$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n$

James Hahn
MATH1080
Homework #4

1) a. $d = Av = \begin{bmatrix} a_1 \\ \cdots \\ a_m \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_1 v \\ \vdots \\ a_m v \end{bmatrix}$

$a_i v = \sum_{j=1}^{n} a_{ij} v_j = (n-1)$ adds $+ (n)$ mults $= n+n-1 = 2n-1$ flops

Perform $a_i v$ m times $\Rightarrow (2n-1)m = 2mn - m$

b. $C = AB = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}[b_1 | \dots | b_n] = C_{m\times n}$

$a_i b_j = \sum_{p=1}^{n} a_{ip} b_{pj} = (n-1)$ adds $+ (n)$ mults $= 2n-1$ flops

C will have $m\times n$ elements $\Rightarrow$ perform $a_i b_j$ $m\times n$ times

$\Rightarrow (2n-1)(mn) = 2mn^2 - mn$

c. $x = v'Bv = [v_1 \dots v_n]\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix}$

~~$v' \Rightarrow \forall i,j \in [1,n]\ v_{ij} = v_{ji} \Rightarrow n$ flops~~

$v'B \Rightarrow$ dot product $a_i b_j = 2n-1$ flops n times $= (2n-1)n = 2n^2 - n$

$\Rightarrow (v'B)_{1\times n}$

$(v'B)v \Rightarrow$ inner product $[v_1' \dots v_n']\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow 2n-1$ flops

$\Rightarrow (2n^2-n)$ flops $+ (2n-1)$ flops

$= 2n^2 - n + 2n - 1 = 2n^2 + n - 1$

d. $x = A(1:n,:)(Bv)$

$\Rightarrow Bv$ is similar to Av in part (a), but replace m by n

$\Rightarrow 2n^2 - n$ where $(Bv) \in \mathbb{R}^{n\times 1}$.

$\Rightarrow$ Let $Bv = c_{n\times 1}$. $A(1:n,:)c$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$ flops

$\Rightarrow (2n^2-n)$ flops $+ (2n^2-n)$ flops $= 2n^2 - n + 2n^2 - n$

$= 4n^2 - 2n$ flops

$\rightarrow$

$A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{n\times n}$, $v\in\mathbb{R}^{n}$

1 cont'd

e. $x = (A(1:n,:)B)v$

$\Rightarrow A(1:n,:)B$ is the same as part (b), but replace m by n

$\Rightarrow 2n^3 - n^2$. Let $(A(1:n,:)B) = C_{n\times n}$

$\Rightarrow Cv$ is the same as part (a), but replace m by n

$\Rightarrow 2n^2 - n$

$\Rightarrow (2n^3-n^2)$ flops $+ (2n^2-n)$ flops $= 2n^3 - n^2 + 2n^2 - n$

$= 2n^3 + n^2 - n$ flops

2 a.

```
for k = 1:n
    a(k) = B(k,k)* v(n-k);
end
```

loop n times, 1 mult

B(k,k): scalar; v(n-k): scalar

$n * 1 =$ n flops

b.

```
for k = 1:n
    x = B(k, n-k+1:n)*v(n-k+1:n)
end
```

loop n times

$$\sum_{k=1}^{n}\underbrace{\sum_{i=n-k+1}^{n} B_{ki}v_i}_{\substack{k-1 \text{ adds} \\ k \text{ mults}}\Big\} 2k-1 \text{ flops}} = \sum_{k=1}^{n} 2k-1 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2\,\frac{n(n+1)}{2} - n$$

$$= n(n+1) - n = n^2$$

c.

```
for k = 1:n-1
    for j = (n-k):n
        C = C + B(:,1:j)A(n-j+1:n, k);
    end
end
```

$B(:,1:j)$: n×j matrix; $A(n-j+1:n,k)$: column vector with j×1 dimensions

$C +$: (n)adds

outer product n×1 ⇒ (j-1) adds + (j) mults for <> n times ⇒ n(2j-1)

$$\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j-1)+n = \sum_{k=1}^{n-1}\sum_{j=n-k}^{n} n(2j) = 2n\sum_{k=1}^{n-1}\sum_{j=n-k}^{n} j = 2n\sum_{k=1}^{n-1}\Bigg(\underbrace{\underbrace{\sum_{j=1}^{n} j}_{\frac{n(n+1)}{2}} - \underbrace{\sum_{j=1}^{n-k-1} j}_{\frac{(n-k-1)(n-k)}{2}}}_{\frac{2n(k+1)-k^2-k}{2}}\Bigg)$$

$$= n\sum_{k=1}^{n-1} 2n(k+1)-k^2-k = 2n^2\sum_{k=1}^{n-1} k + 2n^2\sum_{k=1}^{n-1} 1 - n\sum_{k=1}^{n-1} k^2 - n\sum_{k=1}^{n-1} k$$

$$= 2n^2\frac{(n-1)n}{2} + 2n^2(n-1) - n\frac{(n-1)(n)(2n-1)}{6} - n\frac{(n-1)n}{2}$$

$$= n^4 + n^3 - 2n^2 - \frac{n^2(n-1)(n+1)}{3} \text{ flops}$$