Research and Adaption of Pragmatic Portfolio Optimization Based on Modern Portfolio Theories and Its Adjustments

Programming with State-of-The-Art Portfolio Allocation

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1 Context

Every investor, whether a trader, a mutual fund, or an individual investor, must make a decision on how to use their limited resources. Even the unconscious decision to have a cash-only portfolio is an investment decision in the end. While individuals can depend purely on their instincts, follow trends, or seek advice from others, institutional investors are frequently constrained by rules, prospectuses, and the requirement to outperform a specific benchmark. The idea is to identify a combination of securities that complement each other rather than just one suitable asset. Diversification is the strategy of dispersing risk across many investments.

Asset allocation is the process of distributing different weights (portions of wealth) to various asset categories such as stocks, bonds, or cash. While doing so, the investors are looking for the greatest mix to meet their needs in an uncertain climate. To get at the best allocation, one must estimate, appraise, model, and manage uncertainty. Because this is not a simple task with few to no factors, the investor needs to rely on measurements and function estimates.

One of the best-known optimization models in finance is the portfolio selection model, which was developed by Harry Markowitz [1] and forms the foundation of modern portfolio theory. The mean-variance approach by Markowitz led to significant developments in financial economics, such as Tobin's mutual fund separation theorem [2] and Sharpe's Capital Asset Pricing Model (CAPM) [3]. Markowitz was awarded the 1990 Nobel Prize for Economics for their enormous influence on both theory and practice. Also, the various researchers began to study deeper via the mean-variance approach. For example of modifying the Markowitz-based portfolio allocation, Guangfeng Deng and his research team [4] used particle swarm optimization (PSO), a collaborative population-based meta-heuristic algorithm for solving the Cardinality Constraints Markowitz Portfolio Optimization problem (CCMPO problem). The research on the efficient frontier model, which first formulated by Harry Markowitz and expanded in Robert Merton's research. Robert Merton defined every possible combination of risky assets can be plotted in risk–expected return space and the collection of all such possible portfolios defines a region in this space. In the absence of the opportunity to hold a risk-free asset, this region is the opportunity set (the feasible set). The positively sloped (upward-sloped) top boundary of this region is a portion of a hyperbola [5].

As a result, the original model has undergone numerous improvements and alterations over the years. Only a few complete software models are publicly accessible for usage, study, or modification, despite the fact that theoretical discoveries are well documented and portfolio management companies regularly incorporate these findings into complex models. This thesis attempts to design and build useful Python-based tools for asset allocation as a result. The tools must be adequate to deliver accurate numerical solutions and simple enough to allow interested practitioners to comprehend the underlying theory. We provide the following response to the question of how to effectively and clearly utilize the concepts of the State-of-The-Art portfolio optimizers and the robust optimization mathematics [6] in real-world asset allocation.

1.1 Structure, Focus, and Methods

In Section 1 of this seminar paper, we lay the groundwork for modeling the securities market. Section 2 discusses Markowitz's mean-variance optimization [1], which helped to transform portfolio management from an art to a science. The important finding is that by combining assets with various projected returns and volatility, a mathematically optimal allocation may be determined [7]. In the first section, we use analytical solutions whenever possible. We then use the techniques from Section 3 to answer the issues numerically using Python. Mostly because analytical solutions to most nontrivial optimization issues are impossible to find. Furthermore, we may use historical time series to examine how optimized portfolios perform in empirical backtesting. Python provides an excellent framework for optimizing, backtesting, and visualizing our technique and outcomes.

1.2 Developed package and the potential market size

Robo-advisors and portfolio optimization are directly connected as robo-advisors use portfolio optimization techniques to manage their clients' portfolios. Portfolio optimization involves selecting a combination of investments that will maximize returns while minimizing risk based on the individual's financial goals and risk tolerance.

Robo-advisors automate this process using algorithms and data analytics to create and manage portfolios for their clients. They analyze market data and use modern portfolio theory to determine the optimal allocation of assets for each client's portfolio. This helps to ensure that the portfolio is diversified and well-balanced, which minimizes risk and maximizes returns over the long term.

In this way, robo-advisors have become a significant player in the portfolio optimization market, providing low-cost and automated investment management services to a wide range of investors. As a result, they have disrupted the traditional financial services industry and have forced many traditional financial advisors to adapt their business models to compete. From an academic perspective, robo-advisors are considered to be a form of Algorithmic Trading, which uses mathematical algorithms and models to make investment decisions. They are based on the principles of modern portfolio theory, which aims to maximize the expected return of a portfolio while minimizing its risk.

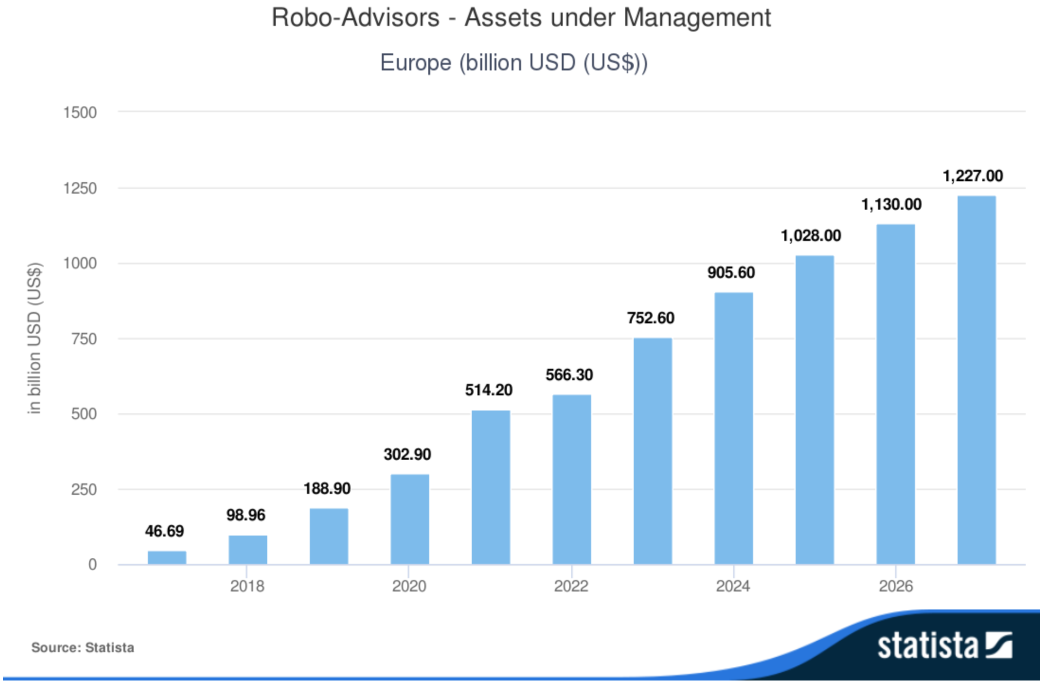

Figure 1: Increasing of Robo Advisors European Market

From a financial perspective, robo-advisors have had a significant impact on the portfolio optimization market. They have disrupted the traditional business model of investment management by offering low-cost, automated, and convenient services to individual investors. This has made professional-level investment advice and portfolio management more accessible and affordable for a wider range of people.

The Stock-Aesthetic Unified Agent (SUA) is a portfolio optimization tool currently in development. SUA has the capability to assess and compare various figures and perform factor analysis based on historical data. The tool utilizes advanced optimizers to allocate the customer's budget such as Effi-

Figure 2: Logo of Stock-aesthetic Unified Agent (SUA)

cient Frontier, Semi Variance, Mean-Variance (MeanVar), Monte Carlo Value at Risk (McVaR), Minimum Variance (MIN Var), and Hierarchical Risk Parity (HRP) [8], mean, and risk moment to manage portfolios. Furthermore, SUA calculates the estimated future returns of each state-of-the-art model by using AutoML, a cutting-edge technology that automates machine learning tasks. The combination of these advanced methods results in a tool that is capable of providing high-quality investment advice to investors and has the potential to significantly impact the portfolio optimization market.

2 Asset Allocations and its Expressions

We consider a financial market where N assets i = 1, 2, ..., N are traded. Typical financial assets such as common stocks, bonds, or domestic and foreign exchanges. Generally, the term "asset" can be associated with any financial instrument that can be bought or sold. In this seminar paper, most of the "assets" will be listed as common stocks by filtered in the following standard:

1. All common stocks were listed in *NASDAQ*, *NYSE*, and *AMEX* exchanges, which located in New York, USA.

2. All common stocks listed in the ticker list

3. Implement the service as a single local client application.

4. Deploy the service and test its operationality (loop back to the previous step if necessary).

5. Carry out a benchmark of attacks on the implementation to assess the security of the service.

6. Use the same benchmark on another service for comparison.

By filtering by upper conditions, we could evaluate the substituent of each market sector.

Market	Market Sector	Amount
NASDAQ	Technology and Telecommunication	69
NASDAQ and NYSE	Energy	37
NASDAQ and NYSE	Consumers (Discrete + Staple)	175
NASDAQ and NYSE	Property	34
NASDAQ and NYSE	Financial	110
NASDAQ and NYSE	Health	80
NASDAQ and NYSE	Industrial	84
NASDAQ and NYSE	Utility	46

Table 1: The constituents by each business sector

2.1 Asset Prices and Returns

Every asset is described by its return as a profit on an investment over a defined period of time, in proportion to the original investment. Let $P_i(t)$ be the Price of an asset i at time t and $P_i(t-1)$ the respective price at time $t-1$. Since stock prices can only take a certain set of values the variable $P_i(t)$ is a positive, discrete random variable. The linear return (r_l) is defined as:

$$r_l = \frac{p_i(t)}{P_i(t-1)} - 1$$

In our developed package, we used the mean historical returns instead of upper mentioned, linear return. Mean historical returns are a measure of the average performance of an asset over a given time period. They are often used as a reference point for investors to gauge the potential returns of an asset. However, it is important to note that past performance does not necessarily indicate future results, and that mean historical returns should be considered within the context of expected returns.

Expected returns are the anticipated profits or losses that an investor expects to realize from an asset. They are based on a range of factors, including the level of risk associated with the asset, the level of supply and demand, economic conditions, and government policies.

The relationship between mean historical returns and expected returns can be described using the following equation:

Expected Return = Mean Historical Return + Risk Premium

Where the risk premium is the additional return that investors expect to receive for taking on additional risk.

It is important for investors to consider both mean historical returns and expected returns when making investment decisions. While mean historical returns can provide a useful reference point, they should not be the sole basis for investment decisions. Instead, investors should carefully consider the full range of expected returns and the associated level of risk. By doing so, they can make more informed decisions and potentially maximize their returns.

Asset prices and returns are central concepts in finance and economics. Asset prices refer to the price at which an asset, such as a stock or a bond, is bought or sold, while returns refer to the profit or loss that an investor realizes from an asset. Understanding the factors that influence asset prices and returns is crucial for investors who wish to make informed decisions and potentially maximize their returns.

One factor that can affect asset prices and returns is the level of supply and demand for the asset. When demand for an asset is high and supply is low, the price of the asset will tend to increase. This can lead to higher returns for investors who own the asset. Conversely, when demand is low and supply is high, the price of the asset will tend to decrease, resulting in lower returns for investors.

Another factor that can influence asset prices and returns is the level of risk associated with the asset. Riskier assets, such as stocks, tend to have higher returns, but also higher price volatility. The relationship between risk and return can be described using the following equation:

Return = Risk Premium + Risk-Free Rate

Where the risk premium is the additional return that investors expect to receive for taking on additional risk, and the risk-free rate is the return that investors would expect to receive on a risk-free asset, such as a U.S. Treasury bond.

It is important for investors to consider all of these factors when making investment decisions. By understanding the factors that influence asset prices and returns, investors can make more informed decisions and potentially maximize their returns.

In current developed package supports the **Mean Historical Return** as the basis return model. For custom settings, we also developed to support the **EMA Historical Return**, **CAPM Return** as the expected return model.

2.1.1 Mean Historical Return

The mean historical return is a statistical measure that represents the average rate of return over a specific period of time for a particular investment or portfolio. It is calculated by summing all the returns over a certain period, such as a year or multiple years, and then dividing that sum by the number of observations (i.e., the number of years or periods in the sample). Mean historical return is used as a benchmark for evaluating the performance of an investment or portfolio, and is often used in conjunction with other measures, such as standard deviation and risk-adjusted returns, to provide a more comprehensive assessment of performance. The mathematical expression for mean historical return is as follows:

$$\text{Mean Historical Return} = \frac{\sum Return for each period}{Number of periods}$$

- Return for each period is the rate of return for a specific period, such as a year or a quarter.
- Sigma represents the sum of all the returns for each period in the sample.
- Number of periods is the total number of periods in the sample

For example, if a portfolio has returns of 10 % in year 1, 5 % in year 2, and -3 % in year 3, the mean historical return would be calculated as:

Mean Historical Return = (10 % + 5% - 3%)/ 3 = 4%

It's worth to note that the sample period is important to calculate the mean historical return. Also, it's important to note that it is an average and not a guarantee of future performance.

2.1.2 EMA Historical Return

Exponential moving average (EMA) historical return is a statistical measure that is used to smooth out fluctuations in an investment or portfolio's historical returns over time. It is similar to the mean historical return, but it gives more weight to recent returns and less weight to older returns. This can help to better capture the recent trend in the investment's or portfolio's performance.

The mathematical expression for EMA historical return is as follows:
n

$$EMA = Price(t) * k + EMA(y) * (1 - k)$$

where:

- t=today
- y=yesterday
- N=number of days in EMA
- k=2÷(N+1)

It's worth noting that the EMA is a recursive calculation and it's more complex than a simple average of returns, it is also sensitive to the weighting factor, which will affect the final result. Also, it's important to note that it is an average and not a guarantee of future performance. EMA is used in technical analysis and it is used to identify trends in the market.

2.1.3 CAPM Return

The Capital Asset Pricing Model (CAPM) is a financial model that aims to explain the relationship between risk and expected returns in a portfolio of assets. It is based on the principle that investors demand a higher expected return for investing in assets with higher risk levels. CAPM relies on historical returns to estimate the expected returns of assets, which is a key input for portfolio management and investment decision-making.

The CAPM formula for estimating expected returns is expressed as follows:

$$R_i = R_f + beta_i * (E(R_m) - R_f)$$

where:

- R_i = Expected Return for asset i via CAPM returns
- R_f = Risk-free rate of borrowing/lending, which defaults to 0.02 in the actual package.
- $beta_i$ = The asset's beta (or systematic risk)
- $E(R_m)$ = The expected return of the market

The formula essentially states that the expected return of an asset is equal to the risk-free rate plus the beta of the asset multiplied by the expected excess return of the market.

To calculate the beta of an asset, historical returns are used to estimate the asset's sensitivity to changes in the market. Beta is defined as the covariance of the asset's returns with the returns of the market, divided by the variance of the market returns. By using historical returns to estimate the beta of an asset, CAPM provides a framework for investors to evaluate the expected returns of assets based on their risk levels and the overall market conditions. While there are limitations to the CAPM model, historical returns play a crucial role in estimating expected returns and are an important consideration for investors seeking to make informed investment decisions.

2.2 Risk models

To perform mean-variance optimization, a risk model is necessary in addition to expected returns. This model quantifies asset risk and the most commonly used one is the covariance matrix, which captures both asset volatilities and co-dependence. Diversification is a core principle that can help mitigate risk, as it involves making many uncorrelated investments, with the correlation being a normalized covariance.

In practice, the subject of risk models may be more critical than that of expected returns because historical variance is often a more stable metric than historical returns. Research by Kritzman et al. (2010) [9] has shown that minimum variance portfolios that optimize without including expected returns can perform better out of the sample.

However, in practice, we do not have direct access to the covariance matrix, just as we do not have access to expected returns. Therefore, estimates based on past data are necessary. While the most common approach is to calculate the sample covariance matrix based on historical returns, recent research suggests that there are more robust statistical estimators for the covariance matrix.

So we decided to use Sample Covariance as the original approach to check whether the developed package operated and adapted advanced risk models such as Shrinkage estimators and the normal Covariance method in the case of the CVaR optimization.

2.2.1 Sample Covariance

The conventional approach in estimating the covariance matrix for mean-variance optimization uses the sample covariance matrix. However, this approach has limitations, such as misspecification errors and a lack of robustness. This poses a significant problem in the optimization process, as it may give undue weight to erroneous values, resulting in suboptimal portfolios.

Sample Covariance

$$Cov(x,y) = \frac{\Sigma(x_i-\bar{x})(y_i-y)}{N-1}$$

Figure 3: Sample Covariance Formula

2.2.2 Shrinkage Estimators

Researchers have developed alternative methods to overcome these limitations, such as shrinkage estimators [10]. These estimators improve the accuracy and robustness of covariance matrix estimates by combining the sample covariance matrix with a structured estimator. The structured estimator can be based on prior information, such as expert knowledge, or on assumptions about the structure of the covariance matrix, such as sparsity or factor models. Shrinkage estimators have been shown to outperform the sample covariance method in terms of accuracy, robustness, and out-of-sample performance. Therefore, they are increasingly used in mean-variance optimization and other portfolio management applications.

The essential idea is that the unbiased but often poorly estimated sample covariance can be combined with a structured estimator F, using the below formula (where $\delta$is the shrinkage constant):

$\widehat{\sum} = \delta F + (1 - \delta) S$

The term "shrinkage" in the context of covariance matrix estimation refers to reducing the variance of the sample covariance matrix by adjusting it towards another estimator, known as the shrinkage target. Although the target estimator may be substantially biased, it typically has a lower estimation error. Various alternatives for the shrinkage target exist, and the optimal value of the shrinkage constant δmay vary depending on the chosen target.

3 Portfolio Weighting and Optimizations

In portfolio theory, a portfolio refers to a distribution of an initial capital amount across various assets, which may or may not be of different asset classes. Risky assets and risk-free assets are distinguished, although all assets are considered to be associated with some degree of risk in this thesis unless otherwise indicated. Additionally, all portfolios are considered self-financing, meaning no additional funds are added or withdrawn over the observation period. Finally, any new asset acquisition must be funded by selling an existing asset.

3.1 Weighting

In portfolio optimization, "weight" refers to the proportion of an investor's portfolio allocated to each asset or security. The weight of each asset is determined by the investor's investment objectives, risk tolerance, and market outlook. Portfolio weights play a critical role in determining the performance of a portfolio, as they determine the level of exposure to individual assets or sectors. The optimal weight of each asset in a portfolio is a critical factor in determining portfolio performance. There are several methods that can be used to determine the optimal weight of each asset, and the choice of method depends on the investor's investment objectives, risk tolerance, and market outlook. Ultimately, the key to successful portfolio optimization is choosing a method consistent with the investor's goals and objectives and regularly monitoring and adjusting the portfolio weights as market conditions change.

In the developed package, we evaluated weights with **Max Sharpe Ratios, Minimizing Volatility, and Minimizing Conditional Value-at-Risk (CVaR)**.

3.1.1 Maximizing Sharpe Ratio

The Max Sharpe Ratio is a widely used method in portfolio optimization to select a combination of assets that offers the highest expected return for a given level of risk. Mathematically, the Sharpe Ratio is defined as the ratio of the expected excess return of the portfolio over the risk-free rate to the standard deviation of the portfolio's excess returns. Let r_p be the return of the portfolio, r_f be the risk-free rate, and σ_p be the standard deviation of the portfolio's excess returns, then the Sharpe Ratio (S) can be written as:

$$S = \frac{E(r_p - r_f)}{\sigma_p}$$

However, in practice, the standard Sharpe Ratio may not provide an accurate estimate of the true risk and return characteristics of the portfolio. For example, asset returns may exhibit non-normal distributions, and the covariance matrix may change over time due to market conditions. Therefore, an adapted version of the Max Sharpe Ratio method is necessary. One such method is the Black-Litterman model, which incorporates investors' views about expected returns and covariance matrices into the portfolio optimization process. The model uses a Bayesian approach to estimate the expected returns and covariance matrices of the assets in the portfolio. Let w be the vector of portfolio weights, E be the expected return vector, V be the covariance matrix of the asset returns, τ be a scalar representing the uncertainty of the prior views, P be the matrix of investor views, and Q be the vector of investor views. Then the Black-Litterman portfolio weights (w_{BL}) can be calculated as:

$$w_{BL} = \frac{1}{\lambda}\Sigma^{-1}(E - \lambda V w_{eq})$$

where λ is a scalar representing the degree of confidence in the investor views, $\Sigma = (1/\tau)V + P^T TP$ is the posterior covariance matrix, w_{eq} is the vector of equilibrium weights based on some benchmark, and $T = \tau V$ is the scaling matrix that reflects the degree of uncertainty in the prior views. The Black-Litterman model is a powerful tool for incorporating investor views into the portfolio optimization process and can be used to generate portfolio weights that better reflect the true risk and return characteristics of the assets.

3.2 General Robust Optimization

Portfolio optimizers such as efficient frontier[11] and CVaR are methods used in finance to optimize the allocation of investments in a portfolio. The efficient frontier is a technique that helps investors determine the optimal portfolio based on expected returns and risks. It calculates the boundary of the set of efficient portfolios that maximize returns for a given level of risk. In contrast, CVaR (Conditional Value-at-Risk) is a measure of risk that estimates the expected loss beyond a certain threshold. It helps investors optimize their portfolio by minimizing the risk beyond a specific threshold while achieving the desired level of returns.

In the developed version of SUA, various efficient frontier models (including Mean-Variance Optimization, CVaR, and Semi-Variance Optimization) and Hierarchical Risk Parity (HRP).

3.3 Mean Variance Optimization

Mathematical optimization can be challenging, especially when dealing with complex objectives and constraints. Nonetheless, convex optimization problems represent a well-defined class of problems highly applicable in finance. The Efficient Frontier object (inheriting from Base Convex Optimizer) contains multiple optimization methods that can be called (corresponding to different objective functions) with various parameters. A convex problem is characterized by a specific structure, as follows in Boyd (2004) [12]:

$$
\begin{aligned}
\min_{x} \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0, i = 1, ..., m \\
& Ax = b, \\
x \in \mathbb{R}^n, &
\end{aligned} \tag{1}
$$

$$\text{and } f(x), g_i(x) = \text{convex function}$$

The good news is that standard portfolio optimization problems exhibit convexity, which facilitates the application of a broad range of theoretical principles and advanced solution methods. The primary challenge lies in providing appropriate inputs for a solver. One of the element packages (PyPortfolioOpt) can simplify this process by offering efficient one-liner functions to generate portfolios that minimize volatility and enable the construction of more complex problems from modular units. Such functionality is made possible by leveraging the powerful cvxpy, a Python-based modeling language for convex optimization that underpins PyPortfolioOpt's efficient frontier features.

As per the definition of a convex problem, two primary specifications are required: the optimization objective and the optimization constraints. A classic illustration is the portfolio optimization problem, which aims to minimize risk while satisfying a return constraint (i.e., the portfolio must achieve a minimum return). In terms of implementation, the distinction between an objective and a constraint is not significant. For instance, consider a related problem of maximizing returns under a risk constraint, where the roles of risk and returns are interchanged.

In the User-Interfaced version, we adapted the max sharping method for weighting portfolio.

3.4 Efficient Frontier Adapted Optimization

As we mentioned in section 3.2, we adapted various optimizations based on the Efficient Frontier method. The methods for mean-variance optimization explained earlier are applicable when you possess an expected returns vector and a covariance matrix. The aim and restrictions blend the portfolio's return and volatility.

But suppose you aim to create an efficient frontier for a risk model that is not contingent on covariance matrices or optimize an objective unrelated to portfolio returns (such as tracking error). In that case, PyPortfolioOpt has multiple popular alternatives available and offers assistance for custom optimization problems.

3.4.1 Semivariance Optimization

In traditional mean-variance optimization, volatility is penalized as a measure of risk. However, mean-semivariance optimization only penalizes downside volatility since upside volatility can be desirable. This approach is useful when investors are more concerned about avoiding losses rather than maximizing gains.[13]

There are two methods to tackle the mean-semivariance optimization problem. The first approach is a heuristic solution that involves treating the semi covariance matrix (implemented in risk models) as a typical covariance matrix and performing standard mean-variance optimization. However, this method does not result in an efficient portfolio in the mean-semivariance space, though it can be a good enough approximation.

The second approach involves writing mean-semivariance optimization as a convex problem, which can be solved exactly (with many variables). For instance, to maximize the return for a long-only target semivariance, the following problem needs to be solved. Efficient Frontier, a graph with portfolios on the x-axis and expected returns on the y-axis is a significant concept in modern portfolio theory. The portfolios on the frontier are efficient since they offer the highest expected return for a given level of risk or the lowest risk for a given level of return. The mean-semivariance optimization approach can also be used to construct the efficient frontier.

Thankfully, the optimization of mean-semivariance can be expressed as a convex problem, although it may have numerous variables. This can provide an accurate solution. To illustrate, if the aim is to maximize the return for a long-only target semivariance s^* , the subsequent problem would need to be resolved [14]:

$$
\begin{aligned}
\max_{x} \quad & w^T \mu \\
\text{s.t.} \quad & n^T n \leq s^* \\
& Bw - p + n = 0, \\
& w^T 1 = 1, \\
& n > 0, \\
& p \geq 0
\end{aligned} \tag{2}
$$

B is the T×N (scaled) matrix of excess returns:
$B = \frac{(returns - benchmark)}{\sqrt{T}}$

3.4.2 Conditional Value-at-Risk Optimization

The conditional value-at-risk (CVaR), or expected shortfall, is a frequently used metric for measuring tail risk. This risk measure can be considered as the average loss incurred during "extremely unfavorable" scenarios, with the degree of extremity being determined by the parameter β. In essence, CVaR estimates the expected loss beyond a certain threshold, Beta, in the event of an adverse occurrence, thereby providing insight into the severity of losses that can be expected.

Although the concept of conditional value-at-risk (CVaR) is relatively straightforward, it necessitates using numerous mathematical notations to express it formally. This risk measure involves computing the average of losses that exceed a specified threshold level, which can be expressed as a mathematical function of the underlying probability distribution. The complexity of the notation arises from the necessity to capture the nonlinearity of the loss function and the probability density of potential outcomes. As such, the formal representation of CVaR requires specialized mathematical terminology and concepts. We will adopt the following notation:

- w for the vector of portfolio weights
- r for a vector of asset returns (daily), with probability distribution $p(r)$.
- $L(w, r) = -w^T r$ for the loss of the portfolio
- α for the portfolio value-at-risk (VaR) with confidence β.

More formally, the CVaR at level β for a random variable X is given by:

$$\text{CVaR}(w, \beta) = \frac{1}{1-\beta} \int_{L(w,r) \geq \alpha(w)} \text{L}(w, r) p(r) dr$$

This is the complicated mathematical expression to optimize because we are essentially integrating over VaR values. The key insight of Rockafellar and Uryasev (2001) [15] is that we can can equivalently optimize the following convex function:

$$F_\beta(w,\alpha) = \alpha + \tfrac{1}{1-\beta}\int[-w^T r - \alpha]^+ \mathrm{p}(r)dr,$$
$$\text{where } [x]^+ = max(x,0)$$

Rockafellar and Uryasev (2001) [15] prove that minimising $F_\beta(w,\alpha)$ over w,α minimises the CVaR. Assume that we are given a set of T daily returns, which can either be generated through simulation or obtained from historical data. In such a scenario, the integral present in the expression can be converted into a summation. The CVaR optimization problem reduces to a linear program by Quaranta (2008)[16]:

$$\begin{aligned} \min_{w,\alpha} \quad & \alpha + \frac{1}{1-\beta}\frac{1}{T}\sum_{i=1}^{T} u_i \\ \text{s.t.} \quad & u_i \geq 0 \\ & u_i \geq -w^T r, \alpha \end{aligned} \tag{3}$$

The aforementioned formulation incorporates a new variable for each data point, similar to the Efficient Semivariance approach. However, this approach may lead to performance issues when dealing with extended returns dataframes. Nonetheless, it is crucial to ensure that the dataset used for optimization includes sufficient samples to cover rare tail events.

3.4.3 Conditional Drawdown-at-Risk Optimization

The Conditional Drawdown at Risk (CDaR) is a less commonly used metric for measuring tail risk. It addresses some of the limitations of other measures such as Efficient Semivariance and Efficient CVaR by considering the duration of significant losses in value. The CDaR can be understood as the average of losses that occur during extended periods of poor performance, with the severity level quantified by the parameter β. The drawdown is the reduction in non-compounded returns from the previous peak. Put differently; the CDaR is the average of all drawdowns so severe that they only occur $(1-\beta)\%$ of the time. When β=1, CDaR is simply the maximum drawdown. Although the concept of drawdown is relatively straightforward, it requires several new mathematical notations to express it formally. We will adopt the following notation:

- w for the vector of portfolio weights
- r for a vector of asset returns (daily), with probability distribution $p(r(t))$.
- $D(w, r, t) = -\max_{\tau<t}(w^T r(\tau)) - w^T r(t)$ for the drawdown of the portfolio
- α for the portfolio drawdown (DaR) with confidence β.

More formally, the CDaR can be given by:

$$\text{CDaR}(w, \beta) = \frac{1}{1-\beta} \int_{D(w,r,t) \geq \alpha(w)} \text{D}(w, r, t) p(r(t)) dr(t)$$

This is a complex expression to optimise because we are essentially integrating over VaR values. The key insight of Chekhlov, Zabarankin and Uryasev (2005) [17] is that we can equivalently optimize a convex function, which can be transformed to a linear problem (in the same manner as for CVaR).

3.5 Hierarchical Risk Parity

Hierarchical Risk Parity (HRP) is a risk parity approach developed by Marcos Lopez de Prado [18] that addresses the limitations of traditional risk parity methods. HRP uses a hierarchical clustering algorithm to group assets into clusters based on their pairwise correlation. The algorithm then constructs a tree-like structure to represent the relationship between the clusters. The resulting structure forms the basis for building the HRP portfolio.

The Hierarchical Risk Parity (HRP) optimizer involves several mathematical expressions to determine the optimal portfolio allocation.

First, a distance matrix is constructed from the correlation matrix of asset returns. The distance matrix creates a hierarchical tree of assets, where assets with similar characteristics are clustered together.

Next, the variance of each cluster is calculated, and a minimum variance portfolio is formed for each cluster. The minimum variance portfolio has the lowest variance within each cluster.

The optimization process then combines these minimum variance portfolios to form the optimal portfolio. This is done by iteratively combining the minimum variance portfolios at each node in the hierarchical tree.

The final portfolio weights are determined using the following mathematical expression:

$$w = D * H^{-1} * \sqrt{v}$$

where w is the vector of portfolio weights, D is a diagonal matrix containing the inverse of the standard deviation of the minimum variance portfolios, H is the hierarchical clustering matrix, and v is the vector of variances of the minimum variance portfolios.

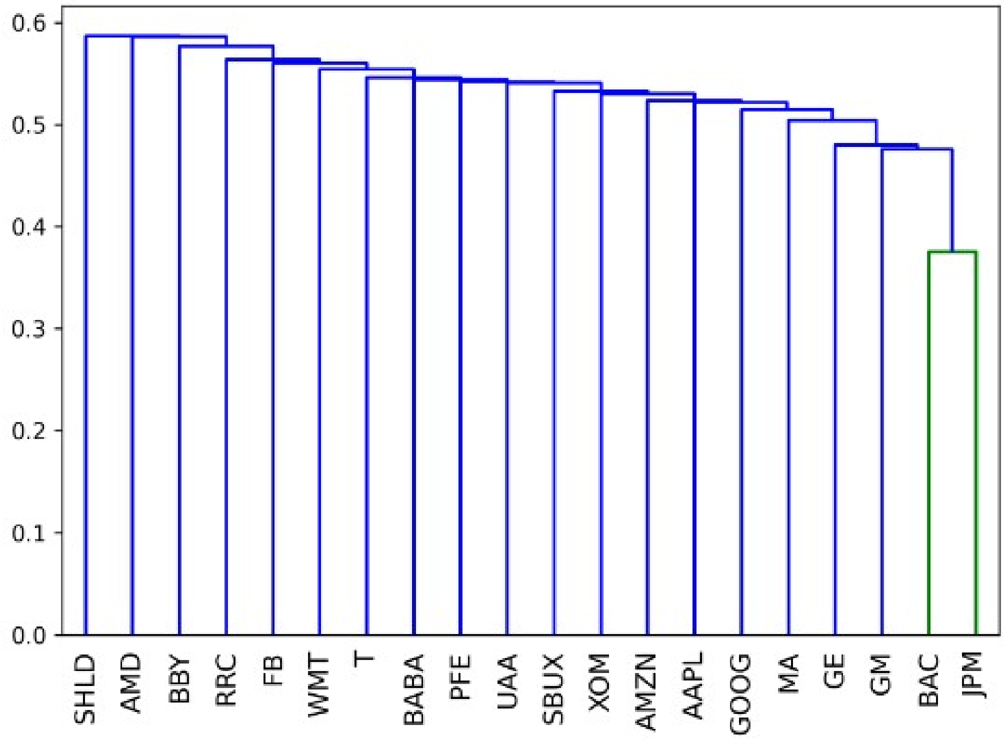

Figure 4: Example of HRP optimizer

The resulting portfolio allocation will be diversified across clusters, with each cluster contributing equally to the overall portfolio's risk. This approach has been shown to produce portfolios with superior risk-adjusted returns compared to traditional mean-variance optimization techniques.

4 Allocation by weights and Generating in reports

In this phase, since the portfolio was formed via the optimization stage, the allocator automatically distributes the recommended amount of stocks based on the calculated weights.

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{
  "ABEV" : 0
  "ACI" : 0.46741
  "BUD" : 0
  "CAG" : 0
  "CPB" : 0
  "DEO" : 0
  "FMX" : 0
  "GIS" : 0
  "HRL" : 0
  "HSY" : 0.31214
  "K" : 0
  "KO" : 0
  "KOF" : 0
  "KR" : 0
  "LW" : 0.05874
  "MKC" : 0
  "SJM" : 0
  "SONY" : 0.16171
  "STZ" : 0
  "TAP" : 0
  "TSN" : 0
  "UL" : 0
}
```


Figure 5: Example of amount allocation from NYSE and NASDAQ Consumer Stocks

After the allocation, we can download as a report format. The report includes Annual return, Cumulative return, Annual volatility, Winning day ratio, Sharpe ratio, Calmar ratio, Information ratio, Stability, Max drawdown, Sortino ratio, Skewness, Kurtosis, Tail ratio, Common sense ratio, Daily value at risk, Alpha score and Beta score. Also, we could see the various visualizations such as returns time series graph, EOY (End of Year) Return vs Benchmark (S&P500), Cumulative returns vs Benchmark, Monthly returns from the set date range, Underwater plot, Worst 5 drawdown plot, Rolling Volatility & Sharpe ratio & beta to benchmark.

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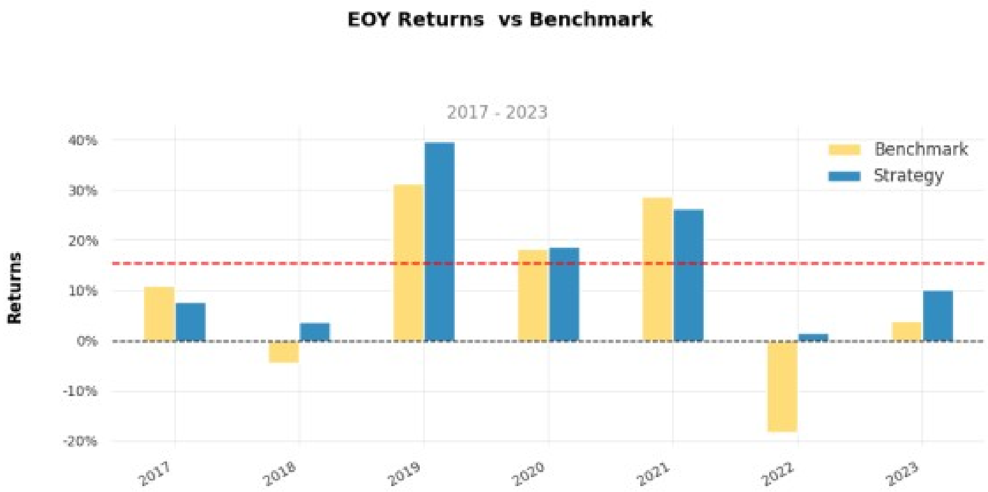

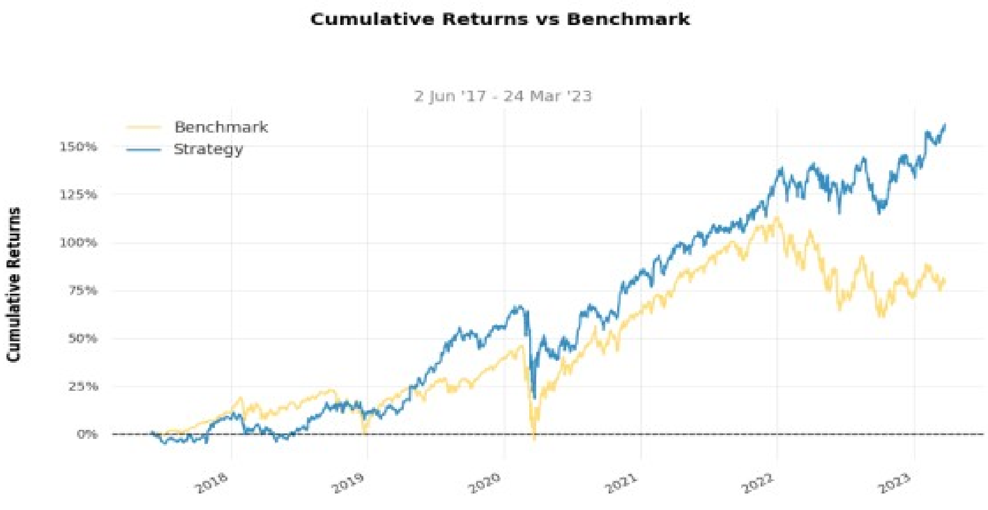

Figure 5: overview of the report from NYSE and NASDAQ Consumer Stocks

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