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Homework 5 - Theory of Linear Programming
Math 1101 - An Introduction to Optimization
The University of Pittsburgh - Fall 2018

Please submit the following problems at the beginning of class Wednesday, November 15. Additionally, please staple this sheet to the front of your solutions.

1. Let $P(\overrightarrow{x}) = P(x_1, x_2, \ldots, x_n)$ be the objective function of a Linear Programming problem that has a feasible area F. Show

$$P(\overrightarrow{x}^*) = \max\{P(\overrightarrow{x}) \mid \overrightarrow{x} \in F\} \Leftrightarrow -P(\overrightarrow{x}^*) = \min\{-P(\overrightarrow{x}) \mid \overrightarrow{x} \in F\},$$

   i.e. the max of $P(\overrightarrow{x})$ occurs at the same location as the min of $-P(\overrightarrow{x})$.

2. Let $F = \{[0,0]^T, [1,2]^T, [4,1]^T\}$. Present (without proof) a graph of
   (a) $aff(F)$.
   (b) $conv(F)$.
   (c) $coni(F)$.

3. Determine for each of the following sets if the set is affine, convex, or a convex cone with vertex at the origin ("determine" means "prove or disprove"). As well, (without proof) state whether each set is a convex polyhedron and/or a convex polytope. If a set is not convex, state (without proof) its convex hull.
   (a) $S_1 = \{[x,y]^T \mid x + y \leq 5, y \geq 0, x \geq 1\}$.
   (b) $S_2 = \{[x,y]^T \mid y \leq |x|, y \geq 0, x \geq 0\}$.
   (c) $S_3 = \{[x,y]^T \mid y - x^2 + 6x \leq 9, y \geq 0, x \geq 0\}$.

4. Determine the extreme (corner) point(s) $P = \{\overrightarrow{P}_1, ..., \overrightarrow{P}_s\}$ of S_2 in the previous question. Then find direction vectors $D = \{\overrightarrow{d}_1, ..., \overrightarrow{d}_t\}$ of S_2 such that

$$S_2 = conv(P) + coni(D)$$

   as in the Finite Basis Theorem and Fundamental Theorem of Linear Programming.

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MATH1101 Optimization
Dr. Wheeler

1.

<u>Proof</u>

First, we want to show $P(x^*) = \max\{ P(x) \mid x \in F \} \implies -P(x^*) = \min\{ -P(x) \mid x \in F \}$
Assume $P(x^*) = \max\{ P(x) \mid x \in F \}$

$\implies \forall x \in F,\ P(x^*) \geq P(x)$

$\implies -P(x^*) \leq -P(x)$ which is the definition of a minimum

$\implies -P(x^*) = \min\{ -P(x) \mid x \in F \}$

So, for $-P(x)$, the given input x^* is a minimum.

Next, we want to show $-P(x^*) = \min\{ -P(x) \mid x \in F \} \implies P(x^*) = \max\{ P(x) \mid x \in F \}$
Assume $-P(x^*) = \min\{ P(x) \mid x \in F \}$

$\implies \forall x \in F,\ -P(x^*) \leq -P(x)$

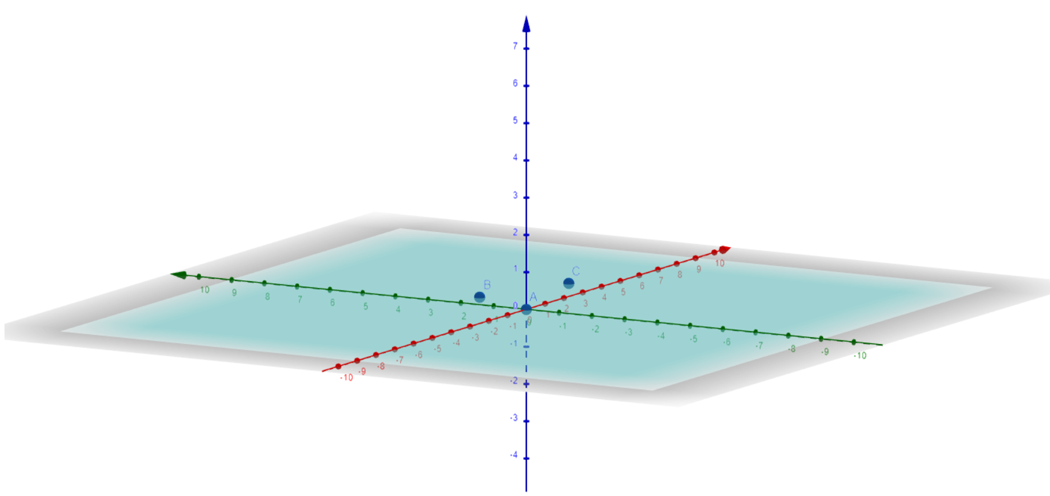
$\implies P(x^*) \geq P(x)$ which is the definition of a maximum

$\implies P(x^*) = \max\{ P(x) \mid x \in F \}$

So, for $P(x)$, the given input x^* is a maximum.

We have shown both directions of the logical statement.
Therefore, $P(x^*) = \max\{ P(x) \mid x \in F \} \iff -P(x^*) = \min\{ -P(x) \mid x \in F \}$. ■

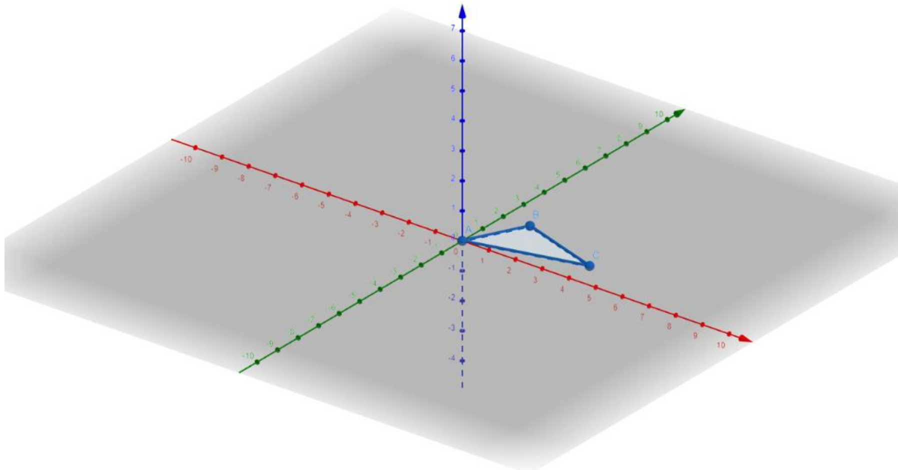
2.



a.

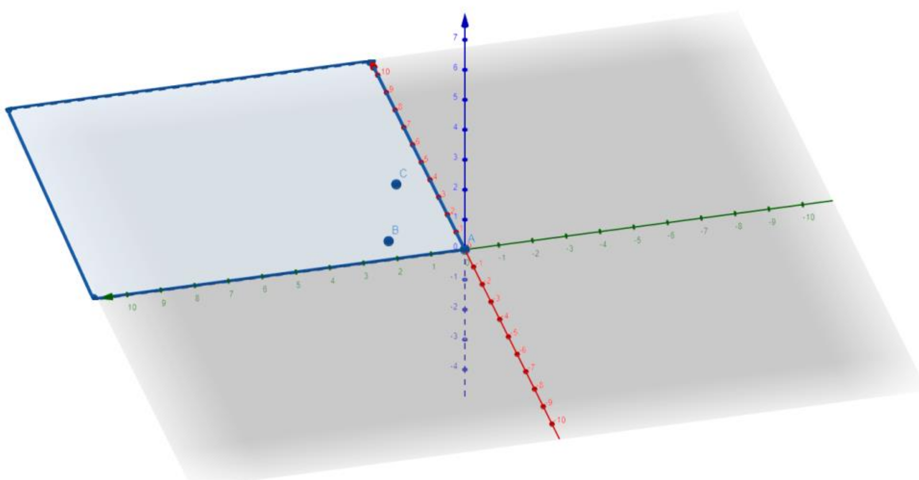
A = (0, 0)
B = (1, 2)
C = (4, 1)
The blue plane is the affine hull. It is located at z = 0. Pretend like the plane is extending outward forever.

b.



A = (0, 0)
B = (1, 2)
C = (4, 1)
The blue polygon is the convex hull. It is located at z = 0.

c.



A = (0, 0)
B = (1, 2)
C = (4, 1)
The blue plane is the conical hull. It is located at z = 0 in the first quadrant. Pretend like the plane is extending outward forever in the first quadrant.

3. Affine set

For any $\alpha_1,\alpha_2 \in \mathbb{R}$ where $\alpha_1+\alpha_2 = 1$ and $(x_1, y_1),(x_2, y_2) \in S$, then $\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) \in S$.

Convex set

For any $\lambda \in [0, 1]$, $\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in S$

Convex cone

For any $\alpha_1, \alpha_2 \in \mathbb{R} \geq 0$, $\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) \in S$.

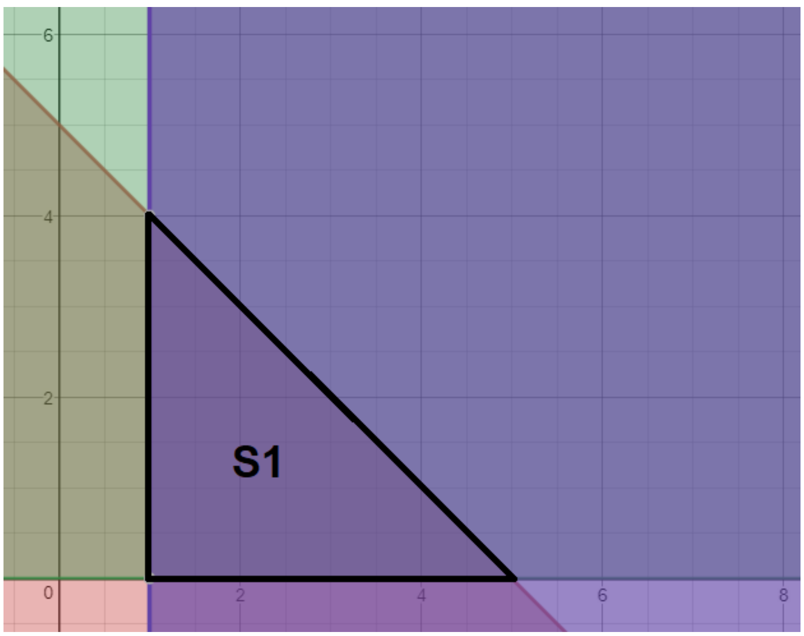
Convex polyhedron

A solid in 3 dimensions with flat faces, straight edges, and sharp corners

Convex polytope

A convex set of points in n-dimensional space. It's basically a convex polyhedron that is bounded.

a.



The pink region is $x + y \leq 5$.
The lighter purple region is $x \geq 1$.
The green region is $y \geq 0$.
The region outlined in black is the region of points included in the set S_1.

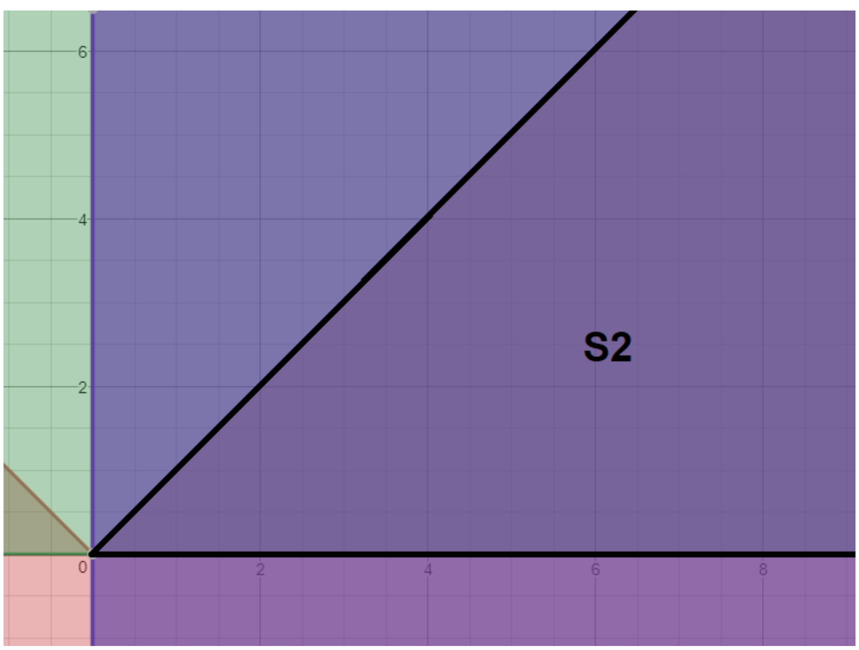
It **is a convex polyhedron** because in $\mathbb{R}^3$, the value of z does not determine whether the inequality holds; only x and y matter. Also, the feasible region clearly has straight edges, sharp corners, and flat faces.
It **is also a convex polytope** because it is a convex polyhedron and bounded ($1 \leq x \leq 5$, $0 \leq y \leq 4$).

So, for S_1, $(2, 2) \in S$ and $(4, 1) \in S$. Take $\alpha_1 = -1/2$, $\alpha_2 = 3/2$. Then $\alpha_1 + \alpha_2 = 1$.
But $\alpha_1(2, 2) + \alpha_2(4, 1) = -1/2(2, 2) + 3/2(4, 1)$
$= (-1, -1) + (6, 3/2)$
$= (5, ½)$
Hence $(5, ½) \notin S$, as $5 + ½ > 5$, so S_1 is **not affine**.

Let $(x_1, y_1), (x_2, y_2) \in S$, $\lambda \in (0, 1)$ s.t. $x_1 + y_1 \leq 5$, $x_1 + y_1 \leq 5$.
Then $\lambda(x_1, y_1) + (1- \lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) = (x_3, y_3)$
Then, $x_3 + y_3 = \lambda(x_1 + y_1) + (1- \lambda)(x_2+y_2)$
$\leq 5\lambda + 5(1- \lambda)$
≤ 5
Also, $y = \lambda y_1 + (1- \lambda)y_2 \geq 0$ since $y_1 \geq 0$, $y_2 \geq 0$, and $\lambda \in (0, 1)$
Also, $x = \lambda x_1 + (1- \lambda)x_2 \geq 1$ since $x_1 \geq 1$, $x_2 \geq 1$, and $\lambda \in (0, 1)$
Since $x_3 + y_3 \leq 5$, $x_3 \geq 1$, and $y_3 \geq 0$, then $(x_3, y_3) \in S_1$, so S_1 is a **convex set**.

Take $(2, 3), (2, 2) \in S_1$ with $\alpha_1 = 3$, $\alpha_2 = 0$.
Then $\alpha_1(2, 3) = (6, 4) \notin S_1$ since $6 + 9 > 5$.
So, S_1 is **not a convex cone**.

b.



The pink region is $y \leq |x|$.
The lighter purple region is $x \geq 0$.
The green region is $y \geq 0$.
The region outlined in black is the region of points included in set S_2, which is unbounded from above on x and y.

S_2 **is a convex polyhedron** because in $\mathbb{R}^3$, the value of z does not determine whether the inequality holds; only x and y matter. Also, the feasible region has straight edges, sharp corners, and flat faces.

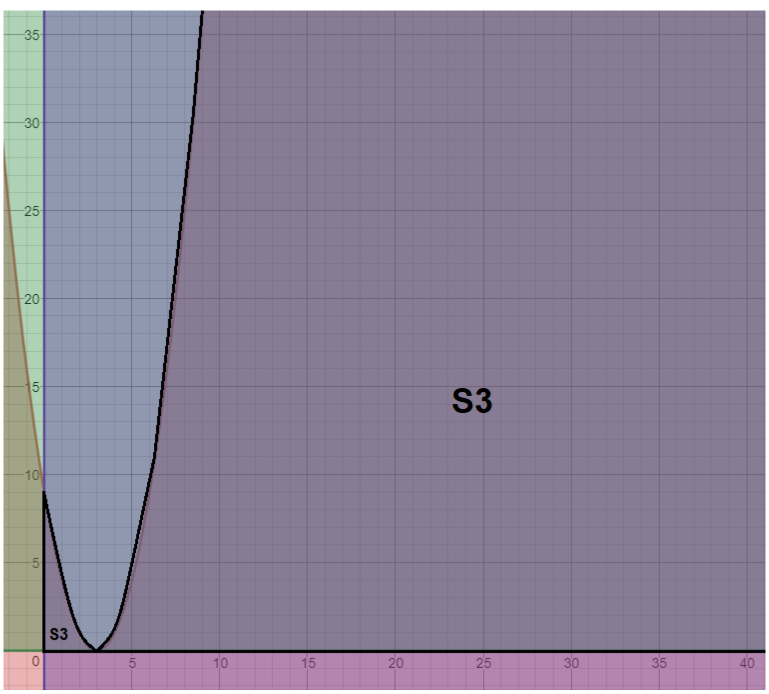
However, it **is not a convex polytope** because x and y are unbounded from above.

Only positive x-axis is on S_2, but to be affine it will contain the whole line.
Take (10, 0), (2, 0) in S_2 with $\alpha_1 = -1/2$, $\alpha_2 = 3/2$ where $\alpha_1 + \alpha_2 = 1$.
Then $\alpha_1(10, 0) + \alpha_2(2, 0)$
$= (-5, 0) + (3, 0)$
$= (-2, 0) \notin S_2$
So, S_2 is **not affine**.

Let $(x_1, y_1), (x_2, y_2) \in S_2$ and $\lambda \in [0, 1]$.
Let $(x_3, y_3) = \lambda(x_1, y_1) + (1- \lambda)(x_2, y_2)$.
Then $y_3 = \lambda y_1 + (1- \lambda)y_2 \leq \lambda|x_1| + (1- \lambda)|x_2|$.
But, $x_1, x_2 \geq 0$, so $y_3 \leq x_3$ and $x_3 \geq 0$, $y_3 \geq 0$.
So, S_2 is a **convex set** (it contains any line segment joining any two points in S_2).

S_2 is a **convex cone with vertex at origin** since any line segment passing through the origin and a point of S_2 lies inside S_2.
If $(x_1, y_1) \in S_2$, then for $\alpha_1 \geq 0$, $\alpha_1 \in \mathbb{R}$, we have $\alpha_1(x_1, y_1) \in S_2$ as $\alpha_1 y_1 \leq \alpha_1 x_1$ and $\alpha_1 x_1 \geq 0$, $\alpha_1 y_1 \geq 0$.

c.



The pink region is $y - x^2 + 6x \leq 9$.
The lighter purple region is $x \geq 0$.
The green region is $y \geq 0$.
The region outlined in black is the region of points included in set S_3, which is unbounded.

It **is not a convex polyhedron** because it's a polynomial function and has curved edges.
It **is not a convex polytope** because it is not a convex polyhedron and it is not bounded (x and y are not bounded from above).

$S_3 = \{ (x, y) \mid y - x^2 + 6x \leq 9, y \geq 0, x \geq 0 \}$
$= \{ (x, y) \mid y \leq x^2 - 6x + 9, y \geq 0, x \geq 0 \}$
$= \{ (x, y) \mid y \leq (x-3)^2, y \geq 0, x \geq 0 \}$.

Let $(1, 4), (1, 1) \in S_3$ with $\alpha_1 = -1/2$ and $\alpha_2 = 3/2$ so $\alpha_1 + \alpha_2 = 1$.
Then $-1/2(1, 4) + 3/2(1, 1)$
$= (1, -1/2) \notin S_3$ since $-1/2 < 0$
So, S_3 is **not an affine set**.

The line segment connecting (0, 0) and (20, 20), $\{ (x, y) \mid y = x, 0 \leq x \leq 20, 0 \leq y \leq 20 \}$ traverses outside of S3.
For example, this statement is true when (x, y) = (3, 3).
In this case, x and y fit within the domain and value range of the line segment, but $(3, 3) \notin S_3$.
Therefore, not all line segments exist in S_3 that connect arbitrary points in S_3.
So, S_3 is **not a convex set**.
The **convex hull** is the first quadrant of the xy plane such that $\{ (x, y) \mid x \geq 0, y \geq 0 \}$.

Let $(1, 1), (1, 2) \in S_3$ with $\alpha_1 = 1$ and $\alpha_2 = 2$ so $\alpha_1, \alpha_2 \geq 0$.
Then $\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2)$
$= 1(1, 1) + 2(1, 2)$
$= (1, 1) + (2, 4)$
$= (3, 5) \notin S_3$
Also, let $(0, 0), (1, 2) \in S_3$ with $\alpha_1 = 1$ and $\alpha_2 = 3$ so $\alpha_1, \alpha_2 \geq 0$.
Then $\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2)$
$= 1(0, 0) + 3(1, 2)$
$= (0, 0) + (3, 6)$
$= (3, 6) \notin S_3$, which originates from the origin
S_3 is **not a convex cone with vertex at origin** because the above two examples, one of which originating from the origin (origin exists in S_3), produce line segments with points nonexistent in S_3.

4.

The extreme points of a convex polygon are the points of intersection of the lines bounding the feasible region, which are $y = |x|$ and $y = 0$ in this case.
These two lines intersect when they equal each other, so $y = |x| = 0 = y$.
They meet exactly and only when $x = 0$.
When $x = 0$, then $y = 0$ as well.
Therefore, the **only existing corner point is (0, 0)**.

Remember from Remark 4.1.4 in the book that any line segment joining corner points produces the optimal solution for a feasible region when that line segment borders the feasible region.
Technically, (∞, ∞) and $(\infty, 0)$ are also corner points since they connect to the lines passing through the origin while bordering the feasible region.
However, we exclude them from the 'optimal solution' since they are not feasible to reach.
Therefore, the line segment connecting $(0, 0)$ to (∞, ∞) and the line segment connecting $(0, 0)$ to $(\infty, 0)$ produce the basis of S_2.
It is clear to see that the **two direction vectors representing these two line segments, from the origin, are $[1, 1]^T$ and $[1, 0]^T$**.
So, the line segment from the origin through the above direction vectors will cross any possible 'corner'/extreme/'optimal' points of the unbounded region.

The two direction vectors, or coni(D), will produce rays from the origin covering the area that is S_2 in the above picture, which is the set coni(D) = $\{ [x, y]^T \mid x = \alpha_1 + \alpha_2, y = \alpha_1, \alpha_1, \alpha_2 \geq 0 \}$ (this is because $(x, y) = \alpha_1(1, 1) + \alpha_2(1, 0)$).
The corner point will create the set conv(P) = $\{ [0, 0]^T \}$.
So, it is easy to see S_2 = conv(P) + coni(D).