NAME: ______________________________ (Instructor: J Wheeler)

Homework 3 - Geometric (Nonlinear) Programming
Math 1101 - an Introduction to Optimization
The University of Pittsburgh - Summer 2018

Please submit the following problems at the beginning of class Monday, October 8. Additionally, please staple these questions to the front of your solutions.

1. (Cobb-Douglas Production Function) By investing x units of labor and y units of capital, a low-end watch manufacturer can Produce $P(x, y) = 50x^{0.4}y^{0.6}$ watches. Find the maximum number of watches that can be produced on a budget of $\$20,000$ if labor costs $100 per unit and capital costs $200 per unit (hence we have the constraint function $g(x, y) = 100x + 200y - 20000 = 0$).

   Please do this problem using

   (a) partial derivatives (include using the Second Derivative Test in order to justify that we do have a minimum), and

   (b) Lagrange multipliers (you may use a calculator, software, etc. to solve the system of equations).

   (c) NOTE: this can also be solved using the **AGM**.

2. (a) Use the AGM to solve the Nonlinear Programming problem (if you are having difficulty determining the δ_i's, you may use Method 6.3.1 in the notes):

$$\begin{aligned} \text{Maximize: } & Q(x, y, z) = xyz \\ \text{Subject to: } & 3x + 4y + 12z = 96 \\ & x, y, z > 0 \end{aligned}$$

   (b) Use Excel's Solver (or Mathematica, etc) to answer part (a). Note on using Solver:
   -since we do not have "> 0" as an option in the constraints but rather ≥ 0, use ≥ 0.00000001 or something similar
   -you must have initial values for x, y, z, so initially try $(4, 15, 2)$ as the starting point. Then try $x = 31.9995, y = z = 0.0001$ and watch what happens. Include a print out of your answer reports with your HW.

3. (a) Use the Method 6.3.1 (MART) from the notes to solve

$$\begin{aligned} \text{Minimize: } & g(x, y) = \frac{1000}{xy} + 2x + 2y + xy \\ \text{Subject to: } & x, y > 0. \end{aligned}$$

   The feasible set in this situation consists of more than 1 vector. You may use a calculator or computer software to solve any system of equations you encounter.

(b) Use Solver (or Mathematica, etc.) to answer part (a). Include a print out of your answer report with your HW.

4. Do the first two iterations of Steepest Descent for the following problems:

   (a) Minimize $f(x, y) = 2x^2 + y^2 - 2xy$ with a starting point of $(2, 3)$.

   (b) Minimize $g(x, y) = x^2+y^2-2x-2y-xy$ with a starting point of $(0, 0)$. Comment on what happens here.

   (c) Use Solver, etc. to find solutions to these two problems with the stated starting point. Repeat this with at least two different starting points of your choice for both problems. Include a print out of your answer reports with your HW.

5. Repeat part a) of the previous problem doing two iterations of Newton's Method.

6. (EXTRA CREDIT) In class, we found $\min P = 5$ where $P(x, y) = \frac{2}{xy} + xy + x + y$ with $x, y \geq 0$ using the **AGM**. Show that the method of Steepest Decent converges to $\min P = 5$ when using the starting point $(2, 3)$.

7. Application: a location problem: AllTalk Communications (from *Ragsdale*)

   - AllTalk Communications provides cellular phone service in several midwestern states.
   - They seek to expand their operations by providing inter-city service between four cities in northern Ohio.
   - To do this, a new tower must be built to accommodate the existing towers in the cities.
   - The tower will have a 40 mile transmission radius.
   - AllTalk would like to minimize the distance between the new tower and the existing towers.

   We can express the location of each tower as an ordered pair (x, y) where x represents the distance east (in miles) the tower is from an arbitrary, but fixed, reference point, and y represents the distance north (in miles) the tower is from the same reference point.

   The Cleveland tower is located at $(5, 45)$, the Akron tower at $(12, 21)$, the Canton tower at $(17, 5)$, and the Youngstown tower at $(52, 21)$. Let (x, y) represent the location of the new tower.

   Clearly state the mathematical model for this problem then use Solver, etc. to locate the optimal position (the one that minimizes the total distance) of the new tower. Recall that the distance between points (a, b) and (x, y) is given by
   $d = \sqrt{(x-a)^2 + (y-b)^2}$.

   Once you have obtained an answer, prepare a short written presentation of your recommendation as a consultant to AllTalk Communications (i.e.

pretend you are a consultant that wants to continue being a consultant and write up a very short presentation of your solution and proposal to AllTalk).

James Hahn
MATH1101 Optimization
Dr. Wheeler

1. a. Max $P(x, y) = 50x^{0.4}y^{0.6}$
Subject to $100x + 200y - 20000 = 0$

We see the constraint can be transformed into:
$x = 200 - 2y$
$y = 100 - 1/2x$

So we have two forms of the equation:
$P(x, y) = 50*(200-2y)^{0.4}y^{0.6}$

$$P_y = \frac{50(120\text{-}2y)}{y^{0.4}(\text{-}2y+200)^{0.6}}$$

$P(x, y) = 50x^{0.4}(100\text{-}1/2x)^{0.6}$

$$P_x = \frac{50(40\text{-}1/2x)}{x^{0.6}(\text{-}1/2x+100)^{0.4}}$$

Therefore:

$$P_{yy} = -\frac{480000}{y^{1.4}(\text{-}2y+200)^{1.6}}$$

And

$$P_{xx} = \frac{50(\text{-}1/2x^{0.6}\,(\text{-}1/2x + 100)^{0.4} - \frac{(\text{-}1/2x \quad 60)(40 - 1/2x)}{x^{0.4}(\text{-}1/2x + 100)^{0.6}}}{x^{1.2}(\text{-}1/2x + 100)^{0.8}}$$

Finally:

$$P_{xy} = P_{yx} = 0$$

We reach critical points of $P_y = 0$ at $y = 60, 0, 100$ and $P_x = 0$ at $x = 200, 80, 0$. We know $x, y \neq 0$, and out of the remaining four values ($y = 60, 100$ and $x = 200, 80$), only the pair (80, 60) satisfies the constraints. As such:

$D(80, 60) = (-.35061)(-1.40244) - 0^2 > 0$ and $P_{xx} = -.35061 < 0$, so this is a maximum

As such, the solution to our problem is **(80, 60), or x = 80 and y = 60**.

b. $\nabla f(x) = < 20x^{-0.6}y^{0.6}, 30x^{0.4}y^{-0.4} >$
$\nabla g(x) = < 100, 200 >$

$20x^{-0.6}y^{0.6} = 100\lambda$
$30x^{0.4}y^{-0.4} = 200\lambda$
$100x + 200y – 20000 = 0$

$1/5x^{-0.6}y^{0.6} = \lambda$
$30x^{0.4}y^{-0.4} = 40x^{-0.6}y^{0.6}$
$x/y = 4/3$
(*) $y = 3/4x$

Substitute (*) into the constraint:
$100x + 200y – 20000 = 0$
$100x + 150x – 20000 = 0$
$250x = 20000$
$x = 80$

By (*), $y = (3/4)*80 = 60$

Therefore, the **solution is (80, 60)**.

2. a. Max $Q = xyz$
Subject to $3x + 4y + 12z = 96$
$x, y, z > 0$

$96 = 3(3x/3 + 4y/3 + 12z/3)$
$\geq 3*(3x)^{1/3} * (4y)^{1/3} * (12z)^{1/3}$
$96 = 3 * 3^{1/3} * 4^{1/3} * 12^{1/3} * Q^{1/3}$
Therefore, $2048/9 = Q$

We reach this max when:
$3x = 4y = 12z = 32$ (this is 96/3)

So:
$x = 32/3$
$y = (1/4)*32 = 8$
$z = (1/12)*32 = 8/3$

Therefore, we reach a maximum **Q of 32/3 at x = 32/3, y = 8, z = 8/3**.

b. **For starting point (4, 15, 2):**

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/6/2018 11:39:57 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.032 Seconds.
Iterations: 5 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$5	Objective y	120	227.5555556

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$3	x	4	10.66666058	Contin
$C$3	y	15	8.000002011	Contin
$D$3	z	2	2.666667517	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$3	Constraints	96	$F$3=$H$3	Binding	0
$B$3	x	10.66666058	$B$3>=$H$4	Not Binding	10.66666057
$C$3	y	8.000002011	$C$3>=$H$4	Not Binding	8.000002001
$D$3	z	2.666667517	$D$3>=$H$4	Not Binding	2.666667507

And for starting point (31.9995, 0.0001):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/6/2018 11:40:30 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.016 Seconds.
Iterations: 1 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$5	Objective y	3.19995E-07	3.19994E-07

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$3	x	31.9995	31.99943467	Contin
$C$3	y	0.0001	1E-04	Contin
$D$3	z	0.0001	1E-04	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$3	Constraints	95.999904	$F$3=$H$3	Binding	0
$B$3	x	31.99943467	$B$3>=$H$4	Not Binding	31.99943466
$C$3	y	1E-04	$C$3>=$H$4	Not Binding	9.999E-05
$D$3	z	1E-04	$D$3>=$H$4	Not Binding	9.999E-05

3. a. Min $g(x, y) = 1000/(xy) + 2x + 2y + xy$
Subject to $x, y > 0$

$\delta' = < \delta_1, \delta_2, \delta_3, \delta_4 >$
Max $v(\delta') = (1000/\delta_1)^{\delta 1}(2/\delta_2)^{\delta 2}(2/\delta_3)^{\delta 3}(1/\delta_4)^{\delta 4}$
Subject to $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$
$- \delta_1 + \delta_2 + \delta_4 = 0$
$- \delta_1 + \delta_3 + \delta_4 = 0$
$\delta_1, \delta_2, \delta_3, \delta_4 > 0$

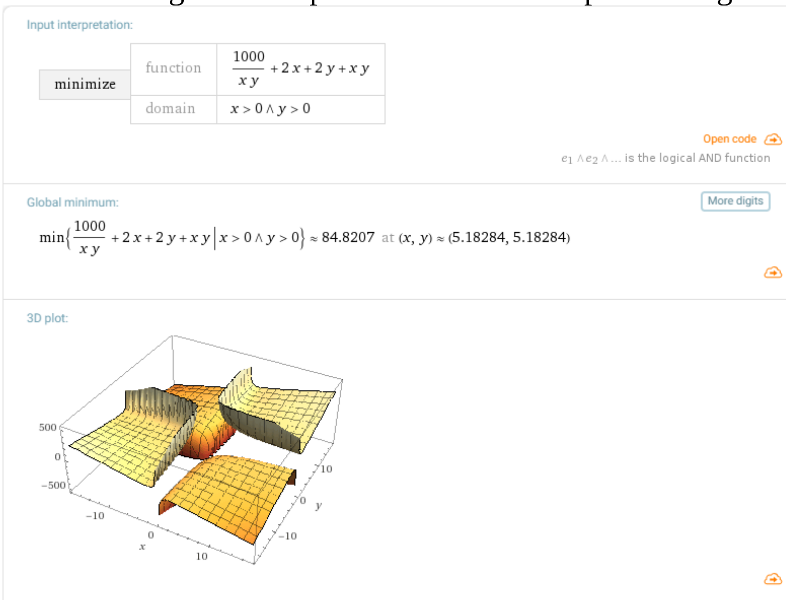
We get:
$\delta_1 = t$
$\delta_2 = 1\text{-}2t$
$\delta_3 = 1\text{-}2t$
$\delta_4 = 3t - 1$

Max $v(t) = (1000/t)^t(2/(1\text{-}2t))^{2\text{-}4t}(1/(3t\text{-}1))^{3t\text{-}1}$
$t = 0.438897$
$\delta_1 = 0.4389$
$\delta_2 = 0.1222$
$\delta_3 = 0.1222$
$\delta_4 = 0.3167$

By multiplying out the equations:
$u_1 = \delta_1 v(\delta') = 5.182 = x$
$u_2 = \delta_2 v(\delta') = 5.182 = y$

Therefore, our solution has been reached at **(5.182, 5.182) with a minimum g(x, y) = 84.8207**.

b. The following is the output from Wolfram Alpha solving the problem:



4. a. Min $f(x, y) = 2x^2 + y^2 – 2xy$ at (2, 3)

$\nabla f(x, y) = < 4x – 2y, 2y – 2x >$

$\nabla f(2, 3) = < 2, 2 >$

Min $\Phi_0(s) = f(2-2s, 3-2s)$

$= 4s^2 – 8s + 5$

$\Phi_0' = 8s – 8$

There is a critical point at s = 1

$\Phi_0'' = 8 > 0$, so the critical point is a minimum

$x_1 = < 2, 3 > - 1< 2, 2 > = < 0, 1 >$

$\nabla f(0, 1) = < -2, 2 >$

Min $\Phi_1(s) = f(2s, 1-2s)$

$= 20s^2 – 8s + 1$

$\Phi_0' = 40s – 8$

There is a critical point at s = 1/5

$\Phi_0'' = 40 > 0$, so the critical point is a minimum

$x_2 = < 0, 1 > - 1/5< -2, 2 > =$ **< 2/5, 3/5 >**

b. Min $g(x, y) = x^2 + y^2 – 2x -2y -xy$ at (0, 0)

$\nabla g(x, y) = < 2x – 2 – y, 2y – 2 - x >$

$\nabla g(0, 0) = < -2,-2 >$

Min $\Phi_0(s) = g(-2s, -2s)$
$= 12s^2 + 8s$
$\Phi_0' = 24s + 8$
There is a critical point at s = -1/3
$\Phi_0'' = 24 > 0$, so the critical point is a minimum

$x_1 = < 0,0 > + 1/3< -2, -2 > = < -2/3, -2/3 >$

$\nabla g(-2/3, -2/3) = < -8/3, -8/3 >$
Min $\Phi_1(s) = g(-2 + 8/3s, -2 + 8/3s)$
$= 64/3s^2 - 128/3s + 20$
$\Phi_0' = 128/3s - 128/3$
There is a critical point at s = 1
$\Phi_0'' = 128/3 > 0$, so the critical point is a minimum

$x_2 = < -2/3, -2/3 > - 1< -8/3, -8/3 > =$ **$< 2, 2 >$.**
Comment: we have converged to the global minimum in two iterations.

c. For part a:

Starting point of (2, 3):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:52:33 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.015 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	5	0

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	2	0	Contin
$C$9	y	3	0	Contin

Constraints

NONE

Starting point of (-5, -5):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:53:55 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0 Seconds.
Iterations: 0 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	25	0

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	-5	0	Contin
$C$9	y	-5	0	Contin

Constraints

NONE

Starting point of (-1010433, 235446):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:56:01 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.031 Seconds.
Iterations: 8 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	2.57319E+12	20.25878612

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	-1010433	3.281661025	Contin
$C$9	y	235446	6.362162126	Contin

Constraints

NONE

For part b:

Starting point of (0, 0):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:57:48 AM
Result: Solver has converged to the current solution. All Constraints are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.031 Seconds.
Iterations: 6 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	0	-4

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	0	1.999999999	Contin
$C$9	y	0	1.999999999	Contin

Constraints

NONE

Starting point of (-10, 10):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:58:34 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.016 Seconds.
Iterations: 5 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	300	-4

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	-10	1.999990928	Contin
$C$9	y	10	1.999987086	Contin

Constraints

NONE

Starting point of (84893484, -35253):

Microsoft Excel 16.0 Answer Report
Worksheet: [Book3]Sheet1
Report Created: 10/9/2018 1:59:44 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.016 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$1	Objective y	7.2099E+15	-1

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$9	x	84893484	0	Contin
$C$9	y	-35253	0.999999995	Contin

Constraints

NONE

5. Min $f(x, y) = 2x^2 + y^2 - 2xy$ with starting point (2, 3)

$\nabla f(x, y) = < 4x – 2y, 2y – 2x >$
$\nabla f(2, 3) = < 2, 2 >$
$f_{xx} = 4$
$f_{yy} = 2$
$f_{xy} = f_{yx} = -2$

$Hf(x, y) = [\ 4 ; -2\ ]$
$[\ -2 ; 2\ ]$

$H^{-1}f(x, y) = [\ ½ ; ½\ ]$
$[\ ½ ; 1\ ]$

$<x_{n+1}, y_{n+1}> = <2, 3> - <2, 2>[\ ½\ ½ ; ½\ 1] = <2, 3> - <2, 3> = <0, 0>$

$\nabla f(0, 0) = < 0, 0 >$
$<x_2, y_2> = <0, 0> - <0, 0>[\ ½\ ½ ; ½\ 1\ ] =$ **<0, 0>, which is our final solution**.

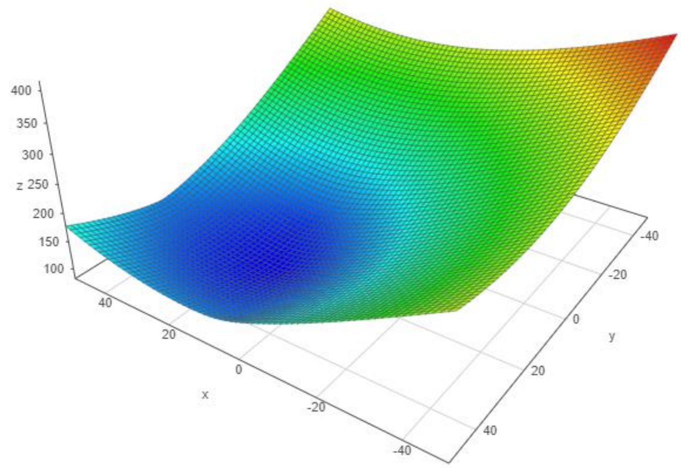
7. The problem can be modelled as the following optimization problem:

Minimize $\sqrt{(x\text{-}5)^2 + (y\text{-}45)^2} + \sqrt{(x\text{-}12)^2 + (y\text{-}21)^2} + \sqrt{(x\text{-}17)^2 + (y\text{-}5)^2} + \sqrt{(x\text{-}52)^2 + (y\text{-}21)^2}$
Subject to $\sqrt{(x\text{-}5)^2 + (y\text{-}45)^2} \leq 40$
$\sqrt{(x\text{-}12)^2 + (y\text{-}21)^2} \leq 40$
$\sqrt{(x\text{-}17)^2 + (y\text{-}5)^2} \leq 40$
$\sqrt{(x\text{-}52)^2 + (y\text{-}21)^2} \leq 40$

This problem has four constraints, namely that the tower must be within a 40-mile transmission radius of each of the towers.

This equation is the sum of the distances from the new tower (x, y) and all of the other towers (5, 45), (12, 21), (17, 5), and (52, 21).

A heatmap of the solution can be found below. The darker the blue, the lower the value. One can easily see (12.2, 21) is the minimum value.



<u>Proposal to AllTalk Communications</u>

Hello ladies and gentlemen. Thank you for providing me with the necessary information for this problem, such as the locations of the existing towers and the future goal, which was to determine the position of a new tower. In addition, you informed us that the tower only has a 40 mile transmission radius. Thank you very much for that well-needed information.

After further investigation, we have decided the most optimal location for the new tower is at (12.2, 21). With this new location, we achieve a minimum distance of 81.76 miles between all of the towers! This will significantly help reduce latency between all the towers in the towns. In addition, we took into account the transmission radius of the tower and can guarantee all of the cities' towers fit well within the 40-mile transmission radius of the new tower. You have nothing to worry about and you will be making a lot of people happy, while saving money, with this new tower!