Name:________________________________ Instructor: JP Wheeler/Grader:

Homework 1 - Intro to Linear Programming and the Simplex Method
Math 1101 - An Introduction to Optimization
The University of Pittsburgh - Fall 2018

Submit the following problems at the beginning of class Friday, September 14. Your work is expected to be clear and legible. Additionally, attach this sheet to the front of your work.

Instructions **For questions 1-3**, do each of the following:
I. Solve the following linear programming problems by graphing the feasible region then evaluating the objective function at each corner point. "Solve" means state the optimal value of the objective function and ***all points*** in the feasible region at which this optimal value occurs. II. Solve each problem a second time using the Simplex Method clearly stating the model after the introduction of slack, surplus, and artificial variables. You may use a calculator or computer to do the row operations, but write down the obtained simplex tableau after each iteration of the method. At each iteration identify the pivot element. III. Check your work using a software package of your choice (Solver, Matlab, etc.). Print and submit your answer screen and please make clear what software you have used.

1.

$$\begin{aligned} \text{Maximize and minimize} P(x,y) &= 5x + 2y \\ \text{Subject to } x + y &\geq 2 \\ 2x + y &\geq 4 \\ x, y &\geq 0 \end{aligned}$$

For this question only (that is, Question 1), when finding the minimum and using the Simplex method (part II above), at each iteration state which variables are basic and which are nonbasic. Also, at each iteration state the value of the objective function.

2.

$$\begin{aligned} \text{Maximize} P(x,y) &= 20x + 10y \\ \text{Subject to } x + y &\geq 2 \\ x + y &\leq 8 \\ 2x + y &\leq 10 \\ x, y &\geq 0 \end{aligned}$$

3.

$$\begin{aligned}
\text{Maximize and minimize} P(x,y) &= 20x + 10y \\
\text{Subject to } 2x + 3y &\geq 30 \\
2x + y &\leq 26 \\
-2x + 5y &\leq 34 \\
x, y &\geq 0
\end{aligned}$$

4. In this problem, there is a tie for the choice of the first pivot column. When you do your work using the simplex method use the method twice to solve the problem two different ways; first by choosing column 1 as the first pivot column and then for your second solution effort, solve by choosing column 2 as the first pivot column. You may use a computer or calculator to perform the Simplex Method, but do write down the results of each iteration.

$$\begin{aligned}
\text{Maximize } P(x,y) &= x + y \\
\text{Subject to } 2x + y &\leq 16 \\
x &\leq 6 \\
y &\leq 10 \\
x, y &\geq 0
\end{aligned}$$

5. In Example 2 in class, we used the dual to solve

$$\begin{aligned}
\text{Minimize } C(x_1, x_2, x_3) &= 40x_1 + 12x_2 + 40x_3 \\
\text{Subject to } 2x_1 + x_2 + 5x_3 &\geq 20 \\
4x_1 + x_2 + x_3 &\geq 30 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

The dual problem has as its first constraint

$$2y_1 + 4y_2 \leq 40. \tag{1}$$

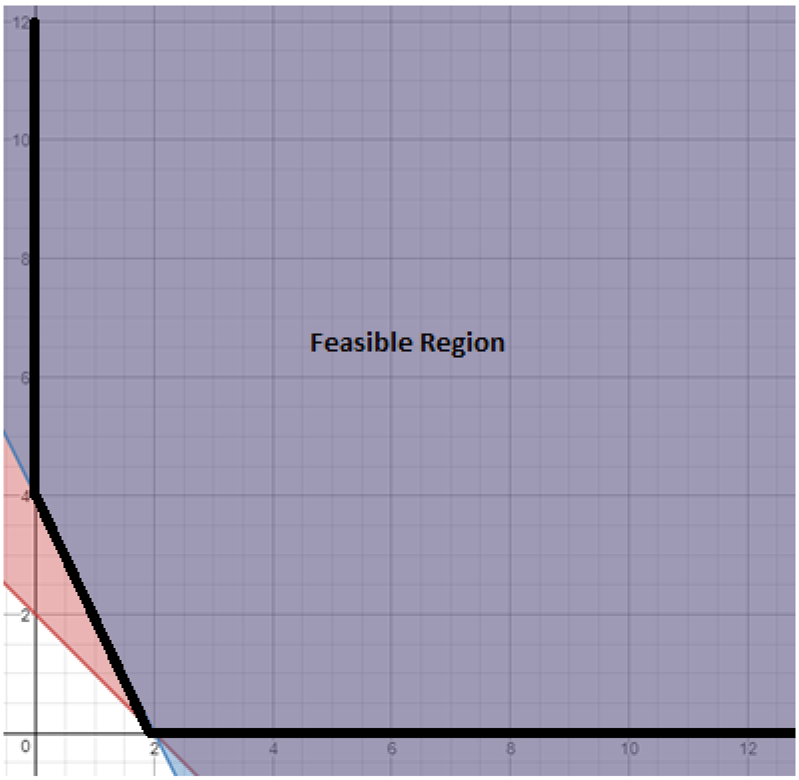
Replace this constraint by its simplified version

$$y_1 + 2y_2 \leq 20 \tag{2}$$

then proceed with the Simplex Method. Compare your answer with the one obtained in class and explain what causes the different answer. Follow the instructions in II from questions 1-3. [Note: the purpose of this question is to illustrate Warning 4.3.2 on page 47 of the notes.]

James Hahn
MATH1101
Homework #1

1. Feasible region is colored in purple below:



I. Corner points = (2, 0) and (0, 4)
P(2, 0) = 5*2 + 2*0 = 10
P(0, 4) = 5*0 + 2*4 = 8
Minimum exists at (0, 4)
Maximum does not exist due to unbounded feasible region

II. Minimize and Maximize P(x, y) = 5x + 2y
Subject to:

$x + y - s_1 = 2$
$2x + y - s_2 = 4$
$x, y, s_1, s_2 \geq 0$

This is a dual problem, so the initial tableau's transpose acts as the initial simplex tableau. Also, by Remark 4.1.3, since the feasible region is unbounded and coefficients of the objective function are positive, there exists a minimum, but no maximum, so below are the tableaus to solve *only* the minimization problem. There exists no maximal solution.

x	y	s1	s2	P	
1	2	1	0	0	5
1	1	0	1	0	2
-2	-4	0	0	1	0

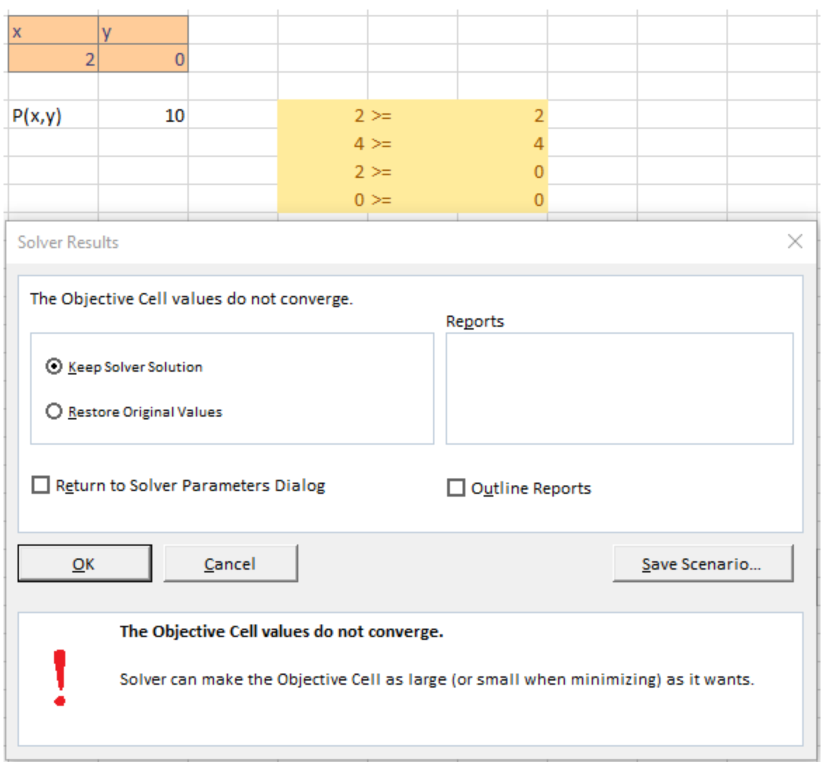
Pivot: **pivot column: 2nd, pivot row: 2nd, pivot element: 1**
Basic variables: **s1, s2, P**
Objective function = **0**

x	y	s1	s2	P	
-1	0	1	-2	0	1
1	1	0	1	0	2
2	0	0	4	1	8

No more negative entries in bottom row. The tableau has converged to a minimum at (0, 4).
Basic variables: **x, y, P**
Objective function = **8**

III. I used Excel's Solver add-in to solve this problem. Below is the answer report:

Results when solving for the **maximum** (not found):



Results when solving for the **minimum**:

Microsoft Excel 16.0 Answer Report
Worksheet: [Book2]Sheet1
Report Created: 9/11/2018 2:43:33 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 6 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$J$5	P(x,y) y	0	8

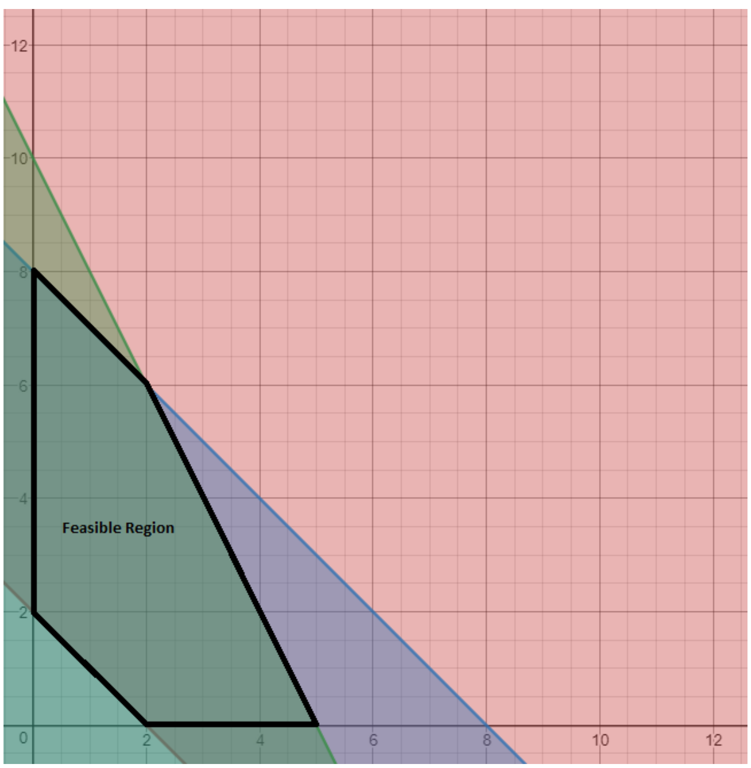
Variable Cells

Cell	Name	Original Value	Final Value	Integer
$I$3	x	0	0	Contin
$J$3	y	0	4	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$L$5	P(x,y)	4	$L$5>=$N$5	Not Binding	2
$L$6	>=	4	$L$6>=$N$6	Binding	0
$L$7	>=	0	$L$7>=$N$7	Binding	0
$L$8	>=	4	$L$8>=$N$8	Not Binding	4

2. Feasible region is marked on the graph below:



I. Corner points = (0, 2) and (2, 0) and (5, 0) and (2, 6) and (0, 8)
P(0, 2) = 20
P(2, 0) = 40
P(5, 0) = 100
P(2, 6) = 100
P(0, 8) = 80

There are two corner points with maximum values. Therefore, the solution is the line connecting both points. So, the solution **S = { (x, y) | y = 2x + 10 for 2 ≤ x ≤ 5 }.**

II. Maximize P(x, y) = 20x + 10y
Subject to:

$x + y - s_1 + a_1 = 2$
$x + y + s_2 = 8$
$2x + y + s_3 = 10$
$x, y, s_1, s_2, s_3, a_1 \geq 0$

Let M = 50 in the initial tableau calculations:

x	y	s1	s2	s3	a1	P	
1	1	-1	0	0	1	0	2
1	1	0	1	0	0	0	8
2	1	0	0	1	0	0	10
-70	-60	50	0	0	0	1	-100

Pivot column: 1st column, Pivot row: 1st row, Pivot element: 1

x	y	s1	s2	s3	a1	P	
1	1	-1	0	0	1	0	2
0	0	1	1	0	-1	0	6
0	-1	2	0	1	-2	0	6
0	10	-20	0	0	70	1	40

Pivot column: 3rd column, Pivot row: 3rd row, Pivot element: 2

x	y	s1	s2	s3	a1	P	
1	0.5	0	0	0.5	0	0	5
0	0.5	0	1	-0.5	0	0	3
0	-0.5	1	0	0.5	-1	0	3
0	0	0	0	10	50	1	100

There are no more negative elements on the last row. We have converged to the **maximum objective function value of 100 at (5, 0)**.

III. I used Excel's Solver add-in to solve this problem. Below is the answer report:

Results when solving for the **maximum**:

Microsoft Excel 16.0 Answer Report
Worksheet: [Book2]Sheet1
Report Created: 9/11/2018 2:21:53 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$15	P(x,y) y	0	100

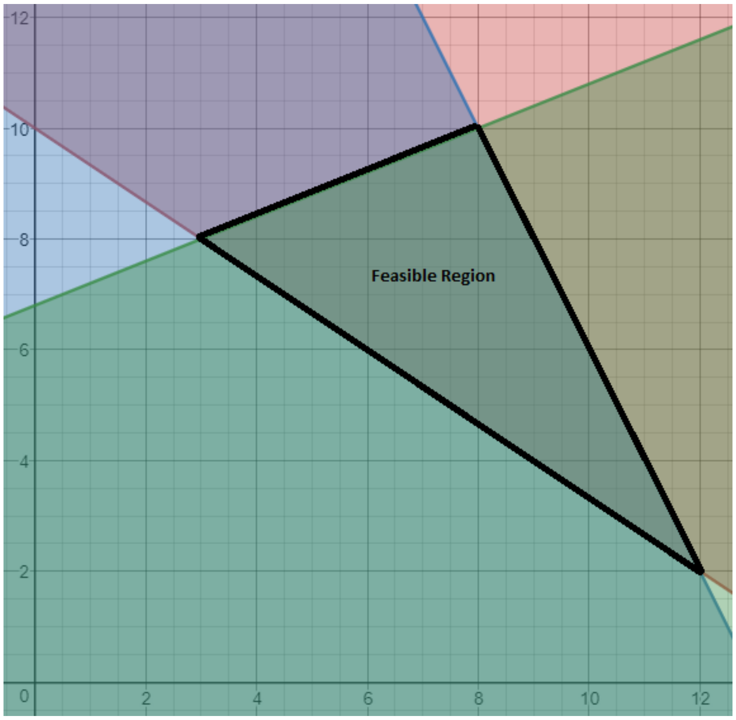
Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$13	x	0	5	Contin
$C$13	y	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$15	P(x,y)	5	$E$15>=$G$15	Not Binding	3
$E$16		5	$E$16<=$G$16	Not Binding	3
$E$17		10	$E$17<=$G$17	Binding	0

3. Feasible region is marked on the graph below:



I. Corner points = (3, 8) and (8, 10) and (12, 2)
P(3, 8) = 140
P(8, 10) = 260
P(12, 2) = 260

There are two coordinates with maximum values. Therefore, the **maximal solution** is the line connecting them. So, the solution **S = { (x, y) | y = -2x + 26 }**. The **minimum solution is located at (3, 8)**.

II. Maximize and Minimize P(x, y) = 20x + 10y
Subject to:

$2x + 3y - s_1 + a_1 = 30$
$2x + y + s_2 = 26$
$-2x + 5y + s_3 = 34$
$x, y, s_1, s_2, s_3, a_1 \geq 0$

Iterations of the tableaus for the maximization problem (let M = 50):

x	y	s1	s2	s3	a1	P	
2	3	-1	0	0	1	0	30
2	1	0	1	0	0	0	26
-2	5	0	0	1	0	0	34
-120	-160	50	0	0	0	1	-1500

Pivot column: 2nd column, Pivot row: 3rd row, Pivot element: 5

x	y	s1	s2	s3	a1	P	
3.2	0	-1	0	-0.6	1	0	9.6
2.4	0	0	1	-0.2	0	0	19.2
-0.4	1	0	0	0.2	0	0	6.8
-184	0	50	0	32	0	1	-412

Pivot column: 1st column, Pivot row: 1st row, Pivot element: 3.2

x	y	s1	s2	s3	a1	P	
1	0	-0.3125	0	-0.1875	0.3125	0	3
0	0	0.75	1	0.25	-0.75	0	12
0	1	-0.125	0	0.125	0.125	0	8
0	0	-7.5	0	-2.5	57.5	1	140

Pivot column: 3rd column, Pivot row: 2nd row, Pivot element; 0.75

x	y	s1	s2	s3	a1	P	
1	0	0	0.416667	-0.0833333	0	0	8
0	0	1	1.33333	0.333333	-1	0	16
0	1	0	0.166667	0.166667	0	0	10
0	0	0	10	0	50	1	260

The tableau has converged to a **maximum objective function value of 260 at (8, 10).**

Iterations of the tableaus for the minimization problem can be found below. We need to maximize the negative objective function, so maximize -P(x, y) = -20x – 10y :

x	y	s1	s2	s3	a1	P	
2	3	-1	0	0	1	0	30
2	1	0	1	0	0	0	26
-2	5	0	0	1	0	0	34
-80	-140	50	0	0	0	1	-1500

Pivot column: 2nd column, Pivot row: 3rd row, Pivot element: 5

x	y	s1	s2	s3	a1	P	
3.2	0	-1	0	-0.6	1	0	9.6
2.4	0	0	1	-0.2	0	0	19.2
-0.4	1	0	0	0.2	0	0	6.8
-136	0	50	0	28	0	1	-548

Pivot column: 1st column, Pivot row: 1st row, Pivot element: 3.2

x	y	s1	s2	s3	a1	P	
1	0	-0.3125	0	-0.1875	0.3125	0	3
0	0	0.75	1	0.25	-0.75	0	12
0	1	-0.125	0	0.125	0.125	0	8
0	0	7.5	0	2.5	42.5	1	-140

There are no more negative entries on the last row. We have converged to an optimal solution for the original minimization problem with objective function value = -140 = -P(x, y), so **P(x, y) = 140 at (3, 8).**

III. I used Excel's Solver add-in to solve this problem. Below are the answer reports:

For the maximization problem:

Microsoft Excel 16.0 Answer Report
Worksheet: [Book2]Sheet1
Report Created: 9/11/2018 2:24:29 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$24	P(x,y) y	0	260

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$22	x	0	8	Contin
$C$22	y	0	10	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$24	P(x,y)	46	$E$24>=$G$24	Not Binding	16
$E$25		26	$E$25<=$G$25	Binding	0
$E$26		34	$E$26<=$G$26	Binding	0

For the minimization problem:

Microsoft Excel 16.0 Answer Report
Worksheet: [Book2]Sheet1
Report Created: 9/11/2018 2:26:14 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$J$24	P(x,y) y	0	140

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$I$22	x	0	3	Contin
$J$22	y	0	8	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$L$24	P(x,y)	30	$L$24>=$N$24	Binding	0
$L$25	<=	14	$L$25<=$N$25	Not Binding	12
$L$26	<=	34	$L$26<=$N$26	Binding	0

4. Solving by choosing column 1 as the first pivot column:

x	y	s1	s2	s3	P	
2	1	1	0	0	0	16
1	0	0	1	0	0	6
0	1	0	0	1	0	10
-1	-1	0	0	0	1	0

Pivot column: 1st column, Pivot row: 2nd row, Pivot element: 1

x	y	s1	s2	s3	P	
0	1	1	-2	0	0	4
1	0	0	1	0	0	6
0	1	0	0	1	0	10
0	-1	0	1	0	1	6

Pivot column: 2nd column, Pivot row, 1st row, Pivot element: 1

x	y	s1	s2	s3	P	
0	1	1	-2	0	0	4
1	0	0	1	0	0	6
0	0	-1	2	1	0	6
0	0	1	-1	0	1	10

Pivot column: 4th column, Pivot row: 3rd row, Pivot element: 2

x	y	s1	s2	s3	P	
0	1	0	0	1	0	10
1	0	0.5	0	-0.5	0	3
0	0	-0.5	1	0.5	0	3
0	0	0.5	0	0.5	1	13

We have arrived at a solution of **P = 13 at (3, 10)**.

Solving by choosing column 2 as the first pivot column:

x	y	s1	s2	s3	P	
2	1	1	0	0	0	16
1	0	0	1	0	0	6
0	1	0	0	1	0	10
-1	-1	0	0	0	1	0

Pivot column: 2nd column, Pivot row: 3rd row, Pivot element: 1

x	y	s1	s2	s3	P	
2	0	1	0	-1	0	6
1	0	0	1	0	0	6
0	1	0	0	1	0	10
-1	0	0	0	1	1	10

Pivot column: 1st column, Pivot row: 1st row, Pivot element: 2

x	y	s1	s2	s3	P	
1	0	0.5	0	-0.5	0	3
0	0	-0.5	1	0.5	0	3
0	1	0	0	1	0	10
0	0	0.5	0	0.5	1	13

We have arrived at a solution of **P = 13 at (3, 10)**.

5. The dual problem becomes:

Maximize $C(y_1, y_2) = 20y_1 + 30y_2$

Subject to:

$y_1 + 2y_2 + x_1 \leq 20$

$y_1 + y_2 + x_2 \leq 12$

$5y_1 + y_2 + x_3 \leq 40$

$y_1, y_2, x_1, x_2, x_3 \geq 0$

y1	y2	x1	x2	x3	P	
1	2	1	0	0	0	20
1	1	0	1	0	0	12
5	1	0	0	1	0	40
-20	-30	0	0	0	1	0

Pivot column: 2nd column, Pivot row: 1st row, Pivot element: 2

y1	y2	x1	x2	x3	P	
0.5	1	0.5	0	0	0	10
0.5	0	-0.5	1	0	0	2
4.5	0	-0.5	0	1	0	30
-5	0	15	0	0	1	300

Pivot column: 1st column, Pivot row: 2nd row, Pivot element: 0.5

y1	y2	x1	x2	x3	P	
0	1	1	-1	0	0	8
1	0	-1	2	0	0	4
0	0	4	-9	1	0	12
0	0	10	10	0	1	320

We have converged to a maximum. The values for the final solution are of **x_1 = 10, x_2 = 10, and x_3 = 0 with C = 320**. The objective function value is the same as we obtained in class, C = 320. However, the final solution values are different. In class, we obtained $x_1 = 5$, $x_2 = 10$, and $x_3 = 0$. As such, by changing the first constraint in the dual problem, we reached different solution parameter values. Lastly, note if we plug $x_1 = 10$, $x_2 = 10$, and x_3 into the original objective function, C = 520, which provides us with an incorrect minimum as well. This perfectly illustrates the warning described in the notes.