Math 1080: Spring 2019

Homework #5

Due Feb 22

Problem 1:

Find the absolute and relative condition number for the following problems. Comment on the values of x for which the problem would be considered well-conditioned or ill-conditioned.

a) $f(x) = (\ln x)^2$

b) $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

c) $f(A) = [trace(A) \quad \det(A)]$ for 2x2 matrix A

Use $\|.\|_\infty$ norm in the formula for the condition number and treat the input as a vector of dimension 4, so your Jacobian becomes a 2 x 4 matrix.

d) $f(x, y) = \begin{bmatrix} xy & x^2 \\ y^2 & xy \end{bmatrix}$

Use $\|.\|_1$ norm in the formula for the condition number and treat the output as a vector of dimension 4, so your Jacobian becomes a 4 x 2 matrix.

Problem 2:

Determine whether the following algorithms are backward stable, stable, or unstable:

a) Computation of $f(x,y) = x^2 - y^2$ as $\tilde{f}(x, y) = [fl(x) \otimes fl(x)] \ominus [fl(y) \otimes fl(y)]$

b) Computation of $f(x,y) = x^2 - y^2$ as $\tilde{f}(x, y) = [fl(x) \oplus fl(y)] \otimes [fl(x) \ominus fl(y)]$

c) Computation of $f(x) = 1/(1 + x)$ as $\tilde{f}(x, y) = 1 \oslash [1 \oplus fl(x)]$

Problem 3:

Determine the accuracy of the algorithms in Problem 2. Which of the algorithms a) and b) is more accurate?

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1 a. absolute condition #

$\hat{K}(x) = |J(x)| = \boxed{\frac{2}{x}\ln(x)}$

relative condition #

$K(x) = |J| \frac{\|x\|}{\|F(x)\|} = \frac{2}{x}\ln(x)\frac{x}{\ln^2 x} = \boxed{\frac{2}{\ln x}}$

Well-conditioned when x is large because $\lim_{x\to\infty} \frac{2}{\ln x} = 0$.
Also, x should be $\gg 10^6$ because of the natural log.
Ill-conditioned for both small and negative x.
I included the negative condition because $\ln(-x)$ is undef.

b. absolute condition #

$\delta f = \|x+\delta x\| - \|x\|$
$\le \|x\| + \|\delta x\| - \|x\|$
$\le \|\delta x\|$

$\hat{K}(x) = \frac{\|\delta f\|}{\|\delta x\|} \le \frac{\|\delta x\|}{\|\delta x\|} = \boxed{1}$

relative condition #

$K(x) = \frac{\frac{\|\delta f\|}{\|F(x)\|}}{\frac{\|\delta x\|}{\|x\|}} \le \frac{\frac{\|\delta x\|}{\|x\|}}{\frac{\|\delta x\|}{\|x\|}} = \boxed{1}$

Well-conditioned for all x because 1 is small.

c. absolute condition #

$J(x) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ a_{22} & -a_{12} & -a_{21} & a_{11} \end{bmatrix}_{2\times4}$

$\hat{K}(x) = \|J(x)\|_\infty = \boxed{\max \sum_{i=1}^{m} J_{i,:}}$

Let $f(x) = [a_{11}+a_{22} \quad a_{11}a_{22} - a_{21}a_{12}]$

Let $x = [a_{11} \; a_{21} \; a_{12} \; a_{22}]$

relative condition #

$K(x) = \|J(x)\|_\infty \frac{\|x\|_\infty}{\|F(x)\|_\infty}$

$= \boxed{\max \sum_{i=1}^{m} J_{i,:} \frac{\max(x)}{\max(tr(x), det(x))}}$

Well-conditioned when all values of x are small
Ill-conditioned when $tr(x) = det(x) = 0$ and when **any value of x is large**

d. absolute condition #

$J(x,y) = \begin{bmatrix} y & x \\ 2x & 0 \\ 0 & 2y \\ y & x \end{bmatrix}$

Let $f(x,y) = [xy \;\; x^2 \;\; y^2 \;\; xy]$

$\hat{K}(x,y) = \|J(x,y)\|_1 = \max\{2x+2y, 2x+2y\} = \boxed{|2x+2y|}$

relative condition #

$K(x,y) = \|J(x,y)\|_1 \frac{\|[x \; y]\|_1}{\|F(x,y)\|_1}$

$= |2x+2y| \frac{|x+y|}{|xy+x^2+y^2+xy|} = 2\frac{(x+y)^2}{(x+y)^2}$

$= \boxed{2}$

Well-conditioned for all values of x and y

2 a. $\tilde{f}(x,y) = (fl(x) \otimes fl(x)) \ominus (fl(y) \otimes fl(y))$

$= \left([x(1+\epsilon_1)\,x(1+\epsilon_1)](1+\epsilon_3) - [y(1+\epsilon_2)\,y(1+\epsilon_2)](1+\epsilon_4)\right)(1+\epsilon_5)$

$= \left(x^2(1+\epsilon_1)^2(1+\epsilon_3)(1+\epsilon_5) - y^2(1+\epsilon_2)^2(1+\epsilon_4)(1+\epsilon_5)\right)$ $\quad |\epsilon_1||\epsilon_2||\epsilon_3||\epsilon_4||\epsilon_5| \le \epsilon_{mach}$

$= (x(1+2\epsilon_6))^2 - (y(1+2\epsilon_7))^2$ $\quad |\epsilon_6||\epsilon_7| \le 2\epsilon_{mach}$

$= \tilde{x}^2 - \tilde{y}^2$

$C = 2$

$= f(\tilde{x},\tilde{y})$ $\boxed{\text{Backwards stable}}$

b. $\tilde{f}(x,y) = [fl(x) \oplus fl(y)] \otimes [fl(x) \ominus fl(y)]$

$= [x(1+\epsilon_1) \oplus y(1+\epsilon_2)] \otimes [x(1+\epsilon_1) \ominus y(1+\epsilon_2)]$ $\quad |\epsilon_1||\epsilon_2| \le \epsilon_{mach}$

$= \left([x(1+\epsilon_1) + y(1+\epsilon_2)](1+\epsilon_3)[x(1+\epsilon_1) - y(1+\epsilon_2)](1+\epsilon_4)\right)(1+\epsilon_5)$

$= [x(1+\epsilon_1) + y(1+\epsilon_2)][x(1+\epsilon_1) - y(1+\epsilon_2)](1+\epsilon_3)(1+\epsilon_4)(1+\epsilon_5)$

$|\epsilon_3||\epsilon_4||\epsilon_5| \le \epsilon_{mach}$

$= \left((x(1+\epsilon_1))^2 - (y(1+\epsilon_2))^2\right)(1+\epsilon_3)(1+\epsilon_4)(1+\epsilon_5)$

$= (x(1+\tfrac{5}{2}\epsilon_6))^2 - (y(1+\tfrac{5}{2}\epsilon_7))^2$ $\quad |\epsilon_6||\epsilon_7| \le \tfrac{5}{2}\epsilon_{mach}$

$C = \frac{5}{2}$

$= \tilde{x}\tilde{x} - \tilde{y}\tilde{y} = \tilde{x}^2 - \tilde{y}^2$

$= f(\tilde{x},\tilde{y})$ $\boxed{\text{Backwards stable}}$

c. $\tilde{f}(x) = 1 \oslash [1 \oplus fl(x)]$

$= 1 \oslash \left([1 + x(1+\epsilon_1)](1+\epsilon_2)\right)$ $\quad |\epsilon_1||\epsilon_2| \le \epsilon_{mach}$

$= \dfrac{1}{(1+x(1+\epsilon_1))(1+\epsilon_2)}(1+\epsilon_3)$ $\quad |\epsilon_3| \le \epsilon_{mach}$

$= \dfrac{1}{1+x(1+\epsilon_1)}\left(\dfrac{1+\epsilon_3}{1+\epsilon_2}\right)$

$= \dfrac{1}{1+\tilde{x}}(1+\epsilon_4)$ $\quad |\epsilon_4| \le 2\epsilon_{mach}$

$= f(\tilde{x})(1+\epsilon_4) \Rightarrow$ NOT Backwords stable

$\|\tilde{f}(x) - f(\tilde{x})\| = f(\tilde{x})\epsilon_4$

$\dfrac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = \epsilon_4 \le 2\epsilon_{mach}$ D

$\dfrac{\|\tilde{x} - x\|}{\|x\|} \le 1\epsilon_{mach}$ C

$\boxed{\text{Stable}}$

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(3) a. $f(x,y) = x^2 - y^2$

$J(x) = [2x \ -2y] \Rightarrow \|J(x)\| = \sqrt{4x^2+4y^2} = 2\|\mathbf{x}\|$

$K(x) = \|J(x)\| \frac{\|x\|}{\|f(x)\|} = 2\|\mathbf{x}\| \frac{\|\mathbf{x}\|}{|x^2-y^2|} = 2\frac{x^2+y^2}{|x^2-y^2|}$

$\boxed{\text{Accuracy} = \Delta \leq C K(x)\varepsilon_{mach} = 2K(x)\varepsilon_{mach}}$ since the algorithm is backwards stable (this is theorem 15.1 in the book).

b. $f(x,y) = x^2 - y^2$

This is the same algorithm as a) with the same conditioning numbers. Since it is also backwards stable, it has the same asymptotic accuracy: $\Delta \leq O(K(x)\varepsilon_{mach})$ where $K(x)$ can be found in part a). Since its C value is different, its actual accuracy is $\boxed{\Delta \leq \frac{5}{2}K(x)\varepsilon_{mach}}$

c. $f(x) = \frac{1}{1+x}$ $\qquad |J(x)| = \frac{1}{(1+x)^2} \Rightarrow K(x) = |J(x)|\frac{|x|}{|f(x)|} = \frac{1}{(1+x)^2}\frac{|x|}{\left|\frac{1}{1+x}\right|}$

In problem 2 part c), we calculated

$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} \leq \underbrace{2}_{D}\varepsilon_{mach}$ and $\frac{\|\tilde{x}-x\|}{\|x\|} \leq \underbrace{1}_{C}\varepsilon_{mach}$

$$\boxed{\Delta \leq D\varepsilon_{mach} + CK(x)\varepsilon_{mach} = 2\varepsilon_{mach} + \frac{1}{(1+x)^2}\frac{|x|}{\left|\frac{1}{1+x}\right|}\varepsilon_{mach}}$$

d. Additional question: Which is more accurate, a) or b)?

Answer: They both have the same asymptotic accuracy, but a) has the **better** true accuracy since it has a lower C value.