/3 & -4/3 \\ -4/3 & 8/3 \end{bmatrix}$

$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix}$ $\quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2\sqrt{6}}{3} & -\frac{\sqrt{6}}{3} \\ 0 & 0 & 1 \end{bmatrix}$ $\quad K_2 = \frac{8}{3} - (-4/3)^2 / \tfrac{8}{3} = 2$

$-4/3 \cdot \frac{\sqrt{3}}{2\sqrt{2}} = -\frac{\sqrt{6}}{3}$

$\frac{\sqrt{24}}{3} = \frac{2\sqrt{6}}{3}$

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2 cont'd

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad \boxed{R = \begin{bmatrix} \sqrt{3} & -\sqrt{3}/3 & -\sqrt{3}/3 \\ 0 & \frac{2\sqrt{3}}{3} & -\sqrt{6}/3 \\ 0 & 0 & \sqrt{2} \end{bmatrix}}$$

b. $$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix}$$

$\boxed{a_{ii} > 0 \ \checkmark}$

$\boxed{\frac{a_{ii} + a_{jj}}{2} > a_{ij} \ \checkmark}$

$\boxed{\text{max element on diagonal } \checkmark}$

$$R_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -3 \end{bmatrix}\begin{bmatrix} 0 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -6 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & -6 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2}/2 \\ 0 & 0 & 1 \end{bmatrix} \quad K_2 = -6 - \tfrac{1}{2}(1) = -6.5$$

$$R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{13}{2} \end{bmatrix}$$ ← not positive definite

3) $$A = \begin{bmatrix} 4 & 2 & 0 & -2 \\ 2 & 10 & -6 & 2 \\ 0 & -6 & 8 & 0 \\ -2 & 2 & 0 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 36 \\ -30 \\ 6 \end{bmatrix}$$

$\boxed{a_{ii} > 0 \ \checkmark}$

$\boxed{\frac{a_{ii} + a_{jj}}{2} > a_{ij} \ \checkmark}$

$\boxed{\text{max element on diagonal } \checkmark}$

$$R_1 = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad K_1 = \begin{bmatrix} 10 & -6 & 2 \\ -6 & 8 & 0 \\ 2 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 8 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & -6 & 3 \\ 0 & -6 & 8 & 0 \\ 0 & 3 & 0 & 3 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad K_2 = \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} -6 \\ 3 \end{bmatrix}\begin{bmatrix} -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

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MATH1080
Homework # 7

3 cont'd

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = 2 - \tfrac{1}{4}(2 \cdot 2) = 1$$

$$R_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow R^T y = b$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -30 \\ 6 \end{bmatrix} \Rightarrow \left. \begin{array}{l} y_1 = 6/2 = 3 \\ y_2 = (36-3)/3 = 11 \\ y_3 = (-30+22)/2 = -4 \\ y_4 = (6+3-11+4) = 2 \end{array} \right\} y = \begin{bmatrix} 3 \\ 11 \\ -4 \\ 2 \end{bmatrix}$$

$Rx = y$

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ -4 \\ 2 \end{bmatrix} \Rightarrow \left. \begin{array}{l} x_4 = 2 \\ x_3 = (-4-2)/2 = -3 \\ x_2 = (11-2-6)/3 = 1 \\ x_1 = (3+2-1)/2 = 2 \end{array} \right\} x = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$